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Hitting probabilities for systems of non-linear stochastic heat equations with additive noise

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Abstract. We consider a system of d coupled non-linear stochastic heat equations in spatial dimension 1 driven by d-dimensional additive space-time white noise. We establish upper and lower bounds on hitting probabilities of the solution $\{u(t\,,x)\}_{t\in\mathbb{R}_+,\, x\in[0\,,1]}$, in terms of respectively Hausdorff measure and Newtonian capacity. We determine the Hausdorff dimensions of level sets and their projections. We also present an anisotropic form of the Kolmogorov continuity theorem.

1. Introduction

Let $\dot W := (\dot W^1\,, \ldots, \dot W^d)$ be a vector of d independent space-time white noises on $[0\,,T]\times[0\,,1]$. For all $1\le i\le d$, let $b_i : \mathbb{R}^d \to \mathbb{R}$ be globally Lipschitz and bounded functions, and $\sigma := (\sigma_{i,j})$ be a deterministic $d\times d$ invertible matrix (ellipticity).

Consider the system of stochastic partial differential equations (*s.p.d.e.*'s)

$$\frac{\partial u_i}{\partial t}(t\,,x) = \frac{\partial^2 u_i}{\partial x^2}(t\,,x) + \sum_{j=1}^{d} \sigma_{i,j} \dot W^j(t\,,x) + b_i(u(t\,,x)), \tag{1.1}$$

for $1\le i\le d$, $t\in[0\,,T]$, and $x\in[0\,,1]$, where $u := (u_1\,,\ldots,u_d)$, with initial conditions $u(0\,,x)=0$ for all $x\in[0\,,1]$, and Neumann boundary conditions

$$\frac{\partial u_i}{\partial x}(t\,,0) = \frac{\partial u_i}{\partial x}(t\,,1) = 0, \qquad 0\le t\le T. \tag{1.2}$$

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Equation (1.1) is formal, but can be interpreted rigorously as follows (Walsh (1986)): Let $W^i = (W^i(s\,,x))_{s\in\mathbb{R}_+,\,x\in[0\,,1]}$, $i = 1,\dots,d$, be independent Brownian sheets, defined on a probability space $(\Omega\,,\mathscr{F},\mathrm{P})$, and set $W = (W^1,\dots,W^d)$. For $t \in [0,T]$, let $\mathscr{F}_t = \sigma\{W(s\,,x),\ s \in [0\,,t],\ x \in [0\,,1]\}$. We say that a process $u = \{u(t\,,x),\ t \in [0\,,T],\ x \in [0\,,1]\}$ is *adapted* to $(\mathscr{F}_t)$ if $u(t\,,x)$ is $\mathscr{F}_t$-measurable for each $(t\,,x) \in [0\,,T] \times [0\,,1]$. We say that $u = (u_1\,,\dots,u_d)$ *is a solution of* (1.1) if u is adapted to $(\mathscr{F}_t)$ and if for $i \in \{1\,,\dots,d\}$, $t \in [0\,,T]$, and $x \in [0\,,1]$,

$$\begin{aligned}u_i(t\,,x) = \int_0^t \int_0^1 G_{t-r}(x\,,v) \sum_{j=1}^d \sigma_{i,j}\, W^j(drdv)& \\ + \int_0^t \int_0^1 G_{t-r}(x\,,v)\, b_i(u(r\,,v))\, dr\, dv.&\end{aligned} \tag{1.3}$$

Here, $G_t(x\,,y)$ denotes the Green kernel for the heat equation with Neumann boundary conditions. See, for example, Walsh (1986) or Bally et al. (1995).

Our goal is to develop aspects of potential theory for the solution to the system of stochastic heat equations (1.1). In particular, given $A \subset \mathbb{R}^d$, we want to determine whether the process $\{u(t\,,x),\ t \geq 0,\ x \in [0\,,1]\}$ visits, or hits, A with positive probability.

Potential theory for single-parameter processes is a mature subject. See, for example Blumenthal and Getoor (1968), Port and Stone (1978), and Doob (1984). There is also a growing literature on the potential theory for multiparameter processes (Khoshnevisan (2002)).

For the linear form of (1.1) ($b \equiv 0, \sigma \equiv I_d$, where I_d denotes the $d \times d$ identity matrix), results on hitting probabilites have been obtained in Mueller and Tribe (2002). In the case $d = 1$, for a particular form of (1.1) with additive noise ($\sigma \equiv I_d$, $b(u) = u^{-\delta}$ for $\delta > 3$ and $b(u) = cu^{-3}$), the issue of whether or not the solution hits 0 has been discussed in Zambotti (2002, 2003) and Dalang et al. (2006).

For *non-linear* s.p.d.e.'s, a general result was obtained in Dalang and Nualart (2004), valid for systems of reduced hyperbolic equations on $\mathbb{R}^2_+$ (essentially equivalent to systems of wave equations in spatial dimension 1) that are driven by two-parameter white noise. In this paper, we will be concerned with obtaining upper and lower bounds on hitting probabilities for the solution of the system (1.1). In a forthcoming paper Dalang et al. (2007), we use quite different techniques from the Malliavin calculus, consider systems of non-linear heat equations with multiplicative noise, and obtain bounds that are slightly different than those in this paper.

Let $\{v(r)\}_{r\in T}$ denote a random field that takes values in $\mathbb{R}^d$, where T is some Borel-measurable subset of $\mathbb{R}^N$. Let $v(T)$ denote the range of T under the random map $r \mapsto v(r)$. We say that a Borel set $A \subseteq \mathbb{R}^d$ is called *polar* for v if $\mathrm{P}\{v(T)\cap A \neq \varnothing\} = 0$; otherwise, A is called *nonpolar*. Two of our main results are the following. They will be proved in Section 5.

Theorem 1.1. *Let u denote the solution to* (1.1) *on* $]0\,,T] \times [0\,,1]$.

(a) *A (nonrandom) Borel set $A \subset \mathbb{R}^d$ is nonpolar for $(t\,,x) \mapsto u(t\,,x)$ if it has positive $(d-6)$-dimensional capacity. On the other hand, if A has zero $(d-6)$-dimensional Hausdorff measure, then A is polar for $(t,x) \mapsto u(t\,,x)$.*

(b) *Fix* $t \in]0, T]$. *A Borel set* $A \subseteq \mathbb{R}^d$ *is nonpolar for* $x \mapsto u(t, x)$ *if* A *has positive* $(d-2)$*-dimensional capacity. If, on the other hand,* A *has zero* $(d-2)$*-dimensional Hausdorff measure, then* A *is polar for* $x \mapsto u(t, x)$.

(c) *Fix* $x \in [0, 1]$. *A Borel set* $A \subseteq \mathbb{R}^d$ *is nonpolar for* $t \mapsto u(t, x)$ *if* A *has positive* $(d-4)$*-dimensional capacity. If, on the other hand,* A *has zero* $(d-4)$*-dimensional Hausdorff measure, then* A *is polar for* $t \mapsto u(t, x)$.

The definitions of capacity and Hausdorff measures will be recalled shortly.

There is a small gap between the conditions of positive capacity and positive Hausdorff measure. In some cases, we know how to bridge that gap. Indeed, the results of Mueller and Tribe (2002) will make this possible in parts (a) and (b) of the following. This reference does not however apply to statement (c).

Corollary 1.2. *Let* u *denote the solution to* (1.1).

(a) *Singletons are polar for* $(t, x) \mapsto u(t, x)$ *if and only if* $d \geq 6$.
(b) *Fix* $t \in]0, T]$. *Singletons are polar for* $x \mapsto u(t, x)$ *if and only if* $d \geq 2$.
(c) *Fix* $x \in [0, 1]$. *Singletons are polar for* $t \mapsto u(t, x)$ *if* $d > 4$ *and are nonpolar when* $d < 4$. *The case* $d = 4$ *is open.*

This corollary is proved in Section 5.

Our work has other, "more geometric," consequences as well. For example, we mention the following.

Corollary 1.3. *Let* u *denote the solution of* (1.1).

(a) *If* $d \geq 6$, *then* $\dim_{\rm H}(u(]0, T] \times]0, 1[)) = 6$ *a.s.*
(b) *Fix* $t \in]0, T]$. *If* $d \geq 2$, *then* $\dim_{\rm H}(u(\{t\} \times]0, 1[)) = 2$ *a.s.*
(c) *Fix* $x \in]0, 1[$. *If* $d \geq 4$, *then* $\dim_{\rm H}(u(\mathbb{R}_+ \times \{x\})) = 4$ *a.s.*

Consequently, when $d \geq 6$, the Hausdorff dimension of the range of the solution to (1.1) is 6 a.s. (Corollary 1.3). On the other hand, when $d < 6$, the range of the solution to (1.1) has full Lebesgue measure a.s. (Corollary 1.2).

This paper is organized as follows. In Section 2 we present general conditions on an $\mathbb{R}^d$-valued random field $(v(t, x))$ that imply *lower* bounds on hitting probabilities (Theorem 2.1). These conditions are stated in terms of a lower bound on the one-point density function of the random vectors $v(t, x)$ and an upper bound on the two-point density function; that is, the density function of $(v(t, x), v(s, y))$ for $(t, x) \neq (s, y)$ (see conditions **A1** and **A2**). These conditions also yield information about level sets of the process and their projections (Theorem 2.4). They are related to, but not identical with, the conditions of Dalang and Nualart (2004).

In Section 3, we isolate properties of the random field that imply *upper* bounds on hitting probabilities (Theorem 3.1), and corresponding properties of level sets and their projections (Theorem 3.2). These conditions are implied by sufficient conditions that are often not too difficult to check, namely that the one-point density function of the random variables $v(t,x)$ is uniformly bounded above and an estimate on L^p-moments of increments of the random field (Theorem 3.3), similar to the condition in the classical Kolmogorov continuity theorem. These conditions are different from those of Dalang and Nualart (2004) which made specific use of the structure of the filtration of the solution to a hyperbolic s.p.d.e. in $\mathbb{R}^2_+$, and, in particular, of Cairoli's maximal inequality for 2-parameter martingales; there is no counterpart to these for the stochastic heat equation.

In Section 4, we verify the conditions of Sections 2 and 3 for the solution of the linear form of (1.1), that is, with $b \equiv 0$ (see Theorem 4.6). In order to obtain the best estimates possible, a careful analysis of moments of increments and of the determinant of the variance/covariance matrix of the (in this case, Gaussian) process $(u(t\,,x))$ is needed. This also requires a version of the Kolmogorov continuity theorem that is tailored to the needs of the stochastic heat equation. This is presented in Appendix A, and may be of independent interest.

Finally, in Section 5, we use Girsanov's theorem to transfer results about hitting probabilities of the solution to the linear form of (1.1) to the general form of (1.1) (Proposition 5.2), and we prove Theorem 1.1 and Corollaries 1.2 and 1.3. Some results on capacity and energy are gathered in Appendix B.

Let us conclude this Introduction by defining the requisite notation and terminology. For all Borel sets $F \subseteq \mathbb{R}^d$ we define $\mathscr{P}(F)$ to be the set of all probability measures with compact support in F. For all integers $k \geq 1$ and $\mu \in \mathscr{P}(\mathbb{R}^k)$, we let $I_\beta(\mu)$ denote the *β-dimensional energy* of μ; that is,

$$I_\beta(\mu) := \iint \mathrm{K}_\beta(\|x-y\|)\,\mu(dx)\,\mu(dy), \tag{1.4}$$

where $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^k$. Here and throughout,

$$\mathrm{K}_\beta(r) := \begin{cases} r^{-\beta} & \text{if } \beta > 0, \\ \log(N_0/r) & \text{if } \beta = 0, \\ 1 & \text{if } \beta < 0, \end{cases} \tag{1.5}$$

where N_0 is a constant whose value will be specified later in the proof of Lemma 2.2.

If $f : \mathbb{R}^d \mapsto \mathbb{R}_+$ is a probability density function, then we will write $I_\beta(f)$ for the β-dimensional energy of the measure $f(x)dx$.

For all $\beta \in \mathbb{R}$, integers $k \geq 1$, and Borel sets $F \subset \mathbb{R}^k$, $\mathrm{Cap}_\beta(F)$ denotes the *β-dimensional capacity of F*; that is,

$$\mathrm{Cap}_\beta(F) := \left[\inf_{\mu \in \mathscr{P}(F)} I_\beta(\mu)\right]^{-1}, \tag{1.6}$$

where $1/\infty := 0$.

Given $\beta \geq 0$, the β-dimensional *Hausdorff measure* of F is defined by

$$\mathscr{H}_\beta(F) = \lim_{\epsilon \to 0^+} \inf \left\{ \sum_{i=1}^\infty (2r_i)^\beta : F \subseteq \bigcup_{i=1}^\infty B(x_i\,, r_i),\ \sup_{i \geq 1} r_i \leq \epsilon \right\}, \tag{1.7}$$

where $B(x\,,r)$ denotes the open (Euclidean) ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. When $\beta < 0$, we define $\mathscr{H}_\beta(F)$ to be infinite.

Throughout, we consider the following *parabolic metric*: For all $s, t \in [0\,,T]$ and $x, y \in [0\,,1]$,

$$\mathbf{\Delta}((t\,,x)\,;(s\,,y)) := |t-s|^{1/2} + |x-y|. \tag{1.8}$$

Clearly, this is a metric on $\mathbb{R}^2$ which generates the usual Euclidean topology on $\mathbb{R}^2$. We associate to this metric the energy form

$$I_\beta^{\mathbf{\Delta}}(\mu) := \iint \mathrm{K}_\beta(\mathbf{\Delta}((t,x)\,;(s,y)))\,\mu(dt\,dx)\,\mu(ds\,dy), \tag{1.9}$$

and its corresponding capacity

$$\mathrm{Cap}_\beta^{\mathbf{\Delta}}(F) := \left[\inf_{\mu\in\mathscr{P}(F)} I_\beta^{\mathbf{\Delta}}(\mu)\right]^{-1}. \tag{1.10}$$

For the Hausdorff measure, we write

$$\mathscr{H}_s^{\mathbf{\Delta}}(F) = \lim_{\epsilon\to 0^+} \inf\left\{\sum_{i=1}^\infty (2r_i)^s : F \subseteq \bigcup_{i=1}^\infty B_{\mathbf{\Delta}}((t_i\,,x_i)\,,r_i),\ \sup_{i\geq 1} r_i \leq \epsilon\right\}, \tag{1.11}$$

where $B_{\mathbf{\Delta}}((t\,,x)\,,r)$ denotes the open $\mathbf{\Delta}$-ball of radius $r>0$ centered at $(t\,,x)\in [0\,,T]\times[0\,,1]$.

2. Lower Bounds on Hitting Probabilities

Fix two compact intervals I and J of $\mathbb{R}$. Suppose that $\{v(t\,,x)\}_{(t,x)\in I\times J}$ is a two-parameter, *continuous* random field with values in $\mathbb{R}^d$, such that $(v(t\,,x)\,,v(s\,,y))$ has a joint probability density function $p_{t,x;s,y}(\cdot\,,\cdot)$, for all $s,t\in I$ and $x,y\in J$ such that $(t\,,x)\neq(s\,,y)$. That is,

$$\mathrm{E}\left[f\left(v(t\,,x)\,,v(s,y)\right)\right] = \iint f(a\,,b)\,p_{t,x;s,y}(a\,,b)\,da\,db, \tag{2.1}$$

for all bounded Borel-measurable functions $f: I\times J\to\mathbb{R}$. We will denote the marginal density function of $v(t\,,x)$ by $p_{t,x}$.

Consider the following hypotheses:

A1.: For all $M>0$, there exists a positive and finite constant $C=C(I,J,M,d)$ such that for all $(t\,,x)\in I\times J$ and all $z\in[-M\,,M]^d$,

$$p_{t,x}(z)\geq C. \tag{2.2}$$

A2.: There exists $\beta>0$ such that for all $M>0$, there exists a constant $c=c(I,J,\beta,M,d)>0$ such that for all $s,t\in I$ and $x,y\in J$ with $(t\,,x)\neq(s\,,y)$, and for every $z_1,z_2\in[-M\,,M]^d$,

$$p_{t,x;s,y}(z_1\,,z_2)\leq \frac{c}{\left[\mathbf{\Delta}((t\,,x)\,;(s\,,y))\right]^{\beta/2}}\exp\left(-\frac{\|z_1-z_2\|^2}{c\mathbf{\Delta}((t\,,x)\,;(s\,,y))}\right). \tag{2.3}$$

Our next theorem discusses lower bounds for various hitting probabilities of the random field v.

Theorem 2.1. *Suppose* **A1** *and* **A2** *are met. Fix $M>0$.*

(1) *There exists a positive and finite constant $a=a(I,J,\beta,M,d)$ such that for all compact sets $A\subseteq[-M\,,M]^d$,*

$$\mathrm{P}\left\{v(I\times J)\cap A\neq\varnothing\right\}\geq a\mathrm{Cap}_{\beta-6}(A). \tag{2.4}$$

(2) *There exists a positive and finite constant $a=a(J,M\,\beta,d)$ such that for all $t\in I$ and for all compact sets $A\subseteq[-M\,,M]^d$,*

$$\mathrm{P}\left\{v(\{t\}\times J)\cap A\neq\varnothing\right\}\geq a\mathrm{Cap}_{\beta-2}(A). \tag{2.5}$$

(3) *There exists a positive and finite constant $a=a(I,M,\beta,d)$ such that for all $x\in J$ and for all compact sets $A\subseteq[-M\,,M]^d$,*

$$\mathrm{P}\left\{v(I\times\{x\})\cap A\neq\varnothing\right\}\geq a\mathrm{Cap}_{\beta-4}(A). \tag{2.6}$$

Before proving this theorem, we need two technical lemmas.

