Critical Brownian sheet does not have double points*

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Abstract

We derive a decoupling formula for the Brownian sheet which has the following ready consequence: An N-parameter Brownian sheet in $\mathbf{R}^d$ has double points if and only if $d < 4N$. In particular, in the critical case where $d = 4N$, Brownian sheet does not have double points. This answers an old problem in the folklore of the subject. We also discuss some of the geometric consequences of the mentioned decoupling, and establish a partial result concerning k-multiple points in the critical case $k(d - 2N) = d$.

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1 Introduction

Let $B := (B^1, \ldots, B^d)$ denote a d-dimensional N-parameter Brownian sheet. That is, B is a d-dimensional, N-parameter, centered Gaussian process with

$$\mathrm{Cov}(B^i(\boldsymbol{s})\,, B^j(\boldsymbol{t})) = \delta_{i,j} \cdot \prod_{k=1}^{N} (s_k \wedge t_k), \tag{1.1}$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise, and $\boldsymbol{s}, \boldsymbol{t} \in \mathbf{R}^N_+$, $\boldsymbol{s} = (s_1, \ldots, s_N)$, $\boldsymbol{t} = (t_1, \ldots, t_N)$. Here and throughout, we define

$$\boldsymbol{\mathcal{T}} := \left\{ (\boldsymbol{s}\,, \boldsymbol{t}) \in (0\,, \infty)^{2N} :\ s_i \neq t_i \text{ for all } i = 1, \ldots, N \right\}. \tag{1.2}$$

The following is the main result of this paper.

Theorem 1.1. *Choose and fix a Borel set $A \subseteq \mathbf{R}^d$. Then,*

$$\mathrm{P}\left\{ \exists\, (\boldsymbol{u}_1\,, \boldsymbol{u}_2) \in \boldsymbol{\mathcal{T}} :\ B(\boldsymbol{u}_1) = B(\boldsymbol{u}_2) \in A \right\} > 0 \tag{1.3}$$

if and only if

$$\mathrm{P}\left\{ \exists\, (\boldsymbol{u}_1\,, \boldsymbol{u}_2) \in \boldsymbol{\mathcal{T}} :\ W_1(\boldsymbol{u}_1) = W_2(\boldsymbol{u}_2) \in A \right\} > 0, \tag{1.4}$$

where W_1 and W_2 are independent N-parameter Brownian sheets in $\mathbf{R}^d$ (unrelated to B).

Theorem 1.1 helps answer various questions about the multiplicities of the random surface generated by the Brownian sheet. We introduce some notation in order to present some of these issues.

Recall that $x \in \mathbf{R}^d$ is a *k-multiple point of B* if there exist distinct points $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_k \in (0\,, \infty)^N$ such that $B(\boldsymbol{s}_1) = \cdots = B(\boldsymbol{s}_k) = x$. We write M_k for the collection of all k-multiple points of B. Note that $M_{k+1} \subseteq M_k$ for all $k \geq 2$.

In this paper, we are concerned mainly with the case $k = 2$; elements of M_2 are the *double points* of B. In Section 5 below, we derive the following ready consequence of Theorem 1.1.

Corollary 1.2. *Let A denote a nonrandom Borel set in $\mathbf{R}^d$. If $d > 2N$, then*

$$\mathrm{P}\{M_2 \cap A \neq \varnothing\} > 0 \quad \textit{if and only if} \quad \mathrm{Cap}_{2(d-2N)}(A) > 0, \tag{1.5}$$

where Cap_β denotes the Bessel–Riesz capacity in dimension $\beta \in \mathbf{R}$; see §2 below. If $d = 2N$, then $\mathrm{P}\{M_2 \cap A \neq \varnothing\} > 0$ if and only if there exists a probability measure μ, compactly supported in A, such that

$$\iint \left| \log_+ \left(\frac{1}{|x-y|} \right) \right|^2 \mu(\mathrm{d}x)\, \mu(\mathrm{d}y) < \infty. \tag{1.6}$$

Finally, if $d < 2N$, then $\mathrm{P}\{M_k \cap A \neq \varnothing\} > 0$ for all $k \geq 2$ and all nonvoid nonrandom Borel sets $A \subset \mathbf{R}^d$.

We apply Corollary 1.2 with $A := \mathbf{R}^d$ and appeal to Taylor's theorem [12, pp. 523–525] to deduce the following.

Corollary 1.3. *An N-parameter, d-dimensional Brownian sheet has double points if and only if $d < 4N$. In addition, M_2 has positive Lebesgue measure almost surely if and only if $d < 2N$.*

When $N = 1$, B is d-dimensional Brownian motion, and this corollary has a rich history in that case: Lévy [18] was the first to prove that Brownian motion has double points [$M_2 \neq \varnothing$] when $d = 2$; this is also true in one dimension, but almost tautologically so. Subsequently, Kakutani [11] proved that Brownian motion in $\mathbf{R}^d$ does not have double points when $d \geq 5$; see also Ville [25]. Dvoretzky, Erdős, and Kakutani [8] then showed that Brownian motion has double points when $d = 3$, but does not have double points in the case that $d = 4$. Later on, Dvoretzky, Erdős, and Kakutani [9] proved that in fact, $M_k \neq \varnothing$ for all $k \geq 2$, when $d = 2$. The remaining case is that $M_3 \neq \varnothing$ if and only if $d \leq 2$; this fact is due to Dvoretzky, Erdős, Kakutani, and Taylor [10].

When $N > 1$ and $k = 2$, Corollary 1.3 is new only in the critical case where $d = 4N$. The remaining [noncritical] cases are much simpler to derive, and were worked out earlier by one of us [14]. In the critical case, Corollary

1.3 asserts that Brownian sheet has no double points. This justifies the title of the paper and solves an old problem in the folklore of the subject. For an explicit mention—in print—of this problem, see B. Fristedt's review of the article of Chen [3] in *The Mathematical Reviews*, where most of the assertion about M_2 (and even M_k) having positive measure was conjectured.

The proof of Theorem 1.1 leads to another interesting property, whose description requires us first to introduce some notation. We identify subsets of $\{1,\ldots,N\}$ with partial orders on $\mathbf{R}^N$ as follows [17]: For all $\boldsymbol{s},\boldsymbol{t}\in\mathbf{R}^N$ and $\pi\subseteq\{1,\ldots,N\}$,

$$\boldsymbol{s}\prec_\pi \mathbf{t} \quad \text{iff} \quad \begin{cases} s_i\le t_i & \text{for all } i\in\pi,\\ s_i\ge t_i & \text{for all } i\notin\pi. \end{cases} \tag{1.7}$$

Clearly every $\boldsymbol{s}$ and $\boldsymbol{t}$ in $\mathbf{R}^N$ can be compared via some π. In fact, $\boldsymbol{s}\prec_\pi\boldsymbol{t}$, where π is the collection of all $i\in\{1,\ldots,N\}$ such that $s_i\le t_i$. We might write $\boldsymbol{s}\prec_\pi\boldsymbol{t}$ and $\boldsymbol{t}\succ_\pi\boldsymbol{s}$ interchangeably. Sometimes, we will also write $\boldsymbol{s}\curlywedge_\pi\boldsymbol{t}$ for the N-vector whose jth coordinate is $\min(s_j,t_j)$ if $j\in\pi$ and $\max(s_j,t_j)$ otherwise.

Given a partial order $\pi\subset\{1,\ldots,N\}$ and $\boldsymbol{s},\boldsymbol{t}\in(0,\infty)^N$ we write $\boldsymbol{s}\ll_\pi\boldsymbol{t}$ if $\boldsymbol{s}\prec_\pi\boldsymbol{t}$ and $s_i\neq t_i$, for all $i\in\{1,\ldots,N\}$. Define

$$\widetilde{M}_k := \left\{ x\in\mathbf{R}^d \;\middle|\; \begin{array}{l} \exists\, \boldsymbol{s}_1,\ldots,\boldsymbol{s}_k\in(0,\infty)^N:\ B(\boldsymbol{s}_1)=\cdots=B(\boldsymbol{s}_k)=x \\ \text{and } \boldsymbol{s}_1\ll_\pi\cdots\ll_\pi\boldsymbol{s}_k \text{ for some } \pi\subset\{1,\ldots,N\} \end{array} \right\}. \tag{1.8}$$

Proposition 1.4. *Let $A\subset\mathbf{R}^d$ be a nonrandom Borel set. Then for all $k\ge 2$,*

$$\mathrm{P}\left\{\widetilde{M}_k\cap A\neq\varnothing\right\}>0 \quad \textit{if and only if} \quad \mathrm{Cap}_{k(d-2N)}(A)>0. \tag{1.9}$$

In particular, there are (strictly) π-ordered k-tuples on which B takes a common value if and only if $k(d-2N)<d$.

Theorem 1.1 can also be used to study various geometric properties of the random set M_2 of double points of B. Of course, we need to study only the case where $M_2 \neq \varnothing$ almost surely. That is, we assume hereforth that $d < 4N$. With this convention in mind, let us start with the following formula:

$$\dim_{_{\rm H}} M_2 = d - 2(d-2N)^+ \quad \text{almost surely.} \tag{1.10}$$

This formula appears in Chen [3] (with a gap in his proof that was filled by Khoshnevisan, Wu, and Xiao [16]). In fact, a formula for $\dim_{_{\rm H}} M_k$ analogous to (1.10) holds for all $k \geq 2$ [3, 16] and has many connections to the well-known results of Orey and Pruitt [21], Mountford [19], and Rosen [24].

As yet another application of Theorem 1.1 we can refine (1.10) by determining the Hausdorff dimension of $M_2 \cap A$ for any nonrandom closed set $A \subset \mathbf{R}^d$. First, let us remark that a standard covering argument (similar to the proof of Part (i) of Lemma 5.2) shows that for any fixed nonrandom Borel set $A \subset \mathbf{R}^d$

$$\dim_{_{\rm H}}(M_2 \cap A) \leq \dim_{_{\rm H}} A - 2(d-2N) \quad \text{almost surely.} \tag{1.11}$$

The following corollary provides an essential lower bound for $\dim_{_{\rm H}}(M_2 \cap A)$. Recall that the essential supremum $\|Z\|_{L^\infty(\mathrm{P})}$ of a nonnegative random variable Z is defined as

$$\|Z\|_{L^\infty(\mathrm{P})} := \inf\left\{\lambda > 0 : \mathrm{P}\{Z > \lambda\} = 0\right\} \qquad (\inf \varnothing := +\infty). \tag{1.12}$$

Corollary 1.5. *Choose and fix a nonrandom closed set $A \subset \mathbf{R}^d$. If $\dim_{_{\rm H}} A < 2(d-2N)$, then with probability one A does not contain any double points of the Brownian sheet. On the other hand, if $\dim_{_{\rm H}} A \geq 2(d-2N)$, then*

$$\left\|\dim_{_{\rm H}}(M_2 \cap A)\right\|_{L^\infty(\mathrm{P})} = \dim_{_{\rm H}} A - 2(d-2N)^+. \tag{1.13}$$

Equation (1.10) follows from Corollary 1.5 and the fact that $\dim_{_{\rm H}} M_2$ is a.s. a constant. The proof of this "zero-one law" follows more-or-less standard methods, which we skip.

There is a rich literature of decoupling, wherein expectation functionals for sums of dependent random variables are analyzed by making clever comparisons to similar expectation functionals that involve only sums of *independent* [sometimes conditionally independent] random variables. For a definitive account, see the recent book of de la Peña and Giné [7].

Theorem 1.1 of the present paper follows the general philosophy of decoupling, but applies it to random fractals rather than random variables [or vectors]. A "one-parameter" version of these ideas appear earlier in the work of Peres [23]. From a technical point of view, Theorem 1.1 is rather different from the results of decoupling theory.

This paper is organized as follows. Section 2 recalls the main notions of potential theory and presents our main technical result concerning conditional laws of the Brownian sheet (Theorem 2.4). In Section 3, we present a sequence of estimates concerning the pinned Brownian sheet. Section 4 contains the proof of Theorem 2.4. Finally, Section 5 contains the proofs of Theorem 1.1, of its corollaries and of Proposition 1.4.

2 Potential theory

In this section, we first introduce some notation for capacities, energies, and Hausdorff dimension, and we also recall some basic facts about them. Then we introduce the main technical result of this paper, which is a theorem of "conditional potential theory," and is of independent interest.

2.1 Capacity, energy, and dimension

For all real numbers β, we define a function $\kappa_\beta : \mathbf{R}^d \to \mathbf{R}_+ \cup \{\infty\}$ as follows:

$$\kappa_\beta(x) := \begin{cases} \|x\|^{-\beta} & \text{if } \beta > 0, \\ \log_+(\|x\|^{-1}) & \text{if } \beta = 0, \\ 1 & \text{if } \beta < 0, \end{cases} \tag{2.1}$$

where, as usual, $1/0 := \infty$ and $\log_+(z) := 1 \vee \log(z)$ for all $z \geq 0$.

Let $\mathcal{P}(G)$ denote the collection of all probability measures that are supported by the Borel set $G \subseteq \mathbf{R}^d$, and define the β-dimensional *capacity* of

G as

$$\mathrm{Cap}_\beta(G) := \left[\inf_{\substack{\mu \in \mathcal{P}(K): \\ K \subset G \text{ is compact}}} \mathrm{I}_\beta(\mu) \right]^{-1}, \tag{2.2}$$

where $\inf \varnothing := \infty$, and $\mathrm{I}_\beta(\mu)$ is the β-dimensional *energy* of μ, defined as follows for all $\mu \in \mathcal{P}(\mathbf{R}^d)$ and $\beta \in \mathbf{R}$:

$$\mathrm{I}_\beta(\mu) := \iint \kappa_\beta(x - y)\, \mu(\mathrm{d}x)\, \mu(\mathrm{d}y). \tag{2.3}$$

In the cases where $\mu(\mathrm{d}x) = f(x)\,\mathrm{d}x$, we may also write $\mathrm{I}_\beta(f)$ in place of $\mathrm{I}_\beta(\mu)$.

Let us emphasize that for all probability measures μ on $\mathbf{R}^d$ and all Borel sets $G \subseteq \mathbf{R}^d$,

$$\mathrm{I}_\beta(\mu) = \mathrm{Cap}_\beta(G) = 1 \qquad \text{when } \beta < 0. \tag{2.4}$$

According to Frostman's theorem [12, p. 521], the Hausdorff dimension of G satisfies

$$\begin{aligned} \dim_{\mathrm{H}} G &= \sup\left\{\beta > 0 :\ \mathrm{Cap}_\beta(G) > 0\right\} \\ &= \inf\left\{\beta > 0 :\ \mathrm{Cap}_\beta(G) = 0\right\}. \end{aligned} \tag{2.5}$$

The reader who is unfamiliar with Hausdorff dimension can use the preceding as its *definition*. The usual definition can be found in Appendix C of Khoshnevisan [12], where many properties of $\dim_{\mathrm{H}}$ are also derived. We will also need the following property:

$$\mathrm{Cap}_n(\mathbf{R}^n) = 0 \qquad \text{for all } n \geq 1. \tag{2.6}$$

See Corollary 2.3.1 of Khoshnevisan [12, p. 525] for a proof.

