Tate conjectures in function field arithmetic

M. Mornev*

ETH Zürich

Upstate New York Online Number Theory Colloquium
2020

* Supported by ETH Zürich Postdoctoral Fellowship Program,
Marie Skłodowska-Curie Actions COFUND program

The setting

- $\mathbb{F}_q$, a field of finite cardinality q.
- $A = \mathbb{F}_q[t]$, the ring of coefficients.
  $F = \mathbb{F}_q(t)$, the fraction field of A.
- K, a field over $\mathbb{F}_q$.

Usual motives have coefficient ring $\mathbb{Z}$: the category is $\mathbb{Z}$-linear.
E.g. abelian varieties, algebraic tori.

Anderson modules and motives

An *A-module scheme* E is

- an abelian group scheme E over $\operatorname{Spec} K$,
- equipped with an action of the ring $A = \mathbb{F}_q[t]$.

Anderson's motive of E

$$M(E) = \operatorname{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_{a,K}).$$

$E \mapsto M(E)$ is a *contravariant* functor.

$$K[t] = K \otimes_{\mathbb{F}_q} A, \quad \sigma : x \otimes a \mapsto x^q \otimes a$$
$$K[t]\{\tau\} = \{ y_0 + y_1\tau + \ldots + y_n\tau^n \mid y_i \in K[t] \}$$
$$\tau \cdot y = \sigma(y) \cdot \tau \quad \forall y \in K[t]$$

- $K\{\tau\} = \operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K})$ acts by composition on the left.
- A acts by composition on the right.

An Anderson A-module is

an A-module scheme E over $\operatorname{Spec} K$ such that

- E is isomorphic to a finite product of copies of $\mathbb{G}_{a,K}$.
- the motive $M(E)$ is finitely generated projective over $K[t]$.

$M(E)$ is finitely generated projective over $K\{\tau\} \subset K[t]\{\tau\}$.
NB: $K[t]\{\tau\} = K[t] \otimes_K K\{\tau\}$.

- The *rank* of E is the rank of $M(E)$ over $K[t]$.
- The *dimension* of E is the rank of $M(E)$ over $K\{\tau\}$.

A *Drinfeld A-module* is an Anderson A-module of dimension 1.

Pick $\alpha \in K$.

Example

- $E = \mathbb{G}_{a,K}$
- Action of t on E is given by $\alpha + \tau + \tau^2$.

$\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K}) = K\{\tau\}$, hence $M(E) = K\{\tau\}$.
Claim: $M(E)$ is generated by 1, τ over $K[t]$.

$$t \cdot \tau^n = \tau^n \cdot (\alpha + \tau + \tau^2) = \alpha^{q^n}\tau^n + \tau^{n+1} + \tau^{n+2}$$
$$\tau^{n+2} = (t - \alpha^{q^n}) \cdot \tau^n - \tau^{n+1}$$

Conclusion: E is an Anderson module of dimension 1 and rank 2.

The tangent space at 0

$\mathrm{Lie}(E)$ is an $K[t]$-module of finite length.

Anderson: $\mathrm{Lie}(E)$ is supported at a rational point of the curve $\operatorname{Spec} K[t]/\operatorname{Spec} K$. *We do not demand this.*

A nonzero prime $\mathfrak{p} \subset A$ is *special* if $\mathrm{Lie}(E)[\mathfrak{p}] \neq 0$.
Otherwise $\mathfrak{p}$ is called *generic*.

- There are only finitely many special primes.
- Special primes always exist if K is finite.
- For Drinfeld modules there is at most one special prime.

In the example: a prime $\mathfrak{p} = (f)$ is special if and only if $f(\alpha) = 0$.
If α is transcendental over $\mathbb{F}_q$ then every prime is generic.

Tate modules

A nonzero prime ideal $\mathfrak{p} \subset A$. Completion $A_\mathfrak{p}$, local field $F_\mathfrak{p}$.
A separable closure K^s/K. The $\mathfrak{p}$-adic Tate module:

$$T_\mathfrak{p} E = \mathrm{Hom}_A(F_\mathfrak{p}/A_\mathfrak{p},\, E(K^s))$$

- Finitely generated free over $A_\mathfrak{p}$.
- Continuous action of $G_K = \mathrm{Gal}(K^s/K)$.
- $\mathrm{rk}\, T_\mathfrak{p} E \leqslant \mathrm{rk}\, M(E)$
- $\mathrm{rk}\, T_\mathfrak{p} E = \mathrm{rk}\, M(E) \Leftrightarrow \mathfrak{p}$ is generic.