Lemma 2.2. *Fix $N > 0$ and $\beta > 0$.*

(1) *There exists a finite and positive constant $C_1 = C_1(I, J, \beta, N)$ such that for all $a \in [-N\,, N]$,*

$$\int_I dt \int_I ds \int_J dx \int_J dy \; \frac{e^{-a^2/\mathbf{\Delta}((t,x);(s,y))}}{\mathbf{\Delta}^{\beta/2}((t,x);(s,y))} \leq C_1\, \mathrm{K}_{\beta-6}(a). \tag{2.7}$$

(2) *Fix $\alpha > 0$. There exists a finite and positive constant $C_2 = C_2(I, \beta, N)$ such that for all $a \in [-N\,, N]$,*

$$\int_I dt \int_I ds \; \frac{e^{-a^2/|t-s|^{\alpha}}}{|t-s|^{\alpha\beta/2}} \leq C_2\, \mathrm{K}_{\beta-(2/\alpha)}(a). \tag{2.8}$$

Proof. We start by proving (1). Using the change of variables $\tilde{u} = t - s$ (t fixed), $\tilde{v} = x - y$ (x fixed), we have

$$\begin{aligned} &\int_I dt \int_I ds \int_J dx \int_J dy \; \frac{e^{-a^2/\mathbf{\Delta}((t,x);(s,y))}}{\mathbf{\Delta}^{\beta/2}((t,x);(s,y))} \\ &\qquad \leq 4|I|\,|J| \int_0^{|I|} d\tilde{u} \int_0^{|J|} d\tilde{v}\, (\tilde{u}^{1/2} + \tilde{v})^{-\beta/2} \exp\left(-\frac{a^2}{\tilde{u}^{1/2}+\tilde{v}}\right). \end{aligned} \tag{2.9}$$

A change of variables $[\tilde{u} = a^4 u^2,\ \tilde{v} = a^2 v]$ implies that this is equal to

$$C a^{6-\beta} \int_0^r du \int_0^m dv\, \frac{u}{(u+v)^{\beta/2}} \exp\left(-\frac{1}{u+v}\right), \tag{2.10}$$

where $r := \sqrt{|I|}/a^2$ and $m := |J|/a^2$. Notice that $r \geq r_1 := \sqrt{|I|}/N^2 > 0$ and $m \geq m_1 := |J|/N^2 > 0$.

Observe that

$$\begin{aligned} &\int_0^r du \int_0^m dv\, \frac{u}{(u+v)^{\beta/2}} \exp\left(-\frac{1}{u+v}\right) \\ &\qquad \leq \int_0^r du \int_0^m dv\, (u+v)^{1-\frac{\beta}{2}} \exp\left(-\frac{1}{u+v}\right). \end{aligned} \tag{2.11}$$

Pass to polar coordinates to deduce that the preceding is bounded above by $I_1 + I_2(r\,, m)$, where

$$\begin{aligned} I_1 &:= \int_0^{\sqrt{r_1^2+m_1^2}} d\rho\, \rho^{2-(\beta/2)} \exp(-c/\rho), \\ I_2(r\,, m) &:= \int_{\sqrt{r_1^2+m_1^2}}^{\sqrt{r^2+m^2}} d\rho\, \rho^{2-(\beta/2)}. \end{aligned} \tag{2.12}$$

Clearly, $I_1 \leq C < \infty$, and if $\beta \neq 6$, then

$$I_2(r\,, m) = \frac{\left(\sqrt{r^2+m^2}\right)^{3-(\beta/2)} - \left(\sqrt{r_1^2+m_1^2}\right)^{3-(\beta/2)}}{3-(\beta/2)}. \tag{2.13}$$

There are three separate cases to consider: (i) If $\beta > 6$, then $3 - (\beta/2) < 0$, and hence $I_2(r\,, m) \leq C$ for all $r \geq r_1$ and $m \geq m_1$. (ii) If $\beta < 6$, then $I_2(r\,, m) \leq$

$c(\sqrt{r^2+m^2})^{3-(\beta/2)} = Ca^{\beta-6}$ for all $r \geq r_1$ and $m \geq m_1$. (iii) Finally, if $\beta = 6$, then

$$\begin{aligned} I_2(r\,,m) &\leq C\left[\ln\left(\sqrt{r^2+m^2}\right) - \ln(r_1)\right] \\ &= c\left[\ln\left(\frac{|I|+|J|^2}{r_1}\right) + 2\ln\left(\frac{1}{a}\right)\right]. \end{aligned} \tag{2.14}$$

We combine these observations to deduce that for all $\beta > 0$ there exists $C > 0$ such that for all $a \in [-N\,,N]$, the expression in (2.10) is bounded above by

$$Ca^{6-\beta}(I_1 + I_2(r\,,m)) \leq c\,\mathrm{K}_{\beta-6}(a), \tag{2.15}$$

provided that N_0 in (1.5) is sufficiently large. This proves (1).

Next we prove (2). Fix t and change variables $[u = t-s]$ to see that

$$\int_I dt \int_I ds\ \frac{e^{-a^2/|t-s|^\alpha}}{|t-s|^{\alpha\beta/2}} \leq 2|I| \int_0^{|I|} du\, u^{-\alpha\beta/2} e^{-a^2/u^\alpha}. \tag{2.16}$$

Another change of variables $[u = a^{2/\alpha}v]$ simplifies this expression to

$$Ca^{(2/\alpha)-\beta}\int_0^r dv\, v^{-\alpha\beta/2}e^{-1/v^\alpha}, \tag{2.17}$$

where $r := |I|\,a^{-2/\alpha}$. Notice that $r \geq r_1 := |I|\,N^{-2/\alpha} > 0$.

Observe that

$$\int_0^r dv\, v^{-\alpha\beta/2}e^{-1/v^\alpha} \leq I_1 + I_2(r), \tag{2.18}$$

where

$$I_1 := \int_0^{r_1} dv\, v^{-\alpha\beta/2}e^{-1/v^\alpha}, \qquad I_2(r) := \int_{r_1}^r dv\, v^{-\alpha\beta/2}. \tag{2.19}$$

Clearly, $I_1 \leq C < \infty$. Moreover, if $\alpha\beta \neq 2$ then

$$I_2(r) = \frac{r^{1-(\alpha\beta/2)} - r_1^{1-(\alpha\beta/2)}}{1-(\alpha\beta/2)}. \tag{2.20}$$

As above, we consider three different cases: (i) If $\alpha\beta > 2$, then $1-(\alpha\beta/2) < 0$, and hence $I_2(r) \leq C$ for all $r \geq r_1$. (ii) If $\alpha\beta < 2$, then $I_2(r) \leq Ca^{-(2/\alpha)+\beta}$ for all $r \geq r_1$. (iii) If $\alpha\beta = 2$, then

$$I_2(r) = \left[\ln\left(\frac{|I|}{r_1}\right) + \frac{2}{\alpha}\ln\left(\frac{1}{a}\right)\right]. \tag{2.21}$$

We combine these observations to deduce that for all $\beta > 0$ and $\alpha > 0$, there exists $C > 0$ such that for all $a \in [-N\,,N]$, the expression in (2.17) is bounded above by

$$Ca^{(2/\alpha)-\beta}(I_1 + I_2(r)) \leq c\,\mathrm{K}_{\beta-(2/\alpha)}(a), \tag{2.22}$$

provided that N_0 in (1.5) is sufficiently large. This proves (2) and completes the proof of the lemma. □

For all $a, \nu, \rho > 0$, define

$$\Psi_{a,\nu}(\rho) := \int_0^a \frac{dx}{\rho + x^\nu}. \tag{2.23}$$

Lemma 2.3. *For all $a, \nu, T > 0$, there exists a finite and positive constant $C = C(a\,,\nu\,,T)$ such that for all $0 < \rho < T$,*

$$\Psi_{a,\nu}(\rho) \leq C\mathrm{K}_{(\nu-1)/\nu}(\rho). \tag{2.24}$$

Proof. If $\nu < 1$, then $\lim_{\rho\to 0} \Psi_{a,\nu}(\rho) = \int_0^a x^{-\nu}\,dx < \infty$. In addition, $\rho \mapsto \Psi_{a,\nu}(\rho)$ is nonincreasing, so $\Psi_{a,\nu}$ is bounded on $\mathbb{R}_+$ when $\nu < 1$. In this case, $\mathrm{K}_{(\nu-1)/\nu}(\rho) = 1$, whence follows the result in the case that $\nu < 1$.

For the case $\nu \geq 1$, we change variables ($y = x\rho^{-1/\nu}$) to find that

$$\Psi_{a,\nu}(\rho) = \rho^{-(\nu-1)/\nu} \int_0^{a\rho^{-1/\nu}} \frac{dy}{1+y^\nu}. \tag{2.25}$$

When $\nu > 1$, this gives the desired result, with $c = \int_0^{+\infty} dy\,(1+y^\nu)^{-1}$. When $\nu = 1$, we simply evaluate the integral in (2.23) explicitly: this gives the result for $0 < \rho < T$, given the choice of $\mathrm{K}_0(r)$ in (1.5). We note that the constraint "$0 < \rho < T$" is needed only in this case. □

On several occasions we use the following classical fact, which we recite for the convenience of the reader Khoshnevisan (2002, Lemma 1.4.1, Chap. 3).

The Paley–Zygmund inequality. *If Z is a nonnegative random variable such that $0 < \mathrm{E}(Z^2) < \infty$, then*

$$\mathrm{P}\{Z > 0\} \geq \frac{(\mathrm{E}[Z])^2}{\mathrm{E}[Z^2]}. \tag{2.26}$$

The proof of this inequality involves only a direct application of the Cauchy–Schwarz inequality. Indeed, $|\mathrm{E}[Z]|^2 = |\mathrm{E}[Z\,1_{\{Z>0\}}]|^2 \leq \mathrm{E}[Z^2]\mathrm{P}\{Z > 0\}$.

Proof of Theorem 2.1. We begin by proving (1). Let $A \subset [-M, M]^d$ be a compact set. Without loss of generality, we assume that $\mathrm{Cap}_{\beta-6}(A) > 0$; otherwise there is nothing to prove. By Taylor's theorem (Khoshnevisan (2002, Appendix C, Corollary 2.3.1, p. 525)) this implies that $\beta - 6 < d$ and $A \neq \varnothing$.

There are separate cases to consider:

Case 1: $\beta - 6 < 0$. Then $\mathrm{Cap}_{\beta-6}(A) = 1$. Hence it suffices to prove that there exists a finite and positive constant a (that does not depend on A) such that

$$\mathrm{P}\left\{v(I \times J) \cap A \neq \varnothing\right\} \geq a. \tag{2.27}$$

Define, for all $z \in \mathbb{R}^d$ and $\epsilon > 0$, $\tilde{B}(z\,,\epsilon) := \{y \in \mathbb{R}^d :\ |y - z| < \epsilon\}$, where $|z| := \max_{1\leq j\leq d} |z_j|$, and

$$J_\epsilon(z) = \frac{1}{(2\epsilon)^d} \int_I dt \int_J dx\, \mathbf{1}_{\tilde{B}(z,\epsilon)}(v(t\,,x)). \tag{2.28}$$

Fix $z \in A \subseteq [-M\,, M]^d$. Hypothesis **A1** implies that for all $\epsilon > 0$,

$$\begin{aligned}\mathrm{E}\left[J_\epsilon(z)\right] &= \frac{1}{(2\epsilon)^d} \int_I dt \int_J dx \int_{\tilde{B}(z,\epsilon)} da\, p_{t,x}(a) \\ &\geq C|I|\,|J|,\end{aligned} \tag{2.29}$$

where $C > 0$ does not depend on z.

On the other hand, **A2** implies that

$$\begin{aligned}&\mathrm{E}\left[(J_\epsilon(z))^2\right] \\ &= \frac{1}{(2\epsilon)^{2d}} \int_I dt \int_J dx \int_I ds \int_J dy \int_{\tilde{B}(z,\epsilon)} dz_1 \int_{\tilde{B}(z,\epsilon)} dz_2\ p_{t,x;s,y}(z_1, z_2) \\ &\leq c \int_I dt \int_J dx \int_I ds \int_J dy\, \frac{1}{[\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^{\beta/2}}.\end{aligned} \tag{2.30}$$

The change of variables $u = t - s$ (t fixed), $v = x - y$ (x fixed), implies that the preceding is bounded above by

$$C \int_0^{|I|} du \int_0^{|J|} dv \, (u^{1/2} + v)^{-\beta/2} \le C' \int_0^{|I|} du \, \Psi_{|J|,\beta/2}(u^{\beta/4}). \tag{2.31}$$

Therefore, Lemma 2.3 implies that for all $\epsilon > 0$,

$$\mathrm{E}\left[(J_\epsilon(z))^2 \right] \le C \int_0^{|I|} du \, \mathrm{K}_{1-(2/\beta)}(u^{\beta/4}). \tag{2.32}$$

In order to bound the preceding integral, consider three different cases: (i) If $0 < \beta < 2$, then $1 - 2/\beta < 0$ and the integral equals $|I|$. (ii) If $2 < \beta < 6$, then $\mathrm{K}_{1-(2/\beta)}(u^{\beta/4}) = u^{(1/2)-(\beta/4)}$ and the integral is finite. (iii) If $\beta = 2$, then $\mathrm{K}_0(u^{\beta/4}) = \log(N_0/u^{1/2})$ and the integral is also finite. This fact, (2.29), and the Paley–Zygmund inequality together imply that

$$\mathrm{P}\left\{ J_\epsilon(z) > 0 \right\} \ge C > 0. \tag{2.33}$$

The left-hand side is bounded above by $\mathrm{P}\{v(I \times J) \cap A^{(\epsilon)} \neq \varnothing\}$, where $A^{(\epsilon)}$ denotes the closed ϵ-enlargement of A. Let $\epsilon \downarrow 0$ and appeal to the continuity of the trajectories of v to find that

$$\mathrm{P}\left\{ v(I \times J) \cap A \neq \varnothing \right\} \ge C > 0. \tag{2.34}$$

This proves (2.27).

Case 2: $0 < \beta - 6 < d$. Define, for all $\mu \in \mathscr{P}(A)$ and $\epsilon > 0$,

$$J_\epsilon(\mu) = \frac{1}{(2\epsilon)^d} \int_{\mathbb{R}^d} \mu(dz) \int_I dt \int_J dx \, \mathbf{1}_{\tilde{B}(z,\epsilon)}(v(t\,, x)). \tag{2.35}$$

Fix $\mu \in \mathscr{P}(A)$ such that

$$I_{\beta-6}(\mu) \le \frac{2}{\mathrm{Cap}_{\beta-6}(A)}. \tag{2.36}$$

Note that **A1** implies, as in (2.29), the existence of a positive and finite constant C_1 —that does not depend on μ— such that for all $\epsilon > 0$,

$$\mathrm{E}\left[J_\epsilon(\mu) \right] \ge C_1. \tag{2.37}$$

Next, we will estimate the second moment of $J_\epsilon(\mu)$. Let

$$g_\epsilon(z) := \frac{1}{(2\epsilon)^d} \mathbf{1}_{\tilde{B}(0,\epsilon)}(z). \tag{2.38}$$

Because

$$J_\epsilon(\mu) = \int_I dt \int_J dx \, (g_\epsilon * \mu)(v(t\,, x)), \tag{2.39}$$

Lemma 2.2(1) and **A2** together imply that there exists a finite and positive constant C_2 such that for all $\epsilon > 0$,

$$\begin{aligned} \mathrm{E}\left[(J_\epsilon(\mu))^2 \right] &= \int_I dt \int_J dx \int_I ds \int_J dy \int_{\tilde{B}(z,\epsilon)} dz_1 \int_{\tilde{B}(z,\epsilon)} dz_2 \\ &\qquad \times p_{t,x;s,y}(z_1, z_2)\, (g_\epsilon * \mu)(z_1)\, (g_\epsilon * \mu)(z_2) \\ &\le C_2 I_{\beta-6}(g_\epsilon * \mu). \end{aligned} \tag{2.40}$$

By appealing to Theorem B.1 in Appendix B, we see that for all $\epsilon > 0$,

$$
\begin{aligned}
\mathrm{E}\left[(J_\epsilon(\mu))^2\right] &\le C_2 I_{\beta-6}(\mu) \\
&\le \frac{2C_2}{\mathrm{Cap}_{\beta-6}(A)},
\end{aligned} \tag{2.41}
$$

by (2.36). The preceding, (2.37), and the Paley–Zygmund inequality together imply that

$$
\mathrm{P}\left\{J_\epsilon(\mu) > 0\right\} \ge \frac{C_1^2}{2C_2}\,\mathrm{Cap}_{\beta-6}(A). \tag{2.42}
$$

The left-hand side is bounded above by $\mathrm{P}\{v(I \times J) \cap A^{(\epsilon)} \neq \varnothing\}$, where $A^{(\epsilon)}$ denotes the closed ϵ-enlargement of A. Let $\epsilon \downarrow 0$ and appeal to the continuity of the trajectories of v to find that for all $\mu \in \mathscr{P}(A)$,

$$
\mathrm{P}\left\{v(I \times J) \cap A \neq \varnothing\right\} \ge \frac{C_1^2}{2C_2}\,\mathrm{Cap}_{\beta-6}(A). \tag{2.43}
$$

Case 3: $\beta - 6 = 0$. We proceed as we did in Case 2, but use (2.40) with $\beta = 6$ and Theorem B.2 in the Appendix to obtain that for all $\epsilon > 0$,

$$
\begin{aligned}
\mathrm{E}\left[(J_\epsilon(\mu))^2\right] &\le C_2 I_0(g_\epsilon * \mu) \\
&\le c I_0(\mu) \\
&\le \frac{c}{\mathrm{Cap}_0(A)}.
\end{aligned} \tag{2.44}
$$

This proves part (1) of the theorem.

We prove (2) similarly. Without loss of generality we assume that $\mathrm{Cap}_{\beta-2}(A) > 0$. This implies that $\beta - 2 < d$ and $A \neq \varnothing$. Again, we need to consider three different cases.

Case (i): $\beta - 2 < 0$. We proceed as we did in Case 1, but instead of $J_\epsilon(z)$, we consider

$$
\hat{J}_{\epsilon,t}(z) := \frac{1}{(2\epsilon)^d} \int_J dx\, \mathbf{1}_{\tilde{B}(z,\epsilon)}(v(t\,,x)), \tag{2.45}
$$

for $t \in I$ fixed. We then use **A1** in order to obtain

$$
\mathrm{E}\left[\hat{J}_{\epsilon,t}(z)\right] \ge C\,|J| > 0. \tag{2.46}
$$

Note that, in this case, the constant C depends on t only through I. We use **A2** to bound the second moment of $\hat{J}_{\epsilon,t}(z)$, that is,

$$
\begin{aligned}
\mathrm{E}\left[(\hat{J}_{\epsilon,t}(z))^2\right] &= \frac{1}{(2\epsilon)^{2d}} \int_J dx \int_J dy \int_{\tilde{B}(z,\epsilon)} dz_1 \int_{\tilde{B}(z,\epsilon)} dz_2\, p_{t,x;s,y}(z_1, z_2) \\
&\le C \int_0^{|J|} dv\, v^{-\beta/2},
\end{aligned} \tag{2.47}
$$

which is finite because $0 < \beta < 2$. The rest of the proof follows exactly as in Case 1.

