

A Sketch of Hodge Theory

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All manifolds are compact, connected, Hausdorff, and second countable, unless explicitly mentioned otherwise. All smooth manifolds are of class C^∞ .

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1 Hodge theory on Riemannian manifolds

1.1 Hodge star

Let V be a \mathbb{R} -vector space of dimension n , and $g: V \times V \rightarrow \mathbb{R}$ a positive-definite inner product. One can naturally extend g to all exterior powers $\Lambda^k V$. Namely, on pure tensors

$$g(v_1 \wedge v_2 \wedge \dots \wedge v_k, w_1 \wedge w_2 \wedge \dots \wedge w_k) = \det g(v_i, w_j).$$

If $\{e_i\}_{i=1}^n$ is an orthonormal basis of V then

$$\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}\}_{1 \leq i_1 < i_2 < \dots < i_k \leq k}$$

is an orthonormal basis of $\Lambda^k V$.

Next, choose a volume form $\text{Vol} \in \Lambda^n V$, i.e. a vector satisfying $g(\text{Vol}, \text{Vol}) = 1$. There are exactly two choices, differing by sign.

Definition 1.1.1. Let $v \in \Lambda^k V$ be a vector. The Hodge star $\star v \in \Lambda^{n-k} V$ of v is the vector which satisfies the equation

$$u \wedge \star v = g(u, v) \text{Vol}.$$

for all $u \in \Lambda^k V$.

Proposition 1.1.2. *The Hodge star has the following properties:*

(1) *For every $k \in \Lambda^k V$ the vector $\star v$ exists and is unique. The map $\star: \Lambda^k V \rightarrow \Lambda^{n-k} V$ is an isomorphism of vector spaces.*

(2) *Choose an orthonormal basis $\{e_k\}_{k=1}^n$ of V . Let $\langle n \rangle$ denote the set $\{1, \dots, n\}$. For $I \subset \langle n \rangle$ of cardinality k define e_I to be the tensor $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}$ where $0 \leq i_1 < i_2 < \dots < i_k \leq n$ are the elements of I .*

With such notation, if $I \subset \langle n \rangle$ then

$$\star e_I = \pm e_{\langle n \rangle \setminus I}.$$

(3) *The map $\star: \Lambda^k V \rightarrow \Lambda^{n-k} V$ is an isometry.*

(4) $\star \star = (-1)^{k(n-k)}$ on $\Lambda^k V$.

Proof. (1) Follows since both pairings $\wedge: \Lambda^k V \times \Lambda^{n-k} V \rightarrow \Lambda^n V$, and $g: \Lambda^k V \times \Lambda^k V \rightarrow \mathbb{R}$ are nondegenerate.

(2) Notice that $\text{Vol} \in \{e_{\langle n \rangle}, -e_{\langle n \rangle}\}$. Hence $e_I \wedge e_{\langle n \rangle \setminus I} = \pm \text{Vol}$. If $J \subset \langle n \rangle$ is a subset of size $n - \#I$ different from $\langle n \rangle \setminus I$, then tensors e_I , and e_J have a basis vector in common, so $e_J \wedge e_{\langle n \rangle \setminus I} = 0$.

(3) Follows from (2) because \star sends an orthonormal basis, to an orthonormal basis.

(4) Indeed, for every $w, v \in \Lambda^k V$

$$\begin{aligned} (w, v) \text{Vol} &= g(\star w, \star v) \text{Vol} = \star w \wedge \star \star v = (-1)^{k(n-k)} \star \star v \wedge \star w = \\ &= (-1)^{k(n-k)} (\star \star v, w) \text{Vol} = (-1)^{k(n-k)} g(w, \star \star v) \text{Vol}. \quad \square \end{aligned}$$

1.2 The Laplacian

Let (X, g) be a compact oriented Riemannian manifold of dimension n with volume form Vol . Extend g to all bundles $\Omega^k X$ of differential forms, and define Hodge stars $\star: \Omega_X^k \rightarrow \Omega_X^{n-k}$ in fiberwise manner. Let $A^k(X)$ be the space of global smooth k -forms. Define an inner product on forms $\alpha, \beta \in A^k(X)$ as

$$(\alpha, \beta) = \int_X \alpha \wedge \star \beta = \int_X g_x(\alpha_x, \beta_x) \text{Vol}.$$

This product is positive-definite. Unless X is a point the \mathbb{R} -vector space $A^k(X)$ is infinite-dimensional, and $(,)$ induces an injective map $A^k(X) \rightarrow A^k(X)^*$ which is not surjective (e.g. consider a linear functional which send a function $f \in A^0(X)$ to its value at a chosen point).