Tate conjectures, first version (generic $\mathfrak{p}$)

Assume that K *is finitely generated.* Then the functor $E \mapsto T_{\mathfrak{p}}E$ is

- (FF) fully faithful after $A_{\mathfrak{p}} \otimes_A -$,
- (SS) preserves semi-simple objects on the rational level.

$$A_{\mathfrak{p}} \otimes_A \mathrm{Hom}(E_1, E_2) \xrightarrow{\sim} \mathrm{Hom}(T_{\mathfrak{p}}E_1, T_{\mathfrak{p}}E_2)$$

Anderson: the functor $E \mapsto M(E)$ is fully faithful.

An (effective) A-motive over K is

a left $K[t]\{\tau\}$-module M such that

- M is finitely generated projective over $K[t]$.
- The submodule $K[t] \cdot \tau(M)$ is of finite K-codimension in M.

The *conormal module* $\Omega_M = M/K[t]\tau(M)$.

$$\Omega_{M(E)} \xrightarrow{\sim} \mathrm{Hom}_K(\mathrm{Lie}(E), K) \text{ over } K[t]$$

The same notion of generic and special primes.

- The category is abelian after $F \otimes_A -$.
- There is a tensor product $M \otimes N$.
- No duality; easy to repair.

NB: not every motive arises from E.

Dieudonné-Manin theory

Local field $\hat{F}$ over $\mathbb{F}_q$, ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{m}$.

$$\mathcal{E}_K = \mathcal{E}_{K,\hat{F}} = (\lim_{n\to\infty} K \otimes_{\mathbb{F}_q} \mathcal{O}/\mathfrak{m}^n) \otimes_{\mathcal{O}} \hat{F}$$

Endomorphism $\sigma\colon \mathcal{E}_K \to \mathcal{E}_K$ induced by the q-Frobenius of K.

Example: $\hat{F} = \mathbb{F}_q((z))$, $\mathcal{E} = K((z))$, $\sigma(\sum x_n z^n) = \sum x_n^q z^n$.

An $\mathcal{E}_K$-isocrystal is

a left $\mathcal{E}_K\{\tau\}$-module M such that

- M is finitely generated projective over $\mathcal{E}_K$.
- $\mathcal{E}_K \cdot \tau(M) = M$.

- The category is abelian $\hat{F}$-linear.
- There is a tensor product *and* duality.

Dieudonné-Manin classification theorem

Assume that K is *algebraically closed*. Then

- The category of $\mathcal{E}_K$-isocrystals is semi-simple.
- Simple objects M_λ are classified by *slope* $\lambda \in \mathbb{Q}$.

In the case $\hat{F} = \mathbb{F}_q((z))$:

- Write $\lambda = \frac{s}{r}$ with $r > 0$ and $(s, r) = 1$.
- $M_\lambda = \langle e_1, \ldots, e_r \rangle$
- $e_1 \xrightarrow{\tau} \ldots \xrightarrow{\tau} e_r \xrightarrow{\tau} z^s e_1$

An isocrystal M is *pure* if at most one slope appears in the DM decomposition over an algebraic closure.

If M, N are pure then so is $M \otimes N$ and $\lambda(M \otimes N) = \lambda(M) + \lambda(N)$. Similarly $\lambda(M^*) = -\lambda(M)$.

Filtration theorem (for arbitrary K)

Every $\mathcal{E}_K$-isocrystal M carries a unique filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that:

- Every M_{i+1}/M_i is pure and not zero.
- The slopes are strictly increasing with i.

This is called the *Harder-Narasimhan filtration*.
Splits if K is perfect (and does not split otherwise).

Let M be a pure $\mathcal{E}$-isocrystal of slope 0.

$$T(M) = (\mathcal{E}_{K^s} \otimes_{\mathcal{E}_K} M)^\tau$$

- Finite-dimensional over $\hat{F}$.
- Carries a continuous action of G_K.

Representation theorem

The functor $M \mapsto T(M)$ is an equivalence of

- the category of pure isocrystals of slope 0,
- the category of continuous G_K-representations in finite-dimensional $\hat{F}$-vector spaces.

Can extend this to pure modules of any slope!
The Weil group W_K appears instead of G_K.
The target category is more complicated.

Rational $\mathfrak{p}$-adic completion of motives

Let M be a motive, and $\mathfrak{p}$ a *place* of $F = \mathbb{F}_q(t)$.

