Simplicial rings, cotangent complex, smooth and étale morphisms

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Overview

Simplicial commutative rings

Simplicial rings and modules

Last time: for every simplicial abelian group G, $\pi_i(G)$ has a natural structure of group for all $i \geq 0$; for $i > 0$ it is moreover abelian.

We denote by $s\,\mathrm{Alg}$ the category $\mathrm{Hom}(\Delta^{op}, \mathrm{Alg})$ of simplicial commutative rings. Objects are simplicial sets $(A_n)_{n \geq 0}$ where the A_n are commutative rings and the degeneracy and face maps are ring homomorphisms.

For $A \in s\,\mathrm{Alg}$, we denote $s\,\mathrm{Mod}_A$ the category of simplicial A-modules: functors $\Delta \to \mathrm{Mod}_{\mathbb{Z}}$ with an action of A. In other words, $M \in s\,\mathrm{Mod}_A$ is a simplicial abelian group (M_n) such that M_n is an A_n-module and the maps $M_n \to M_m$ that are A_n-linear.

Model structure: we say that a morphism in $s\,\mathrm{Alg}$ or $s\,\mathrm{Mod}\text{-}A$ is a weak equivalence (resp. fibration) if the underlying morphism of simplicial sets is. This is known to define model structures on both categories.

Recall from Tobias' talk the Dold-Kan correspondance: there is an equivalence of categories

$$N \colon s\,\mathrm{Mod}\text{-}A \to \mathrm{Ch}_A^+$$

between simplicial A-modules and chain complexes of A-modules supported in non-negative degree. The model structure obtained on Ch_A^+ is the projective model structure that we saw some lectures ago. We have $\pi_n(M) = H_n(N(M))$.

The model structure on $s\,\mathrm{Alg}$

Definition: a morphism $A \to B$ in $s\,\mathrm{Alg}$ is *free* if there exists a sequence of sets $(X_n)_{n\geq 0}$ such that $B_n \cong A_n[X_n]$ and $s_j(X_n) \subset X_{n+1}$.

Lemma: a morphism $A \to B$ in $s\,\mathrm{Alg}$ is:

- a weak equivalence if for all $i \geq 0$, $\pi_i(A) \to \pi_i(B)$ is an isomorphism;
- a fibration if $A \to \pi_0(A) \times_{\pi_0(B)} B$ is surjective;
- a cofibration if it is a retract of a free morphism.

Functorial construction of cofibrant replacement: The functor that to a set S associates the ring $\mathbb{Z}[S]$ is left adjoint to the forgetful functor from rings to sets. So let's start with a classical ring A; we have a surjective adjunction ring homomorphism $\eta_A \colon \mathbb{Z}[A] \to A$. On the other hand we have two maps $\eta_{\mathbb{Z}[A]}, \mathbb{Z}[\eta_A] \colon \mathbb{Z}[\mathbb{Z}[A]] \to \mathbb{Z}[A]$. This process constructs a free simplicial ring augmented over the constant simplicial ring A

$$QA := \left(\dots \mathbb{Z}[\mathbb{Z}[\mathbb{Z}[A]]] \overset{\to}{\underset{\to}{\to}} \mathbb{Z}[\mathbb{Z}[A]] \rightrightarrows \mathbb{Z}[A]\right) \to A$$

which can be checked to be a weak equivalence.

Remark: Dold-Kan correspondence identifies $s\,\mathrm{Alg}$ with category of dg-algebras. The Koszul complex is an example of cofibrant dg-algebra.

$\pi_* A$ as a graded ring

Theorem

Let $A \in s\,\mathrm{Alg}$. The graded abelian group $\pi_* A := \bigoplus_{\geq 0} \pi_n(A)$ has a natural structure of graded ring. In particular:

- *$\pi_0(A)$ is a ring*
- *every $\pi_i(A)$ is a $\pi_0(A)$-module.*

Ring structure: given maps $S^m \to A$ and $S^n \to A$, one gets a map

$$S^m \times S^n \to A \times A \to A \otimes_{\mathbb{Z}} A \xrightarrow{m} A$$

which sends $S^m \times \{0\}$ and $\{0\} \times S^n$ to zero and therefore factors via $S^m \times S^n \to S^{m+n}$.

