Étale sites

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We introduce the notions of sites, sheaves on sites, sheafification, constant sheaves and show that sheaves on a site form a abelian category. The definitions of stalks, pushforward, pullback, the proof of enough injectives and the definition of cohomology will only be done for the étale site of a scheme.
We mostly follow the corresponding sections in the Stacks Project [1] in the chapter on sites and sheaves (tag 00UZ) and the chapter on étale cohomology (tag 03N1).

1 Sites

Definition 1.1 *A* site *consists of a category* $\mathcal{C}$ *and a set* $\mathrm{Cov}(\mathcal{C})$ *of families of morphisms* $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I}$ *called* coverings*, such that*

(i) (isomorphism) if $\varphi : V \xrightarrow{\sim} U$ *is an isomorphism in* $\mathcal{C}$*, then* $\{\varphi : V \xrightarrow{\sim} U\}$ *is a covering,*

(ii) (locality) if $\{\varphi_i : U_i \to U\}$ *is a covering and for all* $i \in I$ *we are given a covering* $\{\psi_{ij} : U_{ij} \to U_i\}_{j\in I_i}$ *then*
$$\{\varphi_i \circ \psi_{ij} : U_{ij} \to U\}_{(i,j)}$$
is also a covering, and

(iii) (base change) if $\{U_i \to U\}_{i\in I}$ *is a covering and* $V \to U$ *is a morphism in* $\mathcal{C}$*, then*

  (a) for all $i \in I$ *the fibre product* $U_i \times_U V$ *exists in* $\mathcal{C}$*, and*
  (b) $\{U_i \times_U V \to V\}_{i\in I}$ *is a covering.*

Example 1.2 *If* X *is a scheme, define the associated zariski site* X_{Zar} *as follows: Let the underlying category be the category of open embeddings* $\varphi_i : U \hookrightarrow X$ *with open embeddings over* X *as morphisms. The coverings are* $\{\varphi_i : U_i \hookrightarrow U\}$ *such that* $X = \bigcup \varphi_i(U_i)$*.*

Example 1.3 *If* X *is a scheme, define the associated* big étale site *as follows:*
Let the underlying category be the category of schemes S *over* X *with morphisms* $S \to S'$ *over* X *such that* $S \to S'$ *is étale. (*$S \to X$ *and* $S' \to X$ *do not have to be étale.) Coverings are* $\{\varphi_i : S_i \to X\}_{i\in I}$ *such that* $X = \bigcup \varphi_i(S_i)$*.*

Example 1.4 *If X is a scheme, define the associated* small étale site $X_{\text{étale}}$ *as follows:*

Consider the category of schemes S over X such that the structure morphism $S \to X$ is étale and as morphisms take morphisms over X that are étale . Coverings are $\{\varphi_i : S_i \to X\}_{i \in I}$ such that $X = \bigcup \varphi_i(S_i)$.

Example 1.5 *If X is a scheme, define the associated* (big) fppf site *as follows:*

Let the underlying category be the category of schemes S over X with morphisms $S \to S'$ over X such that $S \to S'$ is flat and locally of finite presentation. Coverings are $\{\varphi_i : S_i \to X\}_{i \in I}$ such that $X = \bigcup \varphi_i(S_i)$.

2 Sheaves

Now we define abelian presheaves and sheaves on a site. Analogously we can define sheaves of sets or sheaves valued in any category.

Definition 2.1 *Let $\mathcal{C}$ be a site. A* abelian presheaf *on $\mathcal{C}$ is a contravariant functor from the underlying category to the category of abelian groups.*

Definition 2.2 *Let $\mathcal{C}$ be a site and let $\mathcal{F}$ be a abelian presheaf on $\mathcal{C}$. We say that $\mathcal{F}$ is a sheaf if for all coverings $\{U_i \to U\}_{i \in I}$ in $\mathrm{Cov}(\mathcal{C})$ the diagram*

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{(i_0,i_1) \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

is an equivalizer diagram.

Definition 2.3 *A* morphism of presheaves *is a natural transformation between the corresponding functors. A* morphism of sheaves *is a morphism of presheaves between sheaves.*

Definition 2.4 *We get categories* $\mathrm{PShAb}(\mathcal{C})$ *and* $\mathrm{ShAb}(\mathcal{C})$ *associated to the site $\mathcal{C}$.*

3 Sheafification

Definition 3.1 *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf. Define $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$ as the equivalizer of the diagram*

$$\prod_{i \in I} \mathcal{F}(U_i) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{(i_0,i_1) \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

For two coverings $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_i \to U\}_{i \in J}$ a refinement $\mathcal{U} \to \mathcal{V}$ is a map $\sigma : I \to J$ and for $i \in I$ morphisms $U_i \to V_{\sigma(i)}$ such that the diagram

$$\begin{array}{ccc} U_i & \longrightarrow & V_{\sigma(i)} \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

commutes. Now consider the colimit over all the coverings $\mathcal{U} = \{U_i \to U\}_{i \in I}$ *of* U *in* $\mathrm{Cov}(\mathcal{C})$ *cofiltered by refinements.*

Define $\mathcal{F}^+(U) := \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$. $\mathcal{F}^+$ *has a natural presheaf structure.*

We get a functor $\mathrm{PShAb} \to \mathrm{PShAb}$, $\mathcal{F} \mapsto \mathcal{F}^+$.

