Examples of model structures

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Today 2 examples: Top and $Ch(R)$. More time on 2^{nd} one.

Both are cofibrantly generated model categories, which means: there are sets W, J, I of morphisms s.t.

- W = weak eq's
- $R(J)$ = fibrations
- $LR(I)$ = cofibrations

$\left\{\begin{array}{l} R(J) = r(J) \\ L(J) = l(J) \end{array}\right\}$ two eq. notations

1. Let's consider Top, the cat of top spaces.

Set S^n = the n-sphere (and $S^{-1} := \emptyset$)

D^n = the n-disk

<u>Def</u>: $I \overset{df}{=} \{ S^{n-1} \hookrightarrow D^n, \ n \geq 0\}$

$J \overset{df}{=} \{ D^n \times \{0\} \hookrightarrow D^n \times [0,1], \ n \geq 0\}$

$W \overset{df}{=}$ weak homotopy equivalences.

<u>thm</u> I, J, W give a cofibrantly generated model category structure on Top.

<u>Pf</u> We won't give it, but just a sample of interesting arguments: we see why every f factors as

$$\cdot \xrightarrow{\text{cof}} \cdot \xrightarrow{\text{fib} \cap W} \cdot$$

and we characterize $\text{cof} = LR(I)$.

Let's go. We'll use the "small object argument" in a simple form called "compact object argument" which is enough in this case.

<u>Def</u>: An "I-cell attachment" is a pushout

$$\begin{tikzcd} \coprod_{s\in\mathcal{S}} S^{n_s-1} \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod_{s\in\mathcal{S}} D^{n_s} \arrow[r] & Y \end{tikzcd}$$

$\mathcal{S}$: an index set

<u>Def</u>: f is an "relative I-cell complex" if it is decomposable as $f: X = Z_0 \hookrightarrow Z_1 \hookrightarrow \dots \hookrightarrow Z_n \hookrightarrow \dots \hookrightarrow Z$
where $Z = \operatorname{colim} Z_i$
each $Z_i \hookrightarrow Z_{i+1}$ is an I-cell attachment

<u>Def</u> I-cell $\overset{\text{def}}{=}$ $\{$ rel. I-cell complexes$\}$

<u>Rem</u> : $I, J \subseteq$ I-cell.

<u>Key prop</u> (Compact object argument)
Given C compact, $f: X \to Z$ $\in$ I-cell and

$$\begin{tikzcd} & C \arrow[d] \\ X \arrow[r, "f"] & Z \end{tikzcd},$$

there exist $(Z_n)_n$ a decomp. of $X \to Z$, and an n with a factorization

$$\begin{tikzcd} & & C \arrow[d] \arrow[dl, dotted] \\ X \arrow[r] & Z_n \arrow[r] & Z \end{tikzcd}$$

<u>Prop</u> For $f: X \to Y$, there is a factorization

$$X \xrightarrow{i} U \xrightarrow{p} Y, \quad i \in I\text{-cell},\ p \in R(I) \cap W$$

[in fact we'll have $R(I) \subset W$!]

<u>Proof</u>: $U_0 = X$

$$\begin{tikzcd} X = U_0 \arrow[r] \arrow[d] & U_1 \arrow[r] \arrow[dl, dashed] & \cdots \arrow[r] \arrow[dll, dashed] & U \\ Y & & & \end{tikzcd}$$

Given $U_k \to Y$, we construct $U_{k+1} \to Y$ as follows:
(U_n) (U_{n+1})

$$\begin{tikzcd} \coprod_{s \in \mathcal{S}} S^{n-1} \arrow[r] \arrow[d] & U_n \arrow[d] \\ \coprod_{s \in \mathcal{S}} D^n \arrow[r] & U_{n+1} \end{tikzcd}$$

where $\mathcal{S} = \bigcup_{n \geq 0} \mathrm{Maps}(S^{n-1}, U_k) \times_{\mathrm{Maps}(S^{n-1}, Y)} \mathrm{Maps}(D^n, Y)$ i.e. we are attaching <u>all</u> possible cells. (A formal construction!)