A similar theorem holds for $M \in s\,\mathrm{Mod}\text{-}A$: $\pi_*(M)$ has a natural structure of $\pi_*(A)$-graded module. That is, the action satisfies $\pi_i(A)\pi_j(M) \subset \pi_{i+j}M$. In particular each $\pi_i(M)$ is a $\pi_0(A)$-module.

Adjunctions; derived tensor product

For simplicial A-modules M, N, the tensor product $M \otimes_A N$ is defined degreewise.

For $f\colon A \to B$ of simplicial rings, we have the extension of scalars functor $- \otimes_A B\colon s\,\mathrm{Mod}\text{-}A \to s\,\mathrm{Mod}\text{-}B$, left adjoint to the restriction of scalars. The adjunction is Quillen. We denote

$$- \otimes^L_A B\colon\ \mathrm{ho}(s\,\mathrm{Mod}\text{-}A) \to \mathrm{ho}(s\,\mathrm{Mod}\text{-}B)$$

the total derived functor. In other words, $M \otimes^L_A B = QM \otimes_A B$ for a cofibrant replacement $QM \to M$.

Every classical ring R gives a constant simplicial ring cR. The induced functor $\mathrm{Alg} \to \mathrm{ho}(s\,\mathrm{Alg})$ is fully faithful and has a left adjoint in $\pi_0\colon\ \mathrm{ho}(s\,\mathrm{Alg}) \to \mathrm{Alg}$. A similar statement holds for A-modules.

The cotangent complex

Motivation

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes. We all remember from primary school the Jacobi-Zariski exact sequence of sheaves of $\mathcal{O}_X$-modules

$$f^*\Omega^1_{Y|Z} \to \Omega^1_{X|Z} \to \Omega^1_{X|Y} \to 0$$

Question: is it possible to prolong the sequence to a long exact sequence, by taking "derived functors" of Kahler differentials?

Evidence for a positive answer:

- when f is smooth, the sequence is exact on the left as well;
- when f is a closed immersion with ideal sheaf $\mathcal{I}$, then $\Omega^1_{X|Y} = 0$, and there is an exact sequence
$$\mathcal{I}/\mathcal{I}^2 \to f^*\Omega^1_{Y|Z} \to \Omega^1_{X|Z} \to 0$$

So it seems that smooth morphisms are "acyclic" the functor of Kahler differentials; and that the conormal sheaf is the "first derived functor" for a closed immersion. Of course the category of schemes is not abelian, so we cannot use homological algebra. We try using homotopical methods instead.

Quillen homology

Let $\mathcal{C}$ be a category; we define a category $\mathcal{C}^{ab}$ of *abelian group objects*.

If $\mathcal{C}$ has a terminal object and binary products, then $\mathcal{C}^{ab}$ is the category of tuples (X, m, i, e) with $X \in \mathcal{C}$ and a multiplication, inverse, identity making X an abelian group. Arrows are what they should be.

For an equivalent more general definition:

- objects: pairs (X, ϕ, σ) of an object $X \in \mathcal{C}$, a functor $\phi \colon \mathcal{C} \xrightarrow{\phi} \underline{\mathrm{Mod}}(\mathbb{Z})$, and an isomorphism σ from the composition $\mathcal{C} \xrightarrow{\phi} \underline{\mathrm{Mod}}(\mathbb{Z}) \to \underline{\mathrm{Sets}}$ to $\mathrm{Hom}(-, X) \colon \mathcal{C} \to \underline{\mathrm{Sets}}$.
- arrows: pairs $(X \to X', \phi \to \phi')$ so that everything is compatible.