Case (ii): $0 < \beta - 2 < d$. We choose $\mu \in \mathscr{P}(A)$ such that $I_{\beta-2}(\mu) \leq 2/\mathrm{Cap}_{\beta-2}(A)$. We proceed as we did in Case 2, but instead of $J_\epsilon(\mu)$, we consider

$$\hat{J}_{\epsilon,t}(\mu) := \frac{1}{(2\epsilon)^d} \int_{\mathbb{R}^d} \mu(dz) \int_J dx\, \mathbf{1}_{\tilde{B}(z,\epsilon)}(v(t\,,x)), \tag{2.48}$$

for $t \in I$ fixed. We then use **A1** in order to obtain

$$\mathrm{E}[\hat{J}_{\epsilon,t}(\mu)] \geq C_1 > 0. \tag{2.49}$$

Finally, **A2** and Lemma 2.2(2) with $\alpha = 1$ and I replaced by J together imply that there exists a finite and positive constant C such that for all $\epsilon > 0$,

$$\mathrm{E}\left[\left(\hat{J}_{\epsilon,t}(\mu)\right)^2\right] \leq C I_{\beta-2}(g_\epsilon * \mu). \tag{2.50}$$

The remainder of the proof of (ii) follows exactly as we did for Case 2.

Case (iii): $\beta = 2$. We proceed as in (ii) and Case 3. This proves part (2) of the theorem.

We prove (3) by applying the same argument, but instead of $J_\epsilon(\mu)$ and/or $\hat{J}_{\epsilon,t}(\mu)$, consider

$$\bar{J}_{\epsilon,x}(\mu) := \frac{1}{(2\epsilon)^d} \int_{\mathbb{R}^d} \mu(dz) \int_I dt\, \mathbf{1}_{\tilde{B}(z,\epsilon)}(v(t\,,x)), \tag{2.51}$$

for $x \in J$ fixed, and use **A1**, **A2** and Lemma 2.2(2) with $\alpha = 1/2$ to conclude. □

Theorem 2.1 is a result about hitting probabilities of the random sets that are obtained by considering various images of v. Next, we describe similar results for other, related, random sets. Define

$$\begin{aligned}
\mathscr{L}(z\,;v) &:= \{(t\,,x) \in I \times J:\ v(t\,,x) = z\}\,,\\
\mathscr{T}(z\,;v) &= \{t \in I:\ v(t\,,x) = z \text{ for some } x \in J\}\,,\\
\mathscr{X}(z\,;v) &= \{x \in J:\ v(t\,,x) = z \text{ for some } t \in I\}\,,\\
\mathscr{L}_x(z\,;v) &:= \{t \in I:\ v(t\,,x) = z\}\,,\\
\mathscr{L}^t(z\,;v) &:= \{x \in J:\ v(t\,,x) = z\}\,.
\end{aligned} \tag{2.52}$$

We note that $\mathscr{L}(z\,;v)$ is the level set of v at level z, $\mathscr{T}(z\,;v)$ (resp. $\mathscr{X}(z\,;v)$) is the projection of $\mathscr{L}(z\,;v)$ onto I (resp. J), and $\mathscr{L}_x(z\,;v)$ (resp. $\mathscr{L}^t(z\,;v)$) is the x-section (resp. t-section) of $\mathscr{L}(z\,;v)$.

Theorem 2.4. *Assume that* **A1** *and* **A2** *are met. Then, for all $R > 0$, there exists a positive and finite constant $a = a(I, J, \beta, R, d)$ such that the following holds for all compact sets $E \subseteq I \times J$, $F \subseteq I$, and $G \subseteq J$, and for all $z \in B(0\,,R)$:*

(1) $\mathrm{P}\{\mathscr{L}(z\,;v) \cap E \neq \varnothing\} \geq a\mathrm{Cap}^{\mathbf{\Delta}}_{\beta/2}(E)$;
(2) $\mathrm{P}\{\mathscr{T}(z\,;v) \cap F \neq \varnothing\} \geq a\mathrm{Cap}_{(\beta-2)/4}(F)$;
(3) $\mathrm{P}\{\mathscr{X}(z\,;v) \cap G \neq \varnothing\} \geq a\mathrm{Cap}_{(\beta-4)/2}(G)$;
(4) *for all $x \in J$,* $\mathrm{P}\{\mathscr{L}_x(z\,;v) \cap F \neq \varnothing\} \geq a\mathrm{Cap}_{\beta/4}(F)$;
(5) *for all $t \in I$,* $\mathrm{P}\{\mathscr{L}^t(z\,;v) \cap G \neq \varnothing\} \geq a\mathrm{Cap}_{\beta/2}(G)$.

Proof. We begin by proving (1). Without loss of generality we assume that $\mathrm{Cap}^{\boldsymbol{\Delta}}_{\beta/2}(E) > 0$. Choose $\mu \in \mathscr{P}(E)$ such that $I^{\boldsymbol{\Delta}}_{\beta/2}(\mu) \leq 2/\mathrm{Cap}^{\boldsymbol{\Delta}}_{\beta/2}(E)$. For all $\delta > 0$, define

$$Z_\delta(\mu) := \frac{1}{(2\delta)^d} \int_E \mu(dt\, dx)\, \mathbf{1}_{\tilde{B}(z,\delta)}(v(t\,,x)). \tag{2.53}$$

Then, in accord with **A1**, there exists a finite and positive constant C_1 such that for all $\mu \in \mathscr{P}(E)$ and $\delta > 0$,

$$\mathrm{E}[Z_\delta(\mu)] \geq C_1. \tag{2.54}$$

On the other hand, **A2** guarantees the existence of a finite and positive constant C_2 such that for all $\mu \in \mathscr{P}(E)$ and $\delta > 0$,

$$\begin{aligned} &\mathrm{E}\left[(Z_\delta(\mu))^2\right] \\ &= \frac{1}{(2\delta)^{2d}} \int_E \mu(dt\, dx) \int_E \mu(ds\, dy) \int_{\tilde{B}(z\,,\delta)} dz_1 \int_{\tilde{B}(z\,,\delta)} dz_2\; p_{t,x;s,y}(z_1\,, z_2) \\ &\leq C_2 \int_E \int_E \frac{\mu(dt\, dx)\, \mu(ds\, dy)}{\left[\boldsymbol{\Delta}\left((t\,,x)\,;(s\,,y)\right)\right]^{\beta/2}} \\ &\leq \frac{2C_2}{\mathrm{Cap}^{\boldsymbol{\Delta}}_{\beta/2}(E)}. \end{aligned} \tag{2.55}$$

Equations (2.54) and (2.55), together with the Paley–Zygmund inequality, imply that

$$\mathrm{P}\left\{Z_\delta(\mu) > 0\right\} \geq \frac{C_1^2}{2C_2}\, \mathrm{Cap}^{\boldsymbol{\Delta}}_{\beta/2}(E). \tag{2.56}$$

The left-hand side is clearly bounded above by

$$\mathrm{P}\left(\bigcup_{z_1 \in B(z,\delta)} (\mathscr{L}(z_1\,; v) \cap E) \neq \varnothing \right). \tag{2.57}$$

Let $\delta \downarrow 0$ to finish the proof of (1).

In order to prove (2), define, for all $\mu \in \mathscr{P}(F)$, $\delta > 0$ and $z \in B(0, R)$,

$$Z_\delta(\mu) = \frac{1}{(2\delta)^d} \int_F \mu(dt) \int_J dx\, \mathbf{1}_{\tilde{B}(z,\delta)}(v(t\,,x)). \tag{2.58}$$

By **A1**, we can find a constant C — depending only on $(I\,,J\,,R\,,d)$ — such that

$$\inf_{\delta>0}\; \inf_{\mu\in\mathscr{P}(F)} \mathrm{E}\left[Z_\delta(\mu)\right] \geq C. \tag{2.59}$$

On the other hand, let g_δ be as defined in (2.38) with ϵ replaced by δ. By **A2**, there exists $\tilde{C}$—depending only on $(I\,,J\,,\beta\,,R\,,d)$—such that for all $\delta > 0$ and $\mu \in \mathscr{P}(F)$,

$$\begin{aligned} &\mathrm{E}\left[(Z_\delta(\mu))^2\right] \\ &= \int_F \mu(dt) \int_J dx \int_F \mu(ds) \int_J dy \int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 \\ &\qquad\qquad \times g_\delta(z_1 - z) g_\delta(z_2 - z)\, p_{t,x;s,y}(z_1\,, z_2). \end{aligned} \tag{2.60}$$

Since

$$\int_J dx \int_J dy\, \frac{1}{\left[\boldsymbol{\Delta}\left((t\,,x)\,;(s\,,y)\right)\right]^{\beta/2}} \leq 2|J|\Psi_{|J|,\beta/2}\left(|t-s|^{\beta/4}\right), \tag{2.61}$$

where $\Psi_{a,\nu}(\rho)$ is defined in (2.23), we see that

$$\begin{aligned}&\mathrm{E}\left[(Z_\delta(\mu))^2\right]\\&\le C\int_F\mu(dt)\int_F\mu(ds)\int_{\mathbb{R}}dz_1\int_{\mathbb{R}}dz_2\\&\qquad\times g_\delta(z_1-z)g_\delta(z_2-z)\,\Psi_{|J|,\beta/2}(|t-s|^{\beta/4}).\end{aligned}\tag{2.62}$$

Since the two dz_i-integrals are equal to 1, Lemma 2.3 implies that there exists a constant $\bar{C}$ such that for all $\mu\in\mathscr{P}(F)$ and $\delta>0$,

$$\begin{aligned}\mathrm{E}\left[(Z_\delta(\mu))^2\right]&\le\bar{C}\int_F\mu(dt)\int_F\mu(ds)\,\mathrm{K}_{1-(2/\beta)}(|t-s|^{\beta/4})\\&=\bar{C}I_{(\beta-2)/4}(\mu).\end{aligned}\tag{2.63}$$

An application of the Paley–Zygmund inequality implies statement (2) of the theorem.

In order to prove (3), we consider instead $\mu\in\mathscr{P}(G)$ and

$$\bar{Z}_\delta(\mu)=\frac{1}{(2\delta)^d}\int_G\mu(dx)\int_I dt\,\mathbf{1}_{\tilde{B}(z,\delta)}(v(t\,,x)).\tag{2.64}$$

Thanks to **A1**, $\mathrm{E}[\bar{Z}_\delta(\mu)]$ is bounded below, uniformly for all $\delta>0$ and $\mu\in\mathscr{P}(G)$. Also, as above, **A2** implies that there exists a positive and finite constant C such that $\mathrm{E}[(\bar{Z}_\delta(\mu))^2]\le CI_{(\beta-4)/2}(\mu)$ for all $\delta>0$ and $\mu\in\mathscr{P}(G)$. Indeed, this is a consequence of Lemma 2.3 and the fact that

$$\int_I dt\int_I ds\,\frac{1}{[\mathbf{\Delta}\left((t\,,x)\,;(s\,,y)\right)]^{\beta/2}}\le 2|I|\Psi_{|I|,\beta/4}\left(|x-y|^{\beta/2}\right).\tag{2.65}$$

Therefore, statement (3) now follows from the two moment bounds and the Paley–Zygmund inequality.

For (4), we consider instead $z\in B(0,R)$, $x\in J$, $\mu\in\mathscr{P}(F)$ and set

$$Z'_\delta(\mu)=\frac{1}{(2\delta)^d}\int_F\mu(dt)\,\mathbf{1}_{\tilde{B}(z,\delta)}(v(t\,,x)).\tag{2.66}$$

As was the case in (1), (2), and (3), $\mathrm{E}[Z'_\delta(\mu)]$ is bounded below, uniformly for all $\delta>0$, $\mu\in\mathscr{P}(F)$ and $x\in J$. In addition, there exists a positive and finite constant C such that

$$\begin{aligned}&\mathrm{E}[(Z'_\delta(\mu))^2]\\&\le\int_F\mu(dt)\int_F\mu(ds)\int_{\mathbb{R}}dz_1\int_{\mathbb{R}}dz_2\,g_\delta(z_1-z)g_\delta(z_2-z)\,p_{t,x;s,x}(z_1\,,z_2).\end{aligned}\tag{2.67}$$

Since $p_{t,x;s,x}(z_1\,,z_2)\le|t-s|^{-\beta/4}$, and the two dz_i-integrals are equal to 1, we see that

$$\mathrm{E}[(Z'_\delta(\mu))^2]\le CI_{\beta/4}(\mu),\tag{2.68}$$

for all $\delta>0$, $\mu\in\mathscr{P}(F)$ and $x\in J$. Therefore, statement (4) follows from the two moment bounds and the Paley–Zygmund inequality.

Finally, in order to prove (5), we consider instead $\mu\in\mathscr{P}(G)$ and

$$Z''_\delta(\mu)=\frac{1}{(2\delta)^d}\int_G\mathbf{1}_{\tilde{B}(z,\delta)}(v(t\,,x))\,\mu(dx).\tag{2.69}$$

Once again by **A1**, $\mathrm{E}[Z''_\delta(\mu)]$ is bounded below, uniformly for all $\delta > 0$ and $\mu \in \mathscr{P}(F)$. And by **A2**, $\mathrm{E}[(Z''_\delta(\mu))^2] \le CI_{\beta/2}(\mu)$, where $C \in]0\,,\infty[$ does not depend on $(\delta\,,\mu)$. From the two moment bounds, (5) follows, whence the theorem. □

Remark 2.5. (a) Hypothesis **A1** is convenient since, together with **A2**, it leads to all the conclusions of Theorem 2.1 and 2.4. If one is only interested in certain of these conclusions, then weaker assumptions than **A1** are possible, analogous to Hypothesis H1 of Dalang and Nualart (2004). For instance, Theorem 2.1(1) can be obtained if **A1** is replaced by:

A1'. For all $M > 0$, there exists a positive and finite constant $C = C(I, J, M, d)$ such that for all $z \in [-M\,, M]^d$,

$$\int_I dt \int_J dx\, p_{t,x}(z) \ge C. \tag{2.70}$$

Indeed, this assumption would be used to get the lower bound in (2.29) and (2.37).

In the same way, Theorem 2.1(2) can be obtained if **A1** is replaced by:

A1^t. For all $M > 0$, there exists a positive and finite constant $C = C(t, J, M, d)$ such that for all $z \in [-M\,, M]^d$,

$$\int_J dx\, p_{t,x}(z) \ge C. \tag{2.71}$$

Similar considerations apply to Theorem 2.1(3), which can be obtained if **A1** is replaced by:

A1$_x$. For all $M > 0$, there exists a positive and finite constant $C = C(x, I, M, d)$ such that for all $z \in [-M\,, M]^d$,

$$\int_I dt\, p_{t,x}(z) \ge C. \tag{2.72}$$

(b) It is also possible to weaken Hypothesis **A2**. For instance, Theorems 2.1(2) and 2.4(5) can be proved if **A2** is replaced by:

A2^t. There exists $\beta > 0$ such that for all $M > 0$, there exists $c = c(t, I, J, \beta, M, d) > 0$ such that for all $x, y \in J$ with $x \neq y$, and for every $z_1, z_2 \in [-M\,, M]^d$,

$$p_{t,x;t,y}(z_1\,, z_2) \le \frac{c}{|x-y|^{\beta/2}} \exp\left(-\frac{\|z_1 - z_2\|^2}{c|x-y|}\right). \tag{2.73}$$

Similar considerations also apply to Theorem 2.1(3).

3. Upper Bounds on Hitting Probabilities

The results of this section complement those of the preceding by establishing upper bounds for various hitting probabilities.

Consider two compact nonrandom intervals $I \subset [0\,, T]$ and $J \subset [0\,, 1]$, and suppose $v = \{v(t\,, x)\}_{(t,x)\in I\times J}$ is an $\mathbb{R}^d$-valued random field. For all positive integers n, set $t^n_k := k2^{-4n}$, $x^n_\ell := \ell 2^{-2n}$, and

$$I^n_k = [t^n_k, t^n_{k+1}], \qquad J^n_\ell = [x^n_\ell, x^n_{\ell+1}], \qquad R^n_{k,\ell} = I^n_k \times J^n_\ell. \tag{3.1}$$

Theorem 3.1. *Fix $\beta > 0$ and $M > 0$. Suppose that there exists $c > 0$ such that for all $z \in [-M, M]^d$, $\epsilon > 0$, large n and $R^n_{k,\ell} \subseteq I \times J$,*

$$\mathrm{P}\{v(R^n_{k,\ell}) \cap B(z\,,\epsilon) \neq \varnothing\} \leq c\,\epsilon^\beta. \tag{3.2}$$

Then there exists a positive and finite constant a such that for all Borel sets $A \subset [-M, M]^d$:

(1) *$\mathrm{P}\{v(I \times J) \cap A \neq \varnothing\} \leq a\mathscr{H}_{\beta-6}(A)$;*
(2) *for every $t \in I$, $\mathrm{P}\{v(\{t\} \times J) \cap A \neq \varnothing\} \leq a\mathscr{H}_{\beta-2}(A)$;*
(3) *for every $x \in J$, $\mathrm{P}\{v(I \times \{x\}) \cap A \neq \varnothing\} \leq a\mathscr{H}_{\beta-4}(A)$.*

Proof. We begin by proving (1). When $\beta - 6 < 0$, there is nothing to prove, so we assume that $\beta - 6 \geq 0$. Fix $\epsilon \in]0\,,1[$ and $n \in \mathbb{N}$ such that $2^{-n-1} < \epsilon \leq 2^{-n}$, and write

$$\mathrm{P}\left\{v\left(I \times J\right) \cap B(z\,,\epsilon) \neq \varnothing\right\} \leq \sum_{\substack{(k,\ell):\\ R^n_{k,\ell}\cap(I\times J)\neq\varnothing}}\sum \mathrm{P}\{v(R^n_{k,\ell}) \cap B(z\,,\epsilon) \neq \varnothing\}. \tag{3.3}$$

The number of pairs $(k\,,\ell)$ involved in the two sums is at most 2^{6n}. Because $\epsilon \leq 2^{-n}$, the condition (3.2) implies that for all large n and all $z \in A$,

$$\begin{aligned}\mathrm{P}\left\{v\left(I \times J\right) \cap B(z\,,\epsilon) \neq \varnothing\right\} &\leq \tilde{C}2^{-n(\beta-6)}\\ &\leq C\epsilon^{\beta-6}.\end{aligned} \tag{3.4}$$

Note that C does not depend on $(n\,,\epsilon)$. Therefore, (3.4) is valid for all $\epsilon \in]0\,,1[$.