Moreover there is a natural morphism $\vartheta : \mathcal{F} \to \mathcal{F}^+$.

Definition 3.2 *A presheaf* $\mathcal{F}$ *on a site* $\mathcal{C}$ *is called* separated *if, for all coverings* $\{U_i \to U\}$, *the homomorphism* $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ *is injective.*

Proposition 3.3 *(Stacks Project tag 00WB)*

For a presheaf $\mathcal{F}$ *on* $\mathcal{C}$

(i) The presheaf $\mathcal{F}^+$ *is separated.*

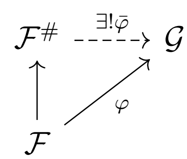
(ii) If $\mathcal{F}$ *is separated, then* $\mathcal{F}^+$ *is a sheaf and the morphism of presheaves* $\vartheta : \mathcal{F} \to \mathcal{F}^+$ *is injective.*

(iii) If $\mathcal{F}$ *is a sheaf, then* $\vartheta : \mathcal{F} \to \mathcal{F}^+$ *is an isomorphism.*

(iv) The presheaf $\mathcal{F}^\# = \mathcal{F}^{++}$ *is always a sheaf.*

Definition 3.4 *Now we can define the sheafification functor* $\mathrm{PShAb} \to \mathrm{ShAb}$, $\mathcal{F} \mapsto \mathcal{F}^\#$.

Proposition 3.5 *(Stacks Project tag 00WH) Let* $\mathcal{F}$ *be presheaf and* $\mathcal{G}$ *a sheaf. Let* $\varphi : \mathcal{F} \to \mathcal{G}$ *be a morphism of presheaves. Then there is a unique morphism of sheaves* $\bar{\varphi} : \mathcal{F}^\# \to \mathcal{G}$ *such that*



commutes.

Proposition 3.6 *Sheafification* $\mathrm{PShAb} \to \mathrm{ShAb}$, $\mathcal{F} \mapsto \mathcal{F}^\#$ *is left adjoint to the forgetful functor* $\iota : \mathrm{ShAb} \to \mathrm{PShAb}$.

$$\mathrm{Hom}_{\mathrm{ShAb}}(\mathcal{F}^\sharp, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{PShAb}}(\mathcal{F}, \iota\mathcal{G}).$$

Proposition 3.7 *(Stacks Project tag 00WJ) Sheafification preserves arbitrary colimits and finite limits.*

4 Constant sheaves

Definition 4.1 *Let $\mathcal{C}$ be a site and let A be an abelian group. The sheafification of the presheaf $\underline{A}_{\text{PShAb}} : U \mapsto A$ is called the* constant sheaf *with value A. Notation $\underline{A}$. Get a functor $\underline{(\text{-})} : \text{Ab} \to \text{ShAb}$.*

Definition 4.2 *If we have a final object X in $\mathcal{C}$ (like X in $X_{\text{étale}}$) we can define the global sections functor as $\Gamma_{\text{PShAb}} : \text{PShAb} \to \text{Ab}, \mathcal{F} \mapsto \mathcal{F}(X)$ and $\Gamma : \text{ShAb} \to \text{Ab}, \mathcal{F} \mapsto \mathcal{F}(X)$. (There is also a definiton for general sites.)*

As in the Algebraic Geometry course it follows:

Proposition 4.3 *$\underline{(\text{-})}_{\text{PShAb}}$ is left adjoint to Γ_{PShAb} and $\underline{(\text{-})}$ is left adjoint to Γ.*

5 The category of sheaves is abelian

Let $\mathcal{C}$ be a site.

Proposition 5.1 *The category* PShAb *has all limits and all colimits.*

Proof. Check that limits and colimits in a presheaf category can be computed object-wise. □

Proposition 5.2 *The category* ShAb *has all limits and all colimits:*
$\varprojlim_{i,\text{ShAb}} \mathcal{F}_i = \varprojlim_{i,\text{PShAb}} \mathcal{F}_i$ *and* $\varinjlim_{i,\text{ShAb}} \mathcal{F}_i = (\varinjlim_{i,\text{PShAb}} \mathcal{F}_i)^{\#}$.

