Proper base change in low degrees

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1 Introduction

Recall the proper base change theorem: Let $f\colon X \to Y$ be a proper morphism and let $\mathcal{F} \in \mathbf{Sh}(X_{\acute{e}t})$ be a torsion sheaf. Then we want to prove that for every cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{e} & X \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

and every $q \geqslant 0$ the base change homomorphism

$$g^{-1}(R^q f_* \mathcal{F}) \to R^q h_*(e^{-1}\mathcal{F})$$

is an isomorphism. In this talk, we consider the special case where

- S is the spectrum of a noetherian strictly henselian local ring $(A, \mathfrak{m})$, and S' is the closed point of $S = \operatorname{Spec} A$,
- $\mathcal{F}$ is a constant sheaf of the form $\underline{\mathbb{Z}/n\mathbb{Z}}$,
- the morphism h, which is the special fiber of f, is projective.

Then by Theorem 5.1 of Emil's talk, the proper base change theorem reduces to showing that for every $q \geqslant 0$,

$$H^q(X, \underline{\mathbb{Z}/n\mathbb{Z}}) \to H^q(X_0, \underline{\mathbb{Z}/n\mathbb{Z}})$$

is an isomorphism. We show that this is an isomorphism for $q = 0$ and surjective for $q = 1$.

2 Formal geometry

We study formal geometry only in the locally noetherian setting. The following proposition hints at the problems one might otherwise encounter.

Proposition 1 (Reminder on completions)**.** *Let A be a noetherian ring and let $\mathfrak{a} \subset A$ be an ideal.*

i) The $\mathfrak{a}$-adic completion of A is complete.

ii) $\mathfrak{a}$-adic completion preserves exactness of sequences of finitely generated A-modules.

iii) Every maximal ideal of $\widehat{A}$ contains $\widehat{\mathfrak{a}}$.

Proof. [2, Chapter 10]. ■

We introduce the notion of *formal schemes*. These are certain *topologically ringed spaces*, i. e., topological spaces endowed with a sheaf of topological rings.
Let A be a noetherian ring and let $\mathfrak{a} \subset A$ be an ideal such that A is complete and Hausdorff with respect to the $\mathfrak{a}$-adic topology. We call $\mathfrak{a}$ an *ideal of definition* for A.

Definition. The *formal spectrum* $\operatorname{Spf} A$ of A is the topologically ringed space with underlying topological space $\mathcal{X} := \operatorname{Spec}(A/\mathfrak{a})$ and structure sheaf of topological rings

$$\mathcal{O}_{\mathcal{X}} := \varprojlim_{n \geqslant 0} \mathcal{O}_{X_n}, \qquad X_n = \operatorname{Spec}(A/\mathfrak{a}^{n+1}).$$

Since we may compute limits of sheaves objectwise,

$$\mathcal{O}_{\mathcal{X}}(U) = \varprojlim_{n} \mathcal{O}_{X_n}(U), \qquad U \subset \mathcal{X}.$$

Definition. A *formal scheme* is a topologically ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ that has an open cover $\mathcal{X} = \bigcup_{i \in I} \mathcal{U}_i$ by topologically ringed spaces $\mathcal{U}_i$ isomorphic to formal spectra.
A *morphism of formal schemes* $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of locally ringed spaces for which the $f^\flat(V)\colon \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ are all continuous.

The formal spectrum comes with a morphism $\operatorname{Spf} A \to \operatorname{Spec} A$, induced by the embeddings $\operatorname{Spec} A/\mathfrak{a}^n \hookrightarrow \operatorname{Spec} A$. By Proposition 1.iii), $\operatorname{Spf} A$ contains all closed points of $\operatorname{Spec} A$.
Here is an example to keep in mind: Let $A = k[[t]]$ be a ring of power series and let $\mathfrak{a} = (t)$. Then as a topological space, $\operatorname{Spf} A$ only contains the one closed point of $\operatorname{Spec} A$. But its functions are all the power series $k[[t]]$!