2.2 Conditional potential theory

Throughout, we assume that our underlying probability space $(\Omega\,, \mathcal{F}, \mathrm{P})$ is complete. Given a partial order π and a point $\boldsymbol{s} \in \mathbf{R}^N_+$, we define $\mathcal{F}_\pi(\boldsymbol{s})$ to be the σ-algebra generated by $\{B(\boldsymbol{u}),\ \boldsymbol{u} \prec_\pi \boldsymbol{s}\}$ and all P-null sets. We then make the filtration $(\mathcal{F}_\pi(\boldsymbol{s}),\ s \in \mathbf{R}_+^N)$ right-continuous in the partial order π, so that $\mathcal{F}_\pi(\boldsymbol{s}) = \cap_{\boldsymbol{t} \succ_\pi \boldsymbol{s}} \mathcal{F}_\pi(\boldsymbol{t})$.

Definition 2.1. Given a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ and a set-valued function A—mapping Ω into subsets of $\mathbf{R}^d$—we say that A is a *$\mathcal{G}$-measurable random set* if $\Omega \times \mathbf{R}^d \ni (\omega\,,x) \mapsto \mathbf{1}_{A(\omega)}(x)$ is $(\mathcal{G} \times \mathcal{B}(\mathbf{R}^d))$-measurable, where $\mathcal{B}(\mathbf{R}^d)$ denotes the Borel σ-algebra on $\mathbf{R}^d$.

We are also interested in two variants of this definition. The first follows:

Definition 2.2. Given a σ-algebra $\mathcal{G}$ of $\mathcal{F}$, we say that $f : \Omega \times \mathbf{R}^d \to \mathbf{R}_+$ is a *$\mathcal{G}$-measurable random probability density function* when f is $(\mathcal{G} \times \mathcal{B}(\mathbf{R}^d))$-measurable and $\mathrm{P}\{\int_{\mathbf{R}^d} f(x)\,\mathrm{d}x = 1\} = 1$.

And the second variant is:

Definition 2.3. Given a σ-algebra $\mathcal{G}$ of $\mathcal{F}$, we say that $\rho : \Omega \times \mathcal{B}(\mathbf{R}^d) \to [0\,,1]$ is a *$\mathcal{G}$-measurable random probability measure* when both of the following hold:

1. $\Omega \ni \omega \mapsto \rho(\omega\,,A)$ is $\mathcal{G}$-measurable for every $A \in \mathcal{B}(\mathbf{R}^d)$; and
2. $A \mapsto \rho(\omega\,,A)$ is a Borel probability measure on $\mathbf{R}^d$ for almost every $\omega \in \Omega$.

For all $\pi \subseteq \{1\,,\ldots,N\}$ and $\boldsymbol{s} \in \mathbf{R}_+^N$, let $\mathrm{P}_{\boldsymbol{s}}^\pi$ be a regular conditional distribution for B given $\mathcal{F}_\pi(\boldsymbol{s})$, with the corresponding expectation operator written as $\mathrm{E}_{\boldsymbol{s}}^\pi$. That is,

$$\mathrm{E}_{\boldsymbol{s}}^\pi f := \int f\,\mathrm{d}\mathrm{P}_{\boldsymbol{s}}^\pi = \mathrm{E}(f \mid \mathcal{F}_\pi(\boldsymbol{s})). \tag{2.7}$$

Consider two nonnegative random variables Z_1 and Z_2. Then we define

$$Z_1 \trianglelefteq Z_2 \quad \text{to mean} \quad \mathrm{P}\left\{\mathbf{1}_{\{Z_1>0\}} \le \mathbf{1}_{\{Z_2>0\}}\right\} = 1, \tag{2.8}$$

and $Z_1 \trianglerighteq Z_2$ to mean $Z_2 \trianglelefteq Z_1$. We also write $Z_1 \asymp Z_2$ when $Z_1 \trianglelefteq Z_2$ and $Z_1 \trianglerighteq Z_2$. That is,

$$Z_1 \asymp Z_2 \quad \text{if and only if} \quad \mathrm{P}\left\{\mathbf{1}_{\{Z_1>0\}} = \mathbf{1}_{\{Z_2>0\}}\right\} = 1. \tag{2.9}$$

The following generalizes Theorem 1.1 of Khoshnevisan and Shi [15]. See also Dalang and Nualart [6, Theorem 3.1]. This is the main technical contribution of the present paper. We use the term *upright box* for a cartesian product $\Theta := \prod_{j=1}^{N}[a_j\,, b_j]$ of intervals, where $a_j < b_j$, for $j = 1, \ldots, N$.

Theorem 2.4. *Choose and fix an upright box* $\Theta := \prod_{j=1}^{N}[a_j\,, b_j]$ *in* $(0\,,\infty)^N$. *For any partial order* $\pi \subseteq \{1\,, \ldots, N\}$*, choose and fix some vector* $\boldsymbol{s} \in (0\,,\infty)^N \backslash \Theta$ *such that* $\boldsymbol{s} \prec_\pi \boldsymbol{t}$ *for every* $\boldsymbol{t} \in \Theta$. *Then for all* $\mathcal{F}_\pi(\boldsymbol{s})$*-measurable bounded random sets* A*,*

$$\mathrm{P}^\pi_{\boldsymbol{s}} \{B(\boldsymbol{u}) \in A \text{ for some } \boldsymbol{u} \in \Theta\} \asymp \mathrm{Cap}_{d-2N}(A). \tag{2.10}$$

We conclude this section with a technical result on "potential theory of random sets." It should be "obvious" and/or well-known. But we know of neither transparent proofs nor explicit references. Therefore, we supply a proof.

Lemma 2.5. *Let* $\mathcal{G}$ *denote a sub-*σ*-algebra on the underlying probability space. Then for all random* $\mathcal{G}$*-measurable closed sets* $A \subseteq \mathbf{R}^d$ *and all non-random* $\beta \in \mathbf{R}$*,*

$$\mathrm{Cap}_\beta(A) \asymp [\inf \mathrm{I}_\beta(\theta)]^{-1}, \tag{2.11}$$

where the infimum is taken over all random $\mathcal{G}$*-measurable probability measures* θ *that are compactly supported in* A*. In addition, there is a* $\mathcal{G}$*-measurable random probability measure* μ *such that* $\mathrm{Cap}_\beta(A) \asymp 1/\mathrm{I}_\beta(\mu)$*.*

Proof. Let $c_\beta(A)$ denote the right-hand side of (2.11). Evidently, $\mathrm{Cap}_\beta(A) \geq c_\beta(A)$ almost surely, and hence $\mathrm{Cap}_\beta(A) \trianglerighteq c_\beta(A)$. It remains to prove that $\mathrm{Cap}_\beta(A) \trianglelefteq c_\beta(A)$. With this in mind, we may—and will—assume without loss of generality that $\mathrm{Cap}_\beta(A) > 0$ with positive probability. In particular, by (2.6), this implies that $\beta < d$.

Let $X_1, \ldots, X_M$ denote M independent isotropic stable processes in $\mathbf{R}^d$ that are independent of the set A, and have a common stability index $\alpha \in (0\,, 2]$. Notice that we can always choose the integer $M \geq 1$ and the real number α such that

$$d - \alpha M = \beta. \tag{2.12}$$

Thus, we choose and fix (M, α).

Define $\boldsymbol{X}$ to be the *additive stable process* defined by

$$\boldsymbol{X}(\boldsymbol{t}) := X_1(t_1) + \cdots + X_M(t_M) \qquad \text{for all } \boldsymbol{t} \in \mathbf{R}_+^M, \tag{2.13}$$

where we write $\boldsymbol{t} = (t_1, \ldots, t_M)$. Theorem 4.1.1 of Khoshnevisan [12, p. 423] tells us that for all nonrandom *compact* sets $E \subseteq \mathbf{R}^d$,

$$\begin{aligned} \mathrm{P}\{\boldsymbol{X}([1\,,2]^M) \cap E \neq \varnothing\} > 0 \ &\Leftrightarrow \ \mathrm{Cap}_{d-\alpha M}(E) > 0 \\ &\Leftrightarrow \ \mathrm{Cap}_\beta(E) > 0; \end{aligned} \tag{2.14}$$

see (2.12) for the final assertion. The proof of that theorem (*loc. cit.*) tells us more. Namely, that whenever $\mathrm{Cap}_\beta(E) > 0$, there exists a random variable $\boldsymbol{T}$, with values in $[1\,,2]^M \cup \{\infty\}$, which has the following properties:

- $\boldsymbol{T} \neq \infty$ if and only if $\boldsymbol{X}([1\,,2]^M) \cap E \neq \varnothing$;
- $\boldsymbol{X}(\boldsymbol{T}) \in E$ almost surely on $\{\boldsymbol{T} \neq \infty\}$; and
- $\mu(\bullet) := \mathrm{P}(\boldsymbol{X}(\boldsymbol{T}) \in \bullet \,|\, \boldsymbol{T} \neq \infty)$ is in $\mathcal{P}(E)$ and $\mathrm{I}_\beta(\mu) < \infty$.

In fact, $\boldsymbol{T}$ can be defined on $\{\boldsymbol{X}([1\,,2]^M) \cap E \neq \varnothing\}$ as follows: First define T_1 to be the smallest $s_1 \in [1\,,2]$ such that there exist $s_2, \ldots, s_M \in [1\,,2]$ that satisfy $\boldsymbol{X}(s_1\,, \ldots, s_M) \in E$. Then, having defined $T_1, \ldots, T_j$ for $j \in \{1\,, \ldots, M-2\}$, we define T_{j+1} to be the smallest $s_{j+1} \in [1\,,2]$ such that there exist $s_{j+2}, \ldots, s_M \in [1\,,2]$ that collectively satisfy

$$\boldsymbol{X}(T_1\,, \ldots, T_j\,, s_{j+1}\,, \ldots, s_M) \in E. \tag{2.15}$$

Finally, we define T_M to be the smallest $s_M \in [1\,,2]$ such that

$$\boldsymbol{X}(T_1\,, \ldots, T_{M-1}\,, s_M) \in E. \tag{2.16}$$

This defines $\boldsymbol{T} := (T_1\,, \ldots, T_M)$ on $\{\boldsymbol{X}([1\,,2]^M) \cap E \neq \varnothing\}$. We also define $\boldsymbol{T} := \infty$ on $\{\boldsymbol{X}([1\,,2]^M) \cap E \neq \varnothing\}$. Then $\boldsymbol{T}$ has the desired properties.

To finish the proof, note that, since $\mathrm{Cap}_\beta(A) > 0$ with positive probability, we can find $n > 0$ such that $\mathrm{Cap}_\beta(A_n) > 0$ with positive probability, where $A_n := A \cap [-n\,,n]^d$ is (obviously) a random $\mathcal{G}$-measurable compact set. Because A_n is independent of $\boldsymbol{X}$, we may apply the preceding with $E := A_n$. The mentioned construction of the resulting [now-random] probability measure μ [on A_n] makes it clear that μ is $\mathcal{G}$-measurable, and $\mathrm{I}_\beta(\mu) < \infty$ almost surely on $\{\mathrm{Cap}_\beta(A_n) > 0\}$. The lemma follows readily from these observations. □

3 Analysis of pinned sheets

For all $\boldsymbol{s} \in (0\,,\infty)^N$ and $\boldsymbol{t} \in \mathbf{R}_+^N$, define

$$B_{\boldsymbol{s}}(\boldsymbol{t}) := B(\boldsymbol{t}) - \delta_{\boldsymbol{s}}(\boldsymbol{t}) B(\boldsymbol{s}), \tag{3.1}$$

where

$$\delta_{\boldsymbol{s}}(\boldsymbol{t}) := \prod_{j=1}^N \left(\frac{s_j \wedge t_j}{s_j} \right). \tag{3.2}$$

One can think of the random field $B_{\boldsymbol{s}}$ as the *sheet pinned to be zero at* $\boldsymbol{s}$. [Khoshnevisan and Xiao [17] called $B_{\boldsymbol{s}}$ a "bridge."]

It is not too difficult to see that

$$B_{\boldsymbol{s}}(\boldsymbol{t}) = B(\boldsymbol{t}) - \mathrm{E}\left[B(\boldsymbol{t}) \,\middle|\, B(\boldsymbol{s}) \right]. \tag{3.3}$$

Next we recall some of the fundamental features of the pinned sheet $B_{\boldsymbol{s}}$.

Lemma 3.1 (Khoshnevisan and Xiao [17, Lemmas 5.1 and 5.2])**.** *Choose and fix a partial order $\pi \subseteq \{1\,,\ldots,N\}$ and a time point $\boldsymbol{s} \in (0\,,\infty)^N$. Then $\{B_{\boldsymbol{s}}(\boldsymbol{t})\}_{\boldsymbol{t} \succ_\pi \boldsymbol{s}}$ is independent of $\mathcal{F}_\pi(\boldsymbol{s})$. Moreover, for every nonrandom upright box $I \subset (0\,,\infty)^N$ and $\pi \subseteq \{1\,,\ldots,N\}$, there exists a finite constant $c > 1$ such that uniformly for all $\boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v} \in I$,*

$$c^{-1} \|\boldsymbol{u} - \boldsymbol{v}\| \le \mathrm{Var}\left(B_{\boldsymbol{s}}^1(\boldsymbol{u}) - B_{\boldsymbol{s}}^1(\boldsymbol{v}) \right) \le c \|\boldsymbol{u} - \boldsymbol{v}\|, \tag{3.4}$$

where $B_{\boldsymbol{s}}^1(\boldsymbol{t})$ denotes the first coordinate of $B_{\boldsymbol{s}}(\boldsymbol{t})$ for all $\boldsymbol{t} \in \mathbf{R}_+^N$.

The next result is the uniform Lipschitz continuity property of the δ's.

Lemma 3.2. *Choose and fix an upright box* $\Theta := \prod_{j=1}^{N}[a_j\,, b_j]$. *Then there exists a constant* $c < \infty$*—depending only on* N, $\min_j a_j$, *and* $\max_j b_j$*—such that*

$$|\delta_{\boldsymbol{s}}(\boldsymbol{u}) - \delta_{\boldsymbol{s}}(\boldsymbol{v})| \leq c\|\boldsymbol{u} - \boldsymbol{v}\| \qquad \text{for all } \boldsymbol{s}, \boldsymbol{u}, \boldsymbol{v} \in \Theta. \tag{3.5}$$

Proof. Notice that $\delta_s(t)$ is the product of N bounded and Lipschitz continuous functions $f_j(t_j) = 1 \wedge (t_j/s_j)$, and the Lipschitz constants of these functions are all bounded by $1/\min_j a_j$. The lemma follows. □

Next, we present a conditional maximal inequality which extends the existing multiparameter-martingale inequalities of the literature in several directions.

Lemma 3.3. *For every* $\pi \subseteq \{1\,, \ldots, N\}$, $\boldsymbol{s} \in \mathbf{R}_+^N$, *and bounded* $\sigma(B)$*-measurable random variable* f,

$$\mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\sup |\mathrm{E}_{\boldsymbol{t}}^{\pi} f|^2\right) \leq 4^N \mathrm{E}_{\boldsymbol{s}}^{\pi}\left(|f|^2\right) \qquad \text{almost surely [P]}, \tag{3.6}$$

where the supremum is taken over all $\boldsymbol{t} \in \mathbf{Q}_+^N$ *such that* $\boldsymbol{t} \succ_\pi \boldsymbol{s}$.

