Étale site and Galois cohomology

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1 Étale site of $\operatorname{Spec} k$

The main reference here is [1].

Proposition 1.1 *The coverings in the étale site of* $\operatorname{Spec} k$ *are the form refined to the form*

$$\left(\coprod_{j\in J_i} \operatorname{Spec} k_{ij} \to k\right)_{i\in I}$$

where k_{ij}/k *are finite separable extensions up to isomorphism.*

Proposition 1.2 *An étale presheaf* $\mathcal{F}$ *on* $(\operatorname{Spec} k)_{\text{ét}}$ *is an étale sheaf if and only if*

1. *for any disjoint union* $\coprod U_i$ *we have*

$$\mathcal{F}\left(\coprod U_i\right) = \prod \mathcal{F}(U_i)$$

2. *for all finite, separable extensions* $k''/k'/k$ *such that* k''/k' *is Galois we have that*

$$\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'') \text{ is injective and } \mathcal{F}(\operatorname{Spec} k') = \mathcal{F}(\operatorname{Spec} k'')^{\operatorname{Gal}(k''/k')}.$$

Proof. Pick an arbitary étale map $U \to \operatorname{Spec} k$ and an étale covering $\{U_i \to U\}_{i\in I}$. Then the equalizer diagram is

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i_0,i_1} \mathcal{F}(U_{i_0} \times_U U_{i_1}).$$

This is equal to $\mathcal{F}(U) \to \mathcal{F}(\coprod_i U_i) \rightrightarrows \mathcal{F}(\coprod U_i \times_k \coprod U_i)$. Thus it suffices to check the equalizer diagram for coverings consisting of a single map $(\coprod U_i \to U)$. One can check that it suffices to check the sheaf condition for U and U' connected. An étale scheme over $\operatorname{Spec} k$ is a disjoint union $\coprod \operatorname{Spec} k_i$ for finite separable extensions k_i/k. If we know that the functor respects disjoint unions, it then suffices to check the equalizer diagram for finite separable extensions. Furthermore, since any finite separable extension is contained in a Galois extension, Galois coverings cover finite separable coverings. Hence assume that k''/k' is Galois. Then, since

$$k'' \otimes_{k'} k'' \cong \prod_{\mathrm{Gal}(k''/k')} k'', \quad x \otimes y \mapsto (xg(y))_g,$$

the exactness of the diagram corresponds to the exactness of

$$\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'') \rightrightarrows \prod_{\mathrm{Gal}(k''/k')} \mathcal{F}(\operatorname{Spec} k'').$$

Thus it suffices to show that $\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'')$ is injective and $\mathcal{F}(\operatorname{Spec} k') = \mathcal{F}(\operatorname{Spec} k'')^{\mathrm{Gal}(k''/k')}$. This shows the sufficiency of the conditions. □

Proposition 1.3 *The functors* $\mathbb{G}_m$ *and* $\mathbb{G}_a$ *are étale sheaves on* $\operatorname{Spec} k$.

Proof. Let $k''/k'/k$ be finite, separable extensions. $\mathbb{G}_m(\operatorname{Spec} k') = k'^\times \to \mathbb{G}_m(\operatorname{Spec} k'') = k''^\times$ is injective. The fact that $\mathbb{G}_m(\operatorname{Spec} k'')^{\mathrm{Gal}(k''/k')} = (k'^\times)^{\mathrm{Gal}(k''/k')} = k'^\times$ follows from Galois theory. Similar argument for $\mathbb{G}_a$. □

Proposition 1.4 *The functor* $\mu_n : (\operatorname{Spec} k)_{\text{ét}} \to \mathrm{Ab}, (T \to \operatorname{Spec} k) \mapsto \mu_n(\Gamma(T, \mathcal{O}_T))$ *is an étale sheaf.*

Example 1.5 (Kummer sequence) *Let* K *be a field containing the all* n*-th roots of unity in* $\bar{K}$*, where* $\mathrm{char} K \nmid n$*. Then we have a short exact sequence of étale sheaves on* $\operatorname{Spec} K$

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{x \mapsto x^n} \mathbb{G}_m \to 0.$$

This is checked at all the geometric points, where the sequence becomes $0 \to \mu_n(\bar{K}) \to \bar{K}^\times \to \bar{K}^\times \to 0$.

2 Relation of Étale cohomology to Galois cohomology

Let G be a topological group.

Proposition-Definition 2.1 *The following are equivalent:*

1. *G is the limit of a cofiltered system of discrete, finite groups,*
2. *G is Hausdorff, compact and totally disconnected.*

A topological group satisfying this condition is called a profinite group.