Suppose that the forgetful functor $\mathcal{C}^{ab} \to \mathcal{C}$ has a left adjoint $\mathrm{Ab} \colon \mathcal{C} \to \mathcal{C}^{ab}$; and that both categories have model structures making the adjunction into a Quillen adjunction. Then, for $X \in \mathcal{C}$, we call the total left derived functor $\mathbb{L}\,\mathrm{Ab}(X)$ the *Quillen homology* of X. To calculate it, take a cofibrant replacement $QX \to X$; then $\mathbb{L}\,\mathrm{Ab}(X) \cong \mathrm{Ab}(QX)$ in $ho(\mathcal{C}^{ab})$.

Example: for $\mathcal{C} = \underline{\mathrm{sSets}}$, we have $\mathcal{C}^{ab} = \underline{\mathrm{sMod}}(\mathbb{Z})$. The abelianization functor is $(X_n)_n \mapsto (\mathbb{Z}[X_n])_n$, the free simplicial abelian group. Since every simplicial set is cofibrant, Quillen homology is the functor $\mathbb{Z}[-]$.

Group objects in the category of rings

We fix a commutative ring k. Let Alg_k be the category of k-algebras. Since a group object must admit a morphism from the terminal object 0, all group objects are zero! We need to change of point of view. We fix a k-algebra A and consider the category $\mathrm{Alg}_{k/A}$ of k-algebras over A; its objects are factorizations $k \to B \to A$. The terminal object is now A.

We define a functor $\Phi \colon \mathrm{Mod}_A \to \mathrm{Alg}_{k/A}$ by sending M to the square zero thickening of A by M, i.e. the k-algebra $A \rtimes M$ having underlying set $A \oplus M$, and ring structure

$$(a_0, m_0) \cdot (a_1, m_1) = (a_0 a_1, a_0 m_1 + a_1 m_0)$$

Notice that $A \rtimes M$ has a natural structure of abelian group object. Indeed, for any $B \in \mathrm{Alg}_{k/A}$

$$\mathrm{Hom}_{\mathrm{Alg}_{k/A}}(B, A \rtimes M) \to \mathrm{Der}_k(B, M)$$
$$f \mapsto pr_2 \circ f$$

is a bijection.

Lemma

The functor Φ induces an equivalence $\mathrm{Mod}_A \to \mathrm{Alg}^{ab}_{k/A}$.

The abelianization functor

Following the philosophy of Quillen homology, we would like to have an abelianization functor $\mathrm{Alg}_{k/A} \to \mathrm{Alg}^{ab}_{k/A} \cong \mathrm{Mod}_A$.

Lemma

The functor

$$\begin{aligned} \mathrm{Alg}_{k/A} &\to \mathrm{Mod}_A \\ B &\mapsto \Omega^1_{B|k} \otimes_B A \end{aligned}$$

is left adjoint to $\Phi\colon \mathrm{Mod}_A \to \mathrm{Alg}_{k/A}$, $M \mapsto A \rtimes M$.

Proof:

$$\mathrm{Hom}_{\mathrm{Alg}_{k/A}}(B, A \rtimes M) = \mathrm{Der}_k(B, M) = \mathrm{Hom}_B(\Omega^1_{B|k}, M) = \mathrm{Hom}_A(\Omega^1_{B|k} \otimes_B A, M).$$

Great! We have the abelianization functor. What about model structures? Here is why we need to pass to simplicial categories.

We take now $k \to A$ of simplicial rings. All the functors constructed before extend degreewise to simplicial categories; moreover for a category $\mathcal{C}$ with terminal object and binary product, $(s\mathcal{C})^{ab} = s(\mathcal{C}^{ab})$. Hence we have an adjunction

$$\begin{aligned} \mathrm{Ab}_{k/A}\colon s\,\mathrm{Alg}_{k/A} &\leftrightarrow s\,\mathrm{Mod}_A \\ B &\mapsto \Omega^1_{B/k} \otimes_B A \\ A \rtimes M &\leftarrow M \end{aligned}$$

The right adjoint $M \mapsto A \rtimes M$ preserves fibrations and weak equivalences, hence the adjunction is Quillen and we may therefore take the total left derived functor.

