Proper Base Change for Curves

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The goal of the talk to is prove the following theorem.

Main Theorem. *Let $f\colon X \to S$ be a proper morphism where $S = \operatorname{Spec}(A)$ for a strictly henselian noetherian local ring $(A, \mathfrak{m}, k)$. Let $p := \operatorname{char}(k)$. Suppose the special fiber X_0 has dimension 1. Then for every torsion sheaf $\mathcal{F}$ on X without p^∞-torsion the base change morphism*

$$H^q_{\acute{e}t}(X, \mathcal{F}) \to H^q_{\acute{e}t}(X_0, \mathcal{F}|_{X_0})$$

is bijective for all $q \geq 0$.

A bijectivity criterion for base change

Definition 1. A an additive fuctor $T\colon \mathcal{A} \to \mathrm{Ab}$ is *effaceable* if for every object $A \in \mathrm{Ob}(\mathcal{A})$ and every element $\alpha \in T(A)$ there is a monomorphism $u\colon A \hookrightarrow M$ in $\mathcal{A}$ such that $T(u)\alpha = 0$.

Lemma 2. *Suppose $\phi^\bullet\colon T^\bullet \to T'^\bullet$ is a morphism of δ-functors from an abelian category $\mathcal{A}$ to the category of abelian groups. Suppose each T^q is effaceable and $\mathcal{E} \subset \mathrm{Ob}(\mathcal{A})$ is a collection of objects such that every object in $\mathcal{A}$ is a subobject of an object in $\mathcal{E}$. Then the following are equivallent.*

(i) $\phi^q_A\colon T(A) \to T'(A)$ *is bijective for all* $q \geq 0$ *and all* $A \in \mathrm{Ob}(\mathcal{A})$.

(ii) $\phi^0(M)$ *is bijective and* $\phi^q(M)$ *is surjective for all* $q > 0$ *and all* $M \in \mathcal{E}$.

(iii) $\phi^0(A)$ *is bijective for all* $A \in \mathrm{Ob}(\mathcal{A})$ *and* T'^q *is effaceable for all* $q > 0$.

Proof. Induct on q. This is a diagram chase best left to the blackboard or a scratch piece of paper. □

Lemma 3. *Let X be a noetherian scheme. The functors $H^q_{\acute{e}t}(X, -)$ are effaceable on the category of constructible étale sheaves on X.*

Proof. Let $\mathcal{F}$ be a constructible sheaf on X. There is an $n > 0$ such that $\mathcal{F}$ is a sheaf of $\mathbb{Z}/n$-modules. Indeed, there is a stratification $X = \coprod_i X_i$ such that $\mathcal{F}|_{X_i}$ is locally constant and hence a sheaf of $\mathbb{Z}/m_i$-modules for some $m_i > 0$. Hence $\mathcal{F}$ is a sheaf of $\mathbb{Z}/\operatorname{lcm}(\{m_i\})$-modules.

For every point $x \in X$ choose a geometric point $\overline{x} \to x$. Consider the acyclic sheaf $\mathcal{G} \coloneqq \prod_{x \in X} \overline{x}_*(\mathcal{F}|_{\overline{x}})$. We have a monomorphism $u \colon \mathcal{F} \hookrightarrow \mathcal{G}$ of $\mathbb{Z}/n$-modules. The sheaf $\mathcal{G}$ is torsion and therefore a colimit of constructible sheaves. We write $\mathcal{G}$ as a colimit of constructible sheaves contatining $\mathcal{F}$.

Since cohomology commutes with colimits we have

$$H^q_{\acute{e}t}(u) \colon H^q_{\acute{e}t}(X, \mathcal{F}) \to \operatorname{colim}_\lambda H^q_{\acute{e}t}(X, C_\lambda) = H^q_{\acute{e}t}(X, \mathcal{G}) = 0.$$

Thus given a cohomology class $\xi \in H^q_{\acute{e}t}(X, \mathcal{F})$ there is a λ for which the representative $H^q_{\acute{e}t}(u)\xi$ in $H^q_{\acute{e}t}(X, C_\lambda)$ is zero. Effaceability follows from $\mathcal{F} \subset C_\lambda$. □

Proposition 4. *Let X be a noetherian scheme and let $X_0 \subset X$ be a subscheme. Let p be a prime number. Suppose that for all $n \geq 0$ with $p \nmid n$ and all finite X-schemes X' the base change morphism*

$$H^q_{\acute{e}t}(X', \mathbb{Z}/n) \to H^q_{\acute{e}t}(X' \times_X X_0, \mathbb{Z}/n)$$

is bijective in degree $q = 0$ and surjective in degrees $q > 0$. Then for all torsion sheaves $\mathcal{F}$ in X with no p^∞-torsion the base change morphism

$$H^q_{\acute{e}t}(X, \mathcal{F}) \to H^q_{\acute{e}t}(X_0, \mathcal{F})$$

is bijective in all degrees $q \geq 0$.

Proof. Since cohomology commutes with colimits and every torsion sheaf is a colimit of constructible sheaves it suffices to prove the proposition for constructible $\mathcal{F}$.