Proposition 2.2 *A subgroup of a compact group is open if and only if it is closed and has finite index.*

Example 2.3 (The examples to keep in mind) *Any finite group is a profinite group. Let k be a field and fix a separable closure k^{sep}. Then the absolute Galois group G_k is a profinite group.*

Definition 2.4 *A* G-module *is an abelian group M with a G acting by group homomorphism. A* discrete G-module *is a G-module such that the action of G is continuous with respect to the discrete topology of M. Equivalently, for all $m \in M$, the stabilizer* $\mathrm{Stab}_G(m) \subset G$ *is open. A* morphism between discrete G-modules *is G-equivariant group homomorphism.*

Theorem 2.5 *Let k be a field and choose a separable closure k^{sep} of k. We denote by $G_k := \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Then the functor*

$$\mathrm{Ab\acute{E}t}(\mathrm{Spec}\, k) \to \{\text{discrete } G_k\text{-sets}\}, \quad \mathcal{F} \mapsto \mathrm{colim}_{k^{\mathrm{sep}} \supset k' \supset k, k''/k' galois} \mathcal{F}(k')$$

has the quasi-inverse

$$\left(\overline{M}\colon \mathrm{Spec}\, k' \mapsto (M^{\mathrm{Gal}(k^{\mathrm{sep}}/k')}\right) \leftarrow\!\shortmid M,$$

i.e. the two categories are equivalent.

Proof. We first show that these functors are well-defined. By functionality of both $\mathcal{F}$ and Spec, the actions of G_k on the $\mathcal{F}(k')$ is compatible, hence we get a G_k-action on $\mathrm{colim}_{k^{\mathrm{sep}} \supset k' \supset k, k''/k' \mathrm{galois}} \mathcal{F}(k')$. Since $\mathcal{F}$ is an abelian sheaf, we know that $\mathrm{colim}_{k^{\mathrm{sep}} \supset k' \supset k, k''/k' \mathrm{galois}} \mathcal{F}(k')$ is a G_k-module. The stabilizer of any element $f = [l, a \in \mathcal{F}(l)] \in \mathrm{colim}_{k^{\mathrm{sep}} \supset k' \supset k} \mathcal{F}(k')$ is

$$\mathrm{Stab}_{G_k}(f) = \ker(G \to \mathrm{Gal}(l/k))$$

which is open. Hence $\mathrm{colim}_{k^{\mathrm{sep}} \supset k' \supset k} \mathcal{F}(k')$ is a discrete G_k-module.

$\overline{M}$ respects disjoint unions by definition. Furthermore for $k''/k'/k$ finite separable and k''/k' Galois we have

$$\overline{M}(k') = M^{\mathrm{Gal}(k^{\mathrm{sep}}/k')} \to \overline{M}(k'')^{\mathrm{Gal}(k^{\mathrm{sep}}/k'')} = M^{\mathrm{Gal}(k^{\mathrm{sep}}/k'')}.$$

Furthermore, $\overline{M}(k'')^{\mathrm{Gal}(k''/k')} = M^{\mathrm{Gal}(k^{\mathrm{sep}}/k')} = \overline{M}(k')$. Hence $\overline{M}$ is an étale sheaf. Then

$$\mathrm{colim}\overline{M}(k') = \mathrm{colim} M^{\mathrm{Gal}(k^{\mathrm{sep}}/k')} \cong M.$$

Furthermore

$$\overline{\mathrm{colim}_{k''}\mathcal{F}(k'')}(k') = (\mathrm{colim}_{k''}\mathcal{F}(k''))^{\mathrm{Gal}(k^{\mathrm{sep}}/k')} \cong \mathcal{F}(k').$$

Thus the two functors are quasi-inverse to each other and thus the categories are equivalent to each other. □

Proposition-Definition 2.6 *The category of discrete G_k-modules is abelian and has enough injectives. We denote by $H^\bullet(G_k, M)$ the right derived functor of the functor of G_k-invariants $M \mapsto M^{G_k}$ and we call $H^\bullet(G_k, M)$ the* galois cohomology *with coefficients in M.*

Proposition 2.7 *Let $\mathcal{F} \in \mathrm{Ab\acute{E}t}(k)$. Then*

$$H^n_{\text{ét}}(\operatorname{Spec} k, \mathcal{F}) \cong H^n(G_k, \operatorname{colim} \mathcal{F}(k')).$$

Proof. The equivalence of categories identifies the functor of G_k-invariants with the global-sections functor. □

3 Group cohomology of profinite groups

The reference here is [5], with details from [2]. The finite case is explained in [4].