Definition 1.2.1. Let $d: A^k(X) \rightarrow A^{k+1}(X)$ be the de Rham differential. Define an operator $d^*: A^{k+1}(X) \rightarrow A^k(X)$ as

$$d^* = -(-1)^{kn} \star d \star.$$

Proposition 1.2.2. For every $\alpha \in A^k(X)$, $\beta \in A^{k+1}(X)$ there is an equality

$$(d\alpha, \beta) = (\alpha, d^*\beta).$$

In other words d^* is adjoint to d .

Proof. By Stokes theorem, and Leibniz rule

$$0 = \int_X d(\alpha \wedge \star \beta) = \int_X d\alpha \wedge \star \beta + (-1)^k \int_X \alpha \wedge d\star \beta.$$

Moreover,

$$(-1)^k \int_X \alpha \wedge d\star \beta = (-1)^{k+k(n-k)} \int_X \alpha \wedge \star \star d\star \beta.$$

Therefore

$$(d\alpha, \beta) = \int_X d\alpha \wedge \star \beta = -(-1)^{kn} \int_X \alpha \wedge \star \star d\star \beta = (\alpha, d^*\beta). \quad \square$$

Definition 1.2.3. The Laplace operator $\Delta: A^k(X) \rightarrow A^k(X)$ is defined as

$$\Delta = dd^* + d^*d.$$

Δ commutes with d , so it is an endomorphism of the de Rham complex $A^\bullet(X)$. Moreover, Δ is homotopy equivalent to zero via d^* by construction. Here is a picture:

$$\begin{array}{ccc} A^{k+1}(X) & & A^{k+1}(X) \\ \uparrow d & \searrow d^* & \uparrow d \\ A^k(X) & \xrightarrow{dd^* + d^*d} & A^k(X) \\ \uparrow d & \searrow d^* & \uparrow d \\ A^{k-1}(X) & & A^{k-1}(X). \end{array}$$

Definition 1.2.4. A form $\alpha \in A^k(X)$ is called harmonic, if $\Delta\alpha = 0$. The space of such forms is denoted $\mathcal{H}^k(X)$.

Let us compute Δ on \mathbb{R}^2 with its standard Riemannian metric. Of course, \mathbb{R}^2 is not compact, so our definition of inner product on $A^\bullet(\mathbb{R}^2)$ does not make sense. But Hodge stars are well-defined.

Let x, y be coordinates on \mathbb{R}^2 , and let $\text{Vol} = dx \wedge dy$. Then

$$\star dx = dy, \quad \star dy = -dx.$$

If $f \in A^0(X)$ then $d^*f = 0$ by reasons of dimension, so $\Delta f = d^*df$. Next,

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy, \\ \star df &= \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx, \\ d\star df &= \frac{\partial^2 f}{\partial x^2} dx \wedge dy + \frac{\partial^2 f}{\partial y^2} dx \wedge dy, \\ \star d\star df &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}, \\ \Delta f &= -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right). \end{aligned}$$

This explains the name ‘‘Laplacian’’, and the term ‘‘harmonic’’.

1.3 Main theorem

Theorem 1.3.1 (Main theorem of Hodge theory). *Consider the Laplace operator $\Delta: A^k(X) \rightarrow A^k(X)$.*

(1) *There is an orthogonal direct sum decomposition*

$$A^k(X) = \mathcal{H}^k(X) \oplus \text{Im } \Delta.$$

(2) $\dim_{\mathbb{R}} \mathcal{H}^k(X) < \infty$.

We will not prove this theorem since its proof uses advanced tools of functional analysis. The key ingredient of the proof is the fact that Δ is a self-adjoint elliptic differential operator. Using theory of such operators one obtains the main theorem as a formal consequence. Those who are interested in more details are invited to read Demailly's excellent account of analytic methods in complex geometry [2].

Corollary 1.3.2. (1) *The de Rham complex $A^\bullet(X)$ decomposes into an orthogonal direct sum*

$$A^\bullet(X) = \mathcal{H}^\bullet(X) \oplus \text{Im } \Delta.$$

(2) $\mathcal{H}^\bullet(X)$ *is a complex of finite-dimensional vector spaces with zero differentials, and d^* is zero on it.*

(3) $\text{Im } \Delta$ *is homotopy equivalent to zero.*

Proof. (1) Indeed, Δ is an endomorphism of $A^\bullet(X)$, so decompositions of individual terms give rise to a decomposition of the whole complex.

(2) Observe that

$$(\alpha, \Delta\alpha) = (\alpha, dd^*\alpha) + (\alpha, d^*d\alpha) = \|d^*\alpha\|^2 + \|d\alpha\|^2.$$

Hence, if $\Delta\alpha = 0$, then $d\alpha = 0$, and $d^*\alpha = 0$.

(3) The kernel of Δ is orthogonal to its image, so $\Delta|_{\text{Im } \Delta}$ is injective. Moreover $\Delta|_{\text{Im } \Delta}$ is surjective, since writing a form α as $\beta + \Delta\gamma$ with β harmonic, we obtain a formula $\Delta\alpha = \Delta(\Delta\gamma)$. The inverse¹ of Δ automatically commutes with d , so Δ is an automorphism of $\text{Im } \Delta$.