Definition

- The *cotangent complex functor* is the total left derived functor

$$\mathbb{L}\,\mathrm{Ab}_{k/A}\colon ho(s\,\mathrm{Alg}_{k/A}) \to ho(s\,\mathrm{Mod}_A)$$

- the *cotangent complex* of $k \to A$, denoted $L_{A/k}$, is $\mathbb{L}\,\mathrm{Ab}_{k/A}(A)$.

In other words, the cotangent complex of $k \to A$ can be calculated by taking a free resolution $QA \to A$ in $s\,\mathrm{Alg}_k$; then $\Omega^1_{QA|k} \otimes_{QA} A$.

We will see the cotangent complex $L_{A/k}$ of $k \to A$ as an object of the derived category of A-modules via the Dold-Kan correspondence.

Lemma: $H_0(L_{A/k}) = \Omega^1_{A/k}$

Proof: the cofibrant replacement $Q \to A$ is a weak equivalence. In particular $\pi_0(Q) = A$. It suffices to show that the diagram

$$\begin{array}{ccc} ho(s\,\mathrm{Alg}_{k/A}) & \xrightarrow{\mathbb{L}\,\mathrm{Ab}} & ho(s\,\mathrm{Mod}_A) \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\ \mathrm{Alg}_{k/A} & \xrightarrow{\mathrm{Ab}} & \mathrm{Mod}_A \end{array}$$

commutes. This happens if and only if the inverted diagram of adjoints commutes

$$\begin{array}{ccc} ho(s\,\mathrm{Alg}_{k/A}) & \xleftarrow[A \rtimes M \mapsfrom M]{} & ho(s\,\mathrm{Mod}_A) \\ {\scriptstyle constant}\uparrow & & {\scriptstyle constant}\uparrow \\ \mathrm{Alg}_{k/A} & \xleftarrow[A \rtimes M \mapsfrom M]{} & \mathrm{Mod}_A \end{array}$$

which is obvious.

Fundamental properties of the cotangent complex

- (Relation with Kahler differentials) $H_0(L_{A/k}) = \Omega^1_{A/k}$
- (base change) let $k \to k'$ be another map; and let $A' := A \otimes_k^{\mathbb{L}} k'$. Then there is a natural isomorphism $L_{A|k} \otimes_A^{\mathbb{L}} k' \to L_{A'|k'}$.
- (Transitivity triangle)Given a composite $k \to A \to B$, there is an exact triangle

$$L_{A|k} \otimes_A^{\mathbb{L}} B \to L_{B|k} \to L_{B|A} \to \left(L_{A|k} \otimes_A^{\mathbb{L}} B\right)[1]$$

  The associated long exact sequence of homology tells us how to extend the Jacobi-Zariski sequence to a long exact sequence.
- (Etaleness) If $k \to A$ is an étale map of classical rings, $L_{A|k} = 0$.
- (Smoothness) If $k \to A$ is a smooth map of classical rings, then $L_{A|k} \to \Omega^1_{A|k}[0]$ is an isomorphism.

Theorem

$A \to B$ in $s\,\mathrm{Alg}_k$ is a weak equivalence if and only if $\pi_0(A) \to \pi_0(B)$ is an isomorphism of classical rings and $L_{B|A} = 0$.

Two proofs (just for the record)

Proof of condition (etale): If $k \to A$ is étale, $L_{A|k} = 0$.