Definition 3.1 *Let G be a profinite group and M a discrete G-module. The* group of n-cochains of G with value in M *is the group of continuous functions $G^n \to M$. We denote this by $C^n(G, M)$. Together with the coboundaries*

$$d : C^n(G, M) \to C^{n+1}(G, M)$$

$$f \mapsto df(g_1, \ldots, g_{n+1}) = g_1 f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n),$$

this forms a complex $C^\bullet(G, M)$. The homology of this complex is called the cohomology of G with coefficients in M.

Proposition 3.2 *Let G be a group. Denote by BG the classifying space of principal G-bundles. Then*

$$H^\bullet(G, \mathbb{Z}) = H^\bullet_{sing}(BG, \mathbb{Z}).$$

Example 3.3 (Ch 2, Section 4, Ex 1 in [4]) *Let S be a set. Then the classifiying space of the free group $F(S)$ is a bouquet of circles indexed by S. Thus*

$$H^i(F(S), \mathbb{Z}) = H^i(BF(S), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}^S, & i = 1, \\ 0, & i > 1 \end{cases}.$$

Proposition 3.4 *For any profinite group G and discrete G-module M, the group $H^0(G, M) = M^G$. Furthermore,*

$$H^1(G, M) = \frac{\{f \in C^1(G, M) | f(xy) = xf(y) + f(x)\}}{\{f \in C^1(G, M) | f(x) = xm - m \text{ for some } m \in M\}}.$$

If G acts trivially on M, then $H^1(G, M) = \mathrm{Hom}_{\mathrm{cont}}(G, M)$.

Proof. The zero-th cohomology group $H^0(G, M)$ is the kernel of $C^0(G, M) = M \to C^1(G, M), m \mapsto (g \mapsto gm - m)$, whose elements are exactly the G-invariant elements of M. The coboundary is the map

$$C^1(G, M) \to C^2(G, M), f \mapsto ((a, b) \mapsto af(b) - f(ab) + f(a).$$

Then

$$H^1(G, M) = \frac{\{f : G \to G | f(xy) = xf(y) + f(x)\}}{\{f : G \to G | f(x) = xm - m \text{ for some } m \in M\}}.$$

If G acts trivially on M, then this translates to $\frac{\{f:G\to M|f(xy)=f(x)+f(y)\}}{\{f:G\to M|f(x)=0\}} = \mathrm{Hom}_{\mathrm{cont}}(G, M)$. □

Example 3.5 *Let L/K be a finite Galois extension. Then $L^\times$ is a discrete $\mathrm{Gal}(L/K)$-module. Then $H^0(\mathrm{Gal}(L/K), L^\times) = K^\times$. We now show that $H^1(\mathrm{Gal}(L/K), L^\times) = 1$. Pick a 1-cocycle f. Then $f(gh) = f(g)gf(h)$ or equivalently $gf(h) = f(g)^{-1}f(gh)$. The coboundary condition is $f'(g) = g(a)/a$ for some $a \in L^\times$. We have*

$$\mathrm{Nm}_{L/L^{\langle h\rangle}}(f(h)) = \prod_{g\in\langle h\rangle} gf(h) \overset{\text{1-cocycle}}{=} \prod_{g\in\langle h\rangle} f(gh)/f(g) = \prod_{i=0}^{n} f(h^{n+1})/f(h^n) = 1.$$

Thus, by Hilbert 90 for $L/L^{\langle h\rangle}$, there exists $b \in L^\times$ such that $f(g) = g(b)/b$.

Proposition 3.6 *Let G be a profinite group. A map $G^n \to M$ into a discrete G-module is a n-cochain if any only if there exists an open normal subgroup $U \subset G$ such that the diagram*

$$\begin{array}{ccc} G^n & \longrightarrow & M \\ \downarrow & & \uparrow \\ (G/U)^n & \longrightarrow & M^U. \end{array}$$

commutes.

Proof. If there exists such an U, then the morphism $G^n \to M$ is the composition of three continuous maps and is thus continuous. To show necessity, assume that f is continuous. Since G^n is compact, the image of G^n in M with the discrete topology must be finite. Thus, since M is a discrete G-module, the cochain f must factor through $M^{\bigcap'_{m\in im(f)} \mathrm{Stab}_G(m)}$. Let V be an open normal subgroup $V \subset \bigcap'_{m\in im(f)} \mathrm{Stab}_G(m)$. Thus f factors as $f : G \to M^V \to M$.