Since Δ commutes with d^* the map d^* defines a homotopy on $\text{Im } \Delta$. By construction,

$$\Delta = dd^* + d^*d.$$

Thus an automorphism of $\text{Im } \Delta$ is homotopy equivalent to zero, which forces the complex itself to be homotopy equivalent to zero. \square

Corollary 1.3.3. *The map $\mathcal{H}^k(X) \rightarrow H^k(X, \mathbb{R})$ sending a harmonic form to its cohomology class is an isomorphism.*

Proof. Omitted. \square

¹This inverse is called the Green operator, and denoted G .

Remark 1.3.4. Informally speaking, with Hodge theory one can transfer structures from the de Rham complex to its cohomology even if these structures do not pass to cohomology directly. For example, Hodge stars give isomorphisms $\star: \mathcal{H}^k(X) \rightarrow \mathcal{H}^{n-k}(X)$, and one can deduce Poincaré duality for X with coefficients in \mathbb{R} from this despite the fact that Hodge stars do not commute with the de Rham differential.

Remark 1.3.5. Wedge product of harmonic forms is not harmonic in general.

2 Hodge theory on complex manifolds

2.1 Almost-complex structures

Definition 2.1.1. An almost-complex structure on an \mathbb{R} -vector space V is an endomorphism $I: V \rightarrow V$ such that $I^2 = -1$.

Proposition 2.1.2. *Let (V, I) be an \mathbb{R} -vector space with almost-complex structure, and let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Extend I to $V_{\mathbb{C}}$ as a \mathbb{C} -vector space endomorphism.*

(1) *There is a decomposition*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

where $V^{1,0}$ is the i -eigenspace, and $V^{0,1}$ the $(-i)$ -eigenspace of I .

(2) *Define a complex conjugation on $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ via \mathbb{C} . Complex conjugation induces an \mathbb{R} -vector space isomorphism $V^{1,0} \rightarrow V^{0,1}$. In particular,*

$$\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1},$$

and $\dim_{\mathbb{R}} V$ is even.

Proof. (1) Consider endomorphisms $\pi^{1,0} = \frac{1}{2}(\text{id} - iI)$, $\pi^{0,1} = \frac{1}{2}(\text{id} + iI)$ of $V_{\mathbb{C}}$, and let $V^{1,0} = \pi^{1,0}(V_{\mathbb{C}})$, $V^{0,1} = \pi^{0,1}(V_{\mathbb{C}})$. A simple computation shows that $\pi^{1,0}, \pi^{0,1}$ define a decomposition of $V_{\mathbb{C}}$, and that $V^{1,0}, V^{0,1}$ so defined are respective eigenspaces of I .

(2) Complex conjugation commutes with I . Hence it sends the i -eigenspace of I to $(-i)$ -eigenspace, and vice versa. Since applying complex conjugation twice gives an identity endomorphism, we obtain what we need. \square

Proposition 2.1.3. *Let (V, I) be an \mathbb{R} -vector space with almost-complex structure.*

(1) Let $V^{p,q} = \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1}$. There is a natural decomposition

$$\Lambda^k V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}.$$

(2) Extend I to $\Lambda^k V_{\mathbb{C}}$ by linearity. Then

$$I|_{V^{p,q}} = i^{p-q}.$$

Proof. Part (1) follows from the fact that given two vector spaces U, W we get a natural decomposition

$$\Lambda^k(U \oplus W) = \bigoplus_{p+q=k} \Lambda^p U \otimes \Lambda^q W.$$

Part (2) is a simple computation. □

Definition 2.1.4. Let (V, I) be an \mathbb{R} -vector space with almost-complex structure. Elements of $\Lambda^k V_{\mathbb{C}}$ belonging to $V^{p,q}$ are said to be of type (p, q) .

If (V, I) is an \mathbb{R} -vector space with almost-complex structure, then one can extend I to the dual V^* by setting $I(\varphi) = \varphi \circ I$. One easily verifies that there are natural isomorphisms $(V^*)^{1,0} = (V^{1,0})^*$, and $(V^*)^{0,1} = (V^{0,1})^*$.

2.2 De Rham differential

Recall that a complex manifold X of complex dimension n is a topological manifold X equipped with a complex structure, that is,

- (1) an open covering $\{U_i\}_{i \in I}$ whose elements are called charts,
- (2) for each chart a homeomorphism $z^i: U_i \rightarrow V_i \subset \mathbb{C}^n$ to an open subset of \mathbb{C}^n called a coordinate system at U_i ,
- (3) such that all the resulting transition maps $z^{ij}: V_i \cap V_j \rightarrow V_i \cap V_j$ are holomorphic.