It is possible to use Lemma 3.5 below in order to remove the restriction that $\boldsymbol{t}$ lies in $\mathbf{Q}_+^N$.

Proof. First we recall Cairoli's inequality:

$$\mathrm{E}\left(\sup_{\boldsymbol{t} \in \mathbf{Q}_+^N} |\mathrm{E}_{\boldsymbol{t}}^{\pi} f|^2\right) \leq 4^N \mathrm{E}\left(|f|^2\right). \tag{3.7}$$

When $\pi = \{1\,, \ldots, N\}$, this was proved by Cairoli and Walsh [2]. The general case is due to Khoshnevisan and Shi [15, Corollary 3.2]. The proof of (3.7) hinges on the following projection property ["commutation"]:

$$\mathrm{P}\left\{\mathrm{E}_{\boldsymbol{u}}^{\pi}\mathrm{E}_{\boldsymbol{t}}^{\pi} f = \mathrm{E}_{\boldsymbol{u} \curlywedge_\pi \boldsymbol{t}}^{\pi} f\right\} = 1, \tag{3.8}$$

where, we recall, $\boldsymbol{u} \curlywedge_\pi \boldsymbol{t}$ denotes the N-vectors whose jth coordinate is $u_j \wedge t_j$ if $j \in \pi$, and $u_j \vee t_j$ if $j \notin \pi$. Now we may observe that if $\boldsymbol{s} \prec_\pi \boldsymbol{u}, \boldsymbol{t}$, then

P-almost surely,

$$\mathrm{P}^\pi_{\boldsymbol{s}}\left\{\mathrm{E}^\pi_{\boldsymbol{u}}\mathrm{E}^\pi_{\boldsymbol{t}} f = \mathrm{E}^\pi_{\boldsymbol{u} \curlywedge_\pi \boldsymbol{t}} f\right\} = 1. \tag{3.9}$$

Thus, we apply the same proof that led to (3.7), but use the regular conditional distribution $\mathrm{P}^\pi_{\boldsymbol{s}}$ in place of P, to finish the proof. □

Next we mention a simple aside on certain Wiener integrals.

Lemma 3.4. *Choose and fix a nonrandom compactly-supported bounded Borel function $h : \mathbf{R}^d \to \mathbf{R}^d$, and a partial order $\pi \subseteq \{1, \ldots, N\}$. Define*

$$G(\boldsymbol{s}) := \int_{\boldsymbol{r} \prec_\pi \boldsymbol{s}} h(\boldsymbol{r})\, B(\mathrm{d}\boldsymbol{r}), \tag{3.10}$$

where the stochastic integral is defined in the sense of Wiener [26, 27], and $\boldsymbol{s}$ ranges over $\mathbf{R}^N_+$. Then G has a continuous modification that is also continuous in $L^2(\mathrm{P})$.

Proof. Define $\mathfrak{S}(\boldsymbol{s})$ to be the π-shadow of $\boldsymbol{s} \in \mathbf{R}^N_+$:

$$\mathfrak{S}(\boldsymbol{s}) := \left\{\boldsymbol{r} \in \mathbf{R}^N_+ : \boldsymbol{r} \prec_\pi \boldsymbol{s}\right\}. \tag{3.11}$$

Then for all $\boldsymbol{s}, \boldsymbol{t} \in \mathbf{R}^N_+$,

$$\begin{aligned} \mathrm{E}\left(|G(\boldsymbol{t}) - G(\boldsymbol{s})|^2\right) &= \int_{\mathfrak{S}(\boldsymbol{s}) \triangle \mathfrak{S}(\boldsymbol{t})} |h(\boldsymbol{r})|^2\, \mathrm{d}\boldsymbol{r} \\ &\leq \sup |h|^2 \times \mathrm{meas}\left(\mathrm{supp}\, h \cap (\mathfrak{S}(\boldsymbol{s}) \triangle \mathfrak{S}(\boldsymbol{t}))\right), \end{aligned} \tag{3.12}$$

where "supp h" denotes the support of h, and "meas" stands for the standard N-dimensional Lebesgue measure. Consequently, $\mathrm{E}(|G(\boldsymbol{t}) - G(\boldsymbol{s})|^2) \leq \mathrm{const} \cdot |\boldsymbol{s} - \boldsymbol{t}|$, where the constant depends only on (N, h). Because G is a Gaussian random field, it follows that

$$\mathrm{E}\left(|G(\boldsymbol{t}) - G(\boldsymbol{s})|^{2p}\right) \leq \mathrm{const} \cdot |\boldsymbol{s} - \boldsymbol{t}|^p \qquad \text{for all } p > 0, \tag{3.13}$$

and the implied constant depends only on (N, h, p). The lemma follows from a suitable form of the Kolmogorov continuity lemma; see, for example, the arguments in Čencov [4] or Proposition A.1 and Remark A.2 of Dalang et al. [5, Proposition A.1 and Remark A.2]. □

Lemma 3.5. *Choose and fix a partial order* $\pi \subseteq \{1, \ldots, N\}$*. If* Z *is* $\sigma(B)$*-measurable and* $\mathrm{E}(Z^2) < \infty$*, then* $\boldsymbol{s} \mapsto \mathrm{E}^\pi_{\boldsymbol{s}} Z$ *has a continuous modification.*

Proof. In the special case that $\pi = \{1, \ldots, N\}$, this is Proposition 2.3 of Khoshnevisan and Shi [15]. Now we adapt the proof to the present setting.

Suppose $h : \mathbf{R}^d \to \mathbf{R}$ is compactly supported and infinitely differentiable. Define $\mathfrak{B}(h) := \int h \, \mathrm{d}B$, and note that

$$\mathrm{E}^\pi_{\boldsymbol{s}} \left(\mathrm{e}^{\mathfrak{B}(h)} \right) = \exp \left\{ \int_{\boldsymbol{r} \prec_\pi \boldsymbol{s}} h(\boldsymbol{r}) \, B(\mathrm{d}\boldsymbol{r}) + \frac{1}{2} \int_{\boldsymbol{r} \not\prec_\pi \boldsymbol{s}} |h(\boldsymbol{r})|^2 \, \mathrm{d}\boldsymbol{r} \right\}. \tag{3.14}$$

Thanks to Lemma 3.4, $\boldsymbol{s} \mapsto \mathrm{E}^\pi_{\boldsymbol{s}}[\exp(\mathfrak{B}(h))]$ is continuous almost surely. We claim that we also have continuity in $L^2(\mathrm{P})$. Indeed, we observe that it suffices to prove that $\boldsymbol{s} \mapsto \exp(J_h(\boldsymbol{s}))$ is continuous in $L^2(\mathrm{P})$, where

$$J_h(\boldsymbol{s}) := \int_{\boldsymbol{r} \prec_\pi \boldsymbol{s}} h(\boldsymbol{r}) \, B(\mathrm{d}\boldsymbol{r}). \tag{3.15}$$

By the Wiener isometry, $\mathrm{E}(\exp(4J_h(\boldsymbol{s}))) \le \exp(8 \int |h(\boldsymbol{r})|^2 \, \mathrm{d}\boldsymbol{r}) < \infty$. By splitting the integral over $\boldsymbol{r} \prec_\pi \boldsymbol{s}$ into an integral over $\boldsymbol{r} \in \mathfrak{S}(\boldsymbol{s}) \setminus \mathfrak{S}(\boldsymbol{t})$ and a remainder term, a direct calculation of $E([\exp(J_h(\boldsymbol{s})) - \exp(J_h(\boldsymbol{t}))]^2)$ using this inequality yields the stated $L^2(\mathrm{P})$ convergence.

We now use the preceding observation, together with an approximation argument, as follows:

Thanks to Lemma 1.1.2 of Nualart [20, p. 5], and by the Stone–Weierstrass theorem, for all integers $m > 0$ we can find nonrandom compactly-supported functions $h_1, \ldots, h_{k_m} \in C^\infty(\mathbf{R}^d)$ and $z_1, \ldots, z_{k_m} \in \mathbf{R}$ such that

$$\mathrm{E}\left(|Z_m - Z|^2 \right) < \mathrm{e}^{-m}, \quad \text{where} \quad Z_m := \sum_{j=1}^{k_m} z_j \, \mathrm{e}^{\mathfrak{B}(h_j)}. \tag{3.16}$$

Because conditional expectations are contractions on $L^2(\mathrm{P})$, it follows that

$$\mathrm{E}\left(|\mathrm{E}^\pi_{\boldsymbol{s}} Z - \mathrm{E}^\pi_{\boldsymbol{t}} Z|^2\right) \le 9\left(2\mathrm{e}^{-m} + \mathrm{E}\left(|\mathrm{E}^\pi_{\boldsymbol{s}} Z_m - \mathrm{E}^\pi_{\boldsymbol{t}} Z_m|^2\right)\right), \tag{3.17}$$

and hence $\boldsymbol{s} \mapsto \mathrm{E}^\pi_{\boldsymbol{s}} Z$ is continuous in $L^2(\mathrm{P})$, therefore continuous in probability.

Thanks to (3.16) and Cairoli's maximal inequality (3.7),

$$\begin{aligned}\mathrm{E}\left(\sup_{\boldsymbol{s}\in\mathbf{Q}^N_+} |\mathrm{E}^\pi_{\boldsymbol{s}} Z_m - \mathrm{E}^\pi_{\boldsymbol{s}} Z|^2\right) &\le 4^N \sup_{\boldsymbol{s}\in\mathbf{R}^N_+} \mathrm{E}\left(|\mathrm{E}^\pi_{\boldsymbol{s}} Z_m - \mathrm{E}^\pi_{\boldsymbol{s}} Z|^2\right) \\ &\le 4^N \mathrm{E}\left(|Z_m - Z|^2\right) \\ &< 4^N \mathrm{e}^{-m}.\end{aligned} \tag{3.18}$$

By the Borel–Cantelli lemma,

$$\lim_{m\to\infty} \sup_{\boldsymbol{s}\in\mathbf{Q}^N_+} |\mathrm{E}^\pi_{\boldsymbol{s}} Z_m - \mathrm{E}^\pi_{\boldsymbol{s}} Z| = 0 \qquad \text{almost surely [P]}. \tag{3.19}$$

Therefore, a.s. [P], the continuous random field $s \mapsto \mathrm{E}^\pi_{\boldsymbol{s}} Z_m$ converges uniformly on $\mathbf{Q}^N_+$ to $s \mapsto \mathrm{E}^\pi_{\boldsymbol{s}} Z$. Therefore, $s \mapsto \mathrm{E}^\pi_{\boldsymbol{s}} Z$ is uniformly continuous on $\mathbf{Q}^N_+$, and so it has a continuous extension to $\mathbf{R}^N_+$. Since $s \mapsto \mathrm{E}^\pi_{\boldsymbol{s}} Z$ is continuous in probability by (3.17), this extension defines a continuous modification of $\boldsymbol{s} \mapsto \mathrm{E}^\pi_{\boldsymbol{s}} Z$. □

Henceforth, we always choose a continuous modification of $\mathrm{E}^\pi_{\boldsymbol{s}} Z$ when Z is square-integrable. With this convention in mind, we immediately obtain the following consequence of Lemmas 3.3 and 3.5.

Lemma 3.6. *For every bounded $\sigma(B)$-measurable random variable f, there exists a* P*-null event off which the following holds: For every $\pi \subseteq \{1, \ldots, N\}$, $\boldsymbol{s} \in \mathbf{R}^N_+$,*

$$\mathrm{E}^\pi_{\boldsymbol{s}}\left(\sup_{\boldsymbol{t}\succ_\pi \boldsymbol{s}} |\mathrm{E}^\pi_{\boldsymbol{t}} f|^2\right) \le 4^N \mathrm{E}^\pi_{\boldsymbol{s}}\left(|f|^2\right). \tag{3.20}$$

For all $\sigma > 0$, $\boldsymbol{t} \in \mathbf{R}^N$, and $z \in \mathbf{R}^d$ define

$$\Gamma_\sigma(\boldsymbol{t}\,; z) := \frac{1}{(2\pi\sigma^2)^{d/2}\|\boldsymbol{t}\|^{d/2}} \exp\left(-\frac{\|z\|^2}{2\sigma^2\|\boldsymbol{t}\|}\right). \tag{3.21}$$

Variants of the next result are well known. We supply a detailed proof because we will need to have good control over the constants involved.

Lemma 3.7. *Let $\Theta := \prod_{j=1}^N [a_j\,, b_j]$ denote an upright box in $(0\,,\infty)^N$, and choose and fix positive constants $\tau_1 < \tau_2$ and $M > 0$. Then there exists a finite constant $c > 1$—depending only on d, N, M τ_1, τ_2, $\min_j a_j$, and $\max_j b_j$—such that for all $\sigma \in [\tau_1\,, \tau_2]$ and $z \in [-M\,, M]^d$,*

$$c^{-1}\kappa_{d-2N}(z) \le \int_{\Theta-\Theta} \Gamma_\sigma(\boldsymbol{t}\,; z)\,\mathrm{d}\boldsymbol{t} \le c\kappa_{d-2N}(z). \tag{3.22}$$

We recall that $\Theta - \Theta$ denotes the collection of all points of the form $\boldsymbol{t} - \boldsymbol{s}$, where $\boldsymbol{s}$ and $\boldsymbol{t}$ range over Θ. Moreover, the proof below shows that the upper bound in (3.22) holds for all $z \in \mathbf{R}^d$.

Proof. Let $D(\rho)$ denote the centered ball in $\mathbf{R}^d$ whose radius is $\rho > 0$. Then we can integrate in polar coordinates to deduce that

$$\begin{aligned}\int_{D(\rho)} \Gamma_\sigma(\boldsymbol{t}\,; z)\,\mathrm{d}\boldsymbol{t} &= \text{const} \cdot \int_0^\rho r^{N-1-(d/2)} \exp\left(-\frac{\|z\|^2}{2\sigma^2 r}\right) \mathrm{d}r \\ &= \frac{\text{const}}{\|z\|^{d-2N}} \cdot \int_0^{2\sigma^2\rho/\|z\|^2} s^{N-1-(d/2)} \mathrm{e}^{-1/s}\,\mathrm{d}s,\end{aligned} \tag{3.23}$$

where the implied constants depend only on the parameters σ, N and d. This proves the result in the case where $\Theta - \Theta$ is a centered ball, since we can consider separately the cases $d < 2N$, $d = 2N$, and $d > 2N$ directly; see the proof of Lemma 3.4 of Khoshnevisan and Shi [15], for instance.

The general case follows from the preceding spherical case, because we can find ρ_1 and ρ_2 such that $D(\rho_1) \subseteq \Theta - \Theta \subseteq D(\rho_2)$, whence it follows that $\int_{D(\rho_1)} \Gamma_\sigma(\boldsymbol{t}\,; z)\,\mathrm{d}\boldsymbol{t} \le \int_{\Theta-\Theta} \Gamma_\sigma(\boldsymbol{t}\,; z)\,\mathrm{d}\boldsymbol{t} \le \int_{D(\rho_2)} \Gamma_\sigma(\boldsymbol{t}\,; z)\,\mathrm{d}\boldsymbol{t}$. □

Now we proceed with a series of "conditional energy estimates" for "continuous additive functionals" of the sheet. First is a lower bound.