Proposition 2. *For every formal scheme $\mathcal{X}$ and every formal spectrum $\mathcal{Y} = \operatorname{Spf} S$ there exists a natural bijection*

$$\operatorname{Mor}(\mathcal{X}, \mathcal{Y}) \cong \operatorname{Hom}_{cont}(S, \mathcal{O}_{\mathcal{X}}(\mathcal{X})), \qquad f \mapsto f^\flat(\mathcal{Y}).$$

Proof. The same as for schemes. First consider the case where $\mathcal{X} = \operatorname{Spf} R$ is a formal spectrum. Then every continuous ring homomorphism $\varphi\colon S \to R$ induces a continuous map $\mathcal{X} \to \mathcal{Y}$ by pullback. ■

Definition. An element $x \in R$ of a topological ring is called *topologically nilpotent* if $\lim_{n\to\infty} x^n = 0$.

Let $\mathcal{X}$ be a formal scheme. Denote by $\mathcal{I}_{\mathcal{X}}$ the ideal sheaf of topologically nilpotent functions. Then on any formal affine subscheme $\operatorname{Spf} A \subset \mathcal{X}$, the ideal $\mathcal{I}_{\mathcal{X}}(\operatorname{Spf} A)$ is an ideal of definition for A. Such an ideal sheaf is called *ideal sheaf of definition* for $\mathcal{X}$. Associate to $\mathcal{X}$ a family of closed embeddings of schemes

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \dots, \qquad X_n = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}_{\mathcal{X}}^{n+1}). \tag{1}$$

We can recover $\mathcal{X}$ as the colimit $\varinjlim_n X_n$.
Conversely, we may *start* with a locally noetherian scheme X and a closed subscheme X_0 cut out by the ideal sheaf $\mathcal{I}$. By setting $X_n := (X_0, \mathcal{O}_X/\mathcal{I}^{n+1})$, we obtain a sequence of schemes like (1). The topologically ringed space $(X_0, \varprojlim_n \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1})$ is then a formal scheme, called the *formal completion* of X along X_0. It comes with a natural morphism $\mathcal{X} \to X$, induced by the inclusions $X_n \hookrightarrow X$.

Proposition 3. *Let $f\colon X \to Y$ be a morphism of locally noetherian schemes and let $X_0 \subset X$ and $Y_0 \subset Y$ be closed subschemes with ideal sheaves $\mathcal{I}$ and $\mathcal{J}$ such that $f(X_0) \subset Y_0$. Let $\mathcal{X}$ be the formal completion of X along X_0 and let $\mathcal{Y}$ be the formal completion of Y along Y_0. Then there exists an induced morphism $\mathcal{X} \to \mathcal{Y}$ making the following diagram commute:*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathcal{Y} & \longrightarrow & Y \end{array}$$

Proof. This follows from functoriality of colimits, because we have commutative diagrams

$$\begin{array}{ccc} X_n & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y_n & \longrightarrow & Y. \end{array}$$

■

We now consider the problem of *algebraization*. Suppose that $\mathcal{Y} = \operatorname{Spf} A$ is the formal spectrum of a complete and Hausdorff ring A, with ideal of definition $\mathfrak{a} \subset A$. Let $Y := \operatorname{Spec} A$. Suppose that $\mathcal{X}$ is a formal scheme over $\mathcal{Y}$. Does there exist a (locally noetherian) scheme X over Y whose $\mathfrak{a}$-adic completion is isomorphic to $\mathcal{X}$?