We want to apply Lemma 2 to the category $\mathcal{A}$ of constructible sheaves, $\phi^\bullet$ the base change morphism $H^\bullet_{\acute{e}t}(X, -) \to H^\bullet_{\acute{e}t}(X_0, -)$ and for $\mathcal{E}$ we take the collection of objects of the form $\bigoplus \pi_* C_i$ for finite morphisms $\pi_* \colon X'_i \to X$ and finite constant sheaves $\mathcal{C}_i$ with no p^∞-torsion on X'_i. In Xiao's talk we saw that every constructible sheaf is a suboject of $\bigoplus \pi_* C_i$.We have surjectivity of the base change morphism for the sheaves $\bigoplus \pi_* C_i$ because cohomology commutes with finite products and we know surjectivity of the C_i by assumtion. Thus we have verified all the hypotheses of Lemma 2 *(ii)*.

□

Surjectivity in degree 2

Proposition 5. *Let $f\colon X \to S$ be a proper morphism with $S = \operatorname{Spec}(A)$ for a strictly henselian local ring $(A, \mathfrak{m}, k)$. Suppose the special fiber $X_0 \coloneqq X \times_S \operatorname{Spec}(k)$ has dimension 1. Then the restriction*

$$\operatorname{Pic}(X) \to \operatorname{Pic}(X_0)$$

is surjective.

Proof. Let $X_n \coloneqq X \times_S \operatorname{Spec}(A/\mathfrak{m}^{n+1})$. We show iteratively that we have surjections $\operatorname{Pic}(X_n) \to \operatorname{Pic}(X_0)$. Consider the ideal sheaf $\mathcal{I}_{n+1} \coloneqq \ker(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n})$. We have the short exact sequence

$$0 \longrightarrow \mathcal{I}_{n+1} \xrightarrow{a \mapsto 1+a} \mathcal{O}_{X_{n+1}}^{\times} \longrightarrow \mathcal{O}_{X_{n+1}}^{\times} \longrightarrow 0.$$

Since $\mathcal{I}_{n+1}$ is coherent on the scheme X_{n+1} of dimension 1 its degree 2 quasicoherent cohomolgoy vanishes and we obtain the surjection $H^1(X_{n+1}, \mathcal{O}_{X_{n+1}}^{\times}) \twoheadrightarrow H^1(X_n, \mathcal{O}_{X_n})$. Passing to the colimit we obtain a surjection $\operatorname{Pic}(\mathfrak{X}) \twoheadrightarrow \operatorname{Pic}(X)$ where $\mathfrak{X}$ is the completion of X along X_0.

By Grothendieck's Existence Theorem we obtain a surjection $\operatorname{Pic}(X \times \operatorname{Spec}(\widehat{A})) \to \operatorname{Pic}(X_0)$. It follows from [2, (8.5.2) (ii) and (8.5.5)] that Pic is locally of finite presentation. Thus we obtain by Artin's Approximation Theorem a surjection $\operatorname{Pic}(X) \to \operatorname{Pic}(X_0)$ □

If suffices to show surjectivity for the sheaves $\mathbb{Z}/l^r$ for some any prime $l \neq \operatorname{char}(k)$. Recall the Kummer sequence

$$0 \longrightarrow \mathbb{Z}/l^r \longrightarrow \mathbb{G}_m \xrightarrow{(-)^{l^r}} \mathbb{G}_m \longrightarrow 1.$$

the base change morphism $H^{\bullet}_{ét}(X, \mathbb{Z}/l^r) \to H^{\bullet}_{ét}(X_0, \mathbb{Z}/l^r)$ yields a morphism of exact sequences

$$\begin{array}{ccccc}
\operatorname{Pic}(X) = H^1_{ét}(X, \mathbb{G}_m) & \longrightarrow & H^2_{ét}(X, \mathbb{Z}/l^r) & \longrightarrow & H^2_{ét}(X, \mathbb{G}_m) \\
\downarrow\!\!\downarrow & & \downarrow & & \downarrow \\
\operatorname{Pic}(X_0) = H^1_{ét}(X_0, \mathbb{G}_m) & \longrightarrow & H^2_{ét}(X_0, \mathbb{Z}/l^r) & \longrightarrow & H^2_{ét}(X_0, \mathbb{G}_m)
\end{array}$$

Fact 6. *For a proper curve X_0 over a seperably closed field the cohomology group $H^2_{ét}(X_0, \mathbb{G}_m)$ is p^∞-torsion.*

Thus $\operatorname{Pic}(X_0) \to H^2_{ét}(X_0, \mathbb{Z}/l^r)$ and $H^2_{ét}(X, \mathbb{Z}/l^r) \to H^2_{ét}(X_0, \mathbb{Z}/l^r)$ are surjective.

Topological invariance of the étale site

Definition 7. A morphism of schemes $f\colon X \to Y$ is *universally injective* if any base change of f is injective.

Lemma 8. *Let $f\colon X \to Y$ be a morphism of schemes. The following are equivalent.*

(a) f is universally injective.

