Local monodromy of Drinfeld modules

M. Mornev*

ETH Zürich

WWU Münster Online Seminar
2021

* Supported by ETH Zürich Postdoctoral Fellowship Program,
Marie Skłodowska-Curie Actions COFUND program

Setting

- K, a local field of residual characteristic p
- K^{sep}, a separable closure of K
- $G = \text{Gal}(K^{\text{sep}}/K)$

Structure of G

$$\begin{array}{ll} G & \text{absolute Galois group} \\ \cup & \\ I & \text{inertia} \\ \cup & \\ I^{0+} & \text{wild inertia} \end{array}$$

$$1 \to I \to G \to \mathrm{Gal}(\bar{k}/k) \to 1$$

$$1 \to I^{0+} \to I \to \prod_{p' \neq p} \mathbb{Z}_{p'}(1) \to 1$$

$$\varprojlim_{(n,p)=1} \left\{ I \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}(1), \quad g \mapsto \frac{g(\sqrt[n]{\pi})}{\sqrt[n]{\pi}} \right\}$$

$\mathbb{Z}/n\mathbb{Z}(1) = \{x \in K^{\mathrm{sep}} \mid x^n = 1\}$, π uniformizes K

Local monodromy of ℓ-adic representations

$\ell \neq p$, a prime; $\mathbb{Q}_\ell$-representations of G

The ℓ-adic monodromy theorem (Grothendieck)

Every ℓ-adic representation $\rho\colon G \to \mathrm{GL}(V)$ has the following properties:

1. The restriction of ρ to an open subgroup of I is unipotent.
2. The image $\rho(I^{0+})$ is finite.

In effect:

- $\rho(I^{0+}) = \{1\}$
- the group $I/I^{0+} = \prod_{p' \neq p} \mathbb{Z}_{p'}(1)$ acts through $\mathbb{Z}_\ell(1)$
- the action of $\mathbb{Z}_\ell(1) \cong \mathbb{G}_a(\mathbb{Z}_\ell)$ is *algebraic*

Weil-Deligne representations

$W \subset G$, the *Weil group* is the preimage of $\mathbb{Z} \subset \widehat{\mathbb{Z}} = \mathrm{Gal}(\bar{k}/k)$ under the reduction homomorphism $G \twoheadrightarrow \mathrm{Gal}(\bar{k}/k)$.

Fix:

- $\Phi \in W$ a Frobenius element;
- $t \in \mathbb{Z}_\ell(1)$ a generator $\rightsquigarrow \chi_t \colon I \twoheadrightarrow \mathbb{Z}_\ell$

WD: $V \mapsto (V, N)$

- $N \colon V(1) \to V, \quad \rho(g) = \exp(\chi_t(g)N), \quad \forall g \in I$ small
- $\mathrm{WD}(\rho) \colon \Phi^n g \mapsto \rho(\Phi^n g) \exp(-\chi_t(g)N)$

Theorem (Deligne)

The functor WD from the category of ℓ-adic representations of G to the category of Weil-Deligne representations in $\mathbb{Q}_\ell$-vector spaces is fully faithful.

(and the essential image of WD is easy to describe).

Application: ℓ-independence

The Weil group W acts on V continuously in the *discrete* topology.

(V, N) is an "algebraic" object. WD-representations make sense for V over any field (of characteristic 0).

$X \to \operatorname{Spec} K$ proper smooth $\rightsquigarrow \{\mathrm{H}^i(\overline{X}, \mathbb{Q}_\ell)\}_\ell$

Pick ℓ, ℓ', embeddings $\mathbb{Q}_\ell, \mathbb{Q}_{\ell'} \hookrightarrow \mathbb{C}$.

ℓ-independence conjecture

The representations $\mathrm{WD}(\mathrm{H}^i(\overline{X}, \mathbb{Q}_\ell))$ and $\mathrm{WD}(\mathrm{H}^i(\overline{X}, \mathbb{Q}_{\ell'}))$ become isomorphic after base change to $\mathbb{C}$.

Known for abelian varieties and some other classes.

Weight-monodromy (1)

Pure ℓ-adic rep V of G:

- V is *unramified*, i.e. arises from $\mathrm{Gal}(\bar{k}/k)$.
- V has *weight* w:
  the eigenvalues of the *geometric* Frobenius are algebraic over $\mathbb{Q}$ and have norm $q^{w/2}$ for every $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. ($q = \#k$)

$X \to \operatorname{Spec} K$ has a smooth proper model over $\mathcal{O}_K \rightsquigarrow$ $\mathrm{H}^i(\overline{X}, \mathbb{Q}_\ell)$ is pure of weight i (Deligne).

