$\mathrm{H}^1(X, \mathbb{G}_m)$ and the Kummer sequence

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Let X be a scheme.

1 $\mathrm{H}^1(X, \mathbb{G}_m)$

Definition 1. *Let $\mathcal{C}$ be a site with final object X and let $\mathcal{G}$ be an abelian sheaf on $\mathcal{C}$. A* $\mathcal{G}$-torsor *is a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$, compatible with restriction, such that*

1. *For each $U \in \mathrm{Ob}(\mathcal{C})$ such that $\mathcal{F}(U) \neq \varnothing$, the action of $\mathcal{G}(U)$ on $\mathcal{F}(U)$ is free and transitive.*
2. *For each $U \in \mathrm{Ob}(\mathcal{C})$, there exists a covering $\{U_i \to U\}_{i \in I}$ such that $\mathcal{F}(U_i) \neq \varnothing$ for all $i \in I$.*

A morphism of $\mathcal{G}$-torsors *is a $\mathcal{G}$-equivariant morphism of sheaves of sets.*

Proposition 2. *Every morphism of $\mathcal{G}$-torsors is an isomorphism, i.e. the category of $\mathcal{G}$-torsors is a groupoid.*

Proof. Let $f : \mathcal{F} \to \mathcal{F}'$ be a morphism of $\mathcal{G}$-torsors and let $\{U_i \to X\}_{i \in I}$ be an covering of X such that $\mathcal{F}(U_i) \neq \varnothing \neq \mathcal{F}'(U_i)$ for all $i \in I$. Let $V_i \to U_i$ be a morphism in $\mathcal{C}$, then $\mathcal{F}(V_i) \neq \varnothing \neq \mathcal{F}'(V_i)$ and $f(V_i) : \mathcal{F}(V_i) \to \mathcal{F}(V_i)$ is bijective as $f(V_i)$ is $\mathcal{G}(V_i)$-invariant and $\mathcal{G}(V_i)$ acts freely and transitively. Hence $f|_{U_i} : \mathcal{F}|_{U_i} \to \mathcal{F}'|_{U_i}$ is an isomorphism and it follows that f is an isomorphism. □

Definition 3. *A torsor is* trivial *if it is isomorphic to $\mathcal{G}$ acting on itself by left-translation.*

Proposition 4. *A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\Gamma(X, \mathcal{F}) \neq \varnothing$.*

Proof. If $\mathcal{F}$ is trivial, then $0 \in \mathcal{F}(X)$. Conversely, suppose that $\mathcal{F}(X) \neq \varnothing$ and let $x \in \mathcal{F}(X)$. We define a morphism of $\mathcal{G}$-torsors

$$\begin{aligned} \mathcal{G}(U) &\to \mathcal{F}(U) \\ g &\mapsto gx|_U. \end{aligned}$$

Since it is an isomorphism, we are done. □

Remark We want to prove that there is a canonical bijection $\mathrm{H}^1(X, \mathbb{G}_m) \cong \mathrm{Pic}(X)$. We do this in two steps. First, we show that there is a canonical bijection between isomorphism classes of $\mathcal{G}$-torsors and $\mathrm{H}^1(X, \mathcal{G})$ and second we show that there is a canonical bijection between isomorphism classes of $\mathbb{G}_m$-torsors and isomorphism classes of invertible sheaves on X_{zar}.

Proposition 5. *Let $\mathcal{G}$ be an abelian sheaf on $X_{\text{ét}}$. There is a canonical bijection between the set of isomorphism classes of $\mathcal{G}$-torsors and* $\mathrm{H}^1(X, \mathcal{G})$.

Proof. We construct inverse maps. Let $\mathcal{F}$ be a $\mathcal{G}$-torsor. Let

$$\mathbb{Z}[\mathcal{F}] := \left(U \mapsto \left\{ {\sum}' n_i [s_i] \;\middle|\; n_i \in \mathbb{Z}, s_i \in \mathcal{F}(U) \right\} \right)^{\#}.$$

denote the free abelian sheaf over $\mathcal{F}$. Let $\sigma : \mathbb{Z}[\mathcal{F}] \to \underline{\mathbb{Z}}$ be the sheafification of ${\sum}' n_i[s_i] \mapsto {\sum}' n_i$. This is a surjective morphism. We have

$$\ker \sigma = \left(U \mapsto \langle [s] - [s'] \mid s, s' \in \mathcal{F}(U) \rangle \right)^{\#}.$$

