Local Kummer theory for Drinfeld modules

M. Mornev*
(joint with Richard Pink)

EPFL

2nd Joint Meeting of UMI – AMS
Palermo 2024

* Supported by Swiss National Science Foundation
(SNSF Ambizione project 202119)

The setting

K local field of characteristic $p > 0$

Algebraic closure K^a, $\quad G_K := \mathrm{Aut}(K^a/K)$, $\quad$ inertia $I_K \subset G_K$

Drinfeld module $\varphi\colon A \to K[\tau]$, $\quad \partial\varphi\colon A \to K$

Assumption: $\partial\varphi(A) \subset \mathcal{O}_K$, i.e. φ has finite residual characteristic

$$\bar{\mathfrak{p}} := \partial\varphi^{-1}(\mathfrak{m}_K)$$

Local monodromy of Drinfeld modules ($\mathfrak{p} \neq \bar{\mathfrak{p}}$)

Aim: Understand the local monodromy representation

$$I_K \longrightarrow \mathrm{GL}(T_{\mathfrak{p}}(\varphi))$$

φ has *stable reduction:* $\qquad \varphi = \psi / M$

$$0 \longrightarrow T_{\mathfrak{p}}(\psi) \longrightarrow T_{\mathfrak{p}}(\varphi) \longrightarrow A_{\mathfrak{p}} \otimes_A M \longrightarrow 0$$

The action differs from the identity by

$$I_K \longrightarrow \mathrm{Hom}_A\big(M,\ T_{\mathfrak{p}}(\psi)\big)$$

$$J_K \ := \ I_K^{\mathrm{ab}} / (I_K^{\mathrm{ab}})^{\times p}$$

The modified Tate module

Need a convenient version of the adelic Tate module:

$$T^{\circ}_{\mathrm{ad}}(\varphi) := \mathrm{Hom}_A\Big(F/A,\ (K^a/\mathfrak{m}_{K^a}, \varphi)\Big).$$

This is a module over $A_{\mathrm{ad}} := \mathrm{End}_A(F/A)$.

Properties:

$$T^{\circ}_{\mathrm{ad}}(\varphi) = \prod\nolimits_{\mathfrak{p}} T^{\circ}_{\mathfrak{p}}(\varphi), \quad T^{\circ}_{\mathfrak{p}}(\varphi) = T_{\mathfrak{p}}(\varphi), \quad \mathfrak{p} \neq \bar{\mathfrak{p}}$$

$$T_{\bar{\mathfrak{p}}}(\varphi) \ \twoheadrightarrow \ T^{\circ}_{\bar{\mathfrak{p}}}(\varphi).$$

The image of inertia

$$\varphi = \psi/M, \quad J_K := I_K^{ab}/(I_K^{ab})^{\times p}, \quad B_{\mathrm{ad}}^{\circ} := \mathbb{F}_p[[G_k]]$$

$$\rho\colon J_K \longrightarrow \mathrm{Hom}_A\big(M,\ T_{\mathrm{ad}}^{\circ}(\psi)\big)$$

Theorem 1 (M. – Pink)

The image $\rho(J_K)$ is a free B_{ad}°-module of rank divisible by $d := [k/\mathbb{F}_p]$ and is a direct summand of $\mathrm{Hom}(\dots)$ up to finite index.

The image of the ramification filtration

$$\varphi = \psi/M, \quad J_K := I_K^{\mathrm{ab}}/(I_K^{\mathrm{ab}})^{\times p}, \quad B_{\mathrm{ad}}^{\circ} := \mathbb{F}_p[[G_k]], \quad d := [k/\mathbb{F}_p]$$

$$\rho\colon J_K \longrightarrow \mathrm{Hom}_A\big(M,\ T_{\mathrm{ad}}^{\circ}(\psi)\big)$$

Ramification subgroup J_K^i, $\quad i \in \mathbb{Z}_{\geqslant 0}$.

Theorem 2 (M. – Pink)

There is a *finite* subset $S \subset \mathbb{Z}_{\geqslant 0} \smallsetminus p\mathbb{Z}_{\geqslant 0}$ such that:

- If $i \notin S$ then $\rho(J_K^i)/\rho(J_K^{i+1})$ is finite.
- If $i \in S$ then $\rho(J_K^i)/\rho(J_K^{i+1})$ is a free B_{ad}°-module of rank d.

The B_{ad}°-module $\rho(J_K^i)$ is free of rank $d \cdot |\{j \in S \mid j \geqslant i\}|$ and is a direct summand of Hom$(\dots)$ up to finite index.