$X \to \operatorname{Spec} K$ smooth proper, but not necessary of good reduction.

Rapoport-Zink (up to technicalities): there is a unique increasing filtration on $\mathrm{H}^i(\overline{X}, \mathbb{Q}_\ell)$ such that

- the subquotients are pure,
- the weights grow strictly with the level.

Weight-monodromy (2)

$V = \mathrm{H}^i(\overline{X}, \mathbb{Q}_\ell)$

Weight-monodromy conjecture

The monodromy operator N determines the weight filtration on V

Jacobson-Morozov: there is a unique increasing filtration $\{M_r\}_{r\in\mathbb{Z}}$ on V such that

- $NM_r \subset M_{r-2}$ for $r \in \mathbb{Z}$,
- the induced map $N^r\colon \mathrm{gr}_r^M V(r) \xrightarrow{\sim} \mathrm{gr}_{-r}^M V$ is an isomorphism for $r \geqslant 0$.

F-representations

From now on: the characteristic of K is equal to p.

Instead of $\mathbb{Q}_\ell$ work with F, a local field of characteristic p.

E a Drinfeld module over $\operatorname{Spec} K \rightsquigarrow V_{\mathfrak{p}}E$
(more generally, t-motives).

Everything breaks down!

1. The image of I^{0+} is **typically** infinite.
2. The group I^{0+} is a free pro-p-group on countably many generators $\rightsquigarrow$ An F-representation of I^{0+} is an essentially arbitrary infinite sequence of elements of $\mathrm{GL}_n(F)$.

End of story? No.

Isocrystals

- $\mathbb{F}_q$, a finite field of cardinality q.
- F, a local field over $\mathbb{F}_q$, ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{m}_F$.
- K, a field over $\mathbb{F}_q$ (not necessarily local).

$$\mathcal{E}_{K,F} = (\lim_{n>0} K \otimes_{\mathbb{F}_q} \mathcal{O}_F/\mathfrak{m}_F^n) \otimes_{\mathcal{O}_F} F$$

$$F = \mathbb{F}_q((z)) \rightsquigarrow \mathcal{E}_{K,F} = K((z))$$

Endomorphism $\sigma \colon \mathcal{E}_{K,F} \to \mathcal{E}_{K,F}$ induced by the q-Frobenius of K.

Definition

An $\mathcal{E}_{K,F}$-*isocrystal* is

- a finitely generated free $\mathcal{E}_{K,F}$-module M
- equipped with an isomorphism $\tau_M^{\mathrm{lin}} \colon \sigma^* M \xrightarrow{\sim} M$.

Morphisms are σ-equivariant morphisms of underlying modules.

Isocrystals and F-representations (1)

Tate module:

$$T(M) = (\mathcal{E}_{K^{\mathrm{sep}},F} \otimes_{\mathcal{E}_{K,F}} M)^{\tau}, \quad \tau\colon x \otimes m \mapsto \sigma(x) \otimes \tau_M^{\mathrm{lin}}(1 \otimes m)$$

Definition

An isocrystal M is *unit-root* if $\dim_F T(M) = \mathrm{rank}_{\mathcal{E}_{K,F}} M$.

Theorem (Katz)

The functor $M \mapsto T(M)$ is an equivalence of categories of

- unit-root $\mathcal{E}_{K,F}$-isocrystals,
- F-representations of G.

Isocrystals and F-representations (2)

Assumption: K is *local*. Ring of integers $\mathcal{O}_K$.

$\mathcal{E}^+_{K,F} = (\lim_{n>0} \mathcal{O}_K \otimes_{\mathbb{F}_q} \mathcal{O}_F/\mathfrak{m}_F^n) \otimes_{\mathcal{O}_F} F$

$F = \mathbb{F}_q((z)) \rightsquigarrow \mathcal{E}^+_{K,F} = \mathcal{O}_K((z))$

Theorem

The functor $M \mapsto T(\mathcal{E}_{K,F} \otimes_{\mathcal{E}^+_{K,F}} M)$ is an equivalence of cat-s of

- unit-root $\mathcal{E}^+_{K,F}$-isocrystals,
- **unramified** F-representations of G.