Now we use a *covering argument*: Choose $\epsilon \in]0\,,1[$ and let $\{B_i\}_{i=1}^\infty$ be a sequence of open balls in $\mathbb{R}^d$ with respective radii $r_i \in]0\,,\epsilon]$ such that

$$A \subseteq \bigcup_{i=1}^\infty B_i \quad \text{and} \quad \sum_{i=1}^\infty (2r_i)^{\beta-6} \leq \mathscr{H}_{\beta-6}(A) + \epsilon. \tag{3.5}$$

Because $\mathrm{P}\{v(I \times J) \cap A \neq \varnothing\}$ is at most $\sum_{i=1}^\infty \mathrm{P}\{v(I \times J) \cap B_i \neq \varnothing\}$, (3.4) and (3.5) together imply that

$$\begin{aligned}\mathrm{P}\left\{v\left(I \times J\right) \cap A \neq \varnothing\right\} &\leq C\sum_{i=1}^\infty r_i^{\beta-6}\\ &\leq \bar{C}(\mathscr{H}_{\beta-6}(A) + \epsilon).\end{aligned} \tag{3.6}$$

Let $\epsilon \to 0^+$ to deduce (1).

In order to prove (2), we can assume that $\beta - 2 \geq 0$ and we fix $\epsilon \in]0\,,1[$. We can find integers n and k such that $2^{-n-1} < \epsilon \leq 2^{-n}$ and $t \in I^n_k$. Then, by (3.2),

$$\begin{aligned}\mathrm{P}\left\{v\left(\{t\} \times J\right) \cap B(z\,,\epsilon) \neq \varnothing\right\} &\leq \sum_{\ell: J^n_\ell \cap J \neq \varnothing} \mathrm{P}\{v(I^n_k \times J^n_\ell) \cap B(z\,,\epsilon) \neq \varnothing\}\\ &\leq C2^{-n\beta}2^{2n}\\ &\leq \tilde{C}\epsilon^{\beta-2}.\end{aligned} \tag{3.7}$$

Now use a covering argument, as we did to prove (1), in order to verify (2).

The proof of (3) follows along similar lines, and is left to the reader. □

Theorem 3.2. *Fix $\beta > 0$ and $M > 0$. If the assumptions of Theorem 3.1 are met, then there exists $a \in]0\,,\infty[$ such that the following holds for all $z \in [-M, M]^d$ and all compact sets $E \subseteq I \times J$, $F \subseteq I$, and $G \subseteq J$:*

(1) $\mathrm{P}\{\mathscr{L}(z\,;v)\cap E\neq\varnothing\}\le a\mathscr{H}^{\boldsymbol{\Delta}}_{\beta/2}(E)$;
(2) $\mathrm{P}\{\mathscr{T}(z\,;v)\cap F\neq\varnothing\}\le a\mathscr{H}_{(\beta-2)/4}(F)$;
(3) $\mathrm{P}\{\mathscr{X}(z\,;v)\cap G\neq\varnothing\}\le a\mathscr{H}_{(\beta-4)/2}(G)$;
(4) *for all* $x\in J$, $\mathrm{P}\{\mathscr{L}_x(z\,;v)\cap F\neq\varnothing\}\le a\mathscr{H}_{\beta/4}(F)$;
(5) *for all* $t\in I$, $\mathrm{P}\{\mathscr{L}^t(z\,;v)\cap G\neq\varnothing\}\le a\mathscr{H}_{\beta/2}(G)$.

Proof. Let $z\in[-M,M]^d$. Fix $r\in]0\,,1[$, $t_0\in I$ and $x_0\in J$. We can find integers n, ℓ and k such that $2^{-2n-2}<r\le 2^{-2n-1}$, $t_0\in I^n_k$, $x_0\in J^n_\ell$. Then condition (3.2) implies that for n large,

$$\mathrm{P}\left\{\inf_{\substack{t_0\le t\le t_0+r^{1/2}\\ x_0\le x\le x_0+r}}|v(t\,,x)-z|\le r^{1/2}\right\}\le\sum_{i=k}^{k+1}\sum_{j=\ell}^{\ell+1}\mathrm{P}\{v(R^n_{i,j})\cap B(z\,,r^{1/2})\neq\varnothing\}\\ \le Cr^{\beta/2}. \tag{3.8}$$

Note that C does not depend on $(n\,,r\,,t_0\,,x_0)$.

Now we use a *covering argument*: Choose $r\in]0\,,1[$ and let $\{E_i\}_{i=1}^\infty$ denote a sequence of open $\boldsymbol{\Delta}$-balls in $I\times J$ with respective radii $r_i\in]0\,,r]$ such that

$$E\subseteq\bigcup_{i=1}^\infty E_i\quad\text{and}\quad\sum_{i=1}^\infty(2r_i)^{\beta/2}\le\mathscr{H}^{\boldsymbol{\Delta}}_{\beta/2}(E)+r. \tag{3.9}$$

Then

$$\begin{aligned}\mathrm{P}\left\{\mathscr{L}(z\,;v)\cap E\neq\varnothing\right\}&=\mathrm{P}\left\{\inf_{(t,x)\in E}|v(t\,,x)-z|=0\right\}\\ &\le\sum_{i=1}^\infty\mathrm{P}\left\{\inf_{(t,x)\in E_i}|v(t\,,x)-z|\le r_i^{1/2}\right\}\\ &\le C\sum_{i=1}^\infty r_i^{\beta/2}\\ &\le\tilde C(\mathscr{H}^{\boldsymbol{\Delta}}_{\beta/2}(E)+r).\end{aligned} \tag{3.10}$$

Let $r\to 0^+$ to deduce (1).

To prove (2), fix $r\in]0\,,1[$ and $t_0\in I$. There exist integers n and k such that $2^{-4n-2}<r\le 2^{-4n-1}$ and $t_0\in I^n_k$. Condition (3.2) implies that for n large,

$$\begin{aligned}&\mathrm{P}\left\{\inf_{t_0\le t\le t_0+r}\ \inf_{x\in J}|v(t\,,x)-z|\le r^{1/4}\right\}\\ &\qquad\le\sum_{i=k}^{k+1}\ \sum_{\ell:J^n_\ell\cap J\neq\varnothing}\mathrm{P}\{v(R^n_{i,\ell})\cap B(z\,,r^{1/4})\neq\varnothing\}\\ &\qquad\le\tilde C2^{-n\beta}2^{2n}\\ &\qquad\le Cr^{(\beta-2)/4},\end{aligned} \tag{3.11}$$

since $r>2^{-4n-2}$. Note that C does not depend on $(n\,,r\,,t_0)$.

Choose $r\in]0\,,1[$ and let $\{F_i\}_{i=1}^\infty$ denote a sequence of open balls in I with respective radii $r_i\in]0\,,r]$ such that

$$F\subseteq\bigcup_{i=1}^\infty F_i\quad\text{and}\quad\sum_{i=1}^\infty(2r_i)^{(\beta-2)/4}\le\mathscr{H}_{(\beta-2)/4}(F)+r. \tag{3.12}$$

Then

$$\begin{aligned}
\mathrm{P}\left\{\mathscr{T}(z\,;v)\cap F\neq\varnothing\right\} &= \mathrm{P}\left\{\inf_{t\in F}\,\inf_{x\in J}|v(t\,,x)-z|=0\right\}\\
&\le \sum_{i=1}^{\infty}\mathrm{P}\left\{\inf_{t\in F_i}\,\inf_{x\in J}|v(t\,,x)-z|\le r_i^{1/4}\right\}\\
&\le C\sum_{i=1}^{\infty} r_i^{(\beta-2)/4}\\
&\le \tilde{C}(\mathscr{H}_{(\beta-2)/4}(E)+r).
\end{aligned}\tag{3.13}$$

Let $r\to 0^+$ to deduce (2).

The proof of (3) follows along similar lines, and is left to the reader.

We now prove (4). Fix $x\in J$, $r\in]0\,,1[$ and $t_0\in I$. There exist integers n, k and ℓ such that $2^{-4n-2}<r\le 2^{-4n-1}$, $t_0\in I_k^n$ and $x\in J_\ell^n$. Condition (3.2) implies that for n large,

$$\begin{aligned}
\mathrm{P}\left\{\inf_{t_0\le t\le t_0+r}|v(t\,,x)-z|\le r^{1/4}\right\} &\le \sum_{i=k}^{k+1}\mathrm{P}\{v(R_{i,\ell}^n)\cap B(z\,,r^{1/4})\neq\varnothing\}\\
&\le Cr^{\beta/4},
\end{aligned}\tag{3.14}$$

Note that C does not depend on $(n\,,r\,,x,t_0)$.

Choose $r\in]0\,,1[$ and let $\{F_i\}_{i=1}^\infty$ denote a sequence of open balls in I with respective radii $r_i\in]0\,,r]$ such that

$$F\subseteq\bigcup_{i=1}^{\infty}F_i\quad\text{and}\quad\sum_{i=1}^{\infty}(2r_i)^{\beta/4}\le\mathscr{H}_{\beta/4}(F)+r.\tag{3.15}$$

Then

$$\begin{aligned}
\mathrm{P}\left\{\mathscr{L}_x(z\,;v)\cap G\neq\varnothing\right\} &= \mathrm{P}\left\{\inf_{t\in F}|v(t\,,x)-z|=0\right\}\\
&\le \sum_{i=1}^{\infty}\mathrm{P}\left\{\inf_{t\in F_i}|v(t\,,x)-z|\le r_i^{1/4}\right\}\\
&\le C\sum_{i=1}^{\infty} r_i^{\beta/4}\\
&\le \tilde{C}(\mathscr{H}_{\beta/4}(E)+r).
\end{aligned}\tag{3.16}$$

Let $r\to 0^+$ to deduce (4).

The proof of (5) follows along similar lines, and is left to the reader. □

The results of this section all assume Condition (3.2). The following provides a useful sufficient condition for (3.2) to hold. This conditions is used for instance in Dalang et al. (2007).

Theorem 3.3. *Fix $M>0$. Assume that the $\mathbb{R}^d$-valued random field v satisfies the following two conditions:*

(i) *For any $(t\,,x)\in I\times J$, the random vector $v(t\,,x)$ has a density $p_{t,x}(z)$ which is uniformly bounded over $z\in[-M,M]^d$ and $(t\,,x)\in I\times J$.*

(ii) *For all $p > 1$, there exists a constant C depending on p, I, J such that for any $(t\,,x),(s\,,y) \in I \times J$,*

$$\mathrm{E}[|v(t\,,x) - v(s\,,y)|^p] \leq C\,[\boldsymbol{\Delta}\,((t\,,x)\,;(s\,,y))]^{p/2}\,. \tag{3.17}$$

Then for any $\beta \in]0\,,d[$, Condition (3.2) is satisfied and therefore, so are the upper bounds on hitting probabilities in Theorems 3.1 and 3.2 for such β.

Proof. Fix $z \in [-M, M]^d$. For $n \in \mathbb{N}$ and $\varepsilon \in]0\,,1[$, set

$$\begin{aligned} Y^n_{k,\ell} &:= |v(t^n_k\,, x^n_\ell) - z|\,, \\ Z^n_{k,\ell} &:= \sup_{(t,x)\in B_{\boldsymbol{\Delta}}((t^n_k,x^n_\ell),\varepsilon^2)} |v(t\,,x) - v(t^n_k\,,x^n_\ell)|\,. \end{aligned} \tag{3.18}$$

Fix $\beta \in]0\,,d[$. We are going to start by showing that

$$\mathrm{P}\left\{Z^n_{k,\ell} \geq \frac{1}{2}Y^n_{k,\ell}\right\} \leq \tilde{c}\varepsilon^\beta. \tag{3.19}$$

Indeed, observe that

$$\mathrm{P}\left\{Z^n_{k,\ell} \geq \frac{1}{2}Y^n_{k,\ell}\right\} \leq \mathrm{P}\left\{Y^n_{k,\ell} \leq \varepsilon^{\beta/d}\right\} + \mathrm{P}\left\{Z^n_{k,\ell} \geq \frac{1}{2}\varepsilon^{\beta/d}\right\}. \tag{3.20}$$

By hypothesis (i), the first term on the right-hand side is bounded by $c\varepsilon^\beta$. By Markov's inequality,

$$\mathrm{P}\left\{Z^n_{k,\ell} \geq \frac{1}{2}\varepsilon^{\beta/d}\right\} \leq \left(\frac{1}{2}\varepsilon^{\beta/d}\right)^{-p}\mathrm{E}[|Z^n_{k,\ell}|^p]. \tag{3.21}$$

Let $p > 6$ and $q = \frac{p}{2} - 3$. Then $q > 0$ and $\frac{q}{p} = \frac{1}{2} - \frac{3}{p} > 0$. Since $\frac{\beta}{2d} < \frac{1}{2}$, we can choose p large enough that $\frac{1}{2} - \frac{3}{p} > \frac{\beta}{2d}$.

Fix $\alpha \in]\frac{\beta}{2d}\,, \frac{q}{p}[$. By hypothesis (ii) and Corollary A.3,

$$\mathrm{E}\left(|Z^n_{k,\ell}|^p\right) \leq (\varepsilon^2)^{\alpha p}, \tag{3.22}$$

and hence,

$$\begin{aligned} \mathrm{P}\left\{Z^n_{k,\ell} \geq \frac{1}{2}Y^n_{k,\ell}\right\} &\leq c\,\varepsilon^\beta + c\,\varepsilon^{2\alpha p - \beta p/d} && (3.23)\\ &\leq c\,\varepsilon^\beta(1 + c\varepsilon^{p(2\alpha-\beta/d)-\beta}). && (3.24) \end{aligned}$$

Since $2\alpha - \beta/d > 0$, it follows that $p(2\alpha - \beta/d) - \beta > 0$ for all sufficiently large p. This proves (3.19).

Now, let $\varepsilon \in]0, 1[$ and $n \in \mathbb{N}$ be such that $2^{-n-1} < \varepsilon \leq 2^{-n}$. According to (3.19),

$$\begin{aligned} \mathrm{P}\left\{v\left(R^n_{k,\ell}\right) \cap B(z\,,\varepsilon) \neq \varnothing\right\} &\leq \mathrm{P}\left\{Y^n_{k,\ell} \leq 2^{-n} + Z^n_{k,\ell}\right\} \\ &\leq \mathrm{P}\left\{Z^n_{k,\ell} \geq \frac{1}{2}Y^n_{k,\ell}\right\} + \mathrm{P}\left\{Y^n_{k,\ell} \leq 2^{1-n}\right\} \\ &\leq c2^{-n\beta} + c2^{(1-n)d}. \end{aligned} \tag{3.25}$$

Therefore, for all large n and all $z \in [-M, M]^d$,

$$\mathrm{P}\left\{v\left(R^n_{k,l}\right) \cap B(z\,,\varepsilon) \neq \varnothing\right\} \leq C2^{-n\beta} \leq \tilde{C}\varepsilon^\beta, \tag{3.26}$$

since $2^{-n-1} < \varepsilon$. This proves (3.2) and whence the theorem. □

4. The Gaussian case

We consider the s.p.d.e. (1.1) in the drift-free case ($b_i \equiv 0$), and write it in vector notation as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sigma \dot{W}. \tag{4.1}$$

The solution is the d-dimensional Gaussian random field $\{u(t\,,x)\}_{t\in[0,T],x\in[0,1]}$ defined by

$$u(t\,,x) = \int_0^t \int_0^1 G_{t-s}(x\,,y)\,\sigma\,W(dsdy), \qquad 0 \le t \le T,\ 0 \le x \le 1. \tag{4.2}$$

The main objective of this section is to show that for $t_0 > 0$, the conclusions of Theorems 2.1, 2.4, 3.1, and 3.2 are satisfied for $(u(t\,,x))$ with $\beta = d$, $I = [t_0\,,T]$, and $J = [0\,,1]$. We point out that it would be much simpler to establish this for $\beta < d$: see the comment just before Proposition 4.4. We begin with the following.

Proposition 4.1. *Fix $t_0 > 0$. Then the solution to* (4.1) *satisfies **A1** and **A2** with $\beta = d$, $I = [t_0\,,T]$ and $J = [0\,,1]$.*

Proof. It suffices to prove that Hypotheses **A1** and **A2** are satisfied for the random field (4.2). We are going to reduce the problem to the case where σ is the $d \times d$ identity matrix by a change of variables. Because σ is invertible,

$$\frac{\partial(\sigma^{-1}u)}{\partial t} = \frac{\partial^2(\sigma^{-1}u)}{\partial x^2} + \dot{W}.$$

Define $v := \sigma^{-1}u$ to find that v solves the following *uncoupled* system of s.p.d.e.'s:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \dot{W}. \tag{4.3}$$

We will prove that Hypotheses **A1** and **A2** hold for the solution of (4.3). Therefore, they also hold for $u = \sigma v$. Note that $v = (v_1, \ldots, v_d)$, where $v_1, \ldots, v_d$ are i.i.d. real-valued processes.

Verification of **A1**. Fix $I = [t_0\,,T]$, $J = [0\,,1]$ and $M > 0$, and let $z \in [-M\,,M]^d$. Then, for all $(t\,,x) \in I \times J$, the probability density function of $v(t\,,x)$ is given by

$$p_{t,x}(z) = \frac{1}{(2\pi\sigma_{t,x}^2)^{d/2}} \exp\left(-\frac{\|z\|^2}{2\sigma_{t,x}^2}\right), \tag{4.4}$$

where

$$\sigma_{t,x}^2 := \mathrm{Var}\ v_i(t\,,x) = \int_0^t dr \int_0^1 dv\,(G_{t-r}(x\,,v))^2. \tag{4.5}$$

Since $(t\,,x) \mapsto \sigma_{t,x}^2$ is a continuous function, it achieves its minimum $\rho_1 > 0$ and its maximum $\rho_2 < \infty$ over $I \times J$. Thus,

$$p_{t,x}(z) \ge \frac{1}{(2\pi\rho_2)^{d/2}} \exp\left(-\frac{M^2 d}{2\rho_1}\right). \tag{4.6}$$

This proves **A1**.