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & Y \end{array}$$

Theorem 4. *If X_0 is projective over Y_0, then $\mathcal{X}$ is algebraizable.*

Proof. [6, Theorem 8.4.10]. ■

3 The case $q = 0$

Lemma 5. *Let A be a noetherian ring and let $X \to \operatorname{Spec} A$ be a proper morphism. Then $\mathcal{O}_X(X)$ is a finite A-algebra.*

Proof. [4, Theorem 3.2.1]. ■

Lemma 6. *Let A be a noetherian ring and let $\mathfrak{a} \subset A$ be an ideal. Let $f\colon X \to \operatorname{Spec} A$ be a proper morphism. Then*

$$\varprojlim_n \Gamma(X, \mathcal{O}_X)/\mathfrak{a}^n\Gamma(X, \mathcal{O}_X) \cong \varprojlim_n \Gamma(X \otimes_A A/\mathfrak{a}^n, \mathcal{O}_{X\otimes_A A/\mathfrak{a}^n}). \tag{2}$$

Proof. [4, Section 4.1]. ■

We can actually understand (2) as a base change homomorphism:

$$\begin{array}{ccccc} \mathcal{X} & \longrightarrow & X & \text{———} & \mathcal{O}_X \\ \downarrow & & \downarrow & & \\ \operatorname{Spf} A & \longrightarrow & \operatorname{Spec} A & & \end{array}$$

Now denote by $\pi_0(T)$ the set of connected components of a scheme T. Since we only consider constant sheaves, it suffices to prove the following theorem:

Theorem 7. *Let A be a noetherian local henselian ring and let $S = \operatorname{Spec} A$. Let $f\colon X \to S$ be a proper morphism and let X_0 be the fiber of f above the closed point of S. Then the map*

$$\pi_0(X) \to \pi_0(X_0), \qquad E \mapsto E \cap X_0, \tag{3}$$

is a bijection.

Proof. Let S' denote the spectrum $\operatorname{Spec} \mathcal{O}_X(X)$. The trivial factorization of A-algebras $A \to \mathcal{O}_X(X) \to \mathcal{O}_X(X)$ induces a factorization $X \to S' \to S$:

$$\begin{array}{ccc} \mathcal{O}_X(X) & \longleftarrow & A \\ & \nwarrow \quad \swarrow & \\ & \mathcal{O}_X(X) & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} X & \longrightarrow & S \\ & \searrow \quad \nearrow & \\ & \operatorname{Spec} \mathcal{O}_X(X) & \end{array}$$

Here the morphism $S' \to S$ is finite (Lemma 5) and $f'\colon X \to S'$ is proper (Cancellation Theorem for proper morphisms, see [8, Theorem 10.1.19]). Since A is henselian, $S' = \coprod_{i=1}^r \operatorname{Spec} A_i$ for henselian local rings A_i.
We now investigate the fibers of f'. Let $s_i \in S'$ be one of the closed points, corresponding to the maximal ideal $\mathfrak{m}_i \subset A_i$. Applying Lemma 6 to the fiber $X_{s_i} \to s_i$ yields

$$\varprojlim_n S_i/\mathfrak{m}_i^n \cong \varprojlim_n \Gamma(X \otimes_{A_i} A_i/\mathfrak{m}_i^n, \mathcal{O}_{X\otimes_{A_i} A_i/\mathfrak{m}_i^n}).$$

On the left is the $\mathfrak{m}_i$-adic completion of S_i; on the right is the functions of the formal completion of X along the fiber X_{s_i}. Thus this fiber is connected. Now it suffices to show that the $X_i := f'^{-1}(S_i)$ are the connected components of X. They certainly form a disjoint union of X. And they are connected, because any closed point of X_i must map to s_i. ■

4 The case $q = 1$

Let R be a ring. A functor

$$F\colon R\text{-algebras} \quad \to \quad \textbf{Sets}$$

is called *locally of finite presentation* if for every filtered system of R-algebras S_i,

$$F(\varinjlim_i S_i) = \varinjlim_i F(S_i).$$

The following lemma motivates this:

Lemma 8. *An R-module M is finitely presented if and only if the functor* $\mathrm{Hom}_R(M, \cdot)$ *is locally of finite presentation.*