Furthermore, for any $m \in f(G^n)$, the fiber $f^{-1}(m)$ is compact. By continuity, for each $x \in f^{-1}(m)$, there exist an open normal subgroup $U_x \subset G$ such that $f(x(U_x)^n) = m$. Since $f^{-1}(m) = \bigcup_{x\in f^{-1}(m)} x(U_x)^n$ is compact, we can cover it by finitely many $x(U_x)^n$. Thus there exits an open normal subgroup $V' = \bigcap'(U_x)$ such that $f(x(V')^n) = f(x)$ for all

$x \in G^n$.

Then set $U = V \cap V'$. Since $M^V \subset M^U$, the map f factors through $G \to M^U$. Furthermore, since $f(x(U)^n) \subset f(x(V')^n) = f(x)$, the map factors as stated in the statement. □

Proposition 3.7 *Let G be a profinite group and M a discrete G-module. Then for each open normal subgroup $U \subset G$, the group of U-invariants M^U is a G/U-module and*

$$\mathrm{colim}_U H^i(G/U, M^U) \cong H^i(G, M).$$

Proof. We claim that the canonical morphism $\lim C^\bullet(G/U, M^U) \to C^\bullet(G, M)$ is an isomorphism. Since every continuous cochain $f : G^n \to M$ factors through some $(G/U)^n \to M^U$ by the previous Proposition, the map is a surjection. Since taking the cofiltered colimit commutes with taking homology, the statement follows. □

Proposition 3.8 *For any short exact sequence of G-modules $0 \to M' \to M \to M'' \to 0$, we get a short exact sequence*

$$0 \to C^\bullet(G, M') \to C^\bullet(G, M) \to C^\bullet(G, M'') \to 0$$

exact in every degree.

Proof. Let U be an open normal subgroup of G. We have a left exact sequence of G/U-modules $0 \to M'^U \to M^U \to M''^U$. Since $C^n(G/U, \cdot) = \mathrm{Hom}_{\mathbb{Z}[G/U]}((G/U)^n, \cdot)$ we know that $C^n(G/U, \cdot)$ is left exact and thus this is a left-exact sequence

$$0 \to C^n(G/U, M'^U) \to C^n(G/U, M^U) \to C^n(G/U, M''^U).$$

Since $\lim C^n(G/U, M^U) = C^n(G, M)$ and filtered limits preserve exactness, it just remains to show that $C^n(G, M) \to C^n(G, M'')$ is surjective. Let $s : M'' \to M$ be a set-theoretic section of the surjection $M \to M''$. The map s is continuous with respect to the discrete topologies on M and M''. Thus, for any cochain $f : G^n \to M''$, the map $s \circ f : G^n \to M$ is continuous and lifts f by construction. □

Proposition 3.9 *The sequence of functors $(M \to H^i(G, M))$ forms a δ-functor whose degree zero one component is $M \to M^G$.*

Proof. The differential $C^0(G, M) \to C^1(G, M)$ is $m \mapsto (g \mapsto (g - 1)m)$. Hence $H^0(C^\bullet(G, M)) = M^G$. The existence of the long exact sequence follows the snake lemma, as for instance in singular cohomology. For any morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

we get a morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^\bullet(G,M') & \longrightarrow & C^\bullet(G,M) & \longrightarrow & C^\bullet(G,M'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^\bullet(G,N') & \longrightarrow & C^\bullet(G,N) & \longrightarrow & C^\bullet(G,N'') & \longrightarrow & 0 \end{array}$$

exact in every degree. Hence we have the necessary commuting diagram

$$\begin{array}{ccc} H^i(G,M'') & \longrightarrow & H^i(G,M') \\ \downarrow & & \downarrow \\ H^{i+1}(G,N'') & \longrightarrow & H^{i+1}(G,N') \end{array}$$

and thus we have a δ-functor. □

Theorem 3.10 *For any profinite group G and discrete G-module we have a unique isomorphism*

$$H^\bullet(C^\bullet(G,M)) \cong H^\bullet(G,M).$$

Lemma 3.11 *Let G be a finite group. Denote by P_n the free $\mathbb{Z}$-module with generators $(g_0,\dots,g_n) \in G^{n+1}$ (i.e. for $n \leqslant -2$, $P_n = 0$). We equip it with the diagonal action of G:*

$$g(g_0,\dots,g_n) = (gg_0,\dots,gg_n).$$

For $n \geqslant 0$, we define

$$d_n : P_n \to P_{n-1}, \quad (g_0,\dots,g_n) \mapsto \sum_{i=0}^{n} (-1)^i (g_0,\dots,\hat{g}_i,\dots,g_n).$$

Then

$$\dots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

is an exact complex of G-modules.