The rational $\mathfrak{p}$-adic completion is

$$M_\mathfrak{p} = \mathcal{E}_{K,F_\mathfrak{p}} \otimes_{K[t]} M$$

- $\mathfrak{p} \subset A$ generic: $M_\mathfrak{p}$ is pure of slope 0.

  For $M = M(E)$ we have a natural isomorphism

  $$T(M_\mathfrak{p}) \xrightarrow{\sim} \mathrm{Hom}_{F_\mathfrak{p}}(V_\mathfrak{p} E, \Omega_\mathfrak{p})$$

  where $\Omega_\mathfrak{p} = F_\mathfrak{p} \otimes_A \Omega^1_{A/\mathbb{F}_q}$.
- $\mathfrak{p} \subset A$ special: $M_\mathfrak{p}$ need not be pure. The slopes are non-negative and at least one is strictly positive.

  $$T(M^0_\mathfrak{p}) \xrightarrow{\sim} \mathrm{Hom}_{F_\mathfrak{p}}(V_\mathfrak{p} E, \Omega_\mathfrak{p})$$

Weights: the ∞-adic completion

Definition (Anderson '86)

The *weights* of M are the slopes of M_∞ taken with the opposite sign. We say that M is *pure* if so is M_∞.

Theorem (Taelman '10)

A motive arises from an Anderson module if and only if its weights are strictly positive.

A *Tate object* L: rank 1, weight 1.
$M \otimes L^{\otimes n}$ is finitely generated over $K\{\tau\}$ for $n \gg 0$.

Theorem (Drinfeld '77)

A motive of rank $r > 0$ arises from a Drinfeld module if and only if it is pure of weight $\frac{1}{r}$.

Tate conjectures

Tate conjectures for A-motives over K

Assume that K is *finitely generated*. Then the functor $M \mapsto M_{\mathfrak{p}}$ is

- (FF) fully faithful after $F_{\mathfrak{p}} \otimes_A -$,
- (SS) preserves semi-simple objects at the rational level.

$$F_{\mathfrak{p}} \otimes_A \mathrm{Hom}(M, N) \xrightarrow{\sim} \mathrm{Hom}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$$

Folklore theorem

Assume that K is finite. Then the Tate conjecture (FF) holds for all motives M and places $\mathfrak{p}$.

Reason: $\mathcal{E}_{K,F_{\mathfrak{p}}} = F_{\mathfrak{p}} \otimes_A K[t]$. Implies injectivity for arbitrary K.

Results

- Taguchi '91, '93: SS for Drinfeld modules, $\mathfrak{p} \neq \infty$.
- Taguchi '95: FF for generic $\mathfrak{p}$ and tr deg $K = 1$.
  Details omitted.
- A. Tamagawa '94, '95, '96: FF+SS for generic $\mathfrak{p}$.
  Details omitted.
- Pink '95: FF+SS for generic $\mathfrak{p}$.
  Deduced from the isogeny conjecture. Never published.
- Watson '03: FF for Drinfeld modules, special $\mathfrak{p}$.
- Stalder '10: FF+SS for generic $\mathfrak{p}$.
- Zywina '16: FF for pure motives, $\mathfrak{p} = \infty$.

M.'20: counterexample to FF for mixed motives, $\mathfrak{p} = \infty$.
The work still continues: $\mathrm{FF}_{\mathfrak{p}=\infty}$ is true for many mixed motives.

Full faithfulness: the algebraic part

Focus on the case $\operatorname{tr\,deg} K = 1$.
$X / \operatorname{Spec} \mathbb{F}_q =$ the smooth projective model of K.

Goal: understand what is $F_{\mathfrak{p}} \otimes_A \operatorname{Hom}(M, N)$.

Gardeyn's theory

$C = \operatorname{Spec} A$, $X \times C$, endomorphism σ.

A *left* $\mathcal{O}_{X\times C}\{\tau\}$*-module*: a pair $(\mathcal{F}, \tau)$ where $\mathcal{F}$ is an $\mathcal{O}_{X\times C}$-module, $\tau\colon \mathcal{F} \to \sigma_* \mathcal{F}$ is a morphism.

An A-motive M gives rise to a coherent sheaf $\tilde{M}$ on $(\operatorname{Spec} K) \times C$ together with a σ-linear endomorphism τ.

Embedding $\iota\colon (\operatorname{Spec} K) \times C \hookrightarrow X \times C$. Pushforward $\iota_* \tilde{M}$.