Let $a : \ker(\sigma) \to \mathcal{G}$ be the sheafification of

$$[s] - [s'] \mapsto h \text{ s.t. } hs' = s.$$

Because the abelian sheaves on $X_{\text{ét}}$ form an abelian category there is the following pushout diagram with exact rows (Stacks Project [Sta18] tags 08N3 and 08N4 for abelian categories)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\sigma) & \longrightarrow & \mathbb{Z}[\mathcal{F}] & \xrightarrow{\sigma} & \underline{\mathbb{Z}} & \longrightarrow & 0 . \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle d} & & \| & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{E} & \xrightarrow{b} & \underline{\mathbb{Z}} & \longrightarrow & 0 \end{array}$$

The long exact sequence in cohomology yields a boundary morphism $\delta : \mathbb{Z} = \mathrm{H}^0(X, \underline{\mathbb{Z}}) \to \mathrm{H}^1(X, \mathcal{G})$. Let $z_{\mathcal{F}} := \delta(1)$.

Claim: Let $\mathcal{F}'$ be the subsheaf of sets of $\mathcal{E}$ defined by

$$U \mapsto b|_U^{-1}(1).$$

Then $\mathcal{F} \cong \mathcal{F}'$.

Proof. We identify $\mathcal{F}$ with the subsheaf of sets $U \mapsto \{[s] \in \mathbb{Z}[\mathcal{F}](U)\}$ of $\mathbb{Z}[\mathcal{F}]$. Its $\mathcal{G}$-action is given by $g[s] := [gs]$. Then $d(\mathcal{F})$ is a subsheaf of $\mathcal{F}'$ and $d|_{\mathcal{F}}$ is $\mathcal{G}$-equivariant. Hence we obtain a morphism, and hence an isomorphism, of $\mathcal{G}$-torsors $\mathcal{F} \xrightarrow{\sim} \mathcal{F}'$. □

In the following, we identify $\mathcal{F}$ with $\mathcal{F}'$.

Conversely, given $z \in \mathrm{H}^1(X, \mathcal{G})$, we construct a $\mathcal{G}$-torsor as follows: choose an embedding $\mathcal{G} \hookrightarrow \mathcal{I}$ of $\mathcal{G}$ into an injective sheaf and let $\mathcal{Q} := \mathcal{I}/\mathcal{G}$ denote the quotient. We obtain a long exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{G}) \to \mathrm{H}^0(X, \mathcal{I}) \xrightarrow{p} \mathrm{H}^0(X, \mathcal{Q}) \xrightarrow{\delta'} \mathrm{H}^1(X, \mathcal{G}) \to 0,$$

as $\mathrm{H}^1(X, \mathcal{I}) = 0$. Pick any $q \in \delta'^{-1}(z)$. Define a subsheaf $\mathcal{F}^z \subset \mathcal{I}$ by

$$\mathcal{F}^z(U) := p|_U^{-1}(q|_U).$$

Because $0 \to \mathcal{G}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U)$ is exact $\mathcal{G}(U)$ acts freely and transitively on $\mathcal{F}(U)$ by translation. Furthermore, as $\mathcal{I} \twoheadrightarrow \mathcal{Q}$ is surjective, for each $U \to X$ we can find a covering $\{U_i \to U\}_{i \in I}$ with $\mathcal{F}(U) \neq \varnothing$. Hence $\mathcal{F}^z$ is a $\mathcal{G}$-torsor. The isomorphism class of $\mathcal{F}^z$ is independent of the choice of q: let q' be a second choice, then $q - q' \mapsto 0 \in \mathrm{H}^1(X, \mathcal{G})$ and is hence the image of a global section $p \in \mathrm{H}^0(X, \mathcal{I})$. Let $\mathcal{F}'^z$ be the subsheaf of $\mathcal{I}$ obtained as the preimage of q'. Then the map $\mathcal{F}^z(U) \to \mathcal{F}'^z(U) : x \mapsto p|_U + x$ defines an isomorphism of $\mathcal{G}$-torsors.

It remains to show that the maps are well-defined and inverses.