Verification of **A2**. We follow the proof of Theorem 2.1 of Dalang and Nualart (2004). The joint probability density function $p^i_{t,x;s,y}(\cdot\,,\cdot)$ of $(v_i(t\,,x)\,,v_i(s\,,y))$—for any two distinct space-time points $(t\,,x)$ and $(s\,,y)$—does not depend on i and can be written as

$$p^i_{t,x;s,y}(z_1\,,z_2) = p^i_{t,x|s,y}(z_1 \mid z_2)p^i_{s,y}(z_2), \tag{4.7}$$

where $z_1\,, z_2 \in \mathbb{R}$, $p^i_{t,x|s,y}(\,\cdot \mid z_2)$ denotes the conditional probability density function of $v_i(t\,,x)$ given $v_i(s\,,y) = z_2$ and $p^i_{s,y}(\cdot)$ denotes the marginal density of $v_i(s\,,y)$. By linear regression,

$$p^i_{t,x|s,y}(z_1 \mid z_2) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{|z_1 - mz_2|^2}{2\tau^2}\right), \tag{4.8}$$

where

$$\begin{aligned} \tau^2 &:= \tau^2_{t,x;s,y} = \sigma^2_{t,x}\left(1-\rho^2_{t,x;s,y}\right), \qquad \rho_{t,x;s,y} = \frac{\sigma_{t,x;s,y}}{\sigma_{t,x}\sigma_{s,y}} \\ m &:= m_{t,x;s,y} = \frac{\sigma_{t,x;s,y}}{\sigma^2_{s,y}}, \qquad \sigma_{t,x;s,y} = \mathrm{Cov}\left(v_i(t\,,x)\,, v_i(s\,,y)\right). \end{aligned} \tag{4.9}$$

As in Dalang and Nualart (2004, (3.8)), the triangle inequality and the elementary bound $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ together yield

$$\begin{aligned} p^i_{t,x;s,y}(z_1, z_2) \leq \frac{1}{2\pi\sigma_{s,y}\tau} &\exp\left(-\frac{|z_1 - z_2|^2}{4\tau^2}\right) \\ &\times \exp\left(\frac{|z_2|^2\,|1-m|^2}{2\tau^2}\right)\exp\left(-\frac{|z_2|^2}{2\sigma^2_{s,y}}\right). \end{aligned} \tag{4.10}$$

We will use the technical estimates in the next two lemmas in order to estimate the right-hand side of (4.10).

Lemma 4.2. *Fix $t_0 > 0$. There exist $c_1, c_2 > 0$ such that for all $s, t \in [t_0\,,T]$, $x, y \in [0\,,1]$ and $i = 1, \ldots, d$,*

$$\frac{1}{c_1}\mathbf{\Delta}((t\,,x)\,;(s\,,y)) \leq \mathrm{E}\left[(v_i(t\,,x) - v_i(s\,,y))^2\right] \leq c_1\mathbf{\Delta}((t\,,x)\,;(s\,,y)) \tag{4.11}$$

and

$$|\sigma_{t,x} - \sigma_{s,y}| \leq c_2\left(|t-s|^{1/2} + |x-y|\log\frac{1}{|x-y|}\right). \tag{4.12}$$

Proof. We assume without loss of generality that $s \leq t$. We start by proving the upper bound in (4.11). We note first that

$$\begin{aligned} &\mathrm{E}[(v_i(t\,,x) - v_i(s\,,y))^2] \\ &= \int_s^t dr \int_0^1 dz\, G^2_{t-r}(x\,,z) + \int_0^s dr\int_0^1 dz\,(G_{t-r}(x\,,z) - G_{s-r}(y\,,z))^2. \end{aligned} \tag{4.13}$$

This can be bounded above by

$$\begin{aligned} \int_s^t dr \int_0^1 dz\, G^2_{t-r}(x\,,z) &+ 2\int_0^s\int_0^1 (G_{t-r}(x\,,z) - G_{s-r}(x\,,z))^2\,dr\,dz \\ &+ 2\int_0^s\int_0^1 (G_{s-r}(x\,,z) - G_{s-r}(y\,,z))^2\,dr\,dz. \end{aligned} \tag{4.14}$$

This and Lemma B.1 of Bally et al. (1995) show that there is $C_0 < \infty$ such that

$$\mathrm{E}[(v_i(t\,,x) - v_i(s\,,y))^2] \leq C_0\mathbf{\Delta}((t\,,x)\,;(s\,,y)), \tag{4.15}$$

which is the desired upper bound.

We now turn to the lower bound in (4.11). We consider three different cases.

Case 1: $s = t$. We follow Walsh (1986, p. 323–326) and express the Green kernel for the heat equation with Neumann boundary conditions as

$$G_t(x\,,y) = \sum_{k=0}^{\infty} e^{-\pi^2 k^2 t} \phi_k(x)\phi_k(y), \tag{4.16}$$

where $\phi_0(x) := 1$ and $\phi_k(x) := 2^{1/2}\cos(k\pi x)$ $[k \geq 1]$. Therefore,

$$\begin{aligned} v_i(t\,,x) &= \int_0^t \int_0^1 G_{t-s}(x\,,y)\, W^i(dsdy) \\ &= \sum_{k=0}^{\infty} \phi_k(x)\, A_t^k, \end{aligned} \tag{4.17}$$

where

$$A_t^k := \int_0^t \int_0^1 e^{-\pi^2 k^2 (t-s)} \phi_k(y) W^i(dsdy). \tag{4.18}$$

We note that t is fixed, and$\{A_t^k\}_{k=0}^{\infty}$ are independent centered Gaussian random variables with variance

$$\begin{aligned} \mathrm{Var}(A_t^k) &= \int_0^t ds \int_0^1 dy\, e^{-2\pi^2 k^2 s} \phi_k^2(y) \\ &= \begin{cases} (1 - e^{-2\pi^2 k^2 t})/(2\pi^2 k^2) & \text{if } k \geq 1, \\ t & \text{if } k = 0. \end{cases} \end{aligned} \tag{4.19}$$

In fact, the A^k's are Ornstein–Uhlenbeck processes if $k \geq 1$, and Brownian motion when $k = 0$. Consequently, for fixed t,

$$v_i(t\,,x) = t^{1/2}\xi_t^0 + \sum_{k=1}^{\infty} \phi_k(x) \left(\frac{1 - e^{-2\pi^2 k^2 t}}{2\pi^2 k^2} \right)^{1/2} \xi_t^k, \tag{4.20}$$

where $\{\xi_t^k\}_{k=0}^{\infty}$ is an i.i.d. sequence of standard Gaussian random variables. Now, recall from Walsh (1986, Exercise 3.9, p. 326) that

$$B_x := x\,\xi_t^0 + \sum_{k=1}^{\infty} \frac{1}{k}\, \phi_k(x)\, \xi_t^k \qquad (0 \leq x \leq 1) \tag{4.21}$$

defines a standard Brownian motion indexed by $[0\,,1]$. Consider

$$\begin{aligned} R_x &:= v_i(t\,,x) - \frac{1}{\pi\sqrt{2}} B_x \\ &= \left(\sqrt{t} - \frac{x}{\pi\sqrt{2}} \right) \xi_t^0 + \sum_{k=1}^{\infty} \phi_k(x)\, \xi_t^k\, r_k. \end{aligned} \tag{4.22}$$

where

$$r_k := \frac{(1 - \exp(-2\pi^2 k^2 t))^{1/2} - 1}{2^{1/2}\pi k}. \tag{4.23}$$

Because $|r_k| = O(k^{-1}\exp(-2\pi^2k^2t_0))$ as $k \to \infty$, $x \mapsto R_x$ is differentiable a.s., and

$$\begin{aligned}
\mathrm{E}\left[(R_x - R_y)^2\right] &\leq 2\frac{|x-y|^2}{2\pi^2} + 2\sum_{k=1}^{\infty}(\phi_k(x) - \phi_k(y))^2 r_k^2 \\
&\leq |x-y|^2 + 4\sum_{k=1}^{\infty}\left(\cos(k\pi x) - \cos(k\pi y)\right)^2 r_k^2 \\
&= |x-y|^2 + 4\sum_{k=1}^{\infty}\left[2\sin\left(k\pi\frac{x-y}{2}\right)\sin\left(k\pi\frac{x+y}{2}\right)\right]^2 r_k^2 \\
&\leq |x-y|^2 + 4\sum_{k=1}^{\infty} k^2\pi^2|x-y|^2 r_k^2 \\
&\leq C|x-y|^2,
\end{aligned} \tag{4.24}$$

where C does not depend on $t \in [t_0\,, T]$ nor on $x, y \in [0\,,1]$. It follows that

$$\begin{aligned}
\mathrm{E}\left[(v_i(t\,,x) - v_i(t\,,y))^2\right] &= \mathrm{E}\left[\left(\frac{B_x - B_y}{\sqrt{2}} + R_x - R_y\right)^2\right] \\
&\geq \frac{1}{4}\mathrm{E}[(B_x - B_y)^2] - \mathrm{E}[(R_x - R_y)^2] \\
&\geq \frac{1}{4}\,|x-y| - C|x-y|^2 \\
&\geq c\,|x-y|,
\end{aligned} \tag{4.25}$$

for $|x-y|$ sufficiently small and for all $t \in [t_0\,, T]$.

Observe that

$$\mathrm{E}\left[(v_i(t\,,x) - v_i(t\,,y))^2\right] = \int_0^t dr \int_0^1 dz\,(G_{t-r}(x\,,z) - G_{t-r}(y\,,z))^2 \tag{4.26}$$

is strictly positive, since the integrand is not identically zero. Because this expression is a continuous function of (t, x, y), it is bounded below on $\{(t, x, y) \in [t_0\,, T] \times [0\,,1]^2 : |x-y| \geq \varepsilon\}$ by a positive constant for every fixed $\varepsilon > 0$. We have proved that (4.25) holds for $s = t \in [t_0\,, T]$ and $|x-y|$ sufficiently small. Therefore, (4.25) holds for all $x, y \in [0\,,1]$ and $t \in [t_0\,, T]$ if c is chosen small enough. We conclude for the moment that there is $c > 0$ such that for all $t \in [t_0\,, T]$ and $x, y \in [0\,,1]$,

$$\mathrm{E}\left[(v_i(t\,,x) - v_i(t\,,y))^2\right] \geq c|x-y|. \tag{4.27}$$

Case 2: $|t-s|^{1/2} \geq \frac{c}{4C_0}|x-y|$, where c and C_0 are the constants appearing in (4.27) and (4.15), respectively.

By Morien (1998, Lemma A1.2),

$$\begin{aligned}
\mathrm{E}\left[(v_i(t\,,x) - v_i(s\,,y))^2\right] &\geq \int_s^t \int_0^1 G_{t-r}^2(x\,,y)\,dr\,dy \\
&\geq \tilde{c}|t-s|^{1/2}.
\end{aligned} \tag{4.28}$$

Because of the inequality that defines this Case 2, this is bounded below by

$$\frac{\tilde{c}}{2}|t-s|^{1/2} + \frac{\tilde{c}}{2}\frac{c}{4C_0}|x-y| \geq c'\boldsymbol{\Delta}((t\,,x)\,;(s\,,y)). \tag{4.29}$$

This proves the lower bound in (4.11) in this Case 2.

Case 3: $|t-s|^{1/2} < \frac{c}{4C_0}|x-y|$, where c and C_0 are the constants appearing in (4.27) and (4.15), respectively.

Using (4.27) and (4.15), we observe that

$$\begin{aligned} &\mathrm{E}\left[(v_i(t\,,x)-v_i(s\,,y))^2\right]\\ &\qquad = \mathrm{E}\left[(v_i(t\,,x)-v_i(t\,,y)+v_i(t\,,y)-v_i(s\,,y))^2\right]\\ &\qquad \geq \frac{1}{2}\mathrm{E}\left[(v_i(t\,,x)-v_i(t\,,y))^2\right]-\mathrm{E}\left[(v_i(t\,,y)-v_i(s\,,y))^2\right]\\ &\qquad \geq \frac{1}{2}c|x-y|-C_0|t-s|^{1/2}. \end{aligned} \tag{4.30}$$

Because of the inequality that defines this Case 3, this is bounded below by

$$\begin{aligned} \frac{c}{2}|x-y|-\frac{c}{4}|x-y| &= \frac{c}{4}|x-y|\\ &\geq \frac{c}{8}|x-y|+\frac{c}{8}\frac{4C_0}{c}|t-s|^{1/2}\\ &\geq \min\left(\frac{c}{8},\frac{C_0}{2}\right)\boldsymbol{\Delta}((t\,,x)\,;(s\,,y)). \end{aligned} \tag{4.31}$$

This completes the proof of Case 3 and of the lower bound in (4.11).

Finally we prove (4.12). When $(t\,,x)=(s\,,y)$, there is nothing to prove. Therefore, by the triangle inequality, it suffices to consider the following two cases.

(i) The case where $s=t$ and $x\neq y$. Note that

$$\begin{aligned} |\sigma_{t,x}-\sigma_{t,y}| &= \frac{|\sigma_{t,x}^2-\sigma_{t,y}^2|}{\sigma_{t,x}+\sigma_{t,y}}\\ &\leq c\,|\sigma_{t,x}^2-\sigma_{t,y}^2|, \end{aligned} \tag{4.32}$$

where c does not depend on $t\in[t_0\,,T]$. Also, by (4.17),

$$\begin{aligned} \sigma_{t,x}^2-\sigma_{t,y}^2 &= \sum_{k=0}^{\infty}\phi_k^2(x)\int_0^t ds\, e^{-2\pi^2k^2(t-s)}-\sum_{k=0}^{\infty}\phi_k^2(y)\int_0^t ds\, e^{-2\pi^2k^2(t-s)}\\ &= \sum_{k=1}^{\infty}\left(\phi_k^2(x)-\phi_k^2(y)\right)\int_0^t ds\, e^{-2\pi^2k^2s}. \end{aligned} \tag{4.33}$$

Therefore,

$$\begin{aligned} |\sigma_{t,x}-\sigma_{t,y}| &\leq c\sum_{k=1}^{\infty}\frac{|\phi_k^2(x)-\phi_k^2(y)|}{k^2}\\ &\leq 2c\sum_{k=1}^{\infty}\frac{|\phi_k(x)-\phi_k(y)|}{k^2}. \end{aligned} \tag{4.34}$$

Now

$$\begin{aligned} |\phi_k(x)-\phi_k(y)| &\leq 4\left|\sin\left(k\,\pi\,\frac{x-y}{2}\right)\right|\\ &\leq 4\left(k\pi\,\frac{|x-y|}{2}\wedge 1\right). \end{aligned} \tag{4.35}$$

Consequently, as long as $|x-y|$ is sufficiently small,

$$\begin{aligned}|\sigma_{t,x}-\sigma_{t,y}| &\le 8c\sum_{k=1}^{\infty}\frac{1}{k^2}\left(k\pi\,\frac{|x-y|}{2}\wedge 1\right)\\ &= \tilde{c}\left(\sum_{1\le k\le 2/|x-y|\pi}\frac{|x-y|}{2k}+\sum_{k>2/|x-y|\pi}\frac{1}{k^2}\right)\\ &\le C_1|x-y|\ln\left(\frac{2}{\pi|x-y|}\right)+C_2|x-y|,\end{aligned} \tag{4.36}$$

where C_1 and C_2 do not depend on $t\in[t_0\,,T]$. This proves (4.12) when $s=t$.

(ii) Case where $x=y$ and $s<t$. As in (4.32),

$$|\sigma_{t,x}-\sigma_{s,x}|\le c\,|\sigma^2_{t,x}-\sigma^2_{s,x}|, \tag{4.37}$$

and

$$\begin{aligned}&\sigma^2_{t,x}-\sigma^2_{s,x}\\ &=\int_s^t\int_0^1 G^2_{t-r}(x\,,y)\,dr\,dy+\int_0^s\int_0^1\left(G^2_{s-r}(x\,,y)-G^2_{t-r}(x\,,y)\right)\,dr\,dy.\end{aligned} \tag{4.38}$$

We appeal to Bally et al. (1995, Lemma B.1) to see that the first term is bounded above in absolute value by $c(t-s)^{\frac{1}{2}}$. Using (4.16), we see that the second term is equal to

$$\begin{aligned}&\sum_{k=1}^{\infty}\phi_k^2(x)\left(\int_0^s dr\,e^{-2\pi^2k^2(s-r)}-\int_0^s dr\,e^{-2\pi^2k^2(t-r)}\right)\\ &\qquad=\sum_{k=1}^{\infty}\phi_k^2(x)\left(1-e^{-2\pi^2k^2(t-s)}\right)\int_0^s dr\,e^{-2\pi^2k^2r}.\end{aligned} \tag{4.39}$$

Using the elementary inequality $0\le 1-e^{-x}\le\min(x,1)$, valid for all $x\ge 0$, evaluating the remaining integral and using the fact that $|\phi_k^2(x)|\le 2$, we see that this is bounded above by

$$\begin{aligned}c\sum_{k=1}^{\infty}\frac{\min(\pi^2k^2(t-s),1)}{\pi^2k^2} &\le C\left(\sum_{k=1}^{\pi^{-1}(t-s)^{-1/2}}(t-s)+\sum_{k>\pi^{-1}(t-s)^{-1/2}}\frac{1}{\pi^2k^2}\right)\\ &\le\tilde{C}(t-s)^{1/2}.\end{aligned} \tag{4.40}$$

This completes the proof of (4.12) and of the lemma. □

Lemma 4.3. *Fix $t_0>0$. There exist $c_1,c_2>0$ such that for all $s,t\in[t_0\,,T]$ and $x,y\in[0\,,1]$,*

$$\frac{1}{c_1}\boldsymbol{\Delta}((t\,,x)\,;(s\,,y))\le\sigma^2_{t,x}\sigma^2_{s,y}-\sigma^2_{t,x;s,y}\le c_1\boldsymbol{\Delta}((t\,,x)\,;(s\,,y)), \tag{4.41}$$

$$|\sigma^2_{t,x}-\sigma_{t,x;s,y}|\le c_2\left[\boldsymbol{\Delta}((t\,,x)\,;(s\,,y))\right]^{1/2}. \tag{4.42}$$

Proof. Let $\gamma^2_{t,x;s,y}:=\mathrm{E}[(v_i(t\,,x)-v_i(s\,,y))^2]$. Then using Mueller and Tribe (2002, (4.3)),

$$\sigma^2_{t,x}\sigma^2_{s,y}-\sigma^2_{t,x;s,y}=\frac{1}{4}\left(\gamma^2_{t,x;s,y}-(\sigma_{t,x}-\sigma_{s,y})^2\right)\left((\sigma_{t,x}+\sigma_{s,y})^2-\gamma^2_{t,x;s,y}\right). \tag{4.43}$$

By Lemma 4.2, $\gamma^2_{t,x,s,y} \leq c\mathbf{\Delta}((t\,,x)\,;(s\,,y))$. Therefore, the second factor of (4.43) is bounded below by a positive constant when $s,t \in [t_0\,,T]$ and $(t\,,x)$ is near $(s\,,y)$. Furthermore, another application of Lemma 4.2 yields

$$\begin{aligned}\gamma^2_{t,x,s,y} - (\sigma_{t,x} - \sigma_{s,y})^2 &\geq c\mathbf{\Delta}((t\,,x)\,;(s\,,y)) - \tilde{c}\,[\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^{3/2} \\ &\geq \tilde{c}\mathbf{\Delta}((t\,,x)\,;(s\,,y)).\end{aligned} \tag{4.44}$$

This proves the lower bound of (4.41) provided $(t\,,x)$ is sufficiently near $(s\,,y)$.