So define $U = \mathrm{colim}\, U_i$, get $U \to Y$.

Now $X \to U$ is a rel I-cell complex (because it is a rel. CW complex)

Suppose
$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & U \arrow[d, "p"] \\ D^n \arrow[r] & Y \end{tikzcd}$$
, then by Key Prop, since S^{n-1} is compact we have a filling
$$\begin{tikzcd} & U_k \arrow[d] \\ S^{n-1} \arrow[r] \arrow[ur, dashed] & U \end{tikzcd}$$
(some k)

So by construction $\exists\, D^n \to U_{k+1}$.

One checks that $p \in W$... Some details omitted ☒

<u>Def</u>: I-Cof = {retracts of rel. I-cell complexes}

<u>Cor</u>: $LR(I) = I$-cof

<u>Pf</u>: "$\subset$" Let $f \in LR(I)$, then we can factor it

$$\begin{tikzcd} X \arrow[r, "\in I\text{-cell}", "g"'] \arrow[d, "f"'] & U \arrow[d, "\in R(I)"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

So $\exists$ lift $Y \to U$, so f is a retract of $g \in I$-cell

"$\supset$" We have $R(I) \supset R(I\text{-cof})$
but also $R(I) \subset R(I\text{-cof})$ by a quite formal check.

It follows that $R(I) = R(I\text{-cof})$
hence $LR(I) = LR(I\text{-cof}) \supset I$-cof ! qed $\boxtimes$

<u>Cor</u> For every space X, we have a diagram

$$\begin{tikzcd} \emptyset \arrow[rr] \arrow[dr, "I\text{-cell complex}"'] & & X \\ & U \arrow[ur, "\text{weak eq}"'] & \end{tikzcd}$$
.

2. Let's now pass to $Ch(R)$, the cat. of chain complexes.

R: a commutative ring

$Ch(R)$: the cat of chain complexes indexed by $\mathbb{Z}$

$$\cdots \to X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots \qquad X_i \in R\text{-Mod}$$
$$d_{n-1} \circ d_n = 0.$$

$Ch(R)$ has all small limits and colimits
($\to$ called "bicomplete")

$0 \in Ch(R)$ is initial & terminal object.

<u>Def</u>: Given $M \in R\text{-Mod}$ and $n \in \mathbb{Z}$, define

$$S^n M := \; \cdots \to 0 \to \overset{n-1}{0} \to \overset{n}{M} \to \overset{n+1}{0} \to 0 \to \cdots$$

$$D^n M := \; \cdots 0 \to M \xrightarrow{id} M \to 0 \to \cdots$$

<u>Rem</u> there is a natural map $S^{n-1}M \to D^n M$

<u>Def</u>: $I = \{ S^{n-1}\mathbf{R} \to D^n\mathbf{R}$, varying n $\}$

$J = \{ 0 \to D^n \mathbf{R}$, varying n $\}$

$W = \{$ quasi-isom's ie $f: X \to Y$ s.t $H_n f : H_n X \xrightarrow{\sim} H_n Y$, all $n \}$

<u>Rem</u> by functoriality of homology, W is stable under retracts; and by long exact sequences of homology we have the 2-out of 3 property.

<u>thm</u>: $Ch(R)$ has a structure of model category called "projective model structure" with $Cof = LR(I)$
$Fib = R(J)$
$W.eq = W$

Again we only comment on <u>some aspects</u> of the proof.