Equivalently, a complex manifold X is a ringed space (X, \mathcal{O}_X) such that

- (1) X is a second countable compact Hausdorff topological space,
- (2) \mathcal{O}_X is a sheaf of \mathbb{C} -algebras, and (X, \mathcal{O}_X) is locally isomorphic to (V, \mathcal{O}_V) over \mathbb{C} , where $V \subset \mathbb{C}^n$ is an open subset, and \mathcal{O}_V is the sheaf of holomorphic functions on V .

A complex manifold X is a fortiori a smooth manifold, since each complex coordinate $z: U \rightarrow \mathbb{C}, U \subset X$, gives rise to a pair of real coordinates $x, y: U \rightarrow \mathbb{R}$ defined by equation $z = x + iy$. Therefore X has a real tangent bundle, which we will denote $T_{X, \mathbb{R}}$.

Proposition 2.2.1. *Let X be a complex manifold of complex dimension n . Its real tangent bundle $T_{X,\mathbb{R}}$ has a canonical almost-complex structure $I: T_{X,\mathbb{R}} \rightarrow T_{X,\mathbb{R}}$ defined as follows.*

Let $x \in X$ be a point. Pick a chart $U \subset X$ containing x , with holomorphic coordinates $\{z_k: U \rightarrow \mathbb{C}\}_{k=1}^n$. Let x_k, y_k be real coordinates defined by equation $z_k = x_k + iy_k$. Define I on the fiber of $T_{X,\mathbb{R}}$ at x via formulas

$$I\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k}, \quad I\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k}.$$

Proof. Consider two identifications $T_{X,x} \cong \mathbb{R}^{2n}$ given by coordinate systems $(z_i)_{i=1}^n$, and $(w_i)_{i=1}^n$. The fact that the transition map from (z_i) to (w_i) is holomorphic at x means precisely that its differential $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ at x commutes with I 's on both sides, and hence the definition of I on $T_x X$ is independent of the choice of coordinate system. The fact that I is smooth follows from the fact that coordinates are smooth. \square

We thus obtain an almost complex structure on $T_{X,\mathbb{R}}$. By duality we get an almost complex structure on $\Omega_{X,\mathbb{R}}^1$, and a decomposition

$$\Omega_{X,\mathbb{C}}^k = \bigoplus_{p+q=k} \Omega_X^{p,q}$$

where $\Omega_{X,\mathbb{C}}^k = \Omega_{X,\mathbb{R}}^k \otimes_{\mathbb{R}} \mathbb{C}$, and $\Omega_X^{p,q} = \Lambda^p \Omega_X^{1,0} \otimes \Lambda^q \Omega_X^{0,1}$. We define $A^{p,q}(X)$ as the space of global smooth sections of $\Omega_X^{p,q}$. Such sections are called forms of type (p,q) .

In the context of complex manifolds we will use notation $A^k(X)$ for the global sections of $\Omega_{X,\mathbb{C}}^k$, i.e. complex-valued forms.

The de Rham differential $d: \Omega_{X,\mathbb{R}}^\bullet \rightarrow \Omega_{X,\mathbb{R}}^{\bullet+1}$ extends complex-linearly to a differential $d: \Omega_{X,\mathbb{C}}^\bullet \rightarrow \Omega_{X,\mathbb{C}}^{\bullet+1}$.

Proposition 2.2.2. (1) *The de Rham differential $d: \Omega_X^{p,q} \rightarrow \Omega_{X,\mathbb{C}}^{p+q+1}$ decomposes as*

$$d = \partial + \bar{\partial},$$

where $\partial: \Omega_X^{p,q} \rightarrow \Omega_X^{p+1,q}$ is of bidegree $(1,0)$, and $\bar{\partial}: \Omega_X^{p,q} \rightarrow \Omega_X^{p,q+1}$ is of bidegree $(0,1)$.

(2) *The following identities hold:*

$$\partial^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0, \quad \bar{\partial}^2 = 0.$$

Proof. (1) We will do it for $(p, q) = (1, 0)$, the rest being done by analogy.

Pick a chart $U \subset X$, and a holomorphic coordinate system $z: U \rightarrow V \subset \mathbb{C}^n$. Let $z_k: U \rightarrow \mathbb{C}$ be the coordinates. Write $z_k = x_k + iy_k$. By definition $dz_k = dx_k + idy_k$. We also have complex-valued smooth functions $\bar{z}_k = x_k - iy_k$, and differentials $d\bar{z}_k$. Observe that

$$\frac{1}{2}(dz_k + d\bar{z}_k) = dx_k, \quad \frac{1}{2i}(dz_k - d\bar{z}_k) = dy_k.$$

Thus $\{dz_k, d\bar{z}_k\}_{k=1}^n$ form a basis of smooth sections of $\Omega_{X, \mathbb{C}}^1$ over U . By definition of I we have equalities

$$I(dx_k) = -dy_k, \quad I(dy_k) = dx_k,$$

so

$$I(dz_k) = idz_k, \quad I(d\bar{z}_k) = -id\bar{z}_k.$$

As a consequence $\Omega_U^{1,0} \subset \Omega_{U, \mathbb{C}}^1$ is precisely the subbundle spanned by $\{dz_k\}_{k=1}^n$. If $f: U \rightarrow \mathbb{C}$ is a smooth function, then df is a section of $\Omega_{X, \mathbb{C}}^1$ over U , so Leibniz rule shows that $d(fdz_k)$ is a section of $\Omega_X^{2,0} \oplus \Omega_X^{1,1}$.

