t-motives (pre-talk)

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Rings of coefficients

Let K be a field.

- Triangulated category of geometric motives over K.
- Original goal: *abelian* category of mixed motives over K.
- Abelian varieties, algebraic tori.

Motives of this kind have *coefficient ring* $\mathbb{Z}$:
the Hom-sets are naturally $\mathbb{Z}$-modules.

t-motives: coefficient ring $\mathbb{F}_q[t]$ (and generalizations).
Only for K over $\mathbb{F}_q$.

The setting

- $\mathbb{F}_q$, a finite field of cardinality q.
- $A = \mathbb{F}_q[t]$, the ring of coefficients.
  $F = \mathbb{F}_q(t)$, the fraction field of A.
- K, a field over $\mathbb{F}_q$.

$$A_K = K \otimes_{\mathbb{F}_q} A, \quad \sigma\colon A_K \to A_K, \quad x \otimes a \mapsto x^q \otimes a.$$

$$A_K\{\tau\} = \{y_0 + y_1\tau + \ldots + y_n\tau^n \mid y_i \in A_K,\ n \geqslant 0\}$$

$$\tau \cdot y = \sigma(y) \cdot \tau$$

Effective A-motives

Definition

An *(effective) A-motive* over K is a left $A_K\{\tau\}$-module M such that

- M is finitely generated projective over A_K.
- The *cotangent module*
$$\Omega_M = M/(A_K) \cdot \tau M$$
is finite-dimensional over K.

- The category is abelian after $F \otimes_A -$.
- There is a tensor product $M \otimes N$.
- Objects are not dualizable; easy to repair $\rightsquigarrow$ tannakian category of A-motives.

Abelian A-modules (1)

An *A-module scheme* E is

- an abelian group scheme E over $\operatorname{Spec} K$,
- equipped with an action of the ring $A = \mathbb{F}_q[t]$.

Anderson's motive of E

$$M(E) = \operatorname{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a).$$

$E \mapsto M(E)$ is a *contravariant* functor.

$M(E)$ carries a left action of the ring $A_K\{\tau\} = A \otimes_{\mathbb{F}_q} (K\{\tau\})$

- $K\{\tau\} = \operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_a)$ acts by composition on the left.
- A acts by composition on the right.

Abelian A-modules (2)

An abelian A-module is

an A-module scheme E over $\operatorname{Spec} K$ such that

- E is isomorphic to a finite product of copies of $\mathbb{G}_a$.
- the motive $M(E)$ is finitely generated projective over A_K.

$M(E)$ is finitely generated projective over $K\{\tau\} \subset A_K\{\tau\}$.
$A_K\{\tau\} = A_K \otimes_K (K\{\tau\})$

$$\Omega_{M(E)} = \operatorname{Hom}_K(\operatorname{Lie} E, K)$$

- The *rank* of E is the rank of $M(E)$ over A_K.
- The *dimension* of E is the rank of $M(E)$ over $K\{\tau\}$.

A *Drinfeld A-module* is an Anderson A-module of dimension 1.

Abelian A-modules (3)

Pick $\alpha \in K$.

Example

- $E = \mathbb{G}_a$
- The action of t on E is given by $\alpha + \tau + \tau^2$.

$\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a) = K\{\tau\}$, hence $M(E) = K\{\tau\}$.
Claim: $M(E)$ is generated by 1, τ over A_K.

$$t \cdot \tau^n = \tau^n \cdot (\alpha + \tau + \tau^2) = \alpha^{q^n}\tau^n + \tau^{n+1} + \tau^{n+2}$$
$$\tau^{n+2} = (t - \alpha^{q^n}) \cdot \tau^n - \tau^{n+1}$$

Conclusion: E is an Anderson module of dimension 1 and rank 2.

Dieudonné-Manin theory (1)

Local field $\hat{F}$ over $\mathbb{F}_q$, ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{m}$.

$$\mathcal{E}_K = \mathcal{E}_{K,\hat{F}} = (\lim_{n\to\infty} K \otimes_{\mathbb{F}_q} \mathcal{O}/\mathfrak{m}^n) \otimes_{\mathcal{O}} \hat{F}$$

Endomorphism $\sigma\colon \mathcal{E}_K \to \mathcal{E}_K$ induced by the q-Frobenius of K.

Example: $\hat{F} = \mathbb{F}_q((z))$, $\mathcal{E} = K((z))$, $\sigma(\sum x_n z^n) = \sum x_n^q z^n$.

An $\mathcal{E}_K$-isocrystal is

a left $\mathcal{E}_K\{\tau\}$-module M such that

- M is finitely generated projective over $\mathcal{E}_K$.
- $\mathcal{E}_K \cdot \tau(M) = M$.

- The category is abelian $\hat{F}$-linear.
- There is a tensor product; every object is dualizable.