In particular, $\rho(J_K^i) = 0$ for $i \gg 0$.

The local Kummer pairing

$$\varphi = \psi/M, \quad J_K := I_K^{\mathsf{ab}}/(I_K^{\mathsf{ab}})^{\times p}, \quad B^{\circ}_{\mathrm{ad}} := \mathbb{F}_p[[G_k]], \quad d := [k/\mathbb{F}_p]$$

$$\rho\colon J_K \longrightarrow \mathrm{Hom}_A\big(M,\ T^{\circ}_{\mathrm{ad}}(\psi)\big)$$

The local Kummer pairing of ψ

$$[\ ,\ )_\psi\colon K \times J_K \longrightarrow T^{\circ}_{\mathrm{ad}}(\psi)$$

$$[\xi,\ g)_\psi\colon\ \left[\tfrac{b}{a}\right] \mapsto g\big(\psi_b(\xi_a)\big) - \psi_b(\xi_a), \quad \psi_a(\xi_a) = \xi$$

$$[\ ,\ )_\psi\colon \mathcal{P}_K \times J_K \longrightarrow T^{\circ}_{\mathrm{ad}}(\psi), \quad \mathcal{P}_K := K/\mathcal{O}_K$$

Perfectness of the Kummer pairing

$B := \mathbb{F}_p[s], \quad \bar{\omega}\colon B \to k[\tau], \quad \bar{\omega}_s := \tau^d, \quad B^\circ := B[s^{-1}], \quad k \hookrightarrow K$

$\mathcal{P}_K := K/\mathcal{O}_K$

$$[\ ,\ )_{\bar{\omega}}\colon \mathcal{P}_K \times J_K \longrightarrow T^\circ_{\mathrm{ad}}(\bar{\omega})$$

$R := \mathrm{End}(\bar{\omega}) = k[\tau]$

Theorem 3 (M. – Pink)

The local Kummer pairing of $\bar{\omega}$ induces an isomorphism

$$J_K \xrightarrow{\sim} \mathrm{Hom}_R\big(\mathcal{P}_K,\ T^\circ_{\mathrm{ad}}(\bar{\omega})\big)$$

that identifies J_K^i with the subgroup of homomorphisms vanishing on

$$W_i\mathcal{P}_K := \langle\, [\xi] \mid \xi \in K \smallsetminus \mathcal{O}_K,\ v(\xi) > -i \,\rangle$$

$W_i\mathcal{P}_K$ is a free left R-module of finite rank

Comparison with the classical theory

For $\mathbb{G}_m$, have the local Kummer pairing

$$(\ ,\ )_{\mathbb{G}_m} \colon \mathcal{V}_K \times T_K \longrightarrow T^{\circ}_{\mathrm{ad}}(\mathbb{G}_m)$$

with

$$\mathcal{V}_K := \mathbb{G}_m(K)/\mathbb{G}_m(\mathcal{O}_K) \xrightarrow{\ \sim\ } \mathbb{Z}.$$

and

$$T^{\circ}_{\mathrm{ad}}(\mathbb{G}_m) = \prod_{\ell \neq p} \mathbb{Z}_\ell(1).$$

Theorem

The local Kummer pairing of $\mathbb{G}_m$ induces an isomorphism

$$T_K \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbb{Z}}\big(\mathcal{V}_K,\ T^{\circ}_{\mathrm{ad}}(\mathbb{G}_m)\big)$$

Some consequences

Corollary 4 (M. – Pink)

$$\mathrm{gr}^i(J_K) \cong \begin{cases} (B^\circ_{\mathrm{ad}})^{\oplus d}, & p \nmid i, \\ 0, & p \mid i. \end{cases}$$

In particular the B°_{ad}-module J_K/J_K^i is finitely generated free for all $i \geqslant 0$.

Reason:

$$\mathrm{gr}^i(J_K) \xrightarrow{\sim} \mathrm{Hom}_R\big(\mathrm{gr}_i^W(\mathcal{P}_K),\ T^\circ_{\mathrm{ad}}(\varpi)\big).$$

The reduced case

$\bar{\varphi}\colon A \to k[\tau], \quad k \hookrightarrow K$

Proposition 5 (M. – Pink)

There is a canonical B°_{ad}-linear isomorphism

$$T^{\circ}_{\mathrm{ad}}(\overline{\omega}) \xrightarrow{\sim} T^{\circ}_{\mathrm{ad}}(\bar{\varphi})$$

that is compatible with the local Kummer pairing.

