Local monodromy of A-motives

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Local Galois groups

K a local field of residual characteristic p, $\quad G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$

inertia

$\downarrow$

$$G \quad \triangleright \quad I \quad \triangleright \quad P$$

$\uparrow$

wild inertia

$$\begin{aligned} G/I &= \widehat{\mathbb{Z}} \\ I/P &= \widehat{\mathbb{Z}}^{(p)}(1) \\ P &= \text{free pro-}p\text{-group on } \aleph_0 \text{ generators} \end{aligned}$$

pro-nilpotent $= \mathrm{pro}_n\{\, [g_1, [g_2, [\ldots, g_n] \cdots] = 1 \,\}$

ℓ-adic local monodromy $\ell \neq p$

$$X \to \operatorname{Spec} K \quad \rightsquigarrow \quad H^i(X_\eta, \mathbb{Z}_\ell) \circlearrowleft G$$

"punctured disk" $\to \operatorname{Spec} K$

$\eta\colon \operatorname{Spec} K^{\mathrm{sep}} \hookrightarrow \operatorname{Spec} K$

ℓ-adic monodromy theorem (Grothendieck)

Up to a finite separable extension L/K each ℓ-adic representation $\rho\colon G \to \mathrm{GL}(V)$ satisfies:

1. $\rho(P) = \{1\}$,
2. $\rho|_I$ is unipotent.

i. *open* ℓ-Sylow subgroup $\subset \mathrm{GL}_n(\mathbb{Z}_\ell)$

ii. $\rho(P) = \{1\}, \quad \rho(I/P) = \rho(\mathbb{Z}_\ell(1))$

iii. Grothendieck's trick: $\rho|_{\mathbb{Z}_\ell(1)}$ unipotent via $\widehat{\mathbb{Z}} \ltimes \mathbb{Z}_\ell(1)$

Weil–Deligne representations

$\mathfrak{g} = \operatorname{Lie} \mathbb{Z}_\ell(1), \quad N\colon \mathfrak{g} \to \mathfrak{gl}(V)$ independent of L/K

Deligne's construction WD: $\rho \mapsto (\widetilde{\rho}, N)$

The representation ρ is

- twisted by $\exp \circ (-N) \circ \pi$,
- restricted to the *Weil group* $W \subset G$.

$\pi\colon G \twoheadrightarrow \mathbb{Z}_\ell(1) \hookrightarrow \mathfrak{g}$.

Theorem (Deligne)

The functor WD is an equivalence of categories of

- ℓ-adic Galois representations,
- ℓ-adic Weil–Deligne representations "of slope 0".

The ℓ-independence conjecture

$\mathrm{WD}(\rho)$ is continuous in the discrete topology on $\mathbb{Q}_\ell$
$\rightsquigarrow$ can replace $\mathbb{Q}_\ell$ by any field of characteristic 0.

Primes $\ell, \ell' \neq p$, field $F \supset \mathbb{Q}_\ell, \mathbb{Q}_{\ell'}$.

Conjecture

For each smooth proper $X / \operatorname{Spec} K$, each $i \geqslant 0$ there is a natural isomorphism

$$F \otimes_{\mathbb{Q}_\ell} \mathrm{WD}\left(H^i(X_\eta, \mathbb{Q}_\ell)\right) \xrightarrow{\sim} F \otimes_{\mathbb{Q}_{\ell'}} \mathrm{WD}\left(H^i(X_\eta, \mathbb{Q}_{\ell'})\right).$$

Some known cases: abelian varieties, K3 surfaces.
Trivial case (*assuming* semi-simplicity): X has good reduction.

Drinfeld modules

$$\begin{array}{c} E \\ \downarrow \\ \operatorname{Spec} K \end{array} \quad \rightsquigarrow \quad T_{\mathfrak{p}} E \circlearrowleft G$$

$\mathfrak{p}$ different from the residual characteristic: "$\ell \neq p$".

$$|\rho(P)| < \infty \Leftrightarrow E \text{ has potential good reduction}$$

1. $|\rho(P)| < \infty$
2. $\mathrm{GL}_n(\mathbb{F}_q[[z]])$ has an open p-Sylow $\Rightarrow |\rho(I)| < \infty$
3. Takahashi: E has potential good reduction

z-adic Galois representations I

Idea: find an "ℓ-like" class of z-adic representations

Ground field $\mathbb{F}_q$, *coefficient field* $\widehat{F}$, ring of integers $\mathcal{O}_{\widehat{F}}$, e.g. $\widehat{F} = \mathbb{F}_q((z))$, $\mathcal{O}_{\widehat{F}} = \mathbb{F}_q[[z]]$

Ring of definition

$$\mathit{\Gamma} = \mathit{\Gamma}_K = K \mathbin{\widehat{\otimes}_{\mathbb{F}_q}} \mathcal{O}_{\widehat{F}}$$

Discrete topology on K, e.g. $\mathit{\Gamma} = K[[z]]$.
Partial Frobenius $\sigma \colon \mathit{\Gamma} \to \mathit{\Gamma}$, e.g. $\sigma(\sum a_n z^n) = \sum a_n^q z^n$.

