Cohomology of Curves

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1 Outline

Let C be a smooth integral qcqs curve over an algebraically closed field k with generic point η, and let n be an integer that is not divisible by the characteristic of k. The goal of this talk is to compute the étale cohomology of C with coefficients in μ_n, following [SGA 4½]. To do so, we use the long exact sequences induced by the Kummer sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \longrightarrow 1, \tag{K}$$

and the divisor sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \eta_* \mathbb{G}_m \xrightarrow{\text{div}} \mathcal{D} \longrightarrow 0, \tag{D}$$

where

$$\mathcal{D} = \bigoplus_{c \in |C|} c_* \mathbb{Z}.$$

We already know that $\mathrm{H}^q(c, \mathbb{Z}) = 0$ for each $q \geqslant 1$; for $\mathrm{H}^q(\eta, \mathbb{G}_m)$ this will be proven in the next couple of talks. But what about $\mathrm{H}^q(C, c_*\mathbb{Z})$ and $\mathrm{H}^q(C, \eta_*\mathbb{G}_m)$? Because the direct image functor associated with a morphism of schemes is in general only left exact, there's no a priori reason for it to preserve cohomology; to tell whether it does, we need to study its right derived functors.

2 Extension by zero

Let $\varphi \colon U \to X$ be an étale morphism. Then the inverse image functor

$$\varphi^* \colon \mathbf{Sh}(X) \to \mathbf{Sh}(U)$$

is given by

$$\varphi^* \mathcal{F}(V \to U) = \mathcal{F}(V \to U \xrightarrow{\varphi} X)$$

on objects; the same formula workes for presheaves.

Lemma 1. *φ^* has an exact left adjoint $\varphi_!$.*

Proof. The construction for presheaves is as follows:

$$\varphi_!^{\mathrm{pre}}\mathcal{G}(V \to X) = \bigoplus_{V \to U} \mathcal{G}(V \to U).$$

Then define $\varphi_!$ by the commutative diagram

$$\begin{array}{ccc} \mathbf{Sh}(U) & \xrightarrow{\varphi_!} & \mathbf{Sh}(X) \\ \big\downarrow & & \big\uparrow{\scriptstyle (-)^{\#}} \\ \mathbf{PSh}(U) & \xrightarrow{\varphi_!^{\mathrm{pre}}} & \mathbf{PSh}(X). \end{array}$$

The functor $\varphi_!^{\mathrm{pre}}$ is exact; $\varphi_!$ is left exact as a composite of left exact functors, and right exact because it is a left adjoint. □

Corollary 2. *φ^* preserves injectives.*

3 Higher direct images

Let $f\colon X \to Y$ be a morphism of schemes.

Definition 3. *The right derived functors*

$$\mathrm{R}^q f_*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y) \quad (q \geqslant 0)$$

are called the higher direct image functors associated with f.

Lemma 4. *Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be a sheaf on X such that $\mathrm{R}^q f_*\mathcal{F} = 0$ for each $q \geqslant 1$. Then $\mathrm{H}^q(Y, f_*\mathcal{F}) = \mathrm{H}^q(X, \mathcal{F})$ for each $q \geqslant 0$.*

Proof. If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$, then $0 \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{I}^\bullet$ is an injective resolution of $f_*\mathcal{F}$. □

Lemma 5. *Let $\mathcal{F}$ be a sheaf on X. If $\mathrm{H}^q(X \times_Y V, \mathrm{pr}_X^*\mathcal{F}) = 0$ for every étale morphism $V \to Y$, then $\mathrm{R}^q f_*\mathcal{F} = 0$.*

Proof. Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Then

$$\mathrm{R}^q f_*\mathcal{F} = \mathrm{H}^q(f_*\mathcal{I}^\bullet);$$

but $\mathrm{H}^q(f_*\mathcal{I}^\bullet)$ is the sheafification of the presheaf cohomology $\mathrm{H}^q(f_*\mathcal{I}^\bullet)^{\mathrm{pre}}$, so it suffices to show that the latter is 0. Let $\psi\colon V \to Y$ be an étale morphism. We have the cartesian diagram

$$\begin{array}{ccc} X \times_Y V & \xrightarrow{\mathrm{pr}_V} & V \\ \big\downarrow{\scriptstyle \mathrm{pr}_X} & & \big\downarrow{\scriptstyle \psi} \\ X & \xrightarrow{f} & Y \end{array}$$