(2) follows since $d^2 = 0$. □

Remark 2.2.3. An almost-complex manifold is a smooth manifold X , necessarily of even \mathbb{R} -dimension, together with an endomorphism $I: T_{X, \mathbb{R}} \rightarrow T_{X, \mathbb{R}}$ satisfying $I^2 = -1$. There are decompositions $\Omega_{X, \mathbb{C}}^k = \bigoplus_{p+q=k} \Omega_X^{p,q}$. The de Rham differential $d: \Omega_X^{1,0} \rightarrow \Omega_{X, \mathbb{C}}^2 = \Omega_X^{2,0} \oplus \Omega_X^{1,1} \oplus \Omega_X^{0,2}$ splits into three parts $d^{2,0} = \partial$, $d^{1,1} = \bar{\partial}$, and $d^{0,2}$. On a general almost-complex manifold the differential $d^{0,2}$ can be nonzero. In fact, $d^{0,2} = 0$ if and only if X has a necessarily unique complex structure compatible with I (Newlander-Nirenberg theorem). The condition $d^{0,2} = 0$ is equivalent to the condition that Lie bracket of two complex vector fields of type $(1, 0)$ is again of type $(1, 0)$.

Definition 2.2.4. The sheaf of holomorphic p -forms Ω_X^p is defined as

$$\Omega_X^p(U) = \{\alpha \in A^{p,0}(U) \mid \bar{\partial}\alpha = 0\},$$

where $U \subset X$ is an open subset.

Lemma 2.2.5 ($\bar{\partial}$ -Poincaré lemma). *The complex of sheaves*

$$0 \rightarrow \Omega_X^p \rightarrow \Omega_X^{p,0} \xrightarrow{\bar{\partial}} \Omega_X^{p,1} \xrightarrow{\bar{\partial}} \dots$$

is exact.

Proof. See [9], p. 60, proposition 2.31. □

The complexes $(A^{p,\bullet}(X), \bar{\partial})$, $0 \leq p \leq n$, are called Dolbeault complexes.

Theorem 2.2.6 (Dolbeault). *There is a natural isomorphism*

$$H^q(A^{p,\bullet}(X), \bar{\partial}) \rightarrow H^q(X, \Omega_X^p).$$

Proof. The sheaves of smooth sections of $\Omega_X^{p,q}$ are fine, and the complex $(\Omega_X^{p,\bullet}, \bar{\partial})$ is a resolution of Ω_X^p . \square

2.3 Hermitian metrics

Let (V, I) be an \mathbb{R} -vector space of dimension $2n$ with almost-complex structure.

Definition 2.3.1. A positive-definite inner product $g: V \times V \rightarrow \mathbb{R}$ is called an hermitian metric on (V, I) if I is an isometry with respect to g .

Remark 2.3.2. If I is not necessarily an isometry, then one can consider an average

$$g_1(u, v) = \frac{1}{4} \sum_{k=0}^3 g(I^k u, I^k v).$$

The average g_1 is still a positive-definite inner product, and I is an isometry with respect to g_1 . \square

Extend the induced metric g on $\Lambda^k V$ to a hermitian form h on $\Lambda^k V_{\mathbb{C}}$ using the rule

$$h(u \otimes z, v \otimes w) = u \bar{w} g(u, v),$$

pick² a volume form $\text{Vol} \in \Lambda^{2n} V$, and extend real Hodge stars \star to $\Lambda^k V_{\mathbb{C}}$ complex-linearly.

Proposition 2.3.3. (1) *The Hodge star $\star: \Lambda^k V_{\mathbb{C}} \rightarrow \Lambda^{2n-k} V_{\mathbb{C}}$ satisfies*

$$u \wedge \star \bar{v} = h(u, v) \text{Vol}.$$

(2) $\star(\Lambda^{p,q} V) = \Lambda^{n-q, n-p} V$.

Proof. (1) Omitted.

(2) Since g is I -invariant, h is I -invariant too. Therefore the decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ is orthogonal. It follows that vectors of different types are orthogonal with respect to h .

Consider a decomposition by types:

$$\star \bar{v} = \sum_{r+s=k} v^{n-r, n-s}.$$

²In a moment we will see that there is a canonical choice of a volume form.