We want to show that $\mathcal{F}^{z_\mathcal{F}} \cong \mathcal{F}$. We obtain a commutative diagram:

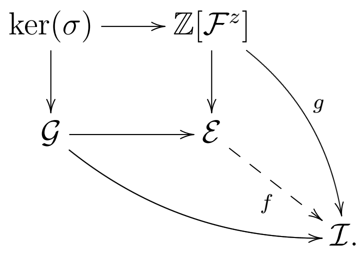
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{E} & \longrightarrow & \underline{\mathbb{Z}} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \end{array} \qquad (*)$$

The morphism $\mathcal{E} \to \mathcal{I}$ exists because $\mathcal{I}$ is injective and the morphism $\underline{\mathbb{Z}} \to \mathcal{Q}$ is induced by the universal property of the cokernel $\underline{\mathbb{Z}}$. This diagram induces a commutative diagram in cohomology

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{H}^0(X, \mathcal{G}) & \longrightarrow & \mathrm{H}^0(X, \mathcal{E}) & \longrightarrow & \mathrm{H}^0(X, \underline{\mathbb{Z}}) & \longrightarrow & \mathrm{H}^1(X, \mathcal{G}) & & \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{H}^0(X, \mathcal{G}) & \longrightarrow & \mathrm{H}^0(X, \mathcal{I}) & \longrightarrow & \mathrm{H}^0(X, \mathcal{Q}) & \longrightarrow & \mathrm{H}^1(X, \mathcal{G}) & \longrightarrow & 0 \end{array}$$

Since the right square commutes, it follows that $1 \in \mathrm{H}^0(X, \underline{\mathbb{Z}})$ is mapped to some $q \in \mathrm{H}^0(X, \mathcal{Q})$ with $q \mapsto z_\mathcal{F}$. Let $\mathcal{F}^{z_\mathcal{F}}$ be the subsheaf of $\mathcal{I}$ constructed above as the preimage of q. Then $f(\mathcal{F}) \subset \mathcal{F}^{z_\mathcal{F}}$ by construction. This induces a morphism of $\mathcal{G}$-torsors and hence $\mathcal{F} \cong \mathcal{F}^{z_\mathcal{F}}$, as desired.

Conversely, let $z \in \mathrm{H}^1(X, \mathcal{G})$. We want to show that $\delta(1) = z$. Construct $0 \to \mathcal{G} \to \mathcal{E} \to \underline{\mathbb{Z}} \to 0$ from $\mathcal{F}^z$ as above. We obtain a commutative diagram



Here g is defined as the sheafification of $[s] \mapsto s$. Let $\mathcal{F}'^z \subset \mathcal{E}$ be the $\mathcal{G}$-torsor isomorphic to $\mathcal{F}^z$ constructed as above. Then $f(\mathcal{F}'^z) = \mathcal{F}^z$, as the diagram commutes.

We again obtain a commutative diagram $(*)$. As $f(\mathcal{F}'^z) = \mathcal{F}^z$, the morphism $\underline{\mathbb{Z}} \to \mathcal{Q}$ maps the global section 1 to q. Hence, by the commutativity of

$$\begin{array}{ccc} 1 \in \mathrm{H}^0(X, \underline{\mathbb{Z}}) & \longrightarrow & \mathrm{H}^1(X, \mathcal{G}) \\ \downarrow & & \| \\ q \in \mathrm{H}^0(X, \mathcal{Q}) & \longrightarrow & \mathrm{H}^1(X, \mathcal{G}) \ni z \end{array}$$

it follows that $1 \mapsto z$, as desired. □

Recall: The abelian presheaves

$$\begin{aligned} \mathbb{G}_a(U) &:= \Gamma(U, \mathcal{O}_U) \\ \mathbb{G}_m(U) &:= \Gamma(U, \mathcal{O}_U^\times) \\ \mu_n(U) &:= \{f \in \Gamma(U, \mathcal{O}_U^\times) \mid f^n = 1\} \end{aligned}$$

are sheaves on $X_{\text{ét}}$.

Definition 6. *We define the* étale structure sheaf $\mathcal{O}_{X,\text{ét}}$ *on* X *to be the sheaf of rings defined by* $U \mapsto \Gamma(U, \mathcal{O}_U)$ *for* U *over* X.

Remark We thus obtain the notions of *sheaf of* $\mathcal{O}_{X,\text{ét}}$*-modules* and *locally free* and *tensor product* in total analogy to the notion of $\mathcal{O}_X$ modules on X_{zar}.