and a natural isomorphism

$$\psi^* f_* = (\mathrm{pr}_V)_* \mathrm{pr}_X^*.$$

Combining this with exactness of ψ^* and Corollary 2, we find

$$\begin{aligned}
\mathrm{H}^q(f_*\mathcal{I}^\bullet)^{\mathrm{pre}}(\psi) &= \psi^*\mathrm{H}^q(f_*\mathcal{I}^\bullet)^{\mathrm{pre}}(\mathrm{id}_V) \\
&= \mathrm{H}^q(\psi^* f_*\mathcal{I}^\bullet)^{\mathrm{pre}}(\mathrm{id}_V) \\
&= \mathrm{H}^q((\mathrm{pr}_V)_*\mathrm{pr}_X^*\mathcal{I}^\bullet)^{\mathrm{pre}}(\mathrm{id}_V) \\
&= (\mathrm{pr}_V)_*\mathrm{H}^q(\mathrm{pr}_X^*\mathcal{I}^\bullet)^{\mathrm{pre}}(\mathrm{id}_V) \\
&= \mathrm{H}^q(\mathrm{pr}_X^*\mathcal{I}^\bullet)^{\mathrm{pre}}(\mathrm{id}_{X\times_Y V}) \\
&= \mathrm{H}^q(X \times_Y V, \mathrm{pr}_X^*\mathcal{F}) \\
&= 0.
\end{aligned}$$

□

4 The restricted étale site

To define the morphism div appearing in (D), it is convenient to restrict our attention to qcqs schemes. A reference for this section is [Tamme, §II.1.5].

Let X be a scheme.

Definition 6. *The* restricted small étale site $\mathbf{pf\acute{E}t}(X)$ of X *consists of the category of all finitely presented étale schemes over X and the topology of jointly surjective morphisms.*

Lemma 7. *If X is qc, then $\mathbf{pf\acute{E}t}(X)$ is a noetherian site.*

Proposition 8. *If X is qs, then the inclusion functor $\mathbf{pf\acute{E}t}(X) \hookrightarrow \mathbf{\acute{E}t}(X)$ induces an equivalence of categories between the associated categories of abelian sheaves.*

Corollary 9. *If X is qcqs, then the functors*

$$\mathrm{H}^q(X, -)\colon \mathbf{Sh}(X) \to \mathbf{Ab}$$

preserve pseudofiltered colimits.

5 Divisor sequence

We work on the restricted small étale site of C.

Lemma 10. *The sheaf $\mathcal{D}$ is given by*

$$\mathcal{D}(U \to C) = \bigoplus_{u\in|U|} \mathbb{Z}.$$

Proof. Define $\mathcal{D}'$ by

$$\mathcal{D}'(U \to C) = \bigoplus_{u \in |U|} \mathbb{Z};$$

then $\mathcal{D}'$ is the coproduct of the sheaves $c_*\mathbb{Z}$ in the category of presheaves, so it suffices to check that it is already a sheaf. Let $(\varphi_i \colon V_i \to U)$ be an arbitrary covering; we need to show that the sequence

$$0 \longrightarrow \mathcal{D}'(U \to X) \xrightarrow{\varrho} \prod_i \mathcal{D}'(V_i \to X) \xrightarrow{\pi} \prod_{i,j} \mathcal{D}'(V_i \times_U V_j \to X)$$

is exact, where

$$\varrho(a_u)_{i,v} = a_{\varphi_i(v)} \quad \text{and} \quad \pi(b_{i,v})_{i,j,v,w} = b_{i,v} - b_{j,w}.$$

The map ϱ is injective because (φ_i) is jointly surjective; $\ker(\pi)$ consists of all $(b_{i,v})$ such that $b_{i,v} = b_{j,w}$ whenever $\varphi_i(v) = \varphi_j(w)$, i.e. exactly $\operatorname{im}(\varrho)$. ☐

Proposition 11. *The sequence (D) is exact.*

Proof. Let $U \to C$ be finitely presented and étale. The inclusion

$$\mathbb{G}_m(U \to C) \to \eta_*\mathbb{G}_m(U \to C)$$

is clearly injective. The morphism div is given by

$$f \mapsto (\operatorname{ord}_u(f)),$$

which is well-defined because U is qc; its kernel is exactly $\mathbb{G}_m(U \to C)$, since U is normal. Finally, div is surjective Zariski-locally on U because U is regular. ☐

Lemma 12. *The direct image functor c_* is exact for every $c \in |C|$.*

Proof. Epimorphisms of sheaves on c are even epimorphisms in the category of presheaves on c, so they are preserved by c_*. ☐

Corollary 13. *For every $c \in |C|$ and every sheaf $\mathcal{F}$ on c,*

$$\mathrm{H}^q(C, c_*\mathcal{F}) = 0$$

for each $q \geqslant 1$.

Corollary 14. *For each $q \geqslant 1$, $\mathrm{H}^q(C, \mathcal{D}) = 0$.*

Theorem 15. *Let K/k be a field extension of transcendence degree 1. Then*

$$\mathrm{H}^q(K, \mathbb{G}_m) = 0$$

for each $q \geqslant 1$.