Proof. Suppose that M is finitely presented, say by $R^m \to R^n \to M \to 0$. Since

$$\mathrm{Hom}_R(R^n, \varinjlim_i N_i) = (\varinjlim_i N_i)^n = \varinjlim_i N_i^n = \varinjlim_i \mathrm{Hom}_R(R^n, N_i),$$

locally finite presentation of $\mathrm{Hom}_R(M, \cdot)$ follows from the five lemma.
Conversely, suppose that $\mathrm{Hom}_R(M, \cdot)$ is locally of finite presentation. Write M as a filtered colimit of finitely presented modules $M = \varinjlim_i M_i$. Since

$$\mathrm{Hom}_R(M, M) = \mathrm{Hom}_R(M, \varinjlim_i M_i) = \varinjlim_i \mathrm{Hom}_R(M, M_i),$$

the identity $M \to M$ factors through some M_i. Thus M is a direct summand of M_i, so is finitely presented. ■

In the following theorem, something called an "excellent ring" appears. We need only know that any localization of $\mathbb{Z}$ is excellent. See [7, 07QS] for more information.

Proposition 9 (Artin's approximation theorem)**.** *Let R be a henselization at a prime ideal of a finitely generated algebra over a field or over an excellent discrete valuation ring A. Let*

$$F\colon R\text{-algebras} \quad \to \quad \textbf{Sets}$$

be a locally finitely presented functor. Then for every $\overline{\xi} \in F(\widehat{R})$ there exists a $\xi \in F(R)$ such that the images of $\overline{\xi}$ and ξ in $F(R/\mathfrak{m})$ are the same.

Proof. [1, Theorem 1.12]. ■

Lemma 10. *Let $S_0 \subset S$ be a closed subscheme defined by a nilpotent ideal sheaf. Then the functor*

$$\text{étale } S\text{-schemes} \to \text{étale } S_0\text{-schemes}, \qquad X \mapsto S_0 \times_S X,$$

is part of an equivalence of categories.

Proof. This functor is fully faithful by Theorem 13 from my last talk. For essential surjectivity, see [7, 039R]. The idea is to use the description of étale ring maps using quotients of polynomial rings with invertible Jacobian determinant. ■

Theorem 11. *Let A be a noetherian local henselian ring and let $S = \operatorname{Spec} A$. Let $f\colon X \to S$ be a proper morphism with projective special fiber X_0. Then*

$$H^1(X, \underline{\mathbb{Z}/n\mathbb{Z}}) \to H^1(X_0, \underline{\mathbb{Z}/n\mathbb{Z}})$$

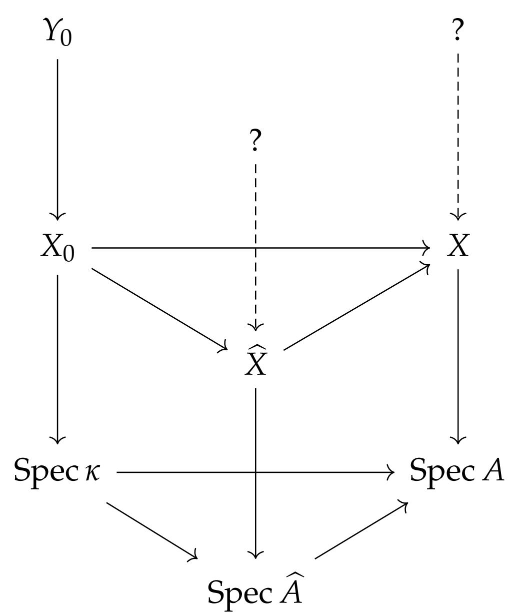
is surjective.

Proof. Recall the theory of torsors from Nicolas' talk last semester. It suffices to show that the map

$$\underline{\mathbb{Z}/n\mathbb{Z}}\text{-torsors on } X \quad \to \quad \underline{\mathbb{Z}/n\mathbb{Z}}\text{-torsors on } X_0$$

is surjective. By Lemma 1.3 from Xiao's talk last week, every torsor $\mathcal{F}$ on X_0 is represented by a finite étale scheme $Y_0 \to X_0$. We show that Y_0 is induced by a finite étale scheme $Y \to X$.