Just as in the Zariski case, we obtain

Fact 7. *Let* $\mathcal{F}$ *be a sheaf of* $\mathcal{O}_{X,\text{ét}}$*-modules. The following are equivalent*

1. $\mathcal{F}$ *is locally free of rank 1*
2. $\mathcal{F}$ *is invertible, i.e. there is a sheaf of* $\mathcal{O}_{X,\text{ét}}$*-modules* $\mathcal{F}'$ *such that* $\mathcal{F} \otimes_{\mathcal{O}_{X,\text{ét}}} \mathcal{F}' \cong \mathcal{O}_{X,\text{ét}}$.

Definition 8. *The* étale Picard group $\mathrm{Pic}(X_{\text{ét}})$ *is the group of invertible* $\mathcal{O}_{X,\text{ét}}$*-modules where the group law is given by the tensor product.*

Theorem 9. *There is a canonical bijection*

$$\mathrm{H}^1(X, \mathbb{G}_m) \cong \mathrm{Pic}\, X.$$

Proof. We define a maps

$$\{\text{Invertible } \mathcal{O}_{X,\text{ét}}\text{-modules up to iso.}\} \to \{\mathbb{G}_m\text{-torsors up to iso.}\}$$

$$[\mathcal{L}] \mapsto [\mathcal{L}^*(U) := (U \mapsto \{s \in \mathcal{L}(U) \mid \mathcal{O}_U \xrightarrow{-\cdot s} \mathcal{L}_U \text{ is an iso.}\})]$$

$$[\underbrace{(U \mapsto (\mathcal{F}(U) \times \mathcal{O}_{X,\text{ét}}(U))/\mathbb{G}_m(U))^\#}_{=:\mathcal{F}\otimes_{\mathbb{G}_m}\mathcal{O}_{X,\text{ét}}}] \leftarrow [\mathcal{F}]$$

We see that $\mathbb{G}_m(U)$ acts on $\mathcal{L}^*(U)$ freely and transitively by multiplication. Moreover, we see that $\mathcal{L}^*$ is already a sheaf. As $\mathcal{L}$ is locally free, we can find a cover of any U étale $\{U_i \to U\}$ over U such that $\mathcal{L}^*(U_i) \cong \mathbb{G}_m(U_i) \neq \varnothing$. Therefore $\mathcal{L}^*$ is a $\mathbb{G}_m$-torsor. The map is clearly defined on isomorphism classes.

In the other direction $g \in \mathbb{G}_m(U)$ acts on $(s, f) \in \mathcal{F}(U) \times \mathcal{O}_{X,\text{ét}}(U)$ by $g(s, f) := (gs, g^{-1}f)$. We set $(s, f) + (s', f') := (s, f + (s'/s)f')$, where $s'/s \in \mathbb{G}_m(U)$ such that $(s'/s)s = s'$. We make it into a sheaf of $\mathcal{O}_{X,\text{ét}}$-modules by setting $h(s, f) := (s, hf)$ for $h \in \mathcal{O}_{X,\text{ét}}(U)$.

Claim: The $\mathcal{O}_{X,\text{ét}}$-module $\mathcal{F} \otimes_{\mathbb{G}_m} \mathcal{O}_{X,\text{ét}}$ is invertible.

Proof. Let $\{U_i \to X\}_{i\in I}$ be a covering of X such that $\mathcal{F}(U_i) \neq \varnothing$ for all $i \in I$. For each i we can pick an isomorphism of sheaves of sets $\varphi_i : \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathbb{G}_m|_{U_i}$. We define a morphism of presheaves

$$
\begin{aligned}
(\mathbb{G}_m(U) \times \mathcal{O}_{X,\text{ét}}(U))/\mathbb{G}_m(U) &\to \mathcal{O}_{X,\text{ét}}(U) \\
(s, f) &\mapsto sf.
\end{aligned}
$$

By a short calculation one verifies that this is an isomorphism of $\mathcal{O}_{X,\text{ét}}(U)$-modules. Therefore $(\mathcal{F} \otimes_{\mathbb{G}_m} \mathcal{O}_{X,\text{ét}})|_{U_i} \cong \mathcal{O}_{U_i,\text{ét}}$, as desired. □