Suppose that for some r, s the component $v^{n-r, n-s}$ is nonzero. Since the wedge product pairing $V^{r, s} \times V^{n-r, n-s} \rightarrow V^{n, n}$ is nondegenerate there exists $u \in \Lambda^{r, s}$ such that $u \wedge v^{n-r, n-s} \neq 0$. If r', s' are such that $r + r' + s + s' = 2n$, but $r + r' \neq n$ or $s + s' \neq n$, then $u \wedge v^{n-r', n-s'} = 0$. Therefore $u \wedge \star \bar{v} = u \wedge v^{n-r, n-s}$. On the other hand, $u \wedge \star \bar{v} = h(u, v) \text{Vol}$ can be nonzero only if $r = p, s = q$. Hence $\star \bar{v} \in V^{n-p, n-q}$. \square

Let X be a complex manifold of complex dimension n , and let g be a Riemannian metric on $T_{X, \mathbb{R}}$ which is hermitian with respect to the canonical almost-complex structure $I: T_{X, \mathbb{R}} \rightarrow T_{X, \mathbb{R}}$. Applying the construction above we obtain hermitian metrics and Hodge stars on $\Omega_X^{p, q}$, and consequently on $A^{p, q}(X)$.

Proposition 2.3.4. *The operators*

$$\partial^* = -\star \bar{\partial} \star, \quad \bar{\partial}^* = -\star \partial \star.$$

are adjoint to ∂ , and $\bar{\partial}$ respectively.

Proof. We will do it for $\bar{\partial}$. If α is of type $(p, q-1)$, and β is of type (p, q) , then the expression

$$d\alpha \wedge \star \bar{\beta} + (-1)^{p+q} \alpha \wedge d\star \bar{\beta} \tag{1}$$

expands to

$$\bar{\partial} \alpha \wedge \star \bar{\beta} + (-1)^{p+q-1} \alpha \wedge \bar{\partial} \star \bar{\beta},$$

because all terms with ∂ vanish by reason of type. Now, (1) is equal to $d(\alpha \wedge \star \bar{\beta})$ by Leibniz rule, so Stokes theorem gives

$$\int_X \bar{\partial} \alpha \wedge \star \bar{\beta} = -(-1)^{p+q-1} \int_X \alpha \wedge \bar{\partial} \star \bar{\beta}.$$

The term on the left is $(\bar{\partial} \alpha, \beta)$. The term on the right is equal to

$$-\int_X \alpha \wedge \star \star \bar{\partial} \star \bar{\beta}.$$

Notice that \star commutes with complex conjugation, and that complex conjugate of $\bar{\partial}$ is ∂ . Therefore the term on the right is equal to

$$-\int_X \alpha \wedge \star \star \overline{\bar{\partial} \star \bar{\beta}},$$

that is $(\alpha, \bar{\partial}^* \beta)$. \square

Definition 2.3.5. Define Laplacians $\Delta, \Delta_\partial, \Delta_{\bar{\partial}}$ as

$$\Delta = dd^* + d^*d, \quad \Delta_\partial = \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial},$$

and let

$$\mathcal{H}^{p, q}(X) = \{\alpha \in A^{p, q}(X) \mid \Delta_{\bar{\partial}} \alpha = 0\}.$$

Theorem 2.3.6. (1) *The Dolbeault complex $A^{p,\bullet}(X)$ decomposes into an orthogonal direct sum*

$$A^{p,\bullet}(X) = \mathcal{H}^{p,\bullet}(X) \oplus \text{Im } \Delta_{\bar{\partial}}.$$

(2) $\mathcal{H}^{p,\bullet}(X)$ *is a complex of finite-dimensional vector spaces with zero differentials, and $\bar{\partial}^*$ is zero on it.*

(3) $\text{Im } \Delta_{\bar{\partial}}$ *is homotopy equivalent to zero.*

Proof. It is the same elliptic operator theory applied to $\Delta_{\bar{\partial}}$, and followed by a computation which we already did for Δ on Riemannian manifolds. \square

2.4 Kähler metrics

Definition 2.4.1. Let (V, I) be an \mathbb{R} -vector space of dimension $2n$ with almost-complex structure I , and hermitian metric g . Define a 2-form $\omega \in \Lambda^2 V^*$ as

$$\omega(u, v) = g(u, Iv).$$

This form is called the Kähler form associated to g .

Observe that ω is indeed alternating since

$$\omega(u, v) = g(u, Iv) = g(Iv, u) = -g(v, Iu) = -\omega(v, u).$$

Moreover, ω is of type $(1, 1)$ when viewed as an element of $\Lambda^2 V_{\mathbb{C}}^*$.