$$J_K \hookrightarrow \mathrm{Hom}_A\big((\mathcal{P}_K, \bar{\varphi}), T^{\circ}_{\mathrm{ad}}(\bar{\varphi})\big)$$

The fundamental isomorphism

Any $\varphi\colon A \to K[\tau]$ extends uniquely to

$$\varphi_\infty\colon F_\infty \longrightarrow K^{\mathrm{perf}}(\!(\tau^{-1})\!)$$

Theorem 6 (M. – Pink)

There is a canonical isomorphism

$$\chi\colon \bar{\varphi}_\infty \xrightarrow{\ \sim\ } \psi_\infty$$

that induces an isomorphism

$$\chi\colon (\mathcal{P}_{K^{\mathrm{perf}}}, \bar{\varphi}) \xrightarrow{\ \sim\ } (\mathcal{P}_{K^{\mathrm{perf}}}, \psi)$$

and is compatible with the local Kummer pairing.

$$\chi = \sum\nolimits_{j\leqslant 0} x_j \tau^j, \quad x_j \in \mathfrak{m}_K \text{ for } j < 0$$

A perfectness theorem in general

Theorem 7 (M. – Pink)

For each $\xi \in K$ of $v(\xi) = -i$ and $p \nmid i$ we have an isomorphism

$$[\xi, \ )_\psi \colon \mathrm{gr}^i(J_K) \xrightarrow{\ \sim\ } T^\circ_{\mathrm{ad}}(\psi).$$

The general case

$\varphi = \psi/M, \quad \bar{M} := R \cdot \chi^{-1}(M)$

$$\begin{array}{ccc} J_K & \xrightarrow{\rho} & \mathrm{Hom}_A\big(M,\ T^{\circ}_{\mathrm{ad}}(\psi)\big) \\ {\scriptstyle \rho_{\bar{M}}}\downarrow & & \downarrow{\scriptstyle \wr} \\ \mathrm{Hom}_R\big(\bar{M},\ T^{\circ}_{\mathrm{ad}}(\bar{\omega})\big) & \hookrightarrow & \mathrm{Hom}_A\big(M,\ T^{\circ}_{\mathrm{ad}}(\bar{\omega})\big) \end{array}$$

$\rho_{\bar{M}}$ is surjective up to finite index, $\quad \hookrightarrow$ is split.

$W_i\bar{M} := W_i\mathcal{P}_{K^{\mathrm{perf}}} \cap \bar{M}, \quad S :=$ breaks of $W_\bullet \bar{M}$

Corollary 8 (M. – Pink)

The image $\rho(J_K)$ is open if and only if $\mathrm{rank}_A(M) = \mathrm{rank}_R(\bar{M})$.

In particular, $\rho(J_K)$ is open if $\mathrm{rank}_A(M) = 1$.

A sufficient condition for open image

$$v(\xi) := -p^n j, \quad j(\xi) := j$$

$$j(M) := \{ j(\xi) : \xi \in M \setminus \{0\} \}$$

Theorem 9 (M. – Pink)

We have

$$|j(M)| \leqslant \mathrm{rank}_R(\bar{M}) \leqslant \mathrm{rank}_A(M)$$

and if $|j(M)| = \mathrm{rank}_A(M)$ then $\rho(J_K)$ is open.

$\mathfrak{p}$-independence of the conductor ($\mathfrak{p} \neq \bar{\mathfrak{p}}$)

$$\rho_{\mathfrak{p}} \colon J_K \longrightarrow \mathrm{GL}\big(T_{\mathfrak{p}}(\varphi)\big)$$

$$\mathfrak{f}_{\mathfrak{p}} := \min\big\{ i \colon \rho_{\mathfrak{p}}(J_K^{i+1}) = \{1\} \big\}$$

Theorem 10 (M. – Pink)

$\mathfrak{f}_{\mathfrak{p}}$ is independent of $\mathfrak{p}$.

Furthermore, for each $i \geqslant 0$ either $\rho_{\mathfrak{p}}(J_K^i) = \{1\}$ or $|\rho_{\mathfrak{p}}(J_K^i)| = \infty$.

Inertia invariants ($\mathfrak{p} \neq \bar{\mathfrak{p}}$)

$\varphi = \psi / M$

$$0 \longrightarrow T_{\mathfrak{p}}(\psi) \longrightarrow T_{\mathfrak{p}}(\varphi) \longrightarrow A_{\mathfrak{p}} \otimes_A M \longrightarrow 0$$

Theorem 11 (M. – Pink)

$$T_{\mathfrak{p}}(\varphi)^{I_K} = T_{\mathfrak{p}}(\psi)$$

Gardeyn: same for coinvariants