In order to extend this inequality to all $(t\,,x)$ and $(s\,,y)$ in $[t_0\,,T] \times [0\,,1]$, it suffices to show that

$$\sigma^2_{t,x}\sigma^2_{s,y} - \sigma^2_{t,x;s,y} > 0 \qquad \text{if } (t\,,x) \neq (s\,,y). \tag{4.45}$$

This could be proved by elementary arguments, but since we are only interested in the conclusion, we use results available in the literature, even if they constitute overkill. Notice that if $s = t$ and $x \neq y$, then this holds because by Bally and Pardoux (1998), the random vector $(v_i(t\,,x), v_i(t\,,y))$ has a density with respect to Lebesgue measure. Since this is a Gaussian random vector, this implies that the determinant of its variance/covariance matrix is non-zero, and this determinant is equal to $\sigma^2_{t,x}\sigma^2_{s,y} - \sigma^2_{t,x;s,y}$.

If $s < t$, and if this determinant were equal to 0, then we would have $|\rho_{t,x;s,y}| = 1$, so there would be $\lambda \in \mathbb{R}$ such that $v_i(t\,,x) = \lambda v_i(s\,,y)$ a.s., and, in particular, we would have

$$\mathrm{E}\left[(v_i(t\,,x) - \lambda v_i(s\,,y))^2\right] = 0. \tag{4.46}$$

However, the left-hand side is equal to

$$\int_s^t \int_0^1 G^2_{t-r}(x\,,z)\,dr\,dz + \int_0^s \int_0^1 (G_{t-r}(x\,,z) - \lambda G_{s-r}(y\,,z))^2 > 0, \tag{4.47}$$

which is a contradiction. Therefore, $\sigma^2_{t,x}\sigma^2_{s,y} - \sigma^2_{t,x;s,y} > 0$ when $s < t$ or $s = t$ and $x \neq y$. This completes the proof of (4.45) and of the lower bound (4.41).

In order to prove the upper bound of (4.41), we use Lemma 4.2, once again, to see that the first factor of (4.43) is bounded above by $c\,\mathbf{\Delta}((t\,,x)\,;(s\,,y))$. Similarly, the second factor is bounded above by a constant. The desired upper bound follows.

It remains to prove (4.42). For this, note that

$$\begin{aligned}|\sigma^2_{t,x} - \sigma_{t,x;s,y}| &= \left|\gamma^2_{t,x;s,y} + \mathrm{Cov}\,(v_i(t\,,x) - v_i(s\,,y)\,, v_i(s\,,y))\right| \\ &\leq \gamma^2_{t,x;s,y} + \gamma_{t,x;s,y}\sigma_{s,y} \\ &\leq c\,[\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^{1/2},\end{aligned} \tag{4.48}$$

where we have used Lemma 4.2 twice in the last inequality. This implies the desired bound. □

By applying Lemmas 4.2 and 4.3 in (4.10), we find, using the independence of the components $v_1, \ldots, v_d$, that for all $z_1, z_2 \in [-M\,,M]^d$,

$$p_{t,x;s,y}(z_1\,,z_2) \leq \frac{c}{\mathbf{\Delta}((t\,,x)\,;(s\,,y))^{d/2}} \exp\left(-\frac{\|z_1 - z_2\|^2}{c\mathbf{\Delta}((t\,,x)\,;(s\,,y))}\right). \tag{4.49}$$

This verifies **A2**, whence follows the proof of Proposition 4.1. □

We now establish an upper bound for hitting small balls. Note that by Lemma 4.2 and the fact that u and v are Gaussian processes, Theorem 3.3 show that (3.2) holds for the solution u of (4.1) and for any $\beta \in]0\,,d[$. The following lemma improves this by establishing (3.2) for $\beta = d$, by using the structure of the Gaussian fields u and v.

Proposition 4.4. *Fix $t_0 > 0$. The solution to (4.1) satisfies (3.2) with $\beta = d$, $I = [t_0\,,T]$ and $J = [0\,,1]$.*

In order to prove Proposition 4.4, we need the following lemma.

Lemma 4.5. *Let $u = (u(t\,,x))$ be as in (4.1). For all $p \geq 1$, there exists $A_p > 0$ such that for all $\epsilon > 0$ and all $(t\,,x)$ fixed,*

$$\mathrm{E}\left[\sup_{[\mathbf{\Delta}((t,x)\,;(s,y))]^{1/2}\leq\epsilon}\|u(t\,,x)-u(s\,,y)\|^p\right]\leq A_p\epsilon^p. \tag{4.50}$$

Proof. It suffices to prove (4.50) for each coordinate u_i, $i = 1,\dots,d$. We plan to apply Proposition A.1 with $S := S_\epsilon = \{(s,y) : [\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^{1/2} < \epsilon\}$, $\rho((t\,,x)\,,(s\,,y)) := [\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^{1/2}$, $\mu(dtdx) := dtdx$, $\Psi(x) := e^{|x|} - 1$, $p(x) := x$, and $f := u_i$. Then, by Lemma 4.2 and the fact that $u = \sigma v$,

$$\mathrm{E}[\mathscr{C}] \leq \mathrm{E}\left[\int_{S_\epsilon} dr\, d\bar{y}\int_{S_\epsilon} ds\, dy\ \exp\left(\frac{|u_i(r\,,\bar{y}) - u_i(s\,,y)|}{\left(|r-s|^{1/2}+|\bar{y}-y|\right)^{1/2}}\right)\right] \leq c_0\epsilon^{12}. \tag{4.51}$$

In accord with Proposition A.1, and by repeated application of Jensen's inequality,

$$\begin{aligned}
&\mathrm{E}\left[\sup_{[\mathbf{\Delta}((t,x)\,;(s,y))]^{1/2}\leq\epsilon}|u_i(t\,,x)-u_i(s\,,y)|^p\right]\\
&\qquad\leq 10^p\mathrm{E}\left[\left(\int_0^{2\epsilon}du\ \ln\left(1+\frac{\mathscr{C}}{[\mu(B_\rho((t\,,x)\,,u/4)]^2}\right)\right)^p\right]\\
&\qquad= 10^p\mathrm{E}\left[\left(\int_0^{2\epsilon}du\ \ln\left(1+\frac{\mathscr{C}}{c_1u^{12}}\right)\right)^p\right]\\
&\qquad\leq 10^p(2\epsilon)^{p-1}\mathrm{E}\left[\int_0^{2\epsilon}du\ \ln^p\left(1+\frac{\mathscr{C}}{c_1u^{12}}\right)\right]\\
&\qquad\leq 10^p(2\epsilon)^{p-1}\int_0^{2\epsilon}du\ \ln^p\left(1+\frac{\mathrm{E}[\mathscr{C}]}{c_1u^{12}}\right)\\
&\qquad\leq 10^p(2\epsilon)^{p-1}\int_0^{2\epsilon}du\ \ln^p\left(1+\frac{c_0}{c_1}\left(\frac{\epsilon}{u}\right)^{12}\right),
\end{aligned} \tag{4.52}$$

and this is manifestly a constant multiple of ϵ^p. □

Proof of Proposition 4.4. Let $u = (u(t\,,x))$ be as in (4.1). Let $R^n_{k,l} := [t^n_k\,,t^n_{k+1}] \times [x^n_\ell\,,x^n_{\ell+1}]$ be as in (3.1). We are going to show that there is $c < \infty$ such that for all $z \in \mathbb{R}^d$ and $\epsilon > 0$,

$$\mathrm{P}\{u(R^n_{k,l}) \cap B(z\,,\epsilon) \neq \varnothing\} \leq c\epsilon^d. \tag{4.53}$$

That is, u satisfies (3.2) with $\beta = d$.

Note that it suffices to prove this with u replaced by v, where v is the solution of (4.3). Without loss of generality, we set $\epsilon := 2^{-n}$. It suffices to prove that there exists $c \in]0\,,\infty[$ such that for all k, ℓ,

$$\mathrm{P}\left\{v(R^n_{k,\ell}) \cap B(z\,,2^{-n}) \neq \varnothing\right\} \leq c\, 2^{-nd}. \tag{4.54}$$

Consider

$$c^n_{k,\ell}(t\,,x) := \frac{\mathrm{E}\left[v_1(t\,,x)v_1(t^n_k\,,x^n_\ell)\right]}{\mathrm{Var}\left[v_1(t^n_k\,,x^n_\ell)\right]}, \tag{4.55}$$

so that

$$\mathrm{E}\left[v(t\,,x)\mid v(t^n_k\,,x^n_\ell)\right] = c^n_{k,\ell}(t\,,x)v(t_k\,,x_\ell). \tag{4.56}$$

Clearly,

$$\begin{aligned}\mathrm{P}\left\{v(R^n_{k,\ell}) \cap B(z\,,2^{-n}) \neq \varnothing\right\} &= \mathrm{P}\left\{\inf_{(t\,,x)\in R^n_{k,\ell}} \|v(t\,,x) - z\| \leq 2^{-n}\right\} \\ &\leq \mathrm{P}\left\{Y^n_{k,\ell} \leq 2^{-n} + Z^n_{k,\ell}\right\},\end{aligned} \tag{4.57}$$

where

$$\begin{aligned}Y^n_{k,\ell} &:= \inf_{(t,x)\in R^n_{k,\ell}} \left\|c^n_{k,\ell}(t\,,x)v(t^n_k\,,x^n_\ell) - z\right\|, \text{ and} \\ Z^n_{k,\ell} &:= \sup_{(t,x)\in R^n_{k,\ell}} \left\|v(t\,,x) - c^n_{k,\ell}(t\,,x)v(t_k\,,x_\ell)\right\|.\end{aligned} \tag{4.58}$$

For $r > 0$,

$$\begin{aligned}\mathrm{P}\{Y^n_{k,\ell} \leq r\} &\leq \mathrm{P}\left(\bigcap_{i=1}^d G^{i,n}_{k,\ell}\right) \\ &= \prod_{i=1}^d \mathrm{P}(G^{i,n}_{k,\ell}),\end{aligned} \tag{4.59}$$

where

$$G^{i,n}_{k,\ell} = \left\{\inf_{(t,x)\in R^n_{k,\ell}} \left|c^n_{k,\ell}(t\,,x)v_i(t^n_k\,,x^n_\ell) - z_i\right| \leq r\right\}. \tag{4.60}$$

The inequality $|c^n_{k,\ell}(t\,,x)v_i(t^n_k\,,x^n_\ell) - z_i| \leq r$ is equivalent to

$$\frac{z_i - r}{c^n_{k,\ell}(t,x)} \leq v_i(t^n_k\,,x^n_\ell) \leq \frac{z_i + r}{c^n_{k,\ell}(t,x)}, \tag{4.61}$$

and the interval $[(z_i - r)/c^n_{k,\ell}(t,x)\,,(z_i + r)/c^n_{k,\ell}(t,x)]$ has length bounded above by $2r/e^n_{k,\ell}$, where

$$e^n_{k,\ell} := \inf_{(t,x)\in R^n_{k,\ell}} c^n_{k,\ell}(t\,,x). \tag{4.62}$$

Therefore,

$$\mathrm{P}(G^{i,n}_{k,\ell}) \leq \sup_{x\in\mathbb{R}} \mathrm{P}\left\{x \leq v_i(t^n_k\,,x^n_\ell) \leq x + \frac{2r}{e^n_{k,\ell}}\right\}. \tag{4.63}$$

Observe that for all $(t\,,x) \in R^n_{k,\ell}$,

$$\begin{aligned}
\left|c^n_{k,\ell}(t\,,x) - 1\right| &= \frac{\left|\mathrm{E}\left[v_1(t^n_k\,,x^n_\ell)\cdot(v_1(t\,,x) - v_1(t^n_k\,,x^n_\ell))\right]\right|}{\mathrm{Var}\left[v_1(t^n_k\,,x^n_\ell)\right]} \\
&\leq \left(\frac{\mathrm{E}\left[(v_1(t\,,x) - v_1(t^n_k\,,x^n_\ell))^2\right]}{\mathrm{Var}\left[v_1(t^n_k\,,x^n_\ell)\right]}\right)^{1/2}.
\end{aligned} \tag{4.64}$$

Lemma 4.2 implies that the numerator is $O(2^{-n})$, whereas the denominator is bounded below by a positive constant. Therefore,

$$\left|c^n_{k,\ell}(t\,,x) - 1\right| \leq \frac{c}{2^n} \qquad \text{for all } (t\,,x) \in R^n_{k,\ell}. \tag{4.65}$$

We emphasize the fact that the constant c does not depend on the choice of (n, k, ℓ). It follows from (4.64) and (4.65) that

$$\frac{r}{e^n_{k,\ell}} \leq c\,r. \tag{4.66}$$

Since $\{v_i(t^n_k\,,x^n_\ell)\}_{i=1,\dots,d}$ are independent, centered, Gaussian random variables with variance bounded below by a positive constant,

$$\mathrm{P}\left\{Y^n_{k,\ell} \leq r\right\} \leq c\,r^d, \tag{4.67}$$

where c does not depend on our choice of (k, ℓ, n, r). Because $Y^n_{k,\ell}$ and $Z^n_{k,\ell}$ are independent, (4.57) and (4.67) together imply that

$$\begin{aligned}
\mathrm{P}\left\{v(R^n_{k,l}) \cap B(z\,,2^{-n}) \neq \varnothing\right\} &\leq c\,\mathrm{E}\left[\left(2^{-n} + Z^n_{k,\ell}\right)^d\right] \\
&\leq c\left(2^{-nd} + \mathrm{E}\left[(Z^n_{k,\ell})^d\right]\right).
\end{aligned} \tag{4.68}$$

We bound $Z^n_{k,\ell}$ by

$$Z^n_{k,\ell} \leq Z^{(1),n}_{k,\ell} + Z^{(2),n}_{k,\ell}, \tag{4.69}$$

where

$$\begin{aligned}
Z^{(1),n}_{k,\ell} &:= \sup_{(t,x)\in R^n_{k,\ell}} \left\|v(t\,,x) - v(t^n_k\,,x^n_\ell)\right\|, \\
Z^{(2),n}_{k,\ell} &:= v(t^n_k\,,x^n_\ell) \times \sup_{(t,x)\in R^n_{k,\ell}} \left|1 - c^n_{k,\ell}(t\,,x)\right|.
\end{aligned} \tag{4.70}$$

On one hand, (4.65) implies that the d-th moment of $Z^{(2),n}_{k,\ell}$ is at most constant times 2^{-nd}. On the other hand, Lemma 4.5 proves that

$$\mathrm{E}\left[\left(Z^{(1),n}_{k,\ell}\right)^d\right] \leq c\,2^{-nd}. \tag{4.71}$$

Therefore, (4.68) implies (4.54), whence the proposition follows. □

The main result of this section is the following theorem, which summarizes the preceding results.

Theorem 4.6. *Let $u = (u(t\,,x))_{t\in[0,T],x\in[0\,,1]}$ be the solution of (4.1). Fix $t_0 > 0$. Then the conclusions of Theorems 2.1, 2.4, 3.1, and 3.2 hold for u, with $I = [t_0\,,T]$, $J = [0\,,1]$, and $\beta = d$.*

Proof. By Proposition 4.1, A.1 and A.2 are satisfied for u with these choices of I, J and β. Therefore, the conclusions of Theorems 2.1, 2.4 are also satisfied. By Proposition 4.4, u satisfies (3.2) with $\beta = d$, $I = [t_0\,, T]$ and $J = [0\,,1]$. Therefore, the conclusions of Theorems 3.1, and 3.2 are also satisfied. □

Remark 4.7. We could have considered the system (1.1) with Dirichlet boundary conditions instead of the Neumann boundary conditions (1.2). In this case, the results and proofs are essentially unchanged, except that one must replace the interval $J = [0\,,1]$ by $J = [\epsilon\,, 1-\epsilon]$, where $\epsilon > 0$ is fixed. Indeed, a lower bound such as (4.28) would obviously not be satisfied at $x = y = 0$ or $x = y = 1$ with Dirichlet boundary conditions.

5. The case of additive noise

The aim of this section is to transfer the results of Section 4 for the Gaussian process (4.1) to the process (1.3). Subsequently, we will establish Theorem 1.1 and Corollaries 1.2 and 1.3 of the Introduction. For this, we will use the following general fact which is a consequence of Girsanov's theorem.