<u>Prop</u> A map $p: X \to Y$ is a fibration iff $\forall n$, p_n surjective

<u>Cor</u> : every X is fibrant $\boxtimes$

<u>Pf(of prop)</u> : a diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d] \\ D^n R \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

is just the datum of an element $y \in Y_n$

A lift gives $x \in X_n$ over y

$Hom(D^n R, -)$ represents $X \mapsto X_n$ $\boxtimes$

Prop X cofibrant $\Rightarrow \forall n$, X_n is projective.
Conversely if $\forall n$, X_n projective and $X_n = 0$ $\forall n \ll 0$ then X is cofibrant

Example of X all whose X_n's are projective, that is not cofibrant: take $R = k[\varepsilon]/\varepsilon^2$, $X = \cdots \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} \cdots$
then $0 \to X$ and if it is a cofibration, then it has left lifting property wrt fibrations e.g. $D^0R \to D^0k$
However

$$\begin{array}{ccc} \mathrm{Hom}_R(X_0/X_1, R) & \xrightarrow{0 \text{ map}} & \mathrm{Hom}_R(X_0/X_1, k) \\ \| & & \| \\ \mathrm{Hom}(X, D^0R) & \longrightarrow & \mathrm{Hom}(X, D^0k) \end{array}$$

$\hookrightarrow$ not surjective! $\lightning$

Prop: $i: A \hookrightarrow B$ is cofibration iff:
" $\forall n$ i_n is split injective, and $\mathrm{coker}(i)$ is cofibrant "

Rem: Given $M \in R$-Mod, a proj. resolution $P_\bullet \to M \to 0$ is a cofibrant replacement of D^0M, because

$$\begin{array}{ccccccc} P_2 & \to & P_1 & \to & P_0 & \to & 0 \\ & & & & \downarrow & & \\ 0 & \to & 0 & \to & M & \to & 0 \end{array} \quad \text{is q-iso}$$

Homotopies

Def the "interval" $\mathrm{Int} := \quad 0 \to \overset{1}{Re_1} \to \overset{0}{Re_0^0 \oplus Re_0^1} \to 0$

$$e_1 \mapsto e_0^0 - e_0^1$$

For $D \in Ch(R)$ we can compute

(✲) $(D \otimes \mathrm{Int})_n = D_n \oplus D_n \oplus D_{n-1}$ ~~with~~ differentials...
with

$$\cdots \quad d_n : D_n \oplus D_n \oplus D_{n-1} \longrightarrow D_{n-1} \oplus D_{n-1} \oplus D_{n-2}$$
$$(x, y, z) \mapsto (dx - z, dy + z, -dz)$$

<u>Lemma</u> We have maps in $\mathrm{Ch}R$: $D \oplus D \longrightarrow D \otimes \mathrm{Int} \longrightarrow D$
$(x, y) \mapsto (x, y, 0)$
$(x, y, z) \mapsto x + y$

whose composition is the sum,
the first is a cofibration, the 2nd an acyclic (trivial) fibration
& they form a cylinder.

<u>Rem</u> Can define $D \otimes \mathrm{Int}$, and hence cylinders, in any ab. category even if we don't have an object Int.
Indeed, use formulas (✡) !

<u>Def</u>: For $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, a <u>chain homotopy</u> $h : f \Rightarrow g$ is a collection of $h_n : X_{n-1} \to Y_n$ s.t. $f - g = dh + hd$.

<u>Lemma</u> Let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in $\mathrm{Ch}(R)$ and $H : X \otimes \mathrm{Int} \to Y$ a homotopy, then $h(z) := H(0, 0, z)$ is a chain homotopy.

[Given $h : f \Rightarrow g$, $H : X \otimes \mathrm{Int} \to Y$ is homotopy
$(x, y, z) \mapsto f(x) + g(y) + h(z)$
and conversely...]

<u>Recall</u> if $\mathcal{C}$ model category, then $W^{-1}\mathcal{C} \simeq \mathcal{C}_{cf} / \text{homotopy}$

Let's assume that $\mathcal{C} = \mathrm{Ch}^-(R)$ bounded below complexes inherits the proj. model cat structure.
then $\mathcal{C}_{cf} = P_R = \{\text{complexes of projectives}\}$

$$D^- R \simeq P_R / \text{homotopy}$$

References used

Hovey, Model categories

Kampf & Porter

Hirschhorn : 2019 article

May & Ponto: More Concise

nLab

Final remark you can dualize everything and get cochain complexes etc.