(b) For every field K the morphism f is injective on K-points, i.e., composition with f

$$\begin{aligned} f \circ -\colon \operatorname{Mor}(\operatorname{Spec}(K), X) &\to \operatorname{Mor}(\operatorname{Spec}(K), Y) \\ g &\mapsto f \circ g \end{aligned}$$

is injective.

(c) f is injective and for every $x \in X$ the field extension $k(x) \supset k(f(x))$ is purely inseperable.

Proof. [3, Tag 01S4] □

Lemma 9. *Any base change of a surjective morphism of schemes is surjective.*

Proof. [3, Tag 01S1] □

Lemma 10. *A morphism is integral if and only if it is affine and universally closed.*

Proof. [3, Tag 01WM] □

Definition 11. A morphism of schemes $f\colon X \to Y$ is a *unverisal homeomorphism* if any base change of f is a homeomorphism of the underlying topological spaces, or, equivalently if f is integral, universally injective, and surjective.

Theorem 12. *Suppose $f\colon X \to Y$ is a morphism of schemes and is a universal homeomorphism. Then the base changing along f*

$$[V \to Y] \mapsto [V \times_Y X \to X]$$

is part of an equivalence of categories

$$Y_{\acute{e}t} \cong X_{\acute{e}t}$$

Proof. [3, Tag 03SI] □

Remark 13. In Ole's talk we saw the theorem for f a closed immersion defined by a nilpotent ideal sheaf.

Vanishing for curves (Surjectivity for degrees greater than 2)

Lemma 14. *Let X be a scheme of finite type over a seperably closed field.*

(i) Suppose $\mathcal{K}$ an étale sheaf on X supported at finitely many points. Then $H^q_{ét}(X, \mathcal{K}) = 0$ for $q > 0$.

(ii) Suppose $\mathcal{F} \to \mathcal{G}$ is an isomorphism of etale sheaves on X outside of finitely many points. Then $H^q_{ét}(X, \mathcal{F}) \cong H^q(X_0, \mathcal{G})$ for $q > 1$.

Proof. Suppose $\mathcal{K}$ is an étale sheaf on X supported on a finite subscheme $i\colon F \hookrightarrow X$. Then $\mathcal{K} \cong i_* i^* \mathcal{K}$. Hence

$$H^q_{ét}(X, \mathcal{K}) = H^q_{ét}(X, i_* i^* \mathcal{K}) = H^q_{ét}(F, i^* \mathcal{K}) = 0$$

for $q > 0$. The last equation follows from Theorem 12 which applies because every algebraic extension of k must be purely inseperable. This proves *(i)*.

By assumption the kernel and cokernel of $\mathcal{F} \to \mathcal{G}$ are finitely supported. Hence *(ii)* follows from *(i)* by looking at the long exact sequence associated to the short exact sequences

$$0 \longrightarrow \ker(\alpha) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\ker(\alpha) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{F}/\ker(\alpha) \longrightarrow j_* j^* \mathcal{F} \longrightarrow \operatorname{coker}(\alpha) \longrightarrow 0.$$

□

Theorem 15. *Let X_0 be a proper curve over a seperably closed field k. Suppose $\operatorname{char}(k) \nmid n$. Then $H^q_{ét}(X_0, \mathbb{Z}/n) = 0$ for all $q > 2$.*

Proof. By the topological invariance of the étale site, we may assume that X_0 is reduced.

We show that we may reduce to the case that X_0 is irreducible. Suppose η_i are the generic points of the irreducible components of X_0. Then $\mathbb{Z}/n \hookrightarrow \sum_i \eta_{i,*}\mathbb{Z}/n$. The cokernel is supported at finitely many points and therefore it's higher cohomology vanishes. It follows that if the cohomology of $\mathbb{Z}/n$ on X_i vanishes for each i then so does the cohomology of $\mathbb{Z}/n$ on X_0.

Assume X_0 is irreducible. Let $\nu\colon \widetilde{X_0} \to X_0$ be the normalization of X_0. The curve $\widetilde{X_0}$ is smooth and proper over k, and hence projective over k. We claim that $H^q_{ét}(X_0, \mathbb{Z}/n) = H^q_{ét}(\widetilde{X_0}, \mathbb{Z}/n)$ for all $q \geq 0$. The morphism ν is a birational morphism, that is an isomorphism outside of finitely many points

$$\nu\colon \widetilde{X_0} \smallsetminus \{\widetilde{P}_1, \dots, \widetilde{P}_r\} \to X_0 \smallsetminus \{P_1, \dots, P_r\}.$$

By Lemma 14 we may replace X_0 with its normalization $\widetilde{X_0}$. So now we need only to prove the theorem for a smooth projective curve over a seperably closed field k. Let $\overline{k}$ be an algebraic closure of k. The extension $k \subset \overline{k}$ is purely inseperable, hence $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$ is a universal homeomorphism. In particular, $X_{0,\overline{k}} \to X_0$ is a universal homeomorphism. Thus by topological invariance we may assume that X_0 is not only smooth and projective over an algebraically closed field k. In this situation, vanishing of $H^q(X_0, \mathbb{Z}/n)$ is part of the computation of cohomology of curves we did last semester. $\square$

References

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