Gardeyn's maximal model

There is a unique left $\mathcal{O}_{X\times C}\{\tau\}$-submodule $\mathcal{M} \subset \iota_* \tilde{M}$ which is

- locally free of finite type over $\mathcal{O}_{X\times C}$,
- maximal with respect to the inclusion relation.

Motives M, $N \rightsquigarrow$ Gardeyn models $\mathcal{M}$, $\mathcal{N}$

Néron property

$$\mathrm{Hom}(M, N) = \mathrm{Hom}(\mathcal{M}, \mathcal{N})$$

$\mathcal{M}_{\mathfrak{p}}$, $\mathcal{N}_{\mathfrak{p}}$: the pullback to $X \times \operatorname{Spec} F_{\mathfrak{p}}$.

Theorem

$$F_{\mathfrak{p}} \otimes_A \mathrm{Hom}(M, N) = \mathrm{Hom}(\mathcal{M}_{\mathfrak{p}}, \mathcal{N}_{\mathfrak{p}})$$

Instant consequence of proper base change.

Full faithfluness: the analytic part

Local field $\hat{F} = \mathbb{F}_q((z))$.

Scheme $\mathcal{X} = X \times \operatorname{Spec} \hat{F}$ with an endomorphism σ. Best viewed as a rigid analytic space over $\operatorname{Spec} \hat{F}$.

We have $\mathcal{M}$, a left $\mathcal{O}_{\mathcal{X}}\{\tau\}$-module which is locally free of finite type over $\mathcal{O}_{\mathcal{X}}$.

Generic fiber functor $\mathcal{M} \mapsto \mathcal{M}_\eta$: base change to $\mathcal{E}_K = K \mathbin{\widehat{\otimes}} \hat{F}$.
We know that $\mathcal{M}_\eta$ is an isocrystal.

When the functor $\mathcal{M} \mapsto \mathcal{M}_\eta$ is fully faithful?

Local analysis

Closed point $x \in X \leftrightarrow$ valuation ring $R \subset K$.

$\mathcal{E}_R = R((z)) = R[[z]][z^{-1}]$, a subring of $\mathcal{E}_K = K((z))$.
NB: $\mathcal{E}_R$ is a PID.

Base change from $\mathcal{X}$ to $\mathcal{E}_R$: $\mathcal{M} \mapsto \mathcal{M}_x$. Produces a left $\mathcal{E}_R\{\tau\}$-module with the following properties:

- $\mathcal{M}_x$ is finitely generated projective over $\mathcal{E}_R$.
- The quotient $\mathcal{M}_x / \mathcal{E}_R \tau(\mathcal{M}_x)$ is of finite length.
- $\mathcal{M}_x$ has a maximality proerty to be discussed later.

To prove full faithfulness it is enough to show that every morphism $\mathcal{M}_\eta \to \mathcal{N}_\eta$ maps $\mathcal{M}_x$ to $\mathcal{N}_x$ for all $x \in X$.

Unramified case (excellent reduction)

An $\mathcal{E}_R$-isocrystal is

a left $\mathcal{E}_R\{\tau\}$-module M such that

- M is finitely generated free over $\mathcal{E}_R$,
- $M = \mathcal{E}_R \tau(M)$.

For almost all points x the module $\mathcal{M}_x$ is an $\mathcal{E}_R$-isocrystal.

Theorem (Watson '03)

The base change functor $\mathcal{E}_K \otimes_{\mathcal{E}_R} -$ is fully faithful on isocrystals.

Open subset $U \subset X \rightsquigarrow$ subspace $\mathcal{U} \subset \mathcal{X}$, a complement of finitely many residue disks. The natural morphism

$$\mathrm{Hom}(\mathcal{M}|_{\mathcal{U}}, \mathcal{N}|_{\mathcal{U}}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{M}_\eta, \mathcal{N}_\eta)$$

is an isomorphism.

Overconvergence

Split the base change problem in two parts: $\mathcal{E}_R \hookrightarrow \mathcal{E}_R^\dagger \hookrightarrow \mathcal{E}_K$

Closed point $x \in X \Leftrightarrow$ normalized valuation $v\colon K^\times \to \mathbb{Z}$.

$\Gamma_R^\dagger \subset K[[z]]$, the subring of series with nonzero radius of convergence w.r.t. v.

The overconvergent ring

$$\mathcal{E}_R^\dagger = \Gamma_R^\dagger[z^{-1}]$$

The z-adic analog of the p-adic overconvergent ring ($\hat{F} = \mathbb{Q}_p$).