Proposition 5.1. *Let u denote the solution of* (1.1) *and let v denote the solution of* (1.1) *with $b \equiv 0$, that is, v is the the solution of* (4.1). *Then for any $\epsilon > 0$, there exists $c > 0$ such that for all be a Borel subsets B of $C([0\,,T] \times [0\,,1], \mathbb{R}^d)$,*

$$\frac{1}{c}(\mathrm{P}\{v \in B\})^{1+\epsilon} \le \mathrm{P}\{u \in B\} \le c\,(\mathrm{P}\{v \in B\})^{1/(1+\epsilon)}. \tag{5.1}$$

Proof. We follow the proof of Corollary 5.3 of Dalang and Nualart (2004) and consider

$$\begin{aligned} L_t &:= \exp\biggl(- \int_0^t \int_0^1 G_{t-s}(x\,,y)\,\sigma^{-1} b(u(s\,,y)) \cdot W(ds\,dy) \\ &\qquad\qquad - \frac{1}{2}\int_0^t \int_0^1 (G_{t-s}(x\,,y))^2 \left\|\sigma^{-1} b(u(s\,,y))\right\|^2 ds\,dy \biggr), \\ J_t &:= \exp\biggl(- \int_0^t \int_0^1 G_{t-s}(x\,,y)\,\sigma^{-1} b(v(s\,,y)) \cdot W(ds\,dy) \\ &\qquad\qquad + \frac{1}{2}\int_0^t \int_0^1 (G_{t-s}(x\,,y))^2 \left\|\sigma^{-1} b(v(s\,,y))\right\|^2 ds\,dy \biggr). \end{aligned} \tag{5.2}$$

Let Q denote the probability measure defined by

$$\frac{d\mathrm{Q}}{d\mathrm{P}} := L_t \qquad \text{on } \mathscr{F}_t, \tag{5.3}$$

where $\mathscr{F}_t$ denotes the σ-algebra defined in the Introduction. Then, by Girsanov's theorem as stated in Proposition 1.6 of Nualart and Pardoux (1994) (see also Dalang and Nualart (2004, Theorem 5.2)),

$$\mathrm{P}\{u \in B\} = \mathrm{E}_{\mathrm{P}}\left[\mathbf{1}_{\{u\in B\}}\right] = \mathrm{E}_{\mathrm{Q}}\left[\mathbf{1}_{\{u\in B\}} L_t^{-1}\right] = \mathrm{E}_{\mathrm{P}}\left[\mathbf{1}_{\{v\in B\}} J_t^{-1}\right]. \tag{5.4}$$

Let $\epsilon > 0$ and apply Hölder's inequality to find that

$$\begin{aligned} \mathrm{P}\{v \in B\} &= \mathrm{E}_{\mathrm{P}}\left[\mathbf{1}_{\{v\in B\}} J_t^{-1/(1+\epsilon)}\, J_t^{1/(1+\epsilon)}\right] \\ &\le \left(\mathrm{E}_{\mathrm{P}}\left[\mathbf{1}_{\{v\in B\}} J_t^{-1}\right]\right)^{1/(1+\epsilon)} \left(\mathrm{E}_{\mathrm{P}}\left[J_t^{1/\epsilon}\right]\right)^{\epsilon/(1+\epsilon)}, \end{aligned} \tag{5.5}$$

and therefore,

$$\mathrm{P}\{u \in B\} \geq (\mathrm{P}\{v \in B\})^{1+\epsilon} \left(\mathrm{E}_{\mathrm{P}} \left[J_t^{1/\epsilon} \right] \right)^{-\epsilon}. \tag{5.6}$$

Let $r = 1/\epsilon$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} &\mathrm{E}_{\mathrm{P}}[J_t^r] \\ &\leq \Bigg(\mathrm{E}_{\mathrm{P}} \Bigg[\exp\Bigg(\int_0^t \int_0^1 -2r\, G_{t-s}(x\,,y)\, \sigma^{-1} b(v(s\,,y)) \cdot W(ds\, dy) \\ &\qquad - \frac{1}{2} \int_0^t \int_0^1 4r^2 (G_{t-s}(x\,,y))^2 \, \|\sigma^{-1} b(v(s\,,y))\|^2 \, ds\, dy \Bigg) \Bigg] \Bigg)^{1/2} \\ &\times \Bigg(\mathrm{E}_{\mathrm{P}} \Bigg[\exp\Bigg(\int_0^t \int_0^1 (2r^2 + r)(G_{t-s}(x\,,y))^2 \, \|\sigma^{-1} b(v(s\,,y))\|^2 \, ds\, dy \Bigg) \Bigg] \Bigg)^{1/2}. \end{aligned} \tag{5.7}$$

The first expectation on the right-hand side equals 1 since it is the expectation of an exponential martingale with bounded quadratic variation. The second factor is bounded by some positive finite constant. This proves the lower bound of (5.1).

In order to prove the upper bound, let $\epsilon > 0$ and apply Hölder's inequality to the right-hand side of (5.4):

$$\mathrm{P}\{u \in B\} \leq (\mathrm{P}\{v \in B\})^{1/1+\epsilon} \left(\mathrm{E}_{\mathrm{P}} \left[J_t^{-(1+\epsilon)/\epsilon} \right] \right)^{\epsilon/1+\epsilon}. \tag{5.8}$$

Let $r = (1+\epsilon)/\epsilon$. Again by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\mathrm{E}_{\mathrm{P}}[J_t^{-r}] \\ &\leq \Bigg(\mathrm{E}_{\mathrm{P}} \Bigg[\exp\Bigg(\int_0^t \int_0^1 2r\, G_{t-s}(x\,,y)\, \sigma^{-1} b(v(s\,,y)) \cdot W(ds\, dy) \\ &\qquad - \frac{1}{2} \int_0^t \int_0^1 4r^2 (G_{t-s}(x\,,y))^2 \, \|\sigma^{-1} b(v(s\,,y))\|^2 \, ds\, dy \Bigg) \Bigg] \Bigg)^{1/2} \\ &\times \Bigg(\mathrm{E}_{\mathrm{P}} \Bigg[\exp\Bigg(\int_0^t \int_0^1 (2r^2 - r)(G_{t-s}(x\,,y))^2 \, \|\sigma^{-1} b(v(s\,,y))\|^2 \, ds\, dy \Bigg) \Bigg] \Bigg)^{1/2}. \end{aligned} \tag{5.9}$$

As above, the first expectation on the right-hand side equals 1 since it is the expectation of an exponential martingale with bounded quadratic variation and the second factor is bounded above by some positive finite constant. This concludes the proof. □

Theorem 1.1 will be a consequence of our next result.

Proposition 5.2. *Let u denote the solution of* (1.1). *Let $I \subset]0\,,T]$ and $J \subset [0\,,1]$ be two fixed non-trivial compact intervals. Fix $M > 0$.*

(1) *For any $\epsilon > 0$, there exists $c > 0$ such that for all Borel sets $A \subseteq [-M\,,M]^d$,*

$$\frac{1}{c}(\mathrm{Cap}_{d-6}(A))^{1+\epsilon} \leq \mathrm{P}\left\{u(I \times J) \cap A \neq \varnothing\right\} \leq c\,(\mathscr{H}_{d-6}(A))^{1/(1+\epsilon)}.$$

(2) *For all $t \in]0\,,T]$ and $\epsilon > 0$, there exists $c > 0$ such that for all Borel sets $A \subseteq [-M\,,M]^d$,*

$$\frac{1}{c}(\mathrm{Cap}_{d-2}(A))^{1+\epsilon} \leq \mathrm{P}\left\{u(\{t\} \times J) \cap A \neq \varnothing\right\} \leq c\,(\mathscr{H}_{d-2}(A))^{1/(1+\epsilon)}.$$

(3) *For all $x \in [0\,,1]$ and $\epsilon > 0$, there exists $c > 0$ such that for all Borel sets $A \subseteq [-M\,,M]^d$,*

$$\frac{1}{c}(\mathrm{Cap}_{d-4}(A))^{1+\epsilon} \leq \mathrm{P}\left\{u(I \times \{x\}) \cap A \neq \varnothing\right\} \leq c\,(\mathscr{H}_{d-4}(A))^{1/(1+\epsilon)}.$$

Proof. In order to prove the upper bound in (1), we apply Proposition 5.1 with $B = \{f \in C([0\,,T] \times [0\,,1], \mathbb{R}^d) : f(I \times J) \cap A \neq \varnothing\}$ and then use Theorem 4.6. When A is compact, we get the lower bound in (1) in the same way. Now consider the case where A is a Borel set. We recall that Cap_β is a Choquet capacity; see Dellacherie and Meyer (1975, Chap. 3). In particular, for any Borel set A,

$$\sup_{F \subset A,\, F \text{ compact}} \mathrm{Cap}_\beta(F) = \mathrm{Cap}_\beta(A). \tag{5.10}$$

Therefore, if $F \subset A$ is compact, then

$$\mathrm{P}\left\{u(I \times J) \cap A \neq \varnothing\right\} \geq \mathrm{P}\left\{u(I \times J) \cap F \neq \varnothing\right\} \geq \frac{1}{c}(\mathrm{Cap}_{d-6}(F))^{1+\epsilon}. \tag{5.11}$$

Taking, on the right-hand side, the supremum over such F and using (5.10) proves the lower bound in (1) for A.

The proofs of (2) and (3) are similar and are left to the reader. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. This theorem is an immediate consequence of Proposition 5.2. □

We prove Corollary 1.2 next.

Proof of Corollary 1.2. We first prove (a). Let $z \in \mathbb{R}^d$. If $d < 6$, then $\mathrm{Cap}_{d-6}(\{z\}) = 1$. Hence, the lower bound of Proposition 5.2(1) implies that $\{z\}$ is not polar. On the other hand, if $d > 6$, then $\mathscr{H}_{d-6}(\{z\}) = 0$ and the upper bound of Proposition 5.2(1) implies that $\{z\}$ is polar. If $d = 6$, we observe that MUELLER AND TRIBE Mueller and Tribe (2002, Corollary 4) show that the law of their *stationary pinned string* Mueller and Tribe (2002, (2.1)) is mutually equivalent, on compact subsets of $]0\,,T[\times]0\,,1[$, to the law of the solution of (1.1) (see Mueller and Tribe (2002, Corollary 4)). In this corollary, Mueller and Tribe consider the heat equation on the circle instead of the heat equation on $[0\,,1]$; however, the Green's functions of these two equations are not very different and the proofs of Mueller and Tribe (2002) apply essentially without changes to our setting. Therefore, from Mueller and Tribe (2002, Theorem 1) and Proposition 5.2, we conclude that when $d = 6$, a.s., the solution of (1.1) does not hit points. This proves (a).

For (b), the cases $d < 2$ and $d > 2$ are proved exactly along the same lines using Proposition 5.2(2). For the case $d = 2$, we again use the mutual equivalence of our process with the stationary pinned string of Mueller and Tribe (2002). For t fixed, the stationary pinned string as a function of x has the same increments as those of a standard Brownian motion with values in $\mathbb{R}^d$ Mueller and Tribe (2002, Section 2). Therefore, points are polar for $x \mapsto u(t\,,x)$ when $d = 2$. This proves (b).

For (c), the statement only concerns the cases $d < 4$ and $d > 4$, which are proved as above using Proposition 5.2(3). □

Next we derive Corollary 1.3. In the special case that $b_i \equiv 0$ and $\sigma_{i,j} \equiv \delta_{i,j}$, Wu and Xiao (2007) find a connection between (1.3) and the theory of local nondeterminism, and hence deduce Corollary 1.3; see their Theorem 2.3 and Proposition 2.4 (*loc. cit.*). Presently, we use an indirect and elementary codimension argument to establish a similar result for the more general $\sigma_{i,j}$ and functions b_i under consideration in this paper.

Proof of Corollary 1.3. Let E be a random set. When it exists, the codimension of E is the real number $\beta \in [0\,,d]$ such that for all compact sets $A \subset \mathbb{R}^d$,

$$\mathrm{P}\{E \cap A \neq \varnothing\} \begin{cases} > 0 & \text{whenever } \dim_{\mathrm{H}}(A) > \beta, \\ = 0 & \text{whenever } \dim_{\mathrm{H}}(A) < \beta. \end{cases} \tag{5.12}$$

See Khoshnevisan (2002, Chap. 11, Section 4). When it is well defined, we write the said codimension as $\mathrm{codim}(E)$. Proposition 5.2 implies that for $d \geq 1$:

$$\mathrm{codim}(u(\mathbb{R}_+ \times]0\,,1[)) = (d-6)^+; \tag{5.13}$$

$$\mathrm{codim}(u(\{t\} \times]0\,,1[)) = (d-2)^+; \tag{5.14}$$

and

$$\mathrm{codim}(u(\mathbb{R}_+ \times \{x\})) = (d-4)^+. \tag{5.15}$$

According to Theorem 4.7.1 of Khoshnevisan (2002, Chap. 11), given a random set E in $\mathbb{R}^d$ whose codimension is strictly between 0 and d,

$$\dim_{\mathrm{H}} E + \mathrm{codim}\, E = d \qquad \text{a.s. on } \{E \neq \varnothing\}. \tag{5.16}$$

When $d > 6$, this implies (a). When $d > 2$, this implies (b), and when $d > 4$ this implies (c) of the corollary.

For the remaining "critical cases" we consider the case $d = 6$ and prove (a) only. The corresponding results for (b) ($d = 2$) and (c) ($d = 4$) are proved analogously.

Because $d = 6$, it follows immediately that $\dim_{\mathrm{H}} u(]0\,,T] \times]0\,,1[) \leq 6$. For the lower bound, we note that $u(]0\,,T] \times]0\,,1[)$ will hit $A \subset \mathbb{R}^6$ as long as A has positive logarithmic capacity (Proposition 5.2). In particular, the codimension of $u(]0\,,T] \times]0\,,1[)$ is zero.

Choose and fix $\beta \in]0\,,6[$. By Peres's Lemma (Khoshnevisan (2002, p. 436)), we can find an independent closed random set $\Lambda_\beta \subset \mathbb{R}^6$ such that for all σ-compact sets $E \subset \mathbb{R}^6$: (i) $\dim_{\mathrm{H}} \Lambda_\beta \cap E = \dim_{\mathrm{H}} E - \beta$ a.s.; (ii) $\mathrm{P}\{\Lambda_\beta \cap E = \varnothing\} = 1$ if $\dim_{\mathrm{H}} E < \beta$; and (iii) $\mathrm{P}\{\Lambda_\beta \cap E \neq \varnothing\} \in \{0\,,1\}$. Because $\dim_{\mathrm{H}} \Lambda_\beta = 6 - \beta$ is positive, Λ_β has positive logarithmic capacity; this follows from Frostman's theorem (Khoshnevisan (2002, p. 521)). Therefore, by Proposition 5.2 and (iii), $u(]0\,,T] \times]0\,,1[) \cap \Lambda_\beta \neq \varnothing$ a.s. But thanks to (ii), $\dim_{\mathrm{H}} u(]0\,,T] \times]0\,,1[) \geq \beta$. Let $\beta \uparrow 6$ to deduce (a) in the case that $d = 6$. This concludes the proof. □

Proposition 5.3. *Let u denote the solution of* (1.1). *Then for all $\epsilon > 0$ and $R > 0$, there exists a positive and finite constant a such that the following holds for all compact sets $E \subset]0\,,T] \times]0\,,1[$, $F \subset]0\,,T]$, and $G \subset]0\,,1[$, and for all $z \in B(0\,,R)$:*

1. $a^{-1}(\mathrm{Cap}^{\mathbf{\Delta}}_{d/2}(E))^{1+\epsilon} \leq \mathrm{P}\{\mathscr{L}(z\,;u) \cap E \neq \varnothing\} \leq a\,(\mathscr{H}^{\mathbf{\Delta}}_{d/2}(E))^{1/(1+\epsilon)}$;
2. $a^{-1}(\mathrm{Cap}_{(d-2)/4}(F))^{1+\epsilon} \leq \mathrm{P}\{\mathscr{T}(z\,;u) \cap F \neq \varnothing\} \leq a\,(\mathscr{H}_{(d-2)/4}(F))^{1/(1+\epsilon)}$;
3. $a^{-1}(\mathrm{Cap}_{(d-4)/2}(G))^{1+\epsilon} \leq \mathrm{P}\{\mathscr{X}(z\,;u) \cap G \neq \varnothing\} \leq a\,(\mathscr{H}_{(d-4)/2}(G))^{1/(1+\epsilon)}$;
4. *For all $x \in]0\,,1[$,*

$$a^{-1}(\mathrm{Cap}_{d/4}(F))^{1+\epsilon} \leq \mathrm{P}\{\mathscr{L}_x(z\,;u) \cap F \neq \varnothing\} \leq a\,(\mathscr{H}_{d/4}(F))^{1/(1+\epsilon)};$$

(5) *For all* $t \in]0\,,T]$,

$$a^{-1}(\mathrm{Cap}_{d/2}(G))^{1+\epsilon} \leq \mathrm{P}\{\mathscr{L}^t(z\,;u) \cap G \neq \varnothing\} \leq a\,(\mathscr{H}_{d/2}(G))^{1/(1+\epsilon)}.$$

Proof. In order to prove (1), it suffices to use Proposition 5.1 with $B = \{f : \mathscr{L}(z\,;u) \cap E \neq \varnothing\}$ and apply Theorem 4.6. The proofs of (2)–(5) follow in exactly the same way. □

Corollary 5.4. *Let u denote the solution of* (1.1). *Choose and fix $z \in \mathbb{R}^d$.*

(a) *If $2 \leq d < 6$, then* $\dim_{\mathrm{H}}\ \mathscr{T}(z\,;u) = \frac{1}{4}(6-d)$ *a.s. on* $\{\mathscr{T}(z\,;u) \neq \varnothing\}$.
(b) *If $4 \leq d < 6$, then* $\dim_{\mathrm{H}}\ \mathscr{X}(z\,;u) = \frac{1}{2}(6-d)$ *a.s. on* $\{\mathscr{X}(z\,;u) \neq \varnothing\}$.
(c) *If $1 \leq d < 4$, then* $\dim_{\mathrm{H}}\ \mathscr{L}_x(z\,;u) = \frac{1}{4}(4-d)$ *a.s. on* $\{\mathscr{L}_x(z\,;u) \neq \varnothing\}$.
(d) *If $d = 1$, then* $\dim_{\mathrm{H}}\ \mathscr{L}^t(z\,;u) = \frac{1}{2}(2-d) = \frac{1}{2}$ *a.s. on* $\{\mathscr{L}^t(z\,;u) \neq \varnothing\}$.

In addition, all four right-most events have positive probability.

Proof. The final positive-probability assertion is an immediate consequence of Proposition 5.3 and Taylor's theorem (Khoshnevisan (2002, Corollary 2.3.1, p. 523)).

For the remainder of the corollary, we proceed as we did in the proof of Corollary 1.3. By Proposition 5.3, for $d \geq 1$, it holds that $\mathrm{codim}(\mathscr{T}(z\,;u)) = \frac{1}{4}(d-2)^+$, $\mathrm{codim}(\mathscr{X}(z\,;u)) = \frac{1}{2}(d-4)^+$, $\mathrm{codim}(\mathscr{L}_x(z;\,u)) = d/4$, $\mathrm{codim}(\mathscr{L}^t(z\,;u)) = d/2$. Hence, (5.16) gives the desired statements of the corollary in all but the critical cases. The critical cases are handled as was done in the proof of Corollary 1.3. □

Remark 5.5. It is natural to expect that if $1 \leq d < 6$, then the $\mathscr{H}^{\boldsymbol{\Delta}}$-Hausdorff dimension of $\mathscr{L}(z\,;u)$ is $(6-d)/2$. Indeed, since the $\mathscr{H}^{\boldsymbol{\Delta}}$-Hausdorff dimension of $]0,T] \times [0,1]$ is 3, this would be compatible with the codimension argument, if it applied.