Isocrystals and F-representations (3)

$$\mathcal{E}^b_{K,F} = K \otimes_{\mathcal{O}_K} \mathcal{E}^+_{K,F}$$

$F = \mathbb{F}_q((z)) \rightsquigarrow \mathcal{E}^b_{K,F} \subset \mathcal{E}_{K,F}$ is the subring of series with bounded coefficients.

Power series $\rightsquigarrow$ functions which are bounded on the *open* unit disk.

Theorem (M.)

The functor $M \mapsto \mathcal{E}_{K,F} \otimes_{\mathcal{E}^b_{K,F}} M$ is fully faithful on unit-root $\mathcal{E}^b_{K,F}$-isocrystals.

Get a full subcategory of F-representations of G.

F-representations arising from A-motives (1)

$A = \mathbb{F}_q[t]$, the ring of coefficients.

$A_K = K \otimes_{\mathbb{F}_q} A, \quad \sigma\colon A_K \to A_K, \quad x \otimes a \mapsto x^q \otimes a.$

Definition

An *(effective) A-motive* over K is an A_K-module M equipped with a morphism $\sigma^* M \to M$ such that

- M is finitely generated projective over A_K.
- The *cotangent module*

$$\Omega_M = \mathrm{coker}(\sigma^* M \to M)$$

  is finite-dimensional over K.

Related to *shtukas*. Anderson: Drinfeld modules $\rightsquigarrow$ A-motives.

F-representations arising from A-motives (2)

$\mathfrak{p} \subset A$ a prime $\rightsquigarrow F_{\mathfrak{p}}$, the local field of A at $\mathfrak{p}$.

The rational $\mathfrak{p}$-adic completion is

$$M_{\mathfrak{p}} = \mathcal{E}_{K,F_{\mathfrak{p}}} \otimes_{A_K} M$$

$\mathfrak{p}$ generic $\rightsquigarrow M_{\mathfrak{p}}$ is a unit-root isocrystal.

Proposition (M.)

For every

- A-motive M over K
- prime $\mathfrak{p} \subset A$, $\neq$ the residual characteristic of M

the isocrystal $M_{\mathfrak{p}}$ arises from $\mathcal{E}^b_{K,F_{\mathfrak{p}}}$.

$\mathfrak{p} \neq \mathrm{res.\,char.}(M) \sim \ell \neq p$

z-adic monodromy (1)

Upper index ramification filtration I^v, $v \in \mathbb{Q}_{\geqslant 0}$
$I^{0+} =$ closure of $\bigcup_{v>0} I^v$

Conjecture (the z-adic monodromy theorem)

An F-representation $\rho\colon G \to \mathrm{GL}(V)$ arises from $\mathcal{E}^b_{K,F}$ if and only if

1. the restriction of ρ to an open subgroup of I is unipotent,
2. $\rho(I^v) = \{1\}$ for $v \gg 0$.

In the ℓ-adic case: $\rho(I^{0+})$ is finite.
Property (1) holds for Anderson modules by Gardeyn's analytic monodromy theorem.

Theorem (M.)

Property (2) holds for Drinfeld modules.

z-adic monodromy (2)

Under development:

- analog of Weil-Deligne construction,
- ℓ-independence conjecture,
- weight-monodromy conjecture.
- z-adic de Rham representations.

Classical theory: ℓ-adic and p-adic representations.

z-adic case: there are analogous types of representations, but they *share the coefficient field*.

Hartl-Kim: Local shtukas $\Leftrightarrow$ a class of z-adic Galois representations (the z-adic analog of crystalline representations).

z-adic de Rham $=$ potentially semi-stable

z-adic monodromy (3)

$\rho\colon G \to \mathrm{GL}(V)$, an F-representation of G

$V_0 \subset V$ an unramified representation such that V/V_0 is also unramified. This holds for Drinfeld modules.

z-adic monodromy theorem: $\rho(I^v) = \{1\}$ for some $v \gg 0$.

- J^v, an abelian quotient of I depending only on v,
- J^v is finitely generated over $\mathbb{F}_p[[\phi]]$ where $1+\phi$ acts as the Frobenius
- Finite-dimensional $\mathbb{F}_p((\phi))$-vector space $H = \mathrm{Hom}_F(V_0, V/V_0)$.
- $\rho|_I$ is encoded by an $\mathbb{F}_p[[\phi]]$-linear homomorphism

$$\chi\colon J^v \to H$$