Suppose first that $k \to A$ is a Zariski localization. Then $A \otimes_k A = A$. One easily gets from the transivity triangle applied to $k \to A \to A \otimes_k A$ and the base change formula that $L_{A|k} = 0$. Now let $k \to A$ be étale. Then $B := A \otimes_k A \to A$ is a Zariski localization, so $L_{A|B} = 0$. By the transitivity triangle for $A \to B \to A$, we get $L_{B|A} \otimes_B A = 0$. But $L_{B|A} = A \otimes_k L_{A|k}$, so we get $L_{A|k} \otimes_k (A \otimes_B A) = 0$. But $A \otimes_B A = A$, and $L_{A|k} \to L_{A|k} \otimes_k A$ has a section given by the A-action so $L_{A|k} = 0$.

Proof of (smoothness): If $k \to A$ is smooth, then $L_{A|k} \to \Omega^1_{A|k}[0]$ is an isomorphism.

Suppose first that A is a polynomial k-algebra. Then A is cofibrant, so that case is okay. We may work locally on A; then there is an étale map $B \to A$ from a polynomial k-algebra. We know that $L_{B|k} = \Omega^1_{B|k}[0]$ and that $L_{A|B} = 0$. By the transitivity triangle, $L_{A|k} = L_{B|k} \otimes^L_B A = L_{B|k} \otimes_B A = \Omega^1_{A|k}[0]$.

Example: the cotangent complex for a complete intersection

Lemma

let R be a ring, $I \subset R$ an ideal generated by a regular sequence, $A := R/I$. Then $L_{A|R} = I/I^2[1]$.

Proof: Suppose first that $R = \mathbb{Z}[x_1, \ldots, x_n]$, $I = (x_1, \ldots, x_n)$. Then $A = \mathbb{Z}$. By the transitivity triangle, $L_{A|R} = L_{R|\mathbb{Z}} \otimes_R A[1] = \Omega^1_{R|\mathbb{Z}} \otimes_R A[1] = I/I^2[1]$.

Now for the general case, we can take $f \colon \mathbb{Z}[x_1, \ldots, x_n]$ sending the x_i to the regular sequence; then $A = R \otimes_{\mathbb{Z}[x_1,\ldots,x_n]} \mathbb{Z}$. Because the sequence is regular, the equality holds also with derived tensor product. It follows by the base change property that

$$L_{A|R} = L_{\mathbb{Z}/\mathbb{Z}[x_1,\ldots,x_n]} \otimes_{\mathbb{Z}} A = I/I^2[1].$$

Deformation theory

In the classical theory, deformations of smooth morphisms of schemes $f\colon X \to Y$ along a square-zero extension of Y by an ideal J, are controlled by the groups $Ext^i(\Omega^1_{X|Y}, f^*J)$. More precisely, the category of deformations is a gerbe on X banded by $\mathcal{H}om(\Omega^1_{X|Y}, f^*J)$.

If $X \to Y$ is not smooth, this fails. However in this case, it is still true that deformations are controlled by

$$Ext^i(L_{X|Y}, f^*J) := \mathrm{Hom}(L_{X|Y}, f^*J[i])$$

See Illusie, *Complexe cotangent; applications à la théorie des déformations*, Thm 1.7 for the precise statement.

Smooth and étale morphisms

Finitely presented maps

Classical notion: a morphism $R \to S$ of rings is *of finite presentation* if for any filtered system $\{T_i\}$ of S-algebras, the natural map

$$\operatorname{colim}_i \operatorname{Hom}_S(T_i, S) \to \operatorname{Hom}_R(\operatorname{colim}_i T_i, S)$$

is a bijection.

Definition

A morphism $A \to B$ of simplicial rings is of *finite presentation* if it is a compact object in $s\,\mathrm{Alg}_A$, i.e. if for any filtered system $\{C_i\}$ of A-algebras, the natural map

$$\operatorname{colim}_i \operatorname{Hom}_A(C_i, B) \to \operatorname{Hom}_A(\operatorname{colim}_i C_i, B)$$

is a weak equivalence.