Proof. Use that the limit in the sheaf condition commutes with limits and that sheafification commutes with colimits as a left adjoint. □

Proposition 5.3 *The category* PShAb *is abelian.*

Proof. Define addition of morphisms object-wise. Use Prop 5.1. Check the isomorphism theorem object-wise. □

Proposition 5.4 *The category* ShAb *is abelian.*

Proof. Define addition of morphisms object-wise. Use Prop 5.2. To verify the isomorphism theorem use that sheafification commutes with the image and the coimage as it commutes with kernels and cokernels. □

Remark 5.5 *Therefore we get a notion of exact sequences of sheaves on sites, of injective sheaves and so on.*

6 Stalks

From now on lets work with the étale site $S_{\text{étale}}$ of some scheme S.

Definition 6.1 *Let $\bar{s}$ be a geometric point of S. An* étale neighbourhood *of $\bar{s}$ is a commutative diagram*

$$\begin{array}{ccc} & & U \\ & \overset{\bar{u}}{\nearrow} & \downarrow \varphi \\ \operatorname{Spec}(\bar{k}) & \xrightarrow[\bar{s}]{} & S \end{array}$$

where φ is étale. Denoted by $(U, \bar{u})$.

A morphism of étale neighbourhoods $(U, \bar{u}) \to (U', \bar{u}')$ *is $h : U \to U'$ such that $\bar{u}' = h\bar{u}$.*

Remark 6.2 *The category of étale neighbourhoods is cofiltered.*

Definition 6.3 *For $\mathcal{F}$ on $Y_{\text{étale}}$ define $\mathcal{F}_{\bar{s}} = \varinjlim_{(U,\bar{u})} \mathcal{F}(U \to X)$. We get a functor* $\mathrm{PShAb}(S_{\text{étale}}) \to \mathrm{Ab}, \mathcal{F} \mapsto \mathcal{F}_{\bar{s}}$

Proposition 6.4 *(Stacks Project tag 03PT) The functor* $\mathrm{PShAb}(S_{\text{étale}}) \to \mathrm{Ab}, \mathcal{F} \mapsto \mathcal{F}_{\bar{s}}$ *is exact.*

Lemma 6.5 *(Stacks Project tag 03PR) Let S be a scheme. Let $\bar{s}$ be a geometric point of S. Let $(U, \bar{u})$ be a étale neighbourhood of $\bar{s}$ and let $\{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Then there exits i in I with $u_i : \bar{s} \to U_i$ such that $\varphi_i : (U_i, \bar{u}) \to (U, \bar{u})$ is a morphism of étale neighbourhoods.*

For this we need the following claim:

Claim 6.6 *(Stacks Project tag 03PC) Let k be a field. Any étale morphism $X \to \operatorname{Spec}(k)$ is of the form $X = \coprod \operatorname{Spec}(k_j) \to \operatorname{Spec}(k)$ for k_j/k finite separable.*

Proposition 6.7 *(Stacks Project tag 03PT) For a presheaf $\mathcal{F}$ we have $\mathcal{F}_{\bar{s}} \cong (\mathcal{F}^{\#})_{\bar{s}}$.*

Proposition 6.8 *(Stacks Project tag 03PT) For every geometric point $\bar{s} \in S$ the functor* $\mathrm{ShAb} \to \mathrm{Ab}, \mathcal{F} \mapsto \mathcal{F}_{\bar{s}}$ *is exact.*

Proposition 6.9 *(Stacks Project tag 03PU)*

(i) A morphism of sheaves is injective if and only if it is injective on the stalks of all geometric points.

(ii) A morphism of sheaves is surjective if and only if it is surjective on the stalks of all geometric points.

(iii) A morphism of sheaves is an isomorphism if and only if it is an isomorphism on the stalks of all geometric points.

(iv) A sequence of sheaves is exact if and only if it is exact on the stalks of all geometric points.

Proposition 6.10 *Stalks at geometric points of constant sheaf $\underline{A}$ are A.*

Example 6.11 *Let $\bar{k}$ be algebraically closed. Then there is an equivalence of categories* $\mathrm{ShAb}(\mathrm{Spec}(\bar{k})_{\text{étale}}) \simeq \mathrm{Ab}, \mathcal{F} \mapsto \mathcal{F}(\mathrm{Spec}(\bar{k})), \underline{A} \hookleftarrow A$

Proof. uses Claim 6.6. □

7 Pushforward

Definition 7.1 *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a presheaf on X. The* direct image, *or* pushforward *of $\mathcal{F}$ (under f) is*

$$\begin{aligned} f_*\mathcal{F} : Y^{\text{opp}}_{\text{étale}} &\to \mathrm{Ab} \\ (V \to Y) &\mapsto \mathcal{F}(X \times_Y V \to X) \end{aligned}$$