Proposition 2.4.2. *The form $\frac{\omega^n}{n!}$ is a volume form.*

Proof. The operator I has no real eigenvalues, so that if $v \in V$ is a nonzero vector, then the subspace spanned by v, Iv is of dimension 2. Moreover, I is an automorphism of this subspace. Thus we can choose a basis of V of the form $\{e_k, Ie_k\}_{k=1}^n$. Since I is an isometry, such a basis can be chosen to be orthonormal.

Let $\alpha_k = e_k^* \wedge (Ie_k)^*$. Observe that $\alpha_k \wedge \alpha_l = \alpha_l \wedge \alpha_k$, $\alpha_k \wedge \alpha_k = 0$, and that $\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_n$ is a volume form. Moreover $\omega = \alpha_1 + \alpha_2 + \dots + \alpha_n$, so

$$\omega^n = n! \cdot \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_n. \quad \square$$

Thus we have a canonical choice of a volume form.

Definition 2.4.3. Let (X, g) be a complex manifold with hermitian metric g . The metric g is called Kähler if the associated Kähler form ω is closed.

Fubini-Study metrics on $\mathbb{C}\mathbb{P}^n$ are Kähler ([9], chapter 3, 3.3.2). As a consequence, one obtains Kähler metrics on closed submanifolds of $\mathbb{C}\mathbb{P}^n$ restricting the Kähler form ω , and reconstructing the metric as $g(-, -) = \omega(I-, -)$. On a complex torus $X = \mathbb{C}^n/\Lambda$ one obtains a Kähler metric extending the standard hermitian metric on $\mathbb{C}^n = (T_{X, \mathbb{R}})_0$ to all of $T_{X, \mathbb{R}}$ by translations. In this case ω is closed because its coefficients are constant. Constructing Kähler metrics is a hard problem in general. It is a deep theorem that every complex surface with even b_1 admits a Kähler metric ([1] IV.3).

Remark 2.4.4. There are topological obstructions to existence of Kähler metrics. For example, if ω^k is exact for some $k \leq n$, then the form $\frac{\omega^n}{n!}$ is also exact, which contradicts the fact that it is a volume form. Hence all even cohomology groups $H^{2k}(X, \mathbb{C})$ must be nonzero.

Remark 2.4.5. A complex manifold with a Kähler metric can be diffeomorphic to a complex manifold which has no Kähler metrics [4].

Theorem 2.4.6. *On a complex manifold (X, g) with Kähler metric one has:*

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.$$

This theorem is an easy consequence of the so-called Kähler identities, relating various operators which act on $A^\bullet(X)$. Kähler identities are deep, and their proof goes beyond the scope of this text.

There are several proofs of Kähler identities in the literature. One can either use the fact that Kähler metrics admit local geodesic coordinates ([9], chapter 6, section 1), or use representation theory, together with Lefschetz decomposition ([5], chapter 3, section 1). There are at least two other coordinate-free proofs: see [7], section 2.2, and [8], lectures 6, and 7 (in Russian). For a conceptual interpretation of Kähler identities see [5], appendix 3.B.

2.5 Symmetries of the diamond

Let (X, g) be a complex manifold with Kähler metric. The de Rham complex $A^\bullet(X)$ is the total complex of the double complex

$$\begin{array}{ccccccc}
 & \bar{\partial} \uparrow & & \bar{\partial} \uparrow & & \bar{\partial} \uparrow & \\
 A^{0,2}(X) & \xrightarrow{\partial} & A^{1,2}(X) & \xrightarrow{\partial} & A^{2,2}(X) & \xrightarrow{\partial} & \\
 \bar{\partial} \uparrow & & \bar{\partial} \uparrow & & \bar{\partial} \uparrow & & \\
 A^{0,1}(X) & \xrightarrow{\partial} & A^{1,1}(X) & \xrightarrow{\partial} & A^{2,1}(X) & \xrightarrow{\partial} & \\
 \bar{\partial} \uparrow & & \bar{\partial} \uparrow & & \bar{\partial} \uparrow & & \\
 A^{0,0}(X) & \xrightarrow{\partial} & A^{1,0}(X) & \xrightarrow{\partial} & A^{2,0}(X) & \xrightarrow{\partial} &
 \end{array}$$

Its columns, the Dolbeault complexes $A^{p,\bullet}(X)$, compute the sheaf cohomology $H^q(X, \Omega_X^p)$.

The Laplacian $\Delta_{\bar{\partial}}$ is an endomorphism of the column complexes, Δ_{∂} is an endomorphism of the row complexes, while Δ is a priori only an endomorphism of the total complex. Since the metric g is Kähler, all three Laplacians coincide up to a constant, so they are endomorphisms of the double complex, in the sense that they send $A^{p,q}(X)$ to $A^{p,q}(X)$, and commute with ∂ , and $\bar{\partial}$.