We show that the two constructions are inverse to each other. Let $\mathcal{L}$ be an invertible $\mathcal{O}_{X,\text{ét}}$-module. Let $\{U_i \to X\}_{i\in I}$ be a covering such that $\mathcal{L}^*(U_i) \neq \varnothing$ for all i. For $U \to U_i$ the map

$$
\begin{aligned}
(\mathcal{L}^*(U) \times \mathcal{O}_{X,\text{ét}}(U))/\mathbb{G}_m(U) &\to \mathcal{L}(U) \\
(s, f) &\mapsto sf
\end{aligned}
$$

defines a canonical isomorphism $(\mathcal{L}^* \otimes_{\mathbb{G}_m} \mathcal{O}_{X,\text{ét}})|_{U_i} \cong \mathcal{L}|_{U_i}$. These isomorphism glue and we obtain a global isomorphism.

Conversely, let $\mathcal{F}$ be a $\mathbb{G}_m$-torsor. We have for U such that $\mathcal{F}(U) \neq \varnothing$

$$
((\mathcal{F}(U) \times \mathcal{O}_{X,\text{ét}}(U))/\mathbb{G}_m(U))^* = ((\mathcal{F}(U) \times \mathcal{O}_{X,\text{ét}}(U)^\times)/\mathbb{G}_m(U)) \cong \mathcal{F}(U),
$$

canonically. Again, we glue these and have $(\mathcal{F} \otimes_{\mathbb{G}_m} \mathcal{O}_{X,\text{ét}})^* \cong \mathcal{F}$.

Using the theorem above, we have shown $\mathrm{H}^1(X, \mathbb{G}_m) \cong \operatorname{Pic} X_{\text{ét}}$. But we have $\operatorname{Pic} X_{\text{ét}} \cong \operatorname{Pic} X_{\text{zar}}$ by descent of quasi-coherent sheaves. □

Remark One can show that this is in fact a group isomorphism.

For more properties and motivation for the $\mathcal{F} \otimes_{\mathbb{G}_m} \mathcal{O}_{X,\text{ét}}$-construction see for example [vB14].

2 The Kummer sequence

Lemma 10. *Let A be a ring, let $P \in A[T]$ be a monic polynomial and let $B := A[T]/(P)$. Suppose that the derivative P' is a unit in B. Then the induced morphism* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is surjective and étale.*

Proof. Since B is free of rank $\deg P$ over A, the inclusion $A \subset B$ is faithfully flat. Clearly the morphism is of finite type. To show that the map is unramified, we just show that $\Omega_{B/A} = 0$. This follows from the second exact sequence for differentials

$$\begin{aligned}(P)/(P^2) &\to B \cdot dT \to \Omega_{B/A} \to 0 \\ P &\mapsto P' dT.\end{aligned}$$

□

Proposition 11 (Kummer sequence)**.** *For every $n > 1$ with $n \in \Gamma(X, \mathcal{O}_X^\times)$, the sequence of étale sheaves*

$$1 \to \mu_{n,X} \to \mathbb{G}_m \xrightarrow{x \mapsto x^n} \mathbb{G}_m \to 1$$

is exact.

Proof. By definition $\mu_{n,X}$ is the kernel of $x \mapsto x^n$. It remains to show the surjectivity of $x \mapsto x^n$ onto $\mathbb{G}_m$. Let U be a scheme over X and let $f \in \Gamma(U, \mathcal{O}_U)^\times$. We need to find an étale covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of U such that $f|_{U_i}$ has an n-th root for all $i \in I$. After covering U with affine opens, we can reduce to the case when $U = \operatorname{Spec} A$ is affine. Let $B := \operatorname{Spec} A[T]/(T^n - f)$ and let $U' := \operatorname{Spec} B$. Let $\pi : U' \to U$ denote the morphism induced by the inclusion $A \subset B$. We calculate $\frac{d(T^n - f)}{dT} = nT^{n-1}$ which is a unit in B. Hence, by Lemma 10, the morphism π is étale and surjective. Furthermore, the section $T \in \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$ is an n-th root for $f|_{\operatorname{Spec} B}$. □

References

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[vB14] R. van Bommel, *Cohomology on quasi-coherents, torsors, h^1 and the picard group*, 2014, https://www.raymondvanbommel.nl/talks/etale3.pdf.