Lemma 3.8. *Choose and fix* $\pi \subseteq \{1, \ldots, N\}$, $\eta > 0$, $\boldsymbol{s} \in (0, \infty)^N$, *and a nonrandom upright box* $\Theta := \prod_{j=1}^N [a_j, b_j]$ *in* $(0, \infty)^N$ *such that* $\boldsymbol{s} \succ_\pi \boldsymbol{t}$ *and* $\eta \le |\boldsymbol{s} - \boldsymbol{t}|_\infty \le \eta^{-1}$ *for every* $\boldsymbol{t} \in \Theta$. *Then there exists a constant* $c > 1$*—depending only on* d, N, η, $\min_j a_j$, *and* $\max_j b_j$*—such that for all* $\mathcal{F}_\pi(\boldsymbol{s})$*-measurable random probability density functions* f *on* $\mathbf{R}^d$,

$$\mathrm{E}_{\boldsymbol{s}}^\pi \left(\int_\Theta f(B(\boldsymbol{u}))\, \mathrm{d}\boldsymbol{u} \right) \ge c^{-1} \mathrm{e}^{-c\|B(\boldsymbol{s})\|^2} \cdot \int_{\mathbf{R}^d} f(z)\, \mathrm{e}^{-c\|z\|^2}\, \mathrm{d}z, \tag{3.24}$$

almost surely [P].

Proof. Thanks to Lemma 3.1, we can write

$$\begin{aligned} \mathrm{E}_{\boldsymbol{s}}^\pi \left(\int_\Theta f(B(\boldsymbol{u}))\, \mathrm{d}\boldsymbol{u} \right) &= \mathrm{E}_{\boldsymbol{s}}^\pi \left(\int_\Theta f\left(B_{\boldsymbol{s}}(\boldsymbol{u}) + \delta_{\boldsymbol{s}}(\boldsymbol{u}) B(\boldsymbol{s})\right)\, \mathrm{d}\boldsymbol{u} \right) \\ &= \int_\Theta \mathrm{d}\boldsymbol{u} \int_{\mathbf{R}^d} \mathrm{d}z\ f(z) g_{\boldsymbol{u}}(z - \delta_{\boldsymbol{s}}(\boldsymbol{u}) B(\boldsymbol{s})), \end{aligned} \tag{3.25}$$

where $g_{\boldsymbol{u}}$ denotes the probability density function of $B_{\boldsymbol{s}}(\boldsymbol{u})$.

According to Lemma 3.1, the coordinatewise variance of $B_{\boldsymbol{s}}(\boldsymbol{u})$ is bounded above and below by constant multiples of $\|\boldsymbol{u} - \boldsymbol{s}\|$. As a result, $g_{\boldsymbol{u}}(z - \delta_{\boldsymbol{s}}(\boldsymbol{u}) B(\boldsymbol{s}))$ is bounded below by an absolute constant multiplied by

$$\begin{aligned} &\frac{1}{\|\boldsymbol{u} - \boldsymbol{s}\|^{d/2}} \exp\left(-\text{const}\ \frac{\|z - \delta_{\boldsymbol{s}}(\boldsymbol{u}) B(\boldsymbol{s})\|^2}{\|\boldsymbol{u} - \boldsymbol{s}\|} \right) \\ &\qquad \ge \eta^{d/2} \exp\left(-\text{const} \cdot \frac{\|z - \delta_{\boldsymbol{s}}(\boldsymbol{u}) B(\boldsymbol{s})\|^2}{\eta} \right). \end{aligned} \tag{3.26}$$

Thus, the inequality

$$\|z - \delta_{\boldsymbol{s}}(\boldsymbol{u}) B(\boldsymbol{s})\|^2 \le 2\|z\|^2 + 2\|B(\boldsymbol{s})\|^2, \tag{3.27}$$

valid because $0 \le \delta_{\boldsymbol{s}}(\boldsymbol{u}) \le 1$, proves that

$$g_{\boldsymbol{u}}(z - \delta_{\boldsymbol{s}}(\boldsymbol{u}) B(\boldsymbol{s})) \ge c_1 \exp\left(-c_2 \left\{ \|z\|^2 + \|B(\boldsymbol{s})\|^2 \right\} \right), \tag{3.28}$$

where c_1 and c_2 are positive and finite constants that depend only on π, d, N, M, η, and $\max_j b_j$. Let $c_1(\pi)$ and $c_2(\pi)$ denote the same constants, but written as such to exhibit their dependence on the partial order π. Apply the preceding for all partial orders π, and let c_1 and c_2 denote respectively the minimum and maximum of $c_1(\pi)$ and $c_2(\pi)$ as π ranges over the various subsets of $\{1\,,\ldots,N\}$. In this way, the preceding display holds without any dependencies on the partial order π. It is now clear that (3.24) follows from (3.25) and (3.28). □

Next we present a delicate joint-density estimate for the pinned sheets. This estimate will be used subsequently to describe a conditional second-moment bound that complements the conditional first-moment bound of Lemma 3.8.

Lemma 3.9. *Choose and fix an upright box* $\Theta := \prod_{j=1}^N [a_j\,, b_j]$ *in* $(0\,,\infty)^N$, *a partial order* $\pi \subseteq \{1\,,\ldots,N\}$, *and* $\boldsymbol{s} \in \mathbf{R}^N_+$ *and* $\eta > 0$ *such that:*

(i) $\boldsymbol{s} \prec_\pi \boldsymbol{t}$ *for all* $\boldsymbol{t} \in \Theta$*; and*

(ii) $\eta \leq |\boldsymbol{s} - \boldsymbol{t}|_\infty \leq \eta^{-1}$ *for all* $\boldsymbol{t} \in \Theta$.

Then there exists a finite constant $c > 1$*—depending only on* d, N, η, $\min_j a_j$, *and* $\max_j b_j$*—such that for all* $x, y \in \mathbf{R}^d$ *and* $\boldsymbol{u}, \boldsymbol{v} \in \Theta$,

$$p_{\boldsymbol{s};\boldsymbol{u},\boldsymbol{v}}(x\,,y) \leq c\Gamma_c(\boldsymbol{u} - \boldsymbol{v}\,; x - y), \tag{3.29}$$

where $p_{\boldsymbol{s};\boldsymbol{u},\boldsymbol{v}}(x\,,y)$ *denotes the probability density function of* $(B_{\boldsymbol{s}}(\boldsymbol{u})\,, B_{\boldsymbol{s}}(\boldsymbol{v}))$.

Proof. The proof is carried out in three steps. We are only going to consider the case where $u \neq u \curlywedge_\pi v \neq v$; indeed, the other cases are simpler and are left to the reader.

Step 1. First consider the case that $\pi = \{1\,,\ldots,N\}$. In this particular case, we respectively write "$\prec$," "$\succ$," and "$\curlywedge$" in place of "$\prec_\pi$," "$\succ_\pi$," and "$\curlywedge_\pi$."

Note that $\boldsymbol{r} \prec \boldsymbol{p}$ if and only if $r_i \leq p_i$ for all $i = 1, \ldots, N$. Furthermore,

$$B_{\boldsymbol{s}}(\boldsymbol{r}) = B(\boldsymbol{r}) - B(\boldsymbol{s}) \quad \text{for all } \boldsymbol{r} \in \Theta. \tag{3.30}$$

Because the joint probability-density function of $(B_{\boldsymbol{s}}(\boldsymbol{u}), B_{\boldsymbol{s}}(\boldsymbol{v}))$ is unaltered if we modify the Brownian sheet, we choose to work with a particularly useful construction of the Brownian sheet. Namely, let $\mathfrak{W}$ denote d-dimensional white noise on $\mathbf{R}_+^N$, and consider the Brownian sheet

$$B(\boldsymbol{t}) := \mathfrak{W}([\mathbf{0}, \boldsymbol{t}]), \quad \text{where } [\mathbf{0}, \boldsymbol{t}] := \prod_{j=1}^{N} [0, t_j]. \tag{3.31}$$

This construction might not yield a continuous random function B, but that is not germane to the discussion.

For the construction cited here,

$$B_{\boldsymbol{s}}(\boldsymbol{r}) = \mathfrak{W}\left([\mathbf{0}, \boldsymbol{r}] \setminus [\mathbf{0}, \boldsymbol{s}]\right) \qquad \text{for all } \boldsymbol{r} \in \Theta. \tag{3.32}$$

For all bounded C^∞ functions $\phi : (\mathbf{R}^d)^2 \to \mathbf{R}_+$ and $\boldsymbol{u}, \boldsymbol{v} \in \Theta$,

$$\mathrm{E}\left[\phi\left(B_{\boldsymbol{s}}(\boldsymbol{u}), B_{\boldsymbol{s}}(\boldsymbol{v})\right)\right] = \iiint \phi(x + y, x + z) g_{\boldsymbol{u} \curlywedge \boldsymbol{v}}(x) F(y) G(z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z, \tag{3.33}$$

where $g_{\boldsymbol{u}\curlywedge\boldsymbol{v}}$ denotes the probability density function of $B_{\boldsymbol{s}}(\boldsymbol{u}\curlywedge\boldsymbol{v}) = \mathfrak{W}([\mathbf{0}, \boldsymbol{u}\curlywedge \boldsymbol{v}] \setminus [\mathbf{0}, \boldsymbol{s}])$ as before, F the probability density function of $\mathfrak{W}([\mathbf{0}, \boldsymbol{u}] \setminus [\mathbf{0}, \boldsymbol{u} \curlywedge \boldsymbol{v}])$, and G the probability density function of $\mathfrak{W}([\mathbf{0}, \boldsymbol{v}] \setminus [\mathbf{0}, \boldsymbol{u}\curlywedge\boldsymbol{v}])$. The integrals are each taken over $\mathbf{R}^d$. The N-dimensional volume of $[\mathbf{0}, \boldsymbol{u} \curlywedge \boldsymbol{v}] \setminus [\mathbf{0}, \boldsymbol{s}]$ is at least $\eta(\min_j a_j)^{N-1}$. Therefore, $g_{\boldsymbol{u}\curlywedge\boldsymbol{v}}$ is bounded above by a constant c_3 that depends only on d, N, η, and $\min_j a_j$. And hence,

$$\begin{aligned} \mathrm{E}\left[\phi\left(B_{\boldsymbol{s}}(\boldsymbol{u}), B_{\boldsymbol{s}}(\boldsymbol{v})\right)\right] &\leq c_3 \iiint \phi(x + y, x + z) F(y) G(z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\ &= c_3 \iint \phi(x, y)\, (F * G)(y - x)\, \mathrm{d}x\, \mathrm{d}y. \end{aligned} \tag{3.34}$$

But $F * G$ is the probability density function of $\mathfrak{W}([\mathbf{0}, \boldsymbol{u}] \triangle [\mathbf{0}, \boldsymbol{v}])$, and the

N-dimensional volume of $[\mathbf{0}\,,\boldsymbol{u}]\triangle[\mathbf{0}\,,\boldsymbol{v}]$ is at least

$$\left(\min_{1\le j\le N} a_j\right)^{N-1}\sum_{k=1}^N |u_k-v_k| \ge \frac{1}{N^{1/2}}\left(\min_{1\le j\le N} a_j\right)^{N-1}\|\boldsymbol{u}-\boldsymbol{v}\|. \tag{3.35}$$

In addition, one can derive an upper bound—using only constants that depend on $\min_j a_j$, $\max_j b_j$ and N—similarly. Therefore, there exists a finite constant $c>1$—depending only on d, N, η, and $\min_j a_j$—such that the following occurs pointwise:

$$(F*G)(y-x)\le c\,\Gamma_c(\boldsymbol{u}-\boldsymbol{v}\,;y-x). \tag{3.36}$$

This proves the lemma in the case where $\pi=\{1\,,\ldots,N\}$.

Step 2. The argument of Step 1 yields in fact a slightly stronger result, which we state next as the following [slightly] **Enhanced Version**: *Choose and fix two positive constants $\nu_1<\nu_2$. Under the conditions of Step 1, there exists a constant ρ—depending only on d, N, η, $\min_j a_j$, $\max_j b_j$, ν_1, and ν_2, such that for all $\boldsymbol{u},\boldsymbol{v}\in\Theta$ and all $\alpha,\beta\in[\nu_1\,,\nu_2]$, the joint probability density function of $(\alpha B_{\boldsymbol{s}}(\boldsymbol{u})\,,\beta B_{\boldsymbol{s}}(\boldsymbol{v}))$—at $(x\,,y)$—is bounded above by $\rho\Gamma_\rho(\boldsymbol{u}-\boldsymbol{v}\,;x-y)$.*

The proof of the enhanced version is the same as the case we expanded on above ($\nu_1=\nu_2=1$). However, a few modifications need to be made: $\phi(x+y\,,x+z)$ is replaced by $\phi(\alpha x+y\,,\beta x+z)$; F is replaced by the probability density function of $\alpha\mathfrak{W}([\mathbf{0}\,,\boldsymbol{u}]\setminus[\mathbf{0}\,,\boldsymbol{u}\curlywedge\boldsymbol{v}])$; G by the probability density function of $\beta\mathfrak{W}([\mathbf{0}\,,\boldsymbol{v}]\setminus[\mathbf{0}\,,\boldsymbol{u}\curlywedge\boldsymbol{v}])$; and $F*G$ is now the probability density function of a centered Gaussian vector with i.i.d. coordinates, the variance of each of which is at least

$$\begin{aligned}\left(\min_{1\le j\le N} a_j\right)^{N-1}\alpha^2\sum_{k=1}^N(u_k-v_k)^+ + \left(\min_{1\le j\le N} a_j\right)^{N-1}\beta^2\sum_{k=1}^N(u_k-v_k)^-\\ \ge(\alpha\wedge\beta)^2\left(\min_{1\le j\le N} a_j\right)^{N-1}\sum_{k=1}^N|u_k-v_k|.\end{aligned} \tag{3.37}$$

The remainder of the proof goes through without incurring major changes.

Step 3. If $\pi = \varnothing$, then the lemma follows from Step 1 and symmetry. Therefore, it remains to consider the case where π and $\{1, \ldots, N\} \setminus \pi$ are both nonvoid. We follow Khoshnevisan and Xiao [17, proof of Proposition 3.1] and define a map $\mathcal{I} : (0, \infty)^N \to (0, \infty)^N$ with coordinate functions $\mathcal{I}_1, \ldots, \mathcal{I}_N$ as follows: For all $k = 1, \ldots, N$,

$$\mathcal{I}_k(\boldsymbol{t}) := \begin{cases} t_k & \text{if } k \in \pi, \\ 1/t_k & \text{if } k \notin \pi. \end{cases} \tag{3.38}$$

Consider any two points $\boldsymbol{u}, \boldsymbol{v} \in \Theta$. We may note that:

(i) $\mathcal{I}(\Theta)$ is an upright box that contains $\mathcal{I}(\boldsymbol{u})$ and $\mathcal{I}(\boldsymbol{v})$;

(ii) $\mathcal{I}(\boldsymbol{s}) \prec \mathcal{I}(\boldsymbol{t})$ for all $\boldsymbol{t} \in \Theta$ [*nota bene*: the partial order!]; and

(iii) $|\mathcal{I}(\boldsymbol{s}) - \mathcal{I}(\boldsymbol{t})|_\infty$ is bounded below by a positive constant η', uniformly for all $\boldsymbol{t} \in \Theta$. Moreover, η' depends only on N, η, $\min_j a_j$, and $\max_j b_j$.