Lemma 16. *For each $q \geqslant 1$, $\mathrm{R}^q\eta_*\mathbb{G}_m = 0$.*

Proof. By Lemma 5, it suffices to show that

$$\mathrm{H}^q(U_\eta, \mathbb{G}_m) = 0 \quad (q \geqslant 1)$$

for every finitely presented étale morphism $U \to X$. Let $\{\eta_1, \ldots, \eta_r\}$ be the points of U_η. Each extension $\kappa(\eta_i)/k$ is of transcendence degree 1; by Theorem 15, $\mathbb{G}_m$ is acyclic on η_i. If $0 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{I}^\bullet$ is an injective resolution of $\mathbb{G}_m$ on U_η, then

$$\big(0 \longrightarrow \mathbb{G}_m(U_\eta) \longrightarrow \mathcal{I}^\bullet(U_\eta)\big) = \prod_{i \in \{1,\ldots,r\}} \big(0 \longrightarrow \mathbb{G}_m(\eta_i) \longrightarrow \mathcal{I}^\bullet(\eta_i)\big).$$

But each $\mathcal{I}^\bullet|_{\eta_i}$ is an injective resolution of the acyclic sheaf $\mathbb{G}_m$ on η_i, so all factors are exact. Thus $0 \longrightarrow \mathbb{G}_m(U_\eta) \longrightarrow \mathcal{I}^\bullet(U_\eta)$ is exact, as desired. □

Corollary 17. *For each* $q \geqslant 1$, $\mathrm{H}^q(C, \eta_*\mathbb{G}_m) = 0$.

Proposition 18 (Cohomology of $\mathbb{G}_m$)**.** *The only potentially nonvanishing cohomology groups of C with coefficients in $\mathbb{G}_m$ are*

$$\mathrm{H}^0(C, \mathbb{G}_m) = \mathcal{O}_C(C)^\times \quad \textit{and} \quad \mathrm{H}^1(C, \mathbb{G}_m) = \mathrm{Pic}(C).$$

Proof. The long exact sequence induced by (D) looks like

$$0 \longrightarrow \mathcal{O}_C(C)^\times \longrightarrow K(C)^\times \xrightarrow{\text{div}} \bigoplus_{c \in |C|} \mathbb{Z} \longrightarrow \mathrm{H}^1(C, \mathbb{G}_m) \longrightarrow 0 \longrightarrow \cdots,$$

with

$$\mathrm{H}^{q-1}(C, \mathcal{D}) = 0 \quad \text{and} \quad \mathrm{H}^q(C, \eta_*\mathbb{G}_m) = 0$$

for each $q \geqslant 2$; hence $\mathrm{H}^q(C, \mathbb{G}_m) = 0$ for each $q \geqslant 2$ and

$$\mathrm{H}^1(C, \mathbb{G}_m) = \mathrm{coker}(\mathrm{div}) = \mathrm{Pic}(C).$$ □

6 Kummer sequence

Proposition 19 (Cohomology of μ_n)**.** *Assume that C is projective over k. Then the only potentially nonvanishing cohomology groups of C with coefficients in μ_n are*

$$\begin{aligned}\mathrm{H}^0(C, \mu_n) &= \mu_n(k),\\ \mathrm{H}^1(C, \mu_n) &= \mathrm{Pic}^0(C)[n],\\ \mathrm{H}^2(C, \mu_n) &= \mathbb{Z}/(n);\end{aligned}$$

moreover, $\mathrm{H}^1(C, \mu_n)$ *is free of rank twice the genus of C as a $\mathbb{Z}/(n)$-module.*

Proof. Since $\mathrm{H}^q(C, \mathbb{G}_m) = 0$ for each $q \geqslant 2$, we have $\mathrm{H}^q(C, \mu_n) = 0$ whenever $q \geqslant 3$. For $q \leqslant 2$, the relevant piece of the long exact sequence induced by (K) is

$$\mathrm{H}^1(C,\mu_n) \hookrightarrow \mathrm{Pic}(C) \xrightarrow{(-)^n} \mathrm{Pic}(C) \twoheadrightarrow \mathrm{H}^2(C,\mu_n).$$

Hence

$$\mathrm{H}^1(C,\mu_n) = \mathrm{Pic}(C)[n] \quad \text{and} \quad \mathrm{H}^2(C,\mu_n) = \mathrm{Pic}(C)/\mathrm{Pic}(C)^n.$$

Note that $\mathrm{Pic}(C)[n] = \mathrm{Pic}^0(C)[n]$. The group $\mathrm{Pic}^0(C)$ is the group of rational points of the Jacobian of C, and is therefore divisible; its n-torsion is free of rank twice the genus of C as a $\mathbb{Z}/(n)$-module (see [Stacks, Tags 0B9Z, 0BA0, 03RP]). To finish the proof, it remains to show that the degree homomorphism

$$\deg\colon \mathrm{Pic}(C) \to \mathbb{Z}$$

induces an isomorphism of $\mathrm{Pic}(C)/\mathrm{Pic}(C)^n$ with $\mathbb{Z}/(n)$. That is so because it is surjective and $\mathrm{Pic}^0(C)$ is divisible. □

References

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