First assume that A is a henselization of a finitely generated algebra over an excellent discrete valuation ring or a field (so we can apply Artin's approximation theorem). Consider the following commutative prism:



It illustrates our strategy of first passing to the completion $\widehat{X} = X \otimes_A \widehat{A}$ and then back to X.
The first step uses formal schemes in an essential way. By Lemma 10, Y_0 is induced by a finite étale X_n-scheme Y_n for every $n \geqslant 1$, where $X_n = X \otimes_A A/\mathfrak{m}^n$. By Proposition 3, this extends to a morphism of formal schemes $\mathcal{Y} \to \mathcal{X}$. Finally, we can use Theorem 4 to obtain an $\widehat{X}$-scheme $\widehat{Y}$ inducing $\mathcal{Y}$:

$$\begin{array}{ccccccc} Y_0 & \hookrightarrow & Y_1 & \hookrightarrow & Y_2 & \hookrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_0 & \hookrightarrow & X_1 & \hookrightarrow & X_2 & \hookrightarrow & \dots \end{array} \qquad \begin{array}{ccc} \mathcal{Y} & \longrightarrow & \widehat{Y} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \widehat{X} \end{array}$$

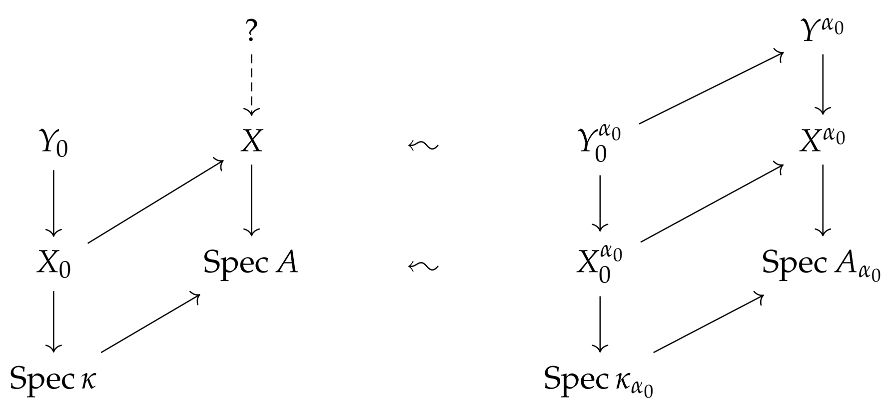
Next, we use Artin's approximation theorem applied to the functor

$$F\colon A\text{-algebras} \quad \to \quad \mathbf{Sets}$$

that associates to each A-algebra B the set of isomorphism classes of finite étale coverings of $X \otimes_A B$. To verify that F is locally of finite presentation, we identify each étale covering in $F(B)$ with the corresponding locally free sheaf of algebras $\mathcal{A}$ on $X \otimes_A B$. Then finite presentation follows from [4, Theorem 8.5.2]. Thus we obtain the desired finite étale covering $Y \to X$.
Finally we discuss the reduction to A being a henselization of a finitely generated algebra over an excellent discrete valuation ring or a field, so now $(A, \mathfrak{m})$ is just any noetherian local henselian ring. We may write A as a filtered colimit $\varinjlim_\alpha B_\alpha$ of finitely generated $\mathbb{Z}$-algebras. Since henselization commutes with colimits,

$$A = A^h = \varinjlim_{\alpha} (B_{\alpha, \mathfrak{m} \cap B_\alpha})^h.$$

Here the $A_\alpha := (B_{\alpha, \mathfrak{m} \cap B_\alpha})^h$ are henselizations of finitely generated algebras over the excellent discrete valuation rings or fields $\mathbb{Z}_{\mathfrak{m} \cap \mathbb{Z}}$, and Artin's approximation theorem applies to them. By [5, Theorem 8.8.2], there exists some index α_0 and a scheme X^α over $\operatorname{Spec} A_{\alpha_0}$ such that $X = X_{\alpha_0} \otimes_{A_{\alpha_0}} A$. Similarly for the covering $Y_0 \to X_0$.



Now the existence of the lifting Y^{α_0} implies the existence of the desired covering Y of X. ■

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