Define

$$W(\boldsymbol{t}) := \left(\prod_{j \notin \pi} t_j \right) \cdot B(\mathcal{I}(\boldsymbol{t})) \qquad \text{for all } \boldsymbol{t} \in (0, \infty)^N. \tag{3.39}$$

Then, according to Khoshnevisan and Xiao (*loco citato*), W is a Brownian sheet. Thus, we have also the corresponding pinned sheet

$$W_{\boldsymbol{s}}(\boldsymbol{t}) = W(\boldsymbol{t}) - \delta_{\boldsymbol{s}}(\boldsymbol{t}) W(\boldsymbol{s}) \qquad \text{for all } \boldsymbol{t} \in (0, \infty)^N. \tag{3.40}$$

It is the case that

$$W_{\boldsymbol{s}}(\boldsymbol{t}) = \left(\prod_{j \notin \pi} t_j \right) \cdot [B(\mathcal{I}(\boldsymbol{t})) - B(\mathcal{I}(\boldsymbol{s}))] \qquad \text{for all } \boldsymbol{t} \in (0, \infty)^N. \tag{3.41}$$

The derivation of this identity requires only a little algebra, which we skip.

Thus, property (ii) above implies the following remarkable identity:

$$W_{\boldsymbol{s}}(\boldsymbol{t}) = \left(\prod_{j\notin\pi} t_j\right) \cdot B_{\mathcal{I}(\boldsymbol{s})}(\mathcal{I}(\boldsymbol{t})) \qquad \text{for all } \boldsymbol{t} \in \Theta. \tag{3.42}$$

As a result of items (i)–(iii), and thanks to Step 1, the joint probability density function—at $(x\,, y)$—of the random vector $(B_{\mathcal{I}(\boldsymbol{s})}(\mathcal{I}(\boldsymbol{u}))\,, B_{\mathcal{I}(\boldsymbol{s})}(\mathcal{I}(\boldsymbol{v})))$ is bounded above by $c_4\Gamma_{c_4}(\mathcal{I}(\boldsymbol{u}) - \mathcal{I}(\boldsymbol{v})\,; x - y)$, where c_4 depends only on d, N, η, $\min_j a_j$, and $\max_j b_j$. Elementary considerations show that $\|\mathcal{I}(\boldsymbol{u}) - \mathcal{I}(\boldsymbol{v})\|$ is bounded above and below by constant multiples of $\|\boldsymbol{u}-\boldsymbol{v}\|$, where the constants have the same parameter dependencies as c_4. These discussions together imply that the joint probability density function—at $(x\,, y)$—of the random vector $(B_{\mathcal{I}(\boldsymbol{s})}(\mathcal{I}(\boldsymbol{u}))\,, B_{\mathcal{I}(\boldsymbol{s})}(\mathcal{I}(\boldsymbol{v})))$ is bounded above by $c_5\Gamma_{c_5}(\boldsymbol{u} - \boldsymbol{v}\,; x - y)$, where c_5 has the same parameter dependencies as c_4. Set $\alpha = \prod_{j\notin\pi} u_j$ and $\beta := \prod_{j\notin\pi} v_j$, and note that α and β are bounded above and below by constants that depend only on $\min_j a_j$ and $\max_j b_j$. Also note that

$$(\alpha B_{\mathcal{I}(\boldsymbol{s})}(\mathcal{I}(\boldsymbol{u}))\ ,\ \beta B_{\mathcal{I}(\boldsymbol{s})}(\mathcal{I}(\boldsymbol{v}))) = (W_{\boldsymbol{s}}(\boldsymbol{u})\,, W_{\boldsymbol{s}}(\boldsymbol{v})), \tag{3.43}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \Theta$. Thus, in accord with Step 2, the joint probability density function—at $(x\,, y)$—of $(W_{\boldsymbol{s}}(\boldsymbol{u})\,, W_{\boldsymbol{s}}(\boldsymbol{v}))$ is bounded above by $c_6\Gamma_{c_6}(\boldsymbol{u} - \boldsymbol{v}\,; x - y)$, where c_6 has the same parameter dependencies as c_4. Because $W_{\boldsymbol{s}}$ has the same finite-dimensional distributions as $B_{\boldsymbol{s}}$, this proves the lemma. □

Lemma 3.10. *Let Θ, $\boldsymbol{s}$, π, and η be as in Lemma 3.9. Then there exists a constant $c > 1$—depending only on d, N, η, $\min_j a_j$, and $\max_j b_j$—such that for all $\mathcal{F}_\pi(\boldsymbol{s})$-measurable random probability density functions f,*

$$\mathrm{E}^\pi_{\boldsymbol{s}}\left(\left|\int_\Theta f(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right|^2\right) \le c\,\mathrm{e}^{c\|B(\boldsymbol{s})\|^2} \cdot \mathrm{I}_{d-2N}(f) \qquad \textit{a.s. } [\mathrm{P}]. \tag{3.44}$$

Proof. Throughout this proof, we define

$$F := \mathrm{E}^{\pi}_{\boldsymbol{s}}\left(\left|\int_{\Theta} f(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right|^2\right). \tag{3.45}$$

A few lines of computation show that with probability one,

$$\begin{aligned} F = \int_{\Theta}\mathrm{d}\boldsymbol{v}\int_{\Theta}\mathrm{d}\boldsymbol{u}\int_{\mathbf{R}^d}\mathrm{d}x\int_{\mathbf{R}^d}\mathrm{d}y \\ f(x+\delta_{\boldsymbol{s}}(\boldsymbol{u})B(\boldsymbol{s}))\ f(y+\delta_{\boldsymbol{s}}(\boldsymbol{v})B(\boldsymbol{s}))\ p_{\boldsymbol{s};\boldsymbol{u},\boldsymbol{v}}(x\,,y), \end{aligned} \tag{3.46}$$

where $p_{\boldsymbol{s};\boldsymbol{u},\boldsymbol{v}}(x\,,y)$ denotes the probability density function of $(B_{\boldsymbol{s}}(\boldsymbol{u})\,,B_{\boldsymbol{s}}(\boldsymbol{v}))$ at $(x\,,y)\in(\mathbf{R}^d)^2$. According to Lemma 3.9, we can find a finite constant $c_7>1$ such that for all $(x\,,y)\in(\mathbf{R}^d)^2$ and $\boldsymbol{u},\boldsymbol{v}\in\Theta$,

$$p_{\boldsymbol{s};\boldsymbol{u},\boldsymbol{v}}(x\,,y)\le c_7\,\Gamma_{c_7}(\boldsymbol{u}-\boldsymbol{v}\,;x-y), \tag{3.47}$$

where Γ_c is the Gaussian density function defined by (3.21). Moreover, c_7 depends only on d, M, N, η, $\min_j a_j$, and $\max_j b_j$. We change variables to deduce that almost surely,

$$F\le c_7\int_{\Theta}\mathrm{d}\boldsymbol{v}\int_{\Theta}\mathrm{d}\boldsymbol{u}\int_{\mathbf{R}^d}\mathrm{d}x\int_{\mathbf{R}^d}\mathrm{d}y\ f(x)\ f(y)\ \Gamma_{c_7}\left(\boldsymbol{u}-\boldsymbol{v}\,;x-y-Q\right), \tag{3.48}$$

where

$$Q := B(\boldsymbol{s})\{\delta_{\boldsymbol{s}}(\boldsymbol{u})-\delta_{\boldsymbol{s}}(\boldsymbol{v})\}. \tag{3.49}$$

Because $\|z\|^2\le 2\|Q\|^2+2\|z-Q\|^2$,

$$\Gamma_{c_7}(\boldsymbol{t}\,;z-Q)\le\frac{1}{(2\pi c_7^2)^{d/2}\|\boldsymbol{t}\|^{d/2}}\exp\left(-\frac{\|z\|^2}{c_7\|\boldsymbol{t}\|}+\frac{\|Q\|^2}{c_7\|\boldsymbol{t}\|}\right). \tag{3.50}$$

According to Lemma 3.2, there exists a constant c_8—with the same parameter dependencies as c_7—such that

$$\begin{aligned}\|Q\|^2 &\le c_8\|\boldsymbol{u}-\boldsymbol{v}\|^2\cdot\|B(\boldsymbol{s})\|^2\\ &\le c_8\max_{1\le j\le N} b_j\|\boldsymbol{u}-\boldsymbol{v}\|\cdot\|B(\boldsymbol{s})\|^2,\end{aligned} \tag{3.51}$$

uniformly for all $\boldsymbol{u}, \boldsymbol{v} \in \Theta$. Therefore, we may apply the preceding display with $\boldsymbol{t} := \boldsymbol{u} - \boldsymbol{v}$ and $z := x - y$ to find that

$$\Gamma_{c_7}(\boldsymbol{u}-\boldsymbol{v}\,; x-y-Q) \le 2^d \exp\left(\frac{\|B(\boldsymbol{s})\|^2}{c_7 c_8 \max_{1\le j\le N} b_j}\right) \cdot \Gamma_{c_9}(\boldsymbol{u}-\boldsymbol{v}\,; x-y). \tag{3.52}$$

Again, c_9 is a positive and finite constant that has the same parameter dependencies as c_7 and c_8. Consequently, the following holds with probability one:

$$\begin{aligned} &\exp\left(-\frac{\|B(\boldsymbol{s})\|^2}{c_7 c_8 \max_{1\le j\le N} b_j}\right) \cdot F \\ &\quad \le 2^d c_7 \int_\Theta \mathrm{d}\boldsymbol{v} \int_\Theta \mathrm{d}\boldsymbol{u} \int_{\mathbf{R}^d} \mathrm{d}x \int_{\mathbf{R}^d} \mathrm{d}y\ f(x)\ f(y)\ \Gamma_{c_9}\left(\boldsymbol{u}-\boldsymbol{v}\,; x-y\right) \\ &\quad = 2^d c_7 \mathrm{meas}(\Theta) \int_{\mathbf{R}^d} \mathrm{d}x \int_{\mathbf{R}^d} \mathrm{d}y\ f(x)\ f(y)\ g(x-y), \end{aligned} \tag{3.53}$$

where $g(z) := \int_{\Theta-\Theta} \Gamma_{c_9}(\boldsymbol{u}\,; z)\,\mathrm{d}\boldsymbol{u}$. Thanks to Lemma 3.7,

$$g(z) \le c_{10} \kappa_{d-2N}(z) \tag{3.54}$$

for all $z \in \mathbf{R}^d$, where c_{10} is a finite constant > 1 that depends only on d, N, M, η, $\min_j a_j$, and $\max_j b_j$. The lemma follows. □

Next we introduce a generalization of Proposition 3.7 of Khoshnevisan and Shi [15].

Lemma 3.11. *Choose and fix an upright box $\Theta := \prod_{j=1}^N [a_j\,, b_j]$ in $(0\,,\infty)^N$ and real numbers $\eta > 0$ and $M > 0$. Then there exists a constant $c_{11} > 0$—depending only on d, N, η, M, $\min_j a_j$, and $\max_j b_j$—such that for all $\pi \subseteq \{1\,,\ldots,N\}$, all $\boldsymbol{s} \in \Theta$ whose distance to the boundary of Θ is at least η, and every $\mathcal{F}_\pi(\boldsymbol{s})$-measurable random probability density function f whose support is contained in $[-M, M]^N$,*

$$\begin{aligned} &\mathrm{E}_{\boldsymbol{s}}^\pi \left(\int_\Theta f(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right) \\ &\qquad\qquad \ge c_{11}\, \mathbf{1}_{\{\|B(s)\|\le M\}} \int_{\mathbf{R}^d} \kappa_{d-2N}(z) f(z + B(\boldsymbol{s}))\,\mathrm{d}z\,, \end{aligned} \tag{3.55}$$

almost surely [P].

Even though both Lemmas 3.8 and 3.11 are concerned with lower bounds for $\mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\int_\Theta f(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right)$, there is a fundamental difference between the two lemmas: In Lemma 3.8, $\boldsymbol{s}$ is at least a fixed distance η away from Θ, whereas Lemma 3.11 considers the case where $\boldsymbol{s}$ belongs to Θ.

Proof. Throughout, we choose and fix an $\boldsymbol{s} \in \Theta$ and a π as per the statement of the lemma.

Consider $\Upsilon := \{\boldsymbol{u} \in \Theta : \boldsymbol{u} \succ_\pi \boldsymbol{s}\}$, which is easily seen to be an upright box. Since $\Upsilon \subseteq \Theta$, it follows that P-almost surely,

$$\begin{aligned}\mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\int_\Theta f(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right) &\geq \mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\int_\Upsilon f(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right) \\ &= \mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\int_\Upsilon f\left(B_{\boldsymbol{s}}(\boldsymbol{u}) + \delta_{\boldsymbol{s}}(\boldsymbol{u})B(\boldsymbol{s})\right)\,\mathrm{d}\boldsymbol{u}\right) \\ &= \int_\Upsilon \mathrm{d}\boldsymbol{u} \int_{\mathbf{R}^d} \mathrm{d}z\ f(z) g_{\boldsymbol{u}}\left(z - \delta_{\boldsymbol{s}}(\boldsymbol{u})B(\boldsymbol{s})\right),\end{aligned} \tag{3.56}$$

where $g_{\boldsymbol{u}}$ denotes the probability density function of $B_{\boldsymbol{s}}(\boldsymbol{u})$, as before. We temporarily use the abbreviated notion $\delta := \delta_{\boldsymbol{s}}(\boldsymbol{u})$ and $y := B(\boldsymbol{s})$. Thanks to Lemma 3.1, for all $z \in \mathbf{R}^d$,

$$g_{\boldsymbol{u}}(z - \delta y) \geq \frac{c_{12}}{\|\boldsymbol{u} - \boldsymbol{s}\|^{d/2}} \exp\left(-\frac{\|z - \delta y\|^2}{c_{12}\|\boldsymbol{u} - \boldsymbol{s}\|}\right), \tag{3.57}$$

where $c_{12} \in (0\,, 1)$ depends only on N, $\min_j a_j$, and $\max_j b_j$. But

$$\|z - \delta y\|^2 \leq 2\|z - y\|^2 + 2\|y\|^2(1 - \delta)^2, \tag{3.58}$$

and

$$0 \leq 1 - \delta = \delta_{\boldsymbol{s}}(\boldsymbol{s}) - \delta_{\boldsymbol{s}}(\boldsymbol{u}) \leq \text{const} \cdot \|\boldsymbol{u} - \boldsymbol{s}\|, \tag{3.59}$$

for a constant that has the same parameter dependencies as c_{12}. Consequently, there exists $c_{13} \in (0\,, 1)$—depending only on N, $\min_j a_j$, and

$\max_j b_j$—such that

$$g_{\boldsymbol{u}}(z - \delta y) \geq c_{13}\, \mathrm{e}^{-\|B(\boldsymbol{s})\|^2/c_{13}} \cdot \Gamma_{c_{13}}(\boldsymbol{u} - \boldsymbol{s}\,; z - y). \tag{3.60}$$

Recall that $\delta = \delta_{\boldsymbol{s}}(\boldsymbol{u})$ and $y := B(\boldsymbol{s})$; it follows from this discussion that P-almost surely,

$$\begin{aligned}&\mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\int_{\Theta} f(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right) \\ &\qquad \geq c_{14}\mathrm{e}^{-\|B(\boldsymbol{s})\|^2/c_{14}} \int_{\mathbf{R}^d} \mathrm{d}z\ f(z + B(\boldsymbol{s})) \left(\int_{\Upsilon - \boldsymbol{s}} \Gamma_{c_{14}}(\boldsymbol{u}\,; z)\,\mathrm{d}\boldsymbol{u}\right).\end{aligned} \tag{3.61}$$

Because the distance between $\boldsymbol{s}$ and the boundary of Θ is at least η, the upright box $\Upsilon - \boldsymbol{s}$ contains $[0\,,\eta]^N$. Therefore, by symmetry,

$$\begin{aligned}&\mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\int_{\Theta} f(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right) \\ &\geq \frac{c_{13}}{2^N}\mathrm{e}^{-\|B(\boldsymbol{s})\|^2/c_{13}} \int_{\mathbf{R}^d} \mathrm{d}z\ f(z + B(\boldsymbol{s})) \left(\int_{[-\eta,\eta]^N} \Gamma_{c_{13}}(\boldsymbol{u}\,; z)\,\mathrm{d}\boldsymbol{u}\right),\end{aligned} \tag{3.62}$$

almost surely [P]. Since the support of f is contained in $[-M, M]^N$, Lemma 3.7 finishes the proof. □

4 Proof of Theorem 2.4

We begin by making two simplifications:

- First, let us note that the upright box Θ is closed, and hence there exists $\eta \in (0, 1)$ such that $\eta \leq |\boldsymbol{s} - \boldsymbol{t}|_{\infty} \leq \eta^{-1}$ for all $\boldsymbol{t} \in \Theta$. This η is held fixed throughout the proof.