Proposition: a morphism $A \to B$ is of finite presentation if and only if $\pi_0(A) \to \pi_0(B)$ is (classically) of finite presentation and $L_{B|A}$ is perfect (i.e. quasi-isomorphic to bounded complex of projective modules).

Remark: a finite presentation map of classical rings may not be of finite presentation as simplicial rings. But for example classical fp+lci implies simplicial fp.

Flatness, formal smoothness/étaleness

A morphism $f\colon R \to S$ of rings is *flat* if the functor $- \otimes_R S$ preserves pullbacks.

Definition

A morphism $f\colon A \to B$ is *flat* if the derived functor $- \otimes^L_A B\colon ho(s\,\mathrm{Mod}_A) \to ho(s\,\mathrm{Mod}_B)$ preserves pullbacks.

A morphism $R \to S$ of rings is *formally étale* if the sheaf of differentials and the conormal sheaf vanish, i.e. if $\tau_{\leq 1} L_{S|R} = 0$.

Definition

A morphism $A \to B$ in $s\,\mathrm{Alg}$ is *formally étale* if $L_{B|A} = 0$;

A morphism $R \to S$ of rings is *formally smooth* if and only if $\tau_{\leq 1} L_{S|R} \cong P[0]$ for a projective S-module; if and only if $\mathrm{Hom}(\tau_{\leq 1} L_{S|R}, M) = 0$ for any $M \in s\,\mathrm{Mod}_S$ with $\pi_0(M) = 0$.

Definition

A morphism $A \to B$ is *formally smooth* if for any $M \in s\,\mathrm{Mod}_B$ with $\pi_0(M) = 0$, $\mathrm{Hom}(L_{B|A}, M) = 0$

Etale and smooth morphisms; definition and characterization

Definition

- A morphism is *étale* if it is formally étale and of finite presentation;
- A morphism is *smooth* if it is formally smooth and of finite presentation;
- A morphism is a *open immersion* if it is flat, of finite presentation and the natural morphism $B \otimes^L_A B \to B$ is an isomorphism.

Charachterization: A morphism $A \to B$ is flat (resp. étale, smooth, open immersion) if and only if:

- $\pi_0(A) \to \pi_0(B)$ is classically flat (resp. étale, smooth, open immersion), and
- for all $i > 0$, the induced map of $\pi_0(B)$-modules

$$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$$

  is an isomorphism.

In particular, for $A \to B$ a morphism of classical rings, the simplicial notion of the above properties is equivalent to the classical one.

Topological invariance of the étale site

Lemma (analogous of infinitesimal lifting property)

Let $A \to B$ be a formally étale morphism, and $T' \to T$ a morphism inducing an isomorphism $\pi_0(T') \to \pi_0(T)$. Then the map $\mathrm{Hom}(B, T') \to \mathrm{Hom}(B, T) \times_{\mathrm{Hom}(A,T)} \mathrm{Hom}(A, T')$ *is an equivalence. This should mean that in ho*$(s\,\mathrm{Alg})$*, for any commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & T' \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & T \end{array}$$

there exists a unique dashed arrow making the diagram commute.

Lemma (analogous of "Deformations of smooth are unobstructed")

Let $A \to B$ be a morphism inducing an isomorphism $\pi_0(A) \to \pi_0(B)$. Then for every smooth B-algebra B' there exists an A-algebra A' such that $B' = B \otimes^L_A A'$.

Corollary

Let $A \to B$ a morphism inducing an isomorphism $\pi_0(A) \to \pi_0(B)$. Then the base change functor $s\,\mathrm{Alg}^{\acute{e}t}_A \to s\,\mathrm{Alg}^{\acute{e}t}_B$ is an equivalence.

Catch: the étale topos of a simplicial ring A is the étale topos of its $\pi_0(A)$. This makes glueing along étale covers easier.

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