Proof. Its a computation to check that this is a complex. To show exactness, we construct a homotopy $(h_n : P_n \to P_{n+1})_{n\in\mathbb{Z}}$ which contracts the identity map. Then the identity and the zero map induce the same map on the cohomology of $(P_n, d_n)_n$. Set

$$h_n = \begin{cases} 0, n \leqslant -2 \\ h_{-1} : \mathbb{Z} \to P_0 \quad x \mapsto x(1), n = -1 \\ h_n : P_n \to P_{n+1}, \quad (g_0,\dots,g_n) \mapsto (1,g_0,\dots,g_n) \end{cases}.$$

It is a computation to check that this is a homotopy. □

Proof. [Proof of 3.10] It suffices to show that the functor $H^\bullet(G, \cdot)$ is effacable. For this fix an arbitrary injective discrete G-module J and $G = \lim_i G_i$.
1. Case: finite group The elements

$$(1, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_n)$$

form a basis of the left $\mathbb{Z}[G]$-module P_n. We thus have an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}[G]}(P_n, I) \xrightarrow{\cong} C^n(G, I), \quad f \mapsto (g_1, \ldots, g_n) \mapsto f(1, g_1, g_1g_2, \ldots, g_1g_2 \cdot g_n), \quad n \geqslant 0.$$

It's a computation to show that the differential $d_{n+1} : P_{n+1} \to P_n$ induces the differential $C^n(G, I) \to C^{n+1}(G, I)$. Thus the applying the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(-, I)$ to the complex $P_\bullet$ we obtain the complex $C^\bullet(G, I)$. The complex is exact away from degree 0. Since I is an injective G-module, the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(-, I)$ is exact. Thus applying it to $P_\bullet$ we get a complex which is exact away from degree 0. Thus $C^\bullet(G, I)$ is exact away from degree 0.

2. Case: Profinite group Let $U \subset G$ be an open normal subgroup. For each left G/U-module M we have

$$\mathrm{Hom}_{\mathbb{Z}[G/U]}(M, I^U) = \mathrm{Hom}_G(M, I).$$

Thus, since I is an injective left G-module, we know that I^U is an injective left G/U-module. Since G/U is finite, we know that $H^n(G/U, I^U) = 0$ for $n > 0$. Then by Prop 3.7 $H^n(G, I) = 0$. □

4 Some computations

Proposition 4.1 (Hilbert 90) *We have* $H^1_{\text{ét}}(\mathrm{Spec}\, k, \mathbb{G}_m) = 0$.

Proof. We know that $H^1_{\text{ét}}(\mathrm{Spec}\, k, \mathbb{G}_m) = H^1(G_k, \mathrm{colim}(k'^\times))$. Hence, by Ex. 3.5, we have $H^1(\mathrm{Spec}\, k, \mathbb{G}_m) = \mathrm{colim} H^1(\mathrm{Gal}(L/k), L^\times)$. In the finite group case, we know that $H^1(\mathrm{Gal}(L/K), L^\times) = 0$. □

Example 4.2 (Kummer Theory) *The associated long exact sequence to the Kummer sequence is*

$$0 \to \mu_n \to K^\times \to K^\times \to H^1_{\text{ét}}(\mathrm{Spec}\, K, \mu_n) \to H^1_{\text{ét}}(\mathrm{Spec}\, K, \mathbb{G}_m) \to \ldots$$

By Hilbert 90, we know that $H^1(\mathrm{Spec}\, K, \mathbb{G}_m) = 0$*. Furthermore, since* G_K *acts trivially on* μ_n*, we have*

$$H^1_{\text{ét}}(\mathrm{Spec}\, K, \mu_n) = H^1(G_K, \mu_n) = \mathrm{Hom}(G_K, \mu_n).$$

Thus we get

$$K^\times/(K^\times)^n \cong \mathrm{Hom}_{cont}(G_K, \mu_n).$$

Example 4.3 *We compute* $H^1(\mathrm{Spec}\, k, \underline{\mathbb{Z}})$*. The stalk of the étale sheaf* $\underline{\mathbb{Z}}$ *is* $\mathbb{Z}$ *with the trivial action of the absolute Galois group. Thus*

$$H^1(\mathrm{Spec}\, k, \underline{\mathbb{Z}}) = H^1(G_k, \mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(G_k, \mathbb{Z}) = 0$$

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