- Thanks to the capacitability theorem of Choquet, we may consider only $\mathcal{F}_{\pi}(\boldsymbol{s})$-measurable *compact* random sets $A \subset [-M\,, M]^d$. Without loss of generality, we may—and will—assume that $M > 1$ is fixed hereforth.

For every nonrandom $\epsilon \in (0\,,1)$, we let A^ϵ denote the closed ϵ-enlargement of A. Let f denote a random $\mathcal{F}_\pi(\boldsymbol{s})$-measurable density function that is supported on A^ϵ. Because we assumed that M is greater than one and ϵ is at most one, $\|z\|^2 \leq \text{const}\cdot M^2$ for all $z \in A^\epsilon$ and $\epsilon \in (0,1)$. Therefore, Lemma 3.8 implies that P-almost surely,

$$\begin{aligned} \mathrm{E}^\pi_{\boldsymbol{s}}\left(\int_\Theta f(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right) &\geq c^{-1}\mathrm{e}^{-c\|B(\boldsymbol{s})\|^2 - cM^2} \\ &\geq c_{14}^{-1}\mathrm{e}^{-c_{14}\|B(\boldsymbol{s})\|^2}. \end{aligned} \tag{4.1}$$

On the other hand, Lemma 3.10 assures us that

$$\mathrm{E}^\pi_{\boldsymbol{s}}\left(\left|\int_\Theta f(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right|^2\right) \leq c_{15}\,\mathrm{e}^{c_{15}\|B(\boldsymbol{s})\|^2}\cdot \mathrm{I}_{d-2N}(f) \qquad \text{a.s. [P].} \tag{4.2}$$

We combine the preceding displays together with the Paley–Zygmund inequality and deduce that P-almost surely,

$$\begin{aligned} \mathrm{P}^\pi_{\boldsymbol{s}}\left\{B(\boldsymbol{u}) \in A^\epsilon \text{ for some } \boldsymbol{u}\in\Theta\right\} &\geq \mathrm{P}^\pi_{\boldsymbol{s}}\left\{\int_\Theta f(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u} > 0\right\} \\ &\geq \frac{\mathrm{e}^{-c_{16}\|B(\boldsymbol{s})\|^2}}{c_{16}\mathrm{I}_{d-2N}(f)}. \end{aligned} \tag{4.3}$$

Let $\mathcal{P}_{\text{ac}}(A^{\epsilon/2})$ denote the collection of all absolutely continuous probability density functions that are supported on $A^{\epsilon/2}$. It is the case that

$$\left[\inf_{f\in\mathcal{P}_{\text{ac}}(A^{\epsilon/2})} \mathrm{I}_{d-2N}(f)\right]^{-1} \asymp \mathrm{Cap}_{d-2N}(A^{\epsilon/2}), \tag{4.4}$$

where the implied constants depend only on d, N, and M [12, Exercise 4.1.4, p. 423]. According to Lemma 2.5, there exists an $\mathcal{F}_\pi(\boldsymbol{s})$-measurable $\mu_\epsilon \in \mathcal{P}(A^{\epsilon/2})$ such that

$$\mathrm{Cap}_{d-2N}(A^{\epsilon/2}) \asymp \left[\mathrm{I}_{d-2N}(\mu_\epsilon)\right]^{-1}, \tag{4.5}$$

where the implied constants depend only on d, N, and M. Let ϕ_ϵ denote a smooth probability density function with support in $B(0\,,\epsilon/2) = \{0\}^{\epsilon/2}$. Then, $f = f_\epsilon := \phi_\epsilon * \mu_\epsilon$ is in $\mathcal{P}_{\rm ac}(A^\epsilon)$ and is $\mathcal{F}_\pi(\boldsymbol{s})$-measurable. We can apply (4.3) with this f, in order to obtain the following: P-almost surely,

$$\mathrm{P}^\pi_{\boldsymbol{s}}\left\{B(\boldsymbol{u}) \in A^\epsilon \text{ for some } \boldsymbol{u} \in \Theta\right\} \geq \frac{\mathrm{e}^{-c_{16}\|B(\boldsymbol{s})\|^2}}{c_{16}\mathrm{I}_{d-2N}(\phi_\epsilon * \mu_\epsilon)}. \tag{4.6}$$

But $\mathrm{I}_{d-2N}(\phi_\epsilon * \mu_\epsilon) \leq C\mathrm{I}_{d-2N}(\mu_\epsilon)$ for a finite nonrandom constant C that depends only on d, N, and $\sup\{|z| : z \in A\}$; see Theorems B.1 and B.2 of [5]. Therefore, we can deduce from (4.5) that

$$\mathrm{P}^\pi_{\boldsymbol{s}}\left\{B(\boldsymbol{u}) \in A^\epsilon \text{ for some } \boldsymbol{u} \in \Theta\right\} \geq c_{17}\,\mathrm{e}^{-c_{17}\|B(\boldsymbol{s})\|^2}\,\mathrm{Cap}_{d-2N}(A^{\epsilon/2}). \tag{4.7}$$

The resulting inequality holds almost surely, simultaneously for all rational $\epsilon \in (0\,,1)$. Therefore, we can let ϵ converge downward to zero, and appeal to Choquet's capacitability theorem to deduce that P-almost surely,

$$\mathrm{P}^\pi_{\boldsymbol{s}}\left\{B(\boldsymbol{u}) \in A \text{ for some } \boldsymbol{u} \in \Theta\right\} \geq c_{18}\,\mathrm{e}^{-c_{17}\|B(\boldsymbol{s})\|^2}\cdot\mathrm{Cap}_{d-2N}(A). \tag{4.8}$$

[Choquet's theorem tells us that the preceding capacities are outer regular, therefore as A^ϵ converges downward to A, so do their respective capacities converge downward to the capacity of A.] Consequently,

$$\mathrm{P}^\pi_{\boldsymbol{s}}\left\{B(\boldsymbol{u}) \in A \text{ for some } \boldsymbol{u} \in \Theta\right\} \trianglerighteq \mathrm{Cap}_{d-2N}(A). \tag{4.9}$$

We complete the theorem by deriving the converse direction; that is,

$$\mathrm{P}^\pi_{\boldsymbol{s}}\left\{B(\boldsymbol{u}) \in A \text{ for some } \boldsymbol{u} \in \Theta\right\} \trianglelefteq \mathrm{Cap}_{d-2N}(A). \tag{4.10}$$

Equation (4.10) holds vacuously unless there is a positive probability that the following happens:

$$\mathrm{P}^\pi_{\boldsymbol{s}}\left\{B(\boldsymbol{u}) \in A \text{ for some } \boldsymbol{u} \in \Theta\right\} > 0. \tag{4.11}$$

Therefore, we may assume that (4.11) holds with positive probability without incurring any further loss in generality.

Define

$$T_1 := \inf \{u_1 \geq 0 : \, B(\boldsymbol{u}) \in A \text{ for some } \boldsymbol{u} = (u_1 \, , \ldots , u_N) \in \Theta\} \, , \qquad (4.12)$$

where $\inf \varnothing := \infty$. Evidently T_1 is a random variable with values in $\pi_1\Theta \cup \{\infty\}$, where π_l denotes the projection map which takes $\boldsymbol{v} \in \mathbf{R}^N$ to v_l. Having constructed $T_1, \ldots, T_j$ for $j \in \{1, \ldots, N-1\}$, with values respectively in $\pi_1\Theta \cup \{\infty\}, \; \ldots, \; \pi_j\Theta \cup \{\infty\}$, we define T_{j+1} to be $+\infty$ almost surely on $\cup_{l=1}^{j}\{T_l = \infty\}$, and

$$T_{j+1} := \inf \left\{u_{j+1} \geq 0 : \, B(T_1 \, , \ldots , T_j, u_{j+1} \, , \ldots , u_N) \in A \quad \text{for some } \boldsymbol{u}^T \in \Theta\right\} ,$$

almost surely on $\cap_{l=1}^{j}\{T_l < \infty\}$, where in the preceding display

$$\boldsymbol{u}^T := (T_1 \, , \ldots , T_j \, , u_{j+1} \, , \ldots , u_N) \, . \qquad (4.13)$$

In this way, we obtain a random variable $\boldsymbol{T}$, with values in $\Theta \cup \{\infty\}^N$, defined as

$$\boldsymbol{T} := (T_1 \, , \ldots , T_N) \, . \qquad (4.14)$$

Because (4.11) holds with positive probability, it follows that

$$\mathrm{P}^{\pi}_{\boldsymbol{s}} \{\boldsymbol{T} \in \Theta\} \asymp \mathrm{P}^{\pi}_{\boldsymbol{s}} \{B(\boldsymbol{u}) \in A \text{ for some } \boldsymbol{u} \in \Theta\} \, . \qquad (4.15)$$

If (4.11) holds for some realization $\omega \in \Omega$, then we define, for all Borel sets $G \subseteq \mathbf{R}^d$,

$$\rho(G)(\omega) := \mathrm{P}^{\pi}_{\boldsymbol{s}} \left(B(\boldsymbol{T}) \in G \mid \boldsymbol{T} \in \Theta \right) (\omega). \qquad (4.16)$$

Otherwise, we choose and fix some point $a \in A$ and define $\rho(G)(\omega) := \delta_a(G)$. It follows that ρ is a random $\mathcal{F}_\pi(\boldsymbol{s})$-measurable probability measure on A.

Let $\phi_1 \in C^\infty(\mathbf{R}^d)$ be a probability density function such that $\phi_1(x) = 0$ if $\|x\| > 1$. We define an approximation to the identity $\{\phi_\epsilon\}_{\epsilon>0}$ by setting

$$\phi_\epsilon(x) := \frac{1}{\epsilon^d}\phi_1\left(\frac{x}{\epsilon}\right) \qquad \text{for all } x \in \mathbf{R}^d \text{ and } \epsilon > 0. \tag{4.17}$$

We plan to apply Lemma 3.11 with $f := \rho * \psi_\epsilon$, where $\psi_\epsilon(x) := \phi_{\epsilon/2} * \phi_{\epsilon/2}(x)$. Furthermore, we can choose a good modification of the conditional expectation in that lemma to deduce that the null set off which the assertion fails can be chosen independently of $\boldsymbol{s}$; see Lemma 3.5.

Note that the support of $\rho*\psi_\epsilon$ is contained in A^ϵ. It follows from Lemma 3.11 that P-almost surely,

$$\begin{aligned}\sup_{\boldsymbol{t}\in\Theta} \mathrm{E}^\pi_{\boldsymbol{t}}\left(\int_{\Theta^\eta} (\rho * \psi_\epsilon)(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right)& \\ \geq c_{11}\,\mathbf{1}_{\{\boldsymbol{T}\in\Theta\}}\int_{\mathbf{R}^d} \kappa_{d-2N}(z)(\rho * \psi_\epsilon)(z + B(\boldsymbol{T}))\,\mathrm{d}z.&\end{aligned} \tag{4.18}$$

The constant c_{11} is furnished by Lemma 3.11. Moreover, Θ^η denotes the closed η-enlargement of Θ. We square both sides and take $\mathrm{E}^\pi_{\boldsymbol{s}}$-expectations. Because $\boldsymbol{s} \prec_\pi \boldsymbol{t}$ for all $\boldsymbol{t} \in \Theta$, Lemma 3.6 tells us that the $\mathrm{E}^\pi_{\boldsymbol{s}}$-expectation of the square of the left-hand side of (4.18) is at most

$$4^N \sup_{\boldsymbol{t}\in\Theta} \mathrm{E}^\pi_{\boldsymbol{s}}\left(\left|\mathrm{E}^\pi_{\boldsymbol{t}}\left[\int_{\Theta^\eta}(\rho * \psi_\epsilon)(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right]\right|^2\right). \tag{4.19}$$

By the conditional Jensen's inequality, $|\mathrm{E}^\pi_{\boldsymbol{t}} Z|^2 \leq \mathrm{E}^\pi_{\boldsymbol{t}}(Z^2)$ [a.s.] for all square-integrable random variables Z. Moreover, $\boldsymbol{s} \prec_\pi \boldsymbol{t}$ implies that $\mathrm{E}^\pi_{\boldsymbol{s}}\mathrm{E}^\pi_{\boldsymbol{t}} = \mathrm{E}^\pi_{\boldsymbol{s}}$; this follows from the tower property of conditional expectations. Consequently,

$$\mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\left|\sup_{\boldsymbol{t}\in\Theta}\mathrm{E}_{\boldsymbol{t}}^{\pi}\left(\int_{\Theta^\eta}(\rho*\psi_\epsilon)(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right)\right|^2\right) \le 4^N\mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\left|\int_{\Theta^\eta}(\rho*\psi_\epsilon)(B(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u}\right|^2\right) \le c\mathrm{e}^{c\|B(\boldsymbol{s})\|^2}\cdot \mathrm{I}_{d-2N}(\rho*\psi_\epsilon), \tag{4.20}$$

where the last inequality follows from Lemma 3.10. This and (4.18) together imply that with probability one [P],

$$\begin{aligned} &c\mathrm{e}^{c\|B(\boldsymbol{s})\|^2}\cdot \mathrm{I}_{d-2N}(\rho*\psi_\epsilon) \\ &\ge \mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\left[\int_{\mathbf{R}^d}\kappa_{d-2N}(z)(\rho*\psi_\epsilon)(z+B(\boldsymbol{T}))\,\mathrm{d}z\cdot \mathbf{1}_{\{\boldsymbol{T}\in\Theta\}}\right]^2\right) \\ &= \mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\left[\int_{\mathbf{R}^d}\kappa_{d-2N}(z)(\rho*\psi_\epsilon)(z+B(\boldsymbol{T}))\,\mathrm{d}z\right]^2 \,\middle|\, \boldsymbol{T}\in\Theta\right) \\ &\qquad\qquad \times \mathrm{P}_{\boldsymbol{s}}^{\pi}\{\boldsymbol{T}\in\Theta\}. \end{aligned} \tag{4.21}$$