Theorem 2.5.1. *Let (X, g) be a complex manifold with Kähler metric.*

(1) *The double complex $(A^{\bullet,\bullet}(X), \partial, \bar{\partial})$ decomposes into a direct sum*

$$A^{\bullet,\bullet}(X) = \mathcal{H}^{\bullet,\bullet}(X) \oplus \text{Im } \Delta.$$

The differentials of $\mathcal{H}^{\bullet,\bullet}(X)$ are zero, while the total complex of $\text{Im } \Delta$ is homotopy equivalent to zero.

(2) *The image of $\mathcal{H}^{p,q}(X)$ in $H^{p+q}(X, \mathbb{C})$ is precisely the subspace $H^{p,q}(X)$ of classes representable by forms of type (p, q) .*

(3) *There is a decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

(4) *This decomposition is independent of the choice of Kähler metric.*

Proof. (1), (1) \Rightarrow (2), (1) + (2) \Rightarrow (3) Omitted.

(4) The type of a form is determined by the almost-complex structure only. \square

Recall that

$$H^{p,q}(X) \cong \mathcal{H}^{p,q}(X) \cong H^q(X, \Omega_X^p).$$

So Hodge theory connects topological cohomology $H^k(X, \mathbb{C})$ with cohomology of geometric objects Ω_X^p .

Remark 2.5.2. The theorem above shows that the double complex $A^{\bullet,\bullet}(X)$ of a Kähler manifold is a special one. Not every double complex of vector spaces can be decomposed into a direct sum of a complex with zero differentials, and an acyclic complex. The easiest example is

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\text{id}} & \mathbb{C} \\ \uparrow & & \uparrow \text{id} \\ 0 & \longrightarrow & \mathbb{C}. \end{array}$$

See [6] for more details.

It is customary to arrange the spaces $H^{p,q} = H^{p,q}(X)$ to a rhombus-shaped figure called Hodge diamond:

$$\begin{array}{ccccccc}
& & & & H^{n,n} & & \\
& & & H^{n,n-1} & & H^{n-1,n} & \\
& & H^{n,n-2} & & H^{n-1,n-1} & & H^{n-2,n} \\
H^{n,0} & \cdot & \cdot & \cdot & \cdot & \cdot & H^{0,n} \\
& \cdot & \cdot & \cdot & \cdot & \cdot & \\
& & H^{2,0} & & H^{1,1} & & H^{0,2} \\
& & & H^{1,0} & & H^{0,1} & \\
& & & & H^{0,0} & &
\end{array}$$

Theorem 2.5.3. *Symmetries of the diamond.*

(1) *Hodge symmetry: reflection along the vertical axis.*

Complex conjugation induces an antilinear isomorphism $H^{p,q}(X) \rightarrow H^{q,p}(X)$.

(2) *Serre duality: rotation by 180° .*

The cup product pairing $H^{p,q}(X) \times H^{n-p,n-q}(X) \rightarrow H^{n,n}(X)$ is nondegenerate.

Proof. (1) Omitted.

(2) If a form α is $\bar{\partial}$ -harmonic then,

$$0 = (\Delta_{\bar{\partial}}\alpha, \alpha) = \|\bar{\partial}\alpha\|^2 + \|\bar{\partial}^*\alpha\|^2,$$

so α is both $\bar{\partial}$ -closed, and $\bar{\partial}^*$ -closed. As a consequence,

$$\bar{\partial}(\star\bar{\alpha}) = \overline{\partial\star\alpha} = -\star^{-1}\bar{\partial}^*\alpha = 0,$$

$$\bar{\partial}^*(\star\bar{\alpha}) = -\star\partial\star(\star\bar{\alpha}) = \pm\overline{\partial\alpha} = 0.$$

Hence the form $\star\bar{\alpha}$ is $\bar{\partial}$ -harmonic, and in particular, $\bar{\partial}$ -closed. Since the integral of $\alpha \wedge \star\bar{\alpha}$ over X is $\|\alpha\|^2 > 0$, Stokes theorem shows that $\alpha \wedge \star\bar{\alpha}$ can not be exact. \square

Remark 2.5.4. Contrary to what its proof suggests, Hodge symmetry is a deep fact. Deligne and Illusie obtained Hodge decomposition for algebraic varieties over \mathbb{C} using reduction to positive characteristic, but to author's knowledge there is no proof of Hodge symmetry which goes around the analytic theory of harmonic forms. It is known that Hodge symmetry fails in positive characteristic.

Remark 2.5.5. Serre duality holds for arbitrary complex manifolds. Indeed, in our proof we only used the Laplacian $\Delta_{\bar{\partial}}$.

Corollary 2.5.6. *If X is a complex Kähler manifold, then the dimension of its odd cohomology groups is even.*

Proof. If $p + q$ is odd then the sum

$$\dim H^{p+q}(X, \mathbb{C}) = \sum \dim H^{p,q}(X)$$

consists of an even number of summands. The claim follows since $\dim H^{p,q} = \dim H^{q,p}$ by Hodge symmetry. \square

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