Definition

A *unit-root $\mathit{\Gamma}$-crystal M* is a pair consisting of

- a finitely generated free $\mathit{\Gamma}$-module M,
- an isomorphism $a \colon \sigma^* M \xrightarrow{\sim} M$, called the *structure isomorphism*.

Morphisms = σ-equivariant morphisms of $\mathit{\Gamma}$-modules.

z-adic Galois representations II

$$T(M) = \{ x \in \Gamma_{K^{\mathrm{sep}}} \otimes_{\Gamma_K} M \mid a_M(1 \otimes x) = x \}$$

Theorem (Katz)

The functor $M \mapsto T(M)$ is an equivalence of categories of

- unit-root Γ_K-crystals,
- continuous representations of G in finite free $\mathcal{O}_{\widehat{F}}$-modules.

$$\Gamma_+ = \mathcal{O}_K \,\widehat{\otimes}_{\mathbb{F}_q}\, \mathcal{O}_{\widehat{F}}$$

Theorem (folklore)

The functor $M \mapsto T(M)$ is an equivalence of categories of

- unit-root Γ_+-crystals,
- *unramified* representations of G in finite free $\mathcal{O}_{\widehat{F}}$-modules.

z-adic Galois representations III

$$\Gamma^b = K \otimes_{\mathcal{O}_K} \Gamma_+$$

Theorem (M.)

The base change functor $\Gamma^b \mapsto \Gamma$ is fully faithful.

A-motive M over $A_K = K \otimes_{\mathbb{F}_q} A \rightsquigarrow$ for all $\mathfrak{p} \neq$ char. a unit-root Γ-crystal

$$M_{\mathfrak{p}} = \Gamma_{K,F_{\mathfrak{p}}} \otimes_{A_K} M$$

$T(M_{\mathfrak{p}}) = \mathfrak{p}$-adic Tate module of M

Proposition (M.)

$M_{\mathfrak{p}}$ is defined over Γ^b for each $\mathfrak{p} \neq$ res. char.

z-adic local monodromy

Extra structure on the inertia group I: *upper index ramification filtration*

$$I^{(\nu)}, \quad \nu \in \mathbb{Q}_{\geqslant 0}$$

$I = I^{(0)}$, $P = I^{(0+)} = \text{closure}\left(\bigcup_{\nu>0} I^{(\nu)}\right)$

z-adic monodromy theorem, case "$\ell \neq p$" (M.)

The functor $M \mapsto T(M)$ is an equivalence of categories of

- unit-root $\mathit{\Gamma}^b$-crystals, and
- Galois representations $\rho\colon G \to \mathrm{GL}(V)$ which satisfy the following up to a finite separable extension L/K:
  1. $\rho(I^{(\nu)}) = \{1\}$ for $\nu \gg 0$,
  2. $\rho|_I$ is unipotent.

Applications I

Theorem (M.)

For each A-motive M, each $\mathfrak{p} \neq$ res. char. there is a finite separable extension L/K such that

$$I \text{ acts unipotently on } T_{\mathfrak{p}}M$$

Theorem (M.)

For every A-motive M there is a number $\nu \geqslant 0$ such that

$$I^{(\nu)} \text{ acts trivially on all } T_{\mathfrak{p}}M, \quad \mathfrak{p} \neq \text{res. char.}$$

Applications II

Corollary

For each Drinfeld A-module E there is a minimal number $\nu \geqslant 0$ such that $I^{(\nu)}$ acts trivially on all $T_{\mathfrak{p}}E$, $\mathfrak{p} \neq$ res. char. Furthermore ν is an integer whenever E has stable reduction.

Object	Monodromy $\rho(I)$
Algebraic variety	virtually cyclic
Drinfeld module	virtually abelian
A-motive	virtually nilpotent

z-adic de Rham representations

$$\Gamma^m = (\Gamma_+)_{(z)}$$

Unit-root Γ^m-crystals generalize local shtukas *and* unit-root Γ^b-crystals.

theorem (M.)

The base change $\Gamma^m \to \Gamma$ is fully faithful.

Definition

A z-adic representation is *de Rham* if it arises from Γ^m.

theorem (M.)

Every unit-root Γ^m-crystal becomes an iterated extension of local shtukas after a finite separable extension L/K.

z-adic de Rham representations are potentially semistable