We apply the Cauchy–Schwarz inequality and the definition of ρ—in this order—to deduce from the preceding that

$$\begin{aligned} &c\mathrm{e}^{c\|B(\boldsymbol{s})\|^2}\cdot \mathrm{I}_{d-2N}(\rho*\psi_\epsilon) \\ &\ge \left[\mathrm{E}_{\boldsymbol{s}}^{\pi}\left(\int_{\mathbf{R}^d}\kappa_{d-2N}(z)\,(\rho*\psi_\epsilon)(z+B(\boldsymbol{T}))\,\mathrm{d}z \,\middle|\, \boldsymbol{T}\in\Theta\right)\right]^2 \times \mathrm{P}_{\boldsymbol{s}}^{\pi}\{\boldsymbol{T}\in\Theta\} \\ &= \left[\int_A \rho(\mathrm{d}x)\int_{\mathbf{R}^d}\mathrm{d}z\;\kappa_{d-2N}(z)\,(\rho*\psi_\epsilon)(z+x)\right]^2 \times \mathrm{P}_{\boldsymbol{s}}^{\pi}\{\boldsymbol{T}\in\Theta\}. \end{aligned} \tag{4.22}$$

The term in square brackets is equal to $\int(\kappa_{d-2N}*\rho*\psi_\epsilon)\,\mathrm{d}\rho$. Since $\psi_\epsilon = \phi_{\epsilon/2}*\phi_{\epsilon/2}$, that same term in square brackets is equal to $\mathrm{I}_{d-2N}(\rho*\phi_{\epsilon/2})$. Thus, the following holds P-almost surely:

$$c\mathrm{e}^{c\|B(\boldsymbol{s})\|^2}\cdot \mathrm{I}_{d-2N}(\rho*\phi_{\epsilon/2}*\phi_{\epsilon/2}) \ge \left[\mathrm{I}_{d-2N}(\rho*\phi_{\epsilon/2})\right]^2 \times \mathrm{P}_{\boldsymbol{s}}^{\pi}\{\boldsymbol{T}\in\Theta\}. \tag{4.23}$$

In order to finish the proof we now consider separately the three cases where $d < 2N$, $d > 2N$, and $d = 2N$. If $d < 2N$, then (4.10) holds because the right-hand side is 1.

If $d > 2N$, then Theorem B.1 of Dalang et al [5] tells us that

$$\mathrm{I}_{d-2N}(\rho * \phi_{\epsilon/2} * \phi_{\epsilon/2}) \leq \mathrm{I}_{d-2N}(\rho * \phi_{\epsilon/2}). \tag{4.24}$$

Since κ_{d-2N} is lower semicontinuous, Fatou's lemma shows that

$$\liminf_{\epsilon \downarrow 0} \mathrm{I}_{d-2N}(\rho * \phi_{\epsilon/2}) \geq \mathrm{I}_{d-2N}(\rho). \tag{4.25}$$

Therefore, (4.23) implies that:

(i) $\mathrm{I}_{d-2N}(\rho) < \infty$ [thanks also to (4.11)]; and

(ii) $\mathrm{P}^{\pi}_{\boldsymbol{s}}\{\boldsymbol{T} \in \Theta\} \leq c \exp(c\|B(\boldsymbol{s})\|^2)/\mathrm{I}_{d-2N}(\rho)$ almost surely.

This proves that P-almost surely,

$$\begin{aligned} \mathrm{P}^{\pi}_{\boldsymbol{s}}\{\boldsymbol{T} \in \Theta\} &\leq \frac{c\mathrm{e}^{c\|B(\boldsymbol{s})\|^2}}{\mathrm{I}_{d-2N}(\rho)} \\ &\leq c\mathrm{e}^{c\|B(\boldsymbol{s})\|^2} \cdot \mathrm{Cap}_{d-2N}(A). \end{aligned} \tag{4.26}$$

Consequently, (4.15) implies the theorem in the case that $d > 2N$.

The final case that $d = 2N$ is handled similarly, but this time we use Theorem B.2 of Dalang et al [5] in place of their Theorem B.1. □

5 Proofs of Theorem 1.1 and its corollaries

We start with the following result which deals with intersections of the images of the Brownian sheet of disjoint boxes that satisfy certain configuration conditions.

Theorem 5.1. *Let $\Theta_1, \ldots, \Theta_k$ in $(0\,,\infty)^N$ be disjoint, closed and nonrandom upright boxes that satisfy the following properties:*

(1) *for all* $j = 1, \ldots, k-1$ *there exists* $\pi(j) \subseteq \{1, \ldots, N\}$ *such that* $\boldsymbol{u} \prec_{\pi(j)} \boldsymbol{v}$ *for all* $\boldsymbol{u} \in \cup_{l=1}^{j} \Theta_l$ *and* $\boldsymbol{v} \in \Theta_{j+1}$*; and*

(2) *there exists a nonrandom* $\eta > 0$ *such that* $|\boldsymbol{r} - \boldsymbol{q}|_\infty \geq \eta$ *for all* $\boldsymbol{r} \in \Theta_i$ *and* $\boldsymbol{q} \in \Theta_j$*, where* $1 \leq i \neq j \leq k$.

Then for any Borel set $A \subseteq \mathbf{R}^d$,

$$\mathrm{P}\left\{\bigcap_{j=1}^{k} B(\Theta_j) \cap A \neq \varnothing\right\} > 0 \;\Leftrightarrow\; \mathrm{P}\left\{\bigcap_{j=1}^{k} W_j(\Theta_j) \cap A \neq \varnothing\right\} > 0, \tag{5.1}$$

where $W_1, \ldots, W_k$ *are* k *independent* N*-parameter Brownian sheets in* $\mathbf{R}^d$ *(which are unrelated to* B*).*

Proof. Under the assumptions **(1)** and **(2)**, we can choose and fix nonrandom time points $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{k-1} \in (0, \infty)^N$ such that for all $l = 1, \ldots, k-1$:

(3) $\boldsymbol{s}_l \prec_{\pi(l)} \boldsymbol{v}$ for all $\boldsymbol{v} \in \Theta_{l+1}$; and

(4) $\boldsymbol{s}_l \succ_{\pi(l)} \boldsymbol{u}$ for all $\boldsymbol{u} \in \cup_{j=1}^{l} \Theta_j$.

Because the elements of Θ_k dominate those of $\Theta_1, \ldots, \Theta_{k-1}$ in partial order $\pi(k-1)$, Theorem 2.4 can be applied [under $\mathrm{P}_{\boldsymbol{s}_{k-1}}^{\pi(k-1)}$] to show that for all nonrandom Borel sets $A \subset \mathbf{R}^d$,

$$\mathrm{P}\left\{[\boldsymbol{B}]_k \cap A \neq \varnothing\right\} > 0 \;\Leftrightarrow\; \mathrm{E}\left[\mathrm{Cap}_{d-2N}\left([\boldsymbol{B}]_{k-1} \cap A\right)\right] > 0, \tag{5.2}$$

where

$$[\boldsymbol{B}]_k := \bigcap_{j=1}^{k} B(\Theta_j). \tag{5.3}$$

The main result of Khoshnevisan and Shi [15] is that $\mathrm{Cap}_{d-2N}(E) > 0$ is necessary and sufficient for $\mathrm{P}\{W_k(\Theta_k) \cap E \neq \varnothing\}$ to be [strictly] positive, where W_k is a Brownian sheet that is independent of B. We apply this with $E := [\boldsymbol{B}]_{k-1} \cap A$ to deduce that

$$\mathrm{P}\left\{[\boldsymbol{B}]_k \cap A \neq \varnothing\right\} > 0 \;\Leftrightarrow\; \mathrm{P}\left\{[\boldsymbol{B}]_{k-1} \cap W_k(\Theta_k) \cap A \neq \varnothing\right\} > 0. \tag{5.4}$$

Because W_k is independent of B, and thanks to **(3)** and **(4)** above, we may apply Theorem 2.4 inductively to deduce that

$$\mathrm{P}\left\{[\boldsymbol{B}]_k \cap A \neq \varnothing\right\} > 0 \quad \Leftrightarrow \quad \mathrm{P}\left\{\bigcap_{j=1}^{k} W_j(\Theta_j) \cap A \neq \varnothing\right\} > 0, \tag{5.5}$$

where $W_1, \ldots, W_k$ are i.i.d. Brownian sheets. This proves Theorem 5.1. □

Note that conditions (1) and (2) in Theorem 5.1 are satisfied for $k = 2$ for two arbitrary upright boxes Θ_1 and Θ_2 that have disjoint projections on each coordinate hyperplane $s_i = 0$, $i = 1, \ldots, N$. Hence we are ready to derive Theorem 1.1.

Proof of Theorem 1.1. Observe that there exist distinct points $\boldsymbol{s}$ and $\boldsymbol{t} \in (0\,,\infty)^N$ with $B(\boldsymbol{s}) = B(\boldsymbol{t}) \in A$, and such that $s_i \neq t_i$, for all $i = 1, \ldots, N$, if and only if we can find disjoint closed upright boxes Θ_1 and Θ_2, with vertices with rational coordinates, such that $[\boldsymbol{B}]_2 \cap A \neq \varnothing$. Moreover, we may require Θ_1 and Θ_2 to be such that the assumptions **(1)** and **(2)** of Theorem 5.1 are satisfied. Since the family of pairs of such closed upright boxes Θ_1 and Θ_2 is countable, it follows that (5.1) implies Theorem 1.1. □

In order to apply Theorem 1.1 to study the nonexistence of double points of the Brownian sheet, we first provide some preliminary results on the following subset of M_2:

$$M_2^{(1)} := \left\{ x \in \mathbf{R}^d \;\middle|\; \begin{array}{l} B(\boldsymbol{s}_1) = B(\boldsymbol{s}_2) = x \text{ for distinct } \boldsymbol{s}_1,\, \boldsymbol{s}_2 \in (0\,,\infty)^N \\ \text{with at least one common coordinate} \end{array} \right\}.$$

Note that $M_2^{(1)}$ cannot be studied by using Theorem 1.1. The next lemma will help us to show that $M_2^{(1)}$ has negligible effect on the properties of M_2.

Lemma 5.2. *The random set $M_2^{(1)}$ has the following properties:*

(i) $\dim_{\mathrm{H}} M_2^{(1)} \leq 4N - 2 - d$ *a.s., and "*$\dim_{\mathrm{H}} M_2^{(1)} < 0$*" means "*$M_2^{(1)} = \varnothing$*."*

(ii) For every nonrandom Borel set $A \subseteq \mathbf{R}^d$,

$$\mathrm{P}\left\{M_2^{(1)} \cap A \neq \varnothing\right\} \leq \text{const} \cdot \mathcal{H}_{2d-2(2N-1)}(A), \tag{5.6}$$

where $\mathcal{H}_\beta$ denotes the β-dimensional Hausdorff measure.

Proof. Part (i) follows from (ii) and a standard covering argument; see for example [1, 5, 29]. We omit the details and only give the following rough outline. We only consider the case where $\boldsymbol{s}_1,\ \boldsymbol{s}_2 \in (0\,,\infty)^N$ are distinct, but have the same first coordinates. This causes little loss of generality.

For a point in a fixed unit cube of $\mathbf{R}^N$, say $[1,2]^N$, there are 2^{2n} possible first coordinates of the form $1 + i2^{-2n}$, $i = 0, \ldots, 2^{2n} - 1$.

For any given such first coordinate, there are $(2^{2n})^{N-1}$ points in $[1,2]^N$ with all other coordinates of the same form as the first coordinate. In another unit cube, such as $[1,2] \times [3,4]^{N-1}$, there are also $(2^{2n})^{N-1}$ points with a given first coordinate and all other coordinates of the form $1 + i2^{-2n}$, $i = 0, \ldots, 2^{2n} - 1$.

We cover the set $M_2^{(1)} \cap [0,1]^d$ by small boxes with sides of length $n2^{-n}$. If we cover $[0,1]^d$ by a grid of small boxes with sides of length $n2^{-n}$, the probability that any small box C in $[0,1]^d$ is needed to help cover $M_2^{(1)} \cap [0,1]^d$ because of the behavior of B near (u_1, u_2) and (u_1, v_2) is approximately

$$\mathrm{P}\{B(u_1,u_2) \in C,\ \|B(u_1,u_2) - B(u_1,v_2)\| \leq n2^{-n}\} \simeq (n2^{-n})^{2d}, \tag{5.7}$$

where $(u_1, u_2) \in [1,2]^N$ and $(u_1, v_2) \in [1,2] \times [3,4]^{N-1}$. Therefore, for $\gamma > 0$,

$$\mathrm{E}\left(\sum (n2^{-n})^\gamma\right) \simeq 2^{nd}(n2^{-n})^\gamma \mathrm{P}\{\text{a given small box is in the covering}\}, \tag{5.8}$$

where the sum on the left-hand side is over all small boxes in a covering of $M_2^{(1)} \cap [0,1]^d$. The probability on the right-hand side is approximately

$$\begin{aligned} &\#\{\text{points } (u_1,u_2) \text{ and } (v_1,v_2) \text{ to be considered}\}\ (n2^{-n})^{2d} \\ &\quad = 2^{2n}\left((2^{2n})^{(N-1)}\right)^2 (n2^{-n})^{2d}. \end{aligned} \tag{5.9}$$

It follows that the left-hand side of (5.8) is approximately equal to

$$n^{\gamma+2d}(2^{-n})^{\gamma-4N+2+d}. \tag{5.10}$$

This converges to 0 if $\gamma > 4N-2-d$, and this explains statement (i).