Dieudonné-Manin theory (2)

Dieudonné-Manin classification theorem

Assume that K is *algebraically closed*. Then

- The category of $\mathcal{E}_K$-isocrystals is semi-simple.
- Simple objects M_λ are classified by *slope* $\lambda \in \mathbb{Q}$.

In the case $\hat{F} = \mathbb{F}_q((z))$:

- Write $\lambda = \frac{s}{r}$ with $r > 0$ and $(s, r) = 1$.
- $M_\lambda = \langle e_1, \ldots, e_r \rangle$
- $e_1 \xrightarrow{\tau} \ldots \xrightarrow{\tau} e_r \xrightarrow{\tau} z^s e_1$

Dieudonné-Manin theory (3)

An isocrystal M is *pure* if at most one slope appears in the DM decomposition over an algebraic closure.

M, N pure $\rightsquigarrow M \otimes N$ is pure, and $\lambda(M \otimes N) = \lambda(M) + \lambda(N)$.
Similarly $\lambda(M^*) = -\lambda(M)$.

Filtration theorem (for arbitrary K)

Every $\mathcal{E}_K$-isocrystal M carries a unique filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that:

- Every M_{i+1}/M_i is pure and not zero.
- The slopes are strictly increasing with i.

This is called the *Harder-Narasimhan filtration*.
Splits if K is perfect (and does not split otherwise).

Dieudonné-Manin theory (4)

Let M be a pure $\mathcal{E}$-isocrystal of slope 0 and K^s a separable closure of K.

$$T(M) = (\mathcal{E}_{K^s} \otimes_{\mathcal{E}_K} M)^\tau$$

- Finite-dimensional over $\hat{F}$.
- Carries a continuous action of G_K.

Representation theorem

The functor $M \mapsto T(M)$ is an equivalence of

- the category of pure isocrystals of slope 0,
- the category of continuous G_K-representations in finite-dimensional $\hat{F}$-vector spaces.

Can extend this to pure modules of any slope!
The Weil group W_K appears instead of G_K.
The target category is more complicated.

Rational $\mathfrak{p}$-adic completion of motives (1)

Let M be a motive, and $\mathfrak{p}$ a *place* of $F = \mathbb{F}_q(t)$.

The rational $\mathfrak{p}$-adic completion is

$$M_{\mathfrak{p}} = \mathcal{E}_{K,F_{\mathfrak{p}}} \otimes_{A_K} M$$

A nonzero prime $\mathfrak{p} \subset A$ is *special* if $\Omega_M[\mathfrak{p}] \neq 0$.
Otherwise $\mathfrak{p}$ is called *generic*.

- There are only finitely many special primes.
- Special primes always exist if K is finite.
- For Drinfeld modules there is at most one special prime.

Rational $\mathfrak{p}$-adic completion of motives (2)

$\mathfrak{p} \subset A$ generic: $M_\mathfrak{p}$ is pure of slope 0.
Galois representation $T(M_\mathfrak{p})$. Dimension $=$ rank of M.

For an abelian A-module E we have the $\mathfrak{p}$-adic Tate module

$$T_\mathfrak{p}(E) = \mathrm{Hom}_A(F_\mathfrak{p}/A_\mathfrak{p},\ E(K^s))$$

Compare: $T_p(E) = \mathrm{Hom}_\mathbb{Z}(\mathbb{Q}_p/\mathbb{Z}_p,\ E(K^s))$

For $M = M(E)$ we have a natural isomorphism

$$T(M_\mathfrak{p}) \xrightarrow{\sim} \mathrm{Hom}_{F_\mathfrak{p}}(V_\mathfrak{p}E,\ \Omega_\mathfrak{p})$$

where $\Omega_\mathfrak{p} = F_\mathfrak{p} \otimes_A \Omega^1_{A/\mathbb{F}_q}$ and $V_\mathfrak{p}(E) = F_\mathfrak{p} \otimes_{A_\mathfrak{p}} T_\mathfrak{p}(E)$.

Weights: the ∞-adic completion

Definition (Anderson '86)

The *weights* of M are the slopes of M_∞ taken with the opposite sign. We say that M is *pure* if so is M_∞.

Theorem (Taelman '10)

A motive arises from an Anderson module if and only if its weights are strictly positive.

A *Tate object* L: rank 1, weight 1.
$M \otimes L^{\otimes n}$ is finitely generated over $K\{\tau\}$ for $n \gg 0$.

Theorem (Drinfeld '77)

A motive of rank $r > 0$ arises from a Drinfeld module if and only if it is pure of weight $\frac{1}{r}$.

Selected results

- Tate conjectures: Y. Taguchi, A. Tamagawa.
- Mumford-Tate conjecture for Drinfeld modules: R. Pink.
- Birch and Swinnerton-Dyer conjecture: L. Taelman.