We get a functor

$$f_* : \mathrm{PShAb}(X_{\text{étale}}) \to \mathrm{PShAb}(Y_{\text{étale}})$$

Proposition 7.2 *(Stacks Project tag 03PX) If $\mathcal{F}$ is a sheaf then $f_*\mathcal{F}$ is a sheaf.*

Definition 7.3 *We get a functor*

$$f_* : \mathrm{ShAb}(X_{\text{étale}}) \to \mathrm{ShAb}(Y_{\text{étale}})$$

8 Pullback

Definition 8.1 *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{G}$ be a presheaf on Y. For every $U \to X$ consider the category I_U where objects are étale $V \to Y$ together with $U \to X \times_Y V$ over X and where the morphisms are étale $V \to V'$ such that*

$$\begin{array}{ccc} U & \longrightarrow & X \times_Y V \\ & \searrow & \downarrow \\ & & X \times_Y V' \end{array}$$

commutes. The inverse image, *or* pullback *of $\mathcal{G}$ (under f) is the coproduct*

$$\begin{aligned} f^+\mathcal{F} : X^{\text{opp}}_{\text{étale}} &\to \mathrm{Ab} \\ (U \to X) &\mapsto \varinjlim_{(V \to Y, U \to X \times_Y V) \in I_U^{\text{opp}}} \mathcal{F}(V \to Y) \end{aligned}$$

Remark 8.2 *This is a filtered colimit.*

Proposition 8.3 *(Stacks Project tag 03PZ)* $f^+ : \mathrm{PShAb}(Y_{\text{étale}}) \to \mathrm{PShAb}(X_{\text{étale}})$ *is left adjoint to* $f_* : \mathrm{PShAb}(X_{\text{étale}}) \to \mathrm{PShAb}(Y_{\text{étale}})$*.*

Definition 8.4 *Define* $f^{-1} : \mathrm{ShAb}(Y_{\text{étale}}) \to \mathrm{ShAb}(X_{\text{étale}})$ *as the composite*

$$\mathrm{ShAb}(Y_{\text{étale}}) \xrightarrow{\iota} \mathrm{PShAb}(Y_{\text{étale}}) \xrightarrow{f^+} \mathrm{PShAb}(X_{\text{étale}}) \xrightarrow{(-)^\#} \mathrm{ShAb}(X_{\text{étale}})$$

Proposition 8.5 *(Stacks Project tag 03PZ)* $f^{-1} : \mathrm{ShAb}(Y_{\text{étale}}) \to \mathrm{ShAb}(X_{\text{étale}})$ *is left adjoint to* $f_* : \mathrm{ShAb}(X_{\text{étale}}) \to \mathrm{ShAb}(Y_{\text{étale}})$.

Proposition 8.6 *(Stacks Project tag 03Q1) Let* $\bar{x} \in X$ *be a geometric point. Then for a presheaf* $\mathcal{F}$*:* $(f^+\mathcal{F})_{\bar{x}} \cong \mathcal{F}_{f\bar{x}}$*. And for a sheaf* $\mathcal{F}$*:* $(f^{-1}\mathcal{F})_{\bar{x}} \cong \mathcal{F}_{f\bar{x}}$

Proposition 8.7 *(Stacks Project tag 03Q1) The functor* $f^{-1} : \mathrm{ShAb}(Y_{\text{étale}}) \to \mathrm{ShAb}(X_{\text{étale}})$ *is exact.*

Proposition 8.8 *(Stacks Project tag 015Z) Pushforward preserves injectivity of sheaves.*

9 Existence of enough injectives

Proposition 9.1 *The category* $\mathrm{ShAb}(X_{\text{étale}})$ *has enough injectives.*

Proof. (Sketch) For a given sheaf $\mathcal{F} \in \mathrm{ShAb}(X_{\text{étale}})$ choose injections into injective groups $\mathcal{F}_{\bar{x}} \hookrightarrow I^x$. Consider $\mathcal{F} \to \prod_x \bar{x}_*(\underline{\mathcal{F}_x}) \to \prod_x \bar{x}_*(\underline{I}^x)$. Use Example 6.11 and Proposition 8.8 to check that $\prod_x \bar{x}_*(\underline{I}^x)$ is injective. Check injectivity of $\mathcal{F} \to \prod_x \bar{x}_*(\underline{\mathcal{F}_x}) \to \prod_x \bar{x}_*(\underline{I}^x)$ on stalks. $\square$

10 Étale cohomology

Since $\mathrm{ShAb}(X_{\text{étale}})$ has enough injectives and since the global sections functor is left exact (as it is right adjoint to the constant sheaf functor) we can apply the construction in section 7 of talk 1.

For $i \in \mathbb{Z}^{\geqslant 0}$ define the etale cohomology functors as the right derived functors

$$H^i(-) = R^i\Gamma(-) : \mathrm{ShAb}(X_{\text{étale}}) \to Ab.$$

References

[1] The Stacks Project Authors, The Stacks project