NB: $\mathcal{E}_R \subset \mathcal{E}_R^\dagger$. Furthermore $\mathcal{E}_R^\dagger$ is a field.

An overconvergent isocrystal is

a left $\mathcal{E}_R^\dagger\{\tau\}$-module M such that

- M is finite-dimensional over $\mathcal{E}_R^\dagger$.
- $M = \mathcal{E}_R^\dagger \cdot \tau(M)$.

For each x the module $\mathcal{M}_x^\dagger = \mathcal{E}_R^\dagger \otimes_{\mathcal{E}_R} \mathcal{M}_x$ is an overconvergent isocrystal.

We shall study the inclusion $\mathrm{Hom}(\mathcal{M}_x, \mathcal{N}_x) \subset \mathrm{Hom}(\mathcal{M}_x^\dagger, \mathcal{N}_x^\dagger)$.

Local maximal models

A *local maximal model* over R is

a left $\mathcal{E}_R\{\tau\}$-module M such that

- M is finitely generated free over $\mathcal{E}_R$,
- the conormal module $M/\mathcal{E}_R\tau(M)$ is of finite length,
- M is the maximal submodule of $K \otimes_R M$ having these properties.

This is a simultaneous generalization of $\mathcal{E}_R$-isocrystals, Gardeyn maximal models and local shtukas of Hartl.

NB: $\mathcal{M}_x$ is a local maximal model for every x.
This follows from the fact that $\mathcal{M}$ is a Gardeyn model.

theorem (M., in progress)

The base change functor $\mathcal{E}_R^\dagger \otimes_{\mathcal{E}_R} -$ is fully faithful on local maximal models.

Corollary

For all A-motives M, N over K and all places $\mathfrak{p}$ of F we have

$$F_\mathfrak{p} \otimes_A \mathrm{Hom}(M, N) = \{f : M_\mathfrak{p} \to N_\mathfrak{p} \mid \forall x \ f(M^\dagger_{\mathfrak{p},x}) \subset N^\dagger_{\mathfrak{p},x}\}.$$

Here $M^\dagger_{\mathfrak{p},x} = \mathcal{E}^\dagger_{R,F_\mathfrak{p}} \otimes_{K[t]} M$. Note that $K(t) \subset \mathcal{E}^\dagger_{R,F_\mathfrak{p}}$ for all R, $\mathfrak{p}$.

By Watson the condition holds for almost all x.

Kedlaya's base change theorem

Consider the base change $\mathcal{E}_R^\dagger \hookrightarrow \mathcal{E}_K$.

In the p-adic setting the $\mathcal{E}_K$-isocrystals carry extra data: a *connection* ∇.

In the p-adic cohomology theory this comes from the Gauß-Manin connection.

∇ is essentially unique (e.g. it is unique on pure objects).

Theorem (Kedlaya '03)

In the p-adic setting the base change functor $\mathcal{E}_K \otimes_{\mathcal{E}_R^\dagger} -$ is fully faithful.

Monodromy

The Robba ring for the valuation v

$\mathcal{R}_v = \{\sum_{n\in\mathbb{Z}} x_n z^n \mid \text{converges on a punctured open disk w.r.t. } v\}$

The p-adic monodromy theorem describes the structure of the Frobenius module $\mathcal{R}_v \otimes_{\mathcal{E}_R^\dagger} M$.

The base change is fully faithful on the level of Frobenius structure **if one assumes that** $\mathcal{R}_v \otimes_{\mathcal{E}_R^\dagger} M$ is as prescribed by the p-adic monodromy theorem.

What if we do not restrict $\mathcal{R}_v \otimes_{\mathcal{E}_R^\dagger} M$?

The base change functor is **not full**, both in the p-adic and the z-adic setting.

This leads to a counterexample to FF for $\mathfrak{p} = \infty$.

Known cases of base change

Theorem (folklore)

The base change functor $\mathcal{E} \otimes_{\mathcal{E}_R^\dagger} -$ is fully faithful on pure isocrystals.

Yields FF for generic $\mathfrak{p}$, and $\mathfrak{p} = \infty$ for pure motives.

Theorem (Ambrus Pal - M. '20)

The base change functor $\mathcal{E} \otimes_{\mathcal{E}_R^\dagger} -$ is fully faithful on isocrystals with "good" monodromy.

"good" $=$ the result of the p-adic monodromy theorem translated to the z-adic setting.

Yields Watson's base change theorem, and FF for Drinfeld modules, special $\mathfrak{p}$. Also applies to $\mathfrak{p} = \infty$ when the motive has potential good reduction everywhere.