In order to prove (ii), we start with a hitting probability estimate for M_2. Let $D := D_1 \times D_2 \times D_3$ denote a compact upright box in $(0\,,\infty)^{1+2(N-1)}$, where D_2, $D_3 \subset (0\,,\infty)^{N-1}$ are disjoint. By using the argument in the proof of Proposition 2.1 in Xiao [28] we can show that simultaneously for all $(a_1, \boldsymbol{a}_2, \boldsymbol{a}_3) \in D$, $r > 0$ and $x \in \mathbf{R}^d$,

$$\mathrm{P}\left\{\begin{array}{l} \exists\,(t_1\,,\boldsymbol{t}_2\,,\boldsymbol{t}_3) \in (a_1 - r^2, a_1 + r^2) \times U_r(\boldsymbol{a}_2) \times U_r(\boldsymbol{a}_3) \\ \text{such that } \left|B(t_1,\boldsymbol{t}_2) - x\right| \le r,\ \left|B(t_1,\boldsymbol{t}_3) - x\right| \le r \end{array}\right\} = O(r^{2d}), \tag{5.11}$$

as $r \downarrow 0$, where $U_r(\boldsymbol{a}) := \{\boldsymbol{t} \in \mathbf{R}^{N-1} : |\boldsymbol{t} - \boldsymbol{a}| \le r^2\}$. The proof of (5.11) is somewhat lengthy. Since it is more or less a standard proof, we omit the details, and offer instead only the following rough outline: (a) For fixed $(t_1\,,\boldsymbol{t}_2\,,\boldsymbol{t}_3)$ in $(a_1 - r^2\,,a_2 + r^2) \times U_r(\boldsymbol{a}_2) \times U_r(\boldsymbol{a}_3)$, we have $\mathrm{P}\{|B(t_1\,,\boldsymbol{t}_2)| \le r\} = O(r^d)$ thanks to direct computation; (b) $\mathrm{P}\{|B(t_1\,,\boldsymbol{t}_3)| \le r \mid |B(t_1\,,\boldsymbol{t}_2)| \le r\} = O(r^d)$ because $B(t_1\,,\boldsymbol{t}_2)$ and $B(t_1\,,\boldsymbol{t}_3)$ are "sufficiently independent"; and (c) B in time-intervals of side length r^2 is roughly "constant" to within at most r units. Part (ii) follows from (5.11) and another covering argument [1, 5]. □

We now show how Theorem 1.1 can be combined with the elegant theory of Peres [22] and Lemma 5.2 to imply the corollaries mentioned in the introduction.

Proof of Corollary 1.2. Theorem 1.1 of Khoshnevisan and Shi [15] asserts that for each $\nu \in \{1\,,2\}$, all nonrandom Borel sets $A \subset \mathbf{R}^d$, contained in a fixed compact subset of $\mathbf{R}^d$, and all upright boxes $\Theta := \prod_{j=1}^N [a_j\,,b_j] \subset (0\,,\infty)^N$, there is a finite constant $R \ge 1$ such that

$$R^{-1}\,\mathrm{Cap}_{d-2N}(A) \le \mathrm{P}\left\{W_\nu(\Theta) \cap A \ne \varnothing\right\} \le R\mathrm{Cap}_{d-2N}(A). \tag{5.12}$$

We first consider the case where $d > 2N$. Because W_1 and W_2 are independent, Corollary 15.4 of Peres [22, p. 240] and (5.12) imply that for all upright boxes $\Theta_1, \Theta_2 \subset (0\,,\infty)^N$

$$\mathrm{P}\left\{W_1(\Theta_1) \cap W_2(\Theta_2) \cap A \neq \varnothing\right\} > 0 \quad \Longleftrightarrow \quad \mathrm{Cap}_{2(d-2N)}(A) > 0. \quad (5.13)$$

Next, let us assume that $\mathrm{Cap}_{2(d-2N)}(A) > 0$. We choose arbitrary upright boxes Θ_1 and Θ_2 that have disjoint projections on each coordinate hyperplane $s_i = 0$, $i = 1, \ldots, N$. It follows that $\mathrm{P}\{M_2 \cap A \neq \varnothing\} > 0$, thanks to (5.13) and Theorem 1.1.

In order to prove the converse, we assume that $\mathrm{Cap}_{2(d-2N)}(A) = 0$. Then $\dim A \leq 2(d-2N)$ which implies $\mathcal{H}_{2d-2(2N-1)}(A) = 0$. It follows from Lemma 5.2 that $\mathrm{P}\{M_2^{(1)} \cap A \neq \varnothing\} = 0$. On the other hand, (5.13) and Theorem 1.1 imply that $\mathrm{P}\{(M_2 \setminus M_2^{(1)}) \cap A \neq \varnothing\} = 0$. This finishes the proof when $d > 2N$.

In the case $d = 2N$, where (1.6) appears, we also use (5.12), Corollary 15.4 of Peres [22, p. 240] and Lemma 5.2.

Finally, if $2N > d$, then B hits points by (5.12). This implies the last conclusion in Corollary 1.2. □

Proof of Corollary 1.3. We appeal to Corollary 1.2 with $A := \mathbf{R}^d$, and use (2.6) to deduce that $\mathrm{P}\{M_2 \neq \varnothing\} > 0$ if and only if $2(d-2N) < d$. Next, we derive the second assertion of the corollary [Fristedt's conjecture].

Choose and fix some $x \in \mathbf{R}^d$. Corollary 1.2 tells us that $\mathrm{P}\{x \in M_2\} > 0$ if and only if $\mathrm{Cap}_{2(d-2N)}(\{x\}) > 0$. Because the only probability measure on $\{x\}$ is the point mass, the latter positive-capacity condition is equivalent to the condition that $d < 2N$. According to the Tonelli theorem, $\mathrm{E}(\mathrm{meas}\, M_2) = \int_{\mathbf{R}^d} \mathrm{P}\{x \in M_2\}\, \mathrm{d}x$, where "meas M_2" denotes the d-dimensional Lebesgue measure of M_2. It follows readily from this discussion that

$$\mathrm{E}(\mathrm{meas}\, M_2) > 0 \quad \Leftrightarrow \quad d < 2N. \quad (5.14)$$

If $d \geq 2N$, then this proves that $\mathrm{meas}\, M_2 = 0$ almost surely.

It only remains to show that if $d < 2N$, then $\mathrm{meas}\, M_2 > 0$ almost surely.

For any integer $\ell \geq 0$, define

$$M_{2,\ell} := \left\{ x \in \mathbf{R}^d : B(\boldsymbol{s}^1) = B(\boldsymbol{s}^2) = x \text{ for distinct } \boldsymbol{s}^1,\, \boldsymbol{s}^2 \in \left[2^\ell, 2^{\ell+1}\right]^N \right\}.$$

Given a fixed point $x \in \mathbf{R}^d$, the scaling properties of the Brownian sheet imply that

$$\mathrm{P}\{x \in M_{2,\ell}\} = \mathrm{P}\{2^{-\ell N/2} x \in M_{2,0}\}. \tag{5.15}$$

By using Theorem 1.1 and (5.13), we see that

$$\inf_{x \in [-q,q]^d} \mathrm{P}\{x \in M_{2,0}\} > 0 \qquad \text{for all } q > 0. \tag{5.16}$$

The scaling property (5.15) then implies that

$$\gamma_q := \inf_{\ell \geq 0} \inf_{x \in [-q,q]^d} \mathrm{P}\{x \in M_{2,\ell}\} > 0 \qquad \text{for all } q > 0. \tag{5.17}$$

In particular,

$$\begin{aligned} \mathrm{P}\left\{x \in M_{2,\ell} \text{ for infinitely many } \ell \geq 0\right\} &\geq \gamma_q \\ &> 0 \quad \text{for all } x \in [-q\,,q]^d. \end{aligned} \tag{5.18}$$

By the zero-one law of Orey and Pruitt [21, pp. 140–141], the left-hand side of (5.18) is identically equal to one. But that left-hand side is at most $\mathrm{P}\{x \in M_2\}$. Because q is arbitrary, this proves that $\mathrm{P}\{x \in M_2\} = 1$ for all $x \in \mathbf{R}^d$ when $d < 2N$. By Tonelli's theorem, $\mathrm{P}\{x \in M_2 \text{ for almost all } x \in \mathbf{R}^d\} = 1$, whence $\operatorname{meas} M_2 = \infty$ almost surely, and in particular $\operatorname{meas} M_2 > 0$ almost surely. □

Proof of Proposition 1.4. According to (5.12) and Corollary 15.4 of Peres [22, p. 240], the following is valid for all k upright boxes $\Theta_1, \ldots, \Theta_k \subset (0\,,\infty)^N$ with vertices with rational coordinates:

$$\mathrm{P}\left(\bigcap_{\nu=1}^k W_\nu(\Theta_\nu) \cap A \neq \varnothing \right) > 0 \quad \iff \quad \mathrm{Cap}_{k(d-2N)}(A) > 0. \tag{5.19}$$

Observe that $\mathrm{P}\{\widetilde{M}_k \cap A \neq \varnothing\} > 0$ if and only if there exists a partial order $\pi \subseteq \{1\,, \ldots, N\}$ together with k disjoint upright boxes $\Theta_1, \ldots, \Theta_k$ in $(0\,,\infty)^N$, with vertices with rational coordinates, such that for $1 \leq i < j \leq k$, $\boldsymbol{s} \in \Theta_i$ and $\boldsymbol{t} \in \Theta_j$ implies $\boldsymbol{s} \ll_\pi \boldsymbol{t}$, and

$$\mathrm{P}\left(\bigcap_{\nu=1}^{k} B(\Theta_\nu) \cap A \neq \varnothing\right) > 0. \tag{5.20}$$

In addition, $\Theta_1, \ldots, \Theta_k$ can be chosen so as to satisfy **(1)** and **(2)** of Theorem 5.1 (with $\pi(j) = \pi$, $j = 1, \ldots, k-1$). It follows from Theorem 5.1 that

$$\mathrm{P}\left\{\widetilde{M}_k \cap A \neq \varnothing\right\} > 0 \quad \Longleftrightarrow \quad \mathrm{P}\left(\bigcap_{\nu=1}^{k} W_\nu(\Theta_\nu) \cap A \neq \varnothing\right) > 0. \tag{5.21}$$

Owing to (5.19), the right-hand side is equivalent to the [strict] positivity of $\mathrm{Cap}_{k(d-2N)}(A)$; this proves the first statement in Proposition 1.4. And the second statement follows by taking (2.6) into account. □

Remark 5.3. The following is a consequence of Proposition 1.4: Fix an integer $k > 2$, and suppose that with positive probability there exist distinct $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in (0\,,\infty)^N$ such that $W_1(\boldsymbol{u}_1) = \cdots = W_k(\boldsymbol{u}_k) \in A$. Then with positive probability there exist distinct $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in (0\,,\infty)^N$ such that $B(\boldsymbol{u}_1) = \cdots = B(\boldsymbol{u}_k) \in A$. We believe the converse is true. But Proposition 1.4, and even (5.5), implies the converse only for special configurations of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$. In particular, the question of the existence of k-multiple points in critical dimensions ($k > 2$ for which $k(d-2N) = d$) remains open. □

Proof of Corollary 1.5. We can combine (5.1) and (5.13) with Corollary 1.2 and deduce that whenever Θ_1 and Θ_2 are the upright boxes of the proof of Theorem 1.1,

$$\mathrm{P}\left\{B(\Theta_1) \cap B(\Theta_2) \cap A \neq \varnothing\right\} > 0 \quad \Leftrightarrow \quad \mathrm{Cap}_{2(d-2N)}(A) > 0. \tag{5.22}$$

This is valid for all nonrandom Borel sets $A \subseteq \mathbf{R}^d$.

By Frostman's theorem, if $\dim_{\mathrm{H}} A < 2(d-2N)$, then $\mathrm{Cap}_{2(d-2N)}(A) = 0$; see (2.5). Consequently, Corollary 1.2 implies that $M_2 \cap A = \varnothing$ almost surely.

Next, consider the case where $\dim_{\mathrm{H}} A \geq 2(d-2N) > 0$. Choose and fix some constant $\rho \in (0\,, d)$. According to Theorem 15.2 and Corollary 15.3 of Peres [22, pp. 239–240], we can find a random set $\boldsymbol{X}_\rho$, independent of the Brownian sheet B, that has the following properties:

- For all nonrandom Borel sets $A \subseteq \mathbf{R}^d$,

$$\mathrm{P}\{\boldsymbol{X}_\rho \cap A \neq \varnothing\} > 0 \quad \Leftrightarrow \quad \mathrm{Cap}_\rho(A) > 0; \quad \text{and} \tag{5.23}$$

- for all nonrandom Borel sets $A \subseteq \mathbf{R}^d$ and all $\beta > 0$,

$$\mathrm{P}\left\{\mathrm{Cap}_\beta(\boldsymbol{X}_\rho \cap A) > 0\right\} > 0 \quad \Leftrightarrow \quad \mathrm{Cap}_{\rho+\beta}(A) > 0. \tag{5.24}$$

[Indeed, $\boldsymbol{X}_\rho$ is the fractal-percolation set $Q_d(\kappa_\rho)$ of Peres (*loc. cit.*).]

Equation (5.22) implies that

$$\mathrm{P}\left\{[\boldsymbol{B}]_2 \cap \boldsymbol{X}_\rho \cap A \neq \varnothing \mid \boldsymbol{X}_\rho\right\} > 0 \quad \Leftrightarrow \quad \mathrm{Cap}_{2(d-2N)}(\boldsymbol{X}_\rho \cap A) > 0, \tag{5.25}$$

where we recall that $[\boldsymbol{B}]_2 := B(\Theta_1) \cap B(\Theta_2)$. Thanks to (5.24),

$$\mathrm{P}\left\{[\boldsymbol{B}]_2 \cap \boldsymbol{X}_\rho \cap A \neq \varnothing \mid \boldsymbol{X}_\rho\right\} \asymp \mathrm{Cap}_{2(d-2N)+\rho}(A). \tag{5.26}$$

holds almost surely. At the same time, (5.23) implies that

$$\mathrm{P}\left\{[\boldsymbol{B}]_2 \cap \boldsymbol{X}_\rho \cap A \neq \varnothing \mid B\right\} \asymp \mathrm{Cap}_\rho\left([\boldsymbol{B}]_2 \cap A\right), \tag{5.27}$$

Therefore, we compare the last two displays to deduce that

$$\mathrm{P}\left\{\mathrm{Cap}_\rho\left([\boldsymbol{B}]_2 \cap A\right) > 0\right\} > 0 \quad \Leftrightarrow \quad \mathrm{Cap}_{2(d-2N)+\rho}(A) > 0. \tag{5.28}$$

Frostman's theorem [12, p. 521] then implies the following:

$$\left\|\dim_{\mathrm{H}}\left([\boldsymbol{B}]_2 \cap A\right)\right\|_{L^\infty(\mathrm{P})} = \dim_{\mathrm{H}} A - 2(d-2N). \tag{5.29}$$

This and (1.11) together imply readily the announced formula for the P-essential supremum of $\dim_{\rm H}(M_2 \cap A)$.

The remaining case is when $d = 2N$. In that case, we define for all measurable functions $\kappa : \mathbf{R}_+ \to \mathbf{R}_+ \cup \{\infty\}$,

$$\mathrm{Cap}_\kappa(A) := \left[\inf \mathrm{I}_\kappa(\mu)\right]^{-1}, \tag{5.30}$$

where the infimum is taken over all compactly supported probability measures μ on A, and

$$\mathrm{I}_\kappa(\mu) := \iint \kappa(\|x-y\|)\,\mu(\mathrm{d}x)\,\mu(dy). \tag{5.31}$$

Then the preceding argument goes through, except we replace:

- $\mathrm{Cap}_{2(d-2N)}(\boldsymbol{X}_\rho \cap A)$ by $\mathrm{Cap}_f(\boldsymbol{X}_\rho \cap A)$ in (5.25), where $f(u) := |\log_+(1/u)|^2$;

- $\mathrm{Cap}_{2(d-2N)+\rho}(A)$ by $\mathrm{Cap}_g(A)$ in (5.26) and (5.28), where $g(u) := |u|^\rho f(u)$;

- $2(d-2N)$ by zero on the right-hand side of (5.29).

The justification for these replacements is the same as for their analogous assertions in the case $d > 2N$. This completes our proof. □

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