Appendix A. Appendix: An anisotropic Kolmogorov Continuity Theorem

We first present an improvement of the classical lemma of Garsia (1972). Recall that $\Psi : \mathbb{R} \to \mathbb{R}_+$ is a *strong Young function* if it is even and convex on $\mathbb{R}$, and strictly increasing on $\mathbb{R}_+$. Its *inverse* is $\Psi^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$.

Proposition A.1. *Let $(S\,,\rho)$ be a metric space, μ a Radon measure on S, and $\Psi : \mathbb{R} \to \mathbb{R}_+$ a strong Young function with $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. Suppose $p : [0\,,\infty[\to \mathbb{R}_+$ is continuous and strictly increasing, with $p(0) = 0$. Define, for any continuous function $f : S \to \mathbb{R}$,*

$$\mathscr{C} := \iint \Psi\left(\frac{f(x)-f(y)}{p(\rho(x\,,y))}\right) \mu(dx)\,\mu(dy). \tag{A.1}$$

Let $B_\rho(s\,,r)$ denote the open d-ball of radius $r > 0$ about $s \in S$. Then, for all $s,t \in S$,

$$\begin{aligned} &|f(t)-f(s)| \\ &\quad \leq 5\int_0^{2\rho(s,t)} \left[\Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(B_\rho(s\,,u/4))]^2}\right) + \Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(B_\rho(t\,,u/4))]^2}\right)\right] p(du). \end{aligned} \tag{A.2}$$

Remark A.2. (a) The following "majorizing-measure condition" is a ready but useful consequence: If $\mathscr{C} < \infty$ then for all $\epsilon > 0$,

$$\sup_{s,t\in S:\ \rho(s,t)\le\epsilon} |f(t) - f(s)| \le 10 \sup_{x\in S} \int_0^{2\epsilon} \Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(B_\rho(x\,,u/4)]^2}\right) p(du). \tag{A.3}$$

Other extensions are found in Arnold and Imkeller (1996) and Heinkel (1981).

(b) Suppose, instead of continuity, that $f \in L^1_{loc}(\mu)$ and

$$\lim_{\epsilon\to 0^+} \frac{1}{\mu(B_\rho(x\,,\epsilon))} \int_{B_\rho(x\,,\epsilon)} f\, d\mu = f(x) \qquad \text{for } \mu\text{-almost all } x. \tag{A.4}$$

Then, a straight-forward modification of our proof shows that there is a μ-null set N such that (A.2) holds for all $s\,,t \in S \setminus N$.

(c) This proposition implies various known Poincaré inequalities and Besov–Morrey–Sobolev embedding theorems in metric spaces. A portion of this assertion in proved in Kassmann (2003) who uses the inequality of Arnold and Imkeller (1996) instead of ours. Buckley and Koskela (1996, 1995) contain some of the recent work on Sobolev embedding theory.

Proof. Throughout, we choose and fix $s, t \in S$, and follow the ideas of Garsia (1972) closely. We may, and will, assume without loss of generality that $\mathscr{C} < \infty$. Otherwise, there is nothing to prove because $\Psi(\infty) = \infty$.

Define, for any bounded set $Q \subset S$ with $\mu(Q) > 0$,

$$\bar{f}_Q := \frac{1}{\mu(Q)} \int_Q f\, d\mu. \tag{A.5}$$

We borrow from Garsia (1972) the following observation: If A and B are bounded measurable subsets of S with $\mu(A), \mu(B) > 0$, then for all $\alpha > 0$,

$$\begin{aligned}\Psi\left(\frac{\bar{f}_A - \bar{f}_B}{\alpha}\right) &= \Psi\left(\frac{1}{\mu(A)\cdot\mu(B)} \int_A \mu(dx) \int_B \mu(dy)\ \frac{f(x) - f(y)}{\alpha}\right) \\ &\le \frac{1}{\mu(A)\cdot\mu(B)} \int_A \mu(dx) \int_B \mu(dy) \Psi\left(\frac{f(x) - f(y)}{\alpha}\right).\end{aligned} \tag{A.6}$$

[The preceding uses Jensen's inequality only.] It follows from this that if $\alpha \ge p(\rho(x\,,y))$ for all $x \in A$ and $y \in B$, then $\Psi((\bar{f}_A - \bar{f}_B)/\alpha)$ is bounded above by $\mathscr{C}/[\mu(A)\mu(B)]$. Thus, we are led to the basic inequality,

$$\left|\bar{f}_A - \bar{f}_B\right| \le \Psi^{-1}\left(\frac{\mathscr{C}}{\mu(A)\cdot\mu(B)}\right) \sup_{x\in A,\, y\in B} p(\rho(x\,,y)). \tag{A.7}$$

Let $r_0 := \frac{1}{2}\rho(s\,,t)$ and define r_n by $p(2r_n) := 2^{-n}p(2r_0)$ for all $n \ge 1$. Notice that as n tends to infinity, both r_n and $p(2r_n)$ decrease to 0.

Define $A_n := B_\rho(s\,,r_n)$ and $B_n := B_\rho(t\,,r_n)$ for all $n \ge 0$, and apply (A.7) to find that

$$\left|\bar{f}_{A_n} - \bar{f}_{A_{n-1}}\right| \le \Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(A_n)]^2}\right) p(2r_{n-1}). \tag{A.8}$$

Because $p(2r_n) - p(2r_{n+1}) = \frac{1}{4}p(2r_{n-1})$,

$$\left|\bar{f}_{A_n} - \bar{f}_{A_{n-1}}\right| \le 4\Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(A_n)]^2}\right) [p(2r_n) - p(2r_{n+1})]\,. \tag{A.9}$$

Note that $\cap_{n=1}^{\infty} A_n = \{s\}$, whence $\lim_{n\to\infty} \bar{f}_{A_n} = f(s)$ by continuity. Therefore, we can add the preceding over all $n \geq 1$, and use the elementary bound $r_1 \leq r_0$, to find that

$$\left|f(s) - \bar{f}_{A_0}\right| \leq 4 \int_0^{2r_0} \Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(B_\rho(s\,,u/2))]^2}\right) p(du). \tag{A.10}$$

Similarly,

$$\left|f(t) - \bar{f}_{B_0}\right| \leq 4 \int_0^{2r_0} \Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(B_\rho(t\,,u/2))]^2}\right) p(du). \tag{A.11}$$

A third application of (A.7) reveals that $|\bar{f}_{A_0} - \bar{f}_{B_0}|$ is at most

$$\begin{aligned}
&\Psi^{-1}\left(\frac{\mathscr{C}}{\mu(A_0)\cdot\mu(B_0)}\right) p(4r_0) \\
&\leq \int_0^{4r_0} \Psi^{-1}\left(\frac{\mathscr{C}}{\mu(B_\rho(s\,,u/4))\cdot\mu(B_\rho(t\,,u/4))}\right) p(du) \qquad\qquad (A.12)\\
&\leq \int_0^{4r_0} \Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(B_\rho(s\,,u/4))]^2}\right) p(du) + \int_0^{4r_0} \Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(B_\rho(t\,,u/4))]^2}\right) p(du).
\end{aligned}$$

Since $r_0 = \frac{1}{2}\rho(s\,,t)$, equations (A.10), (A.11), and (A.12) together imply the proposition. □

Corollary A.3. *Choose and fix two nonrandom compact intervals $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$, and let $\{v(t\,,x)\}_{t\in I, x\in J}$ denote a real-valued stochastic process. Suppose that there exist finite constants $p > 1$, $q > 0$, and $c > 0$ such that for all $(t\,,x) \in I \times J$ and $(s\,,y) \in I \times J$,*

$$\mathrm{E}(|v(t\,,x) - v(s\,,y)|^p) \leq c[\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^{3+q}. \tag{A.13}$$

Then v has a continuous version $\tilde{v}$, and for any $\alpha \in [0\,,q/p[$,

$$\mathrm{E}\left[\left(\sup_{(t,x)\neq(s,y)} \frac{|\tilde{v}(t\,,x) - \tilde{v}(s\,,y)|}{[\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^\alpha}\right)^p\right] < \infty. \tag{A.14}$$

In particular, there is a non-negative random variable C with $\mathrm{E}[C] < \infty$ such that a.s.,

$$|\tilde{v}(t\,,x) - \tilde{v}(s\,,y)| \leq C[\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^\alpha. \tag{A.15}$$

A similar statement can be found in Theorem 1.4.1 of Kunita (1991, p. 31). We include a proof for convenience of the reader.

Proof. We observe that (A.13) implies that $(t\,,x) \mapsto v(t\,,x)$ is continuous in probability, and therefore, has a measurable version (Dellacherie and Meyer (1975, Chap. IV, Théorème 30)), which we continue to denote by v. We note that thanks to (A.13), for any fixed $(t_0\,,x_0) \in I \times J$, $v(\cdot\,,\cdot) - v(t_0\,,x_0) \in L^p_{loc}(dtdx)$ a.s. Since this shifted process has the same increments as $v(\cdot\,,\cdot)$, we may as well assumed that $v \in L^p_{loc}(dtdx)$ almost surely.

We apply Proposition A.1 to this version of v with

$$S = I \times J, \qquad \rho((t\,,x)\,;(s\,,y)) = \mathbf{\Delta}((t\,,x)\,;(s\,,y)), \qquad \mu(dt\,dx) = dtdx, \tag{A.16}$$

and

$$\Psi(x) = |x|^p, \qquad \Psi^{-1}(y) = y^{1/p}, \qquad p(x) = |x|^{\alpha+(6/p)}. \tag{A.17}$$

Let

$$\mathscr{C} = \int_S dtdx \int_S dsdy \ \frac{|v(t\,,x) - v(s\,,y)|^p}{[\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^{6+\alpha p}}. \tag{A.18}$$

By (A.13),

$$\begin{aligned} \mathrm{E}[\mathscr{C}] &\leq \int_S dtdx \int_S dsdy \ [\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^{q-3-\alpha p} \\ &\leq 4|I|\,|J| \int_0^{|I|} d\tilde{u} \int_0^{|J|} dv \ (\tilde{u}^{1/2} + v)^{q-3-\alpha p}. \end{aligned} \tag{A.19}$$

We can check readily that the preceding integral is finite using only the fact that $\alpha \in [0\,, q/p[$. Therefore,

$$\mathrm{E}[\mathscr{C}] < \infty. \tag{A.20}$$

Since $v \in L^p_{loc}(dtdx)$ a.s., and because $p > 1$, a well-known theorem of Jessen, Marcinkiewicz, and Zygmund implies that the following holds with probability one:

$$\lim_{\epsilon,\delta \downarrow 0} \frac{1}{4\epsilon\delta} \int_{t-\epsilon}^{t+\epsilon} \int_{x-\delta}^{x+\delta} v(a\,,b)\, da\, db = v(t\,,x), \tag{A.21}$$

for almost all $(t\,,x) \in I \times J$. See Khoshnevisan (2002, Theorem 2.2.1, Chap. 2, p. 58). In particular, (A.4) holds in the present setting.

We now take into account Remark A.2(b), and deduce that for a.a. ω there exists a set $D(\omega) \subset S$ with full Lebesgue measure such that for all $(t\,,x), (s\,,y) \in D(\omega)$,

$$\begin{aligned} &|v(t\,,x)(\omega) - v(s\,,y)(\omega)| \\ &\quad \leq 10 \sup_{(r,\bar{y})} \int_0^{2\mathbf{\Delta}((t\,,x)\,;(s\,,y))} \Psi^{-1}\left(\frac{\mathscr{C}}{[\mu(B_\rho((r\,,\bar{y})\,, u/4)]^2} \right) u^{\alpha-1+(6/p)}\, du. \end{aligned} \tag{A.22}$$

One can check directly that there exists a $c > 0$ such that $\mu(B_\rho((r\,,\bar{y})\,, u/4) \geq cu^3$ for all $u > 0$ and $(r\,,\bar{y}) \in S$. Therefore,

$$\begin{aligned} |v(t\,,x)(\omega) - v(s\,,y)(\omega)| &\leq 10c^{-2/p} \int_0^{2\mathbf{\Delta}((t,x)\,;(s,y))} \mathscr{C}^{1/p} u^{\alpha-1} du \\ &= c_1^{1/p}\, \mathscr{C}^{1/p}\, [\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^{\alpha}. \end{aligned} \tag{A.23}$$

Define

$$\tilde{v}(t\,,x)(\omega) := \limsup_{(s,y)\in D(\omega):\,(s,y)\to(t,x)} v(s\,,y)(\omega). \tag{A.24}$$

Since $v(\cdot)(\omega)$ is uniformly continuous on D, $\tilde{v}(\cdot)(\omega)$ is continuous on $\bar{D}(\omega) = S$ and coincides with $v(\cdot)(\omega)$ in $D(\omega)$. In addition, by (A.13), $v(s\,,y)$ converges to $v(t\,,x)$ in L^p as $(s\,,y)$ converges to $(t\,,x)$. Therefore, $\tilde{v}(t\,,x) = v(t\,,x)$ a.s. for all $(t\,,x) \in S$, and hence $\tilde{v}$ is a continuous version of v. By (A.23),

$$\left(\sup_{(t,x)\neq(s,y)} \frac{|\tilde{v}(t\,,x) - \tilde{v}(s\,,y)|}{[\mathbf{\Delta}((t\,,x)\,;(s\,,y))]^{\alpha}} \right)^p \leq c_1 \mathscr{C}. \tag{A.25}$$

Equation (A.14) now follows from (A.20). □

Appendix B. Appendix: On Energy Reduction for Smoothed Measures

The goal of this appendix is to prove precise versions of the statement, "if we smooth a measure then we lower its energy."

Theorem B.1. *Let $0 < \alpha < d$ and μ be a probability measure on $\mathbb{R}^d$. Then for all probability density functions $g : \mathbb{R}^d \to \mathbb{R}_+$ with compact support,*

$$I_\alpha(g * \mu) \leq I_\alpha(\mu). \tag{B.1}$$

Theorem B.2. *Choose and fix $n > 1$. Then there exists a positive and finite constant c—depending only on $(d\,, n)$—such that for all probability measures μ on $[-n\,, n]^d$ and all probability density functions $g : \mathbb{R}^d \to \mathbb{R}_+$ with compact support,*

$$I_0(g * \mu) \leq c\, I_0(\mu). \tag{B.2}$$

The proof requires some terminology from harmonic analysis. A function $\kappa : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is called a *potential kernel* if: (i) $\kappa(x) \geq 0$ for all $x \neq 0$; (ii) κ is continuous on $\mathbb{R}^d \setminus \{0\}$; and (iii) $\kappa(0) = \infty$; κ is called *of positive type* if its Fourier transform $\hat{\kappa}$ — viewed in the sense of distributions — is a nonnegative function. We choose the following normalization of Fourier transforms: $\hat{\kappa}(\xi) = \int_{\mathbb{R}^d} \exp(i\xi \cdot x)\kappa(x)\, dx$ for all $\xi \in \mathbb{R}^d$ and $\kappa \in L^1(\mathbb{R}^d)$.

The following is well known; see for example Kahane (1968, Remark 2, p. 133).

Proposition B.3. *If $\kappa : \mathbb{R}^d \to \mathbb{R}_+ \cup \{\infty\}$ is a potential kernel of positive type, then for all Borel probability measures μ on $\mathbb{R}^d$,*

$$\iint \kappa(x-y)\, \mu(dx)\, \mu(dy) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2\, \hat{\kappa}(\xi)\, d\xi. \tag{B.3}$$

First we prove Theorem B.1; it is technically simpler than Theorem B.2, and yet affords us the chance to discuss the reasons for the veracity of both theorems.

Proof of Theorem B.1. Define $\kappa(x) := \|x\|^{-\alpha}$, where $1/0 := \infty$, to find that κ is a potential kernel of positive type with $\hat{\kappa}(\xi) = c\|\xi\|^{-d+\alpha}$; see Stein (1970, Chap. V, §1, Lemma 2(b)), or Kahane (1968, p. 134), for example. Define $\nu(dx) := (g*\mu)(dx)$ and apply Proposition B.3 with ν in place of μ to find that

$$I_\alpha(g * \mu) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{g}(\xi)|^2\, |\hat{\mu}(\xi)|^2\, \hat{\kappa}(\xi)\, d\xi. \tag{B.4}$$

Because $|\hat{g}(\xi)| \leq 1$, another appeal to Proposition B.3 completes the proof. □

Now we work to prove the more difficult Theorem B.2. We define a function $\rho : \mathbb{R}^d \to \mathbb{R}_+ \cup \{\infty\}$ by

$$\rho(x) := \frac{\exp(-\|x\|)}{\|x\|^{d/2}}, \tag{B.5}$$

where $\rho(0) := \infty$. Define $\kappa : \mathbb{R}^d \to \mathbb{R}_+ \cup \{\infty\}$ by

$$\kappa(x) := (\rho * \rho)(x) = \int_{\mathbb{R}^d} \rho(x-y)\, \rho(y)\, dy. \tag{B.6}$$

Lemma B.4. *The function κ is an integrable potential kernel of positive type.*

Proof. Because $\rho(x) \geq 0$ is measurable the convolution is a well-defined nonnegative Borel-measurable function on $\mathbb{R}^d$. Standard arguments show that κ is at least as smooth as ρ. Since ρ is continuous on $\mathbb{R}^d \setminus \{0\}$, then so is κ. Because $\kappa(0)$

is manifestly infinite, this proves that κ is a potential kernel. We may note that $\|\kappa\|_1 = \|\rho\|_1^2 < \infty$. Therefore, $\hat{\kappa}$ is the L^1-form of the Fourier transform of κ. Finally, since ρ is even, $\hat{\rho}$ is real-valued, and therefore $\hat{\kappa}(\xi) = |\hat{\rho}(\xi)|^2 \geq 0$. The lemma follows. □

Lemma B.5. *Let N_0 be as in* (1.5)*. Then there exist positive and finite constants c_1 and c_2—depending only on $(d\,, N_0)$—such that for all $x \in B(0\,, N_0/2)$,*

$$c_1 \mathrm{K}_0(\|x\|) \leq \kappa(x) \leq c_2 \mathrm{K}_0(\|x\|). \tag{B.7}$$

Proof. Choose and fix x with $0 < \|x\| \leq N_0/2$, and write

$$\kappa(x) = T_1 + T_2 + T_3, \tag{B.8}$$

where

$$\begin{aligned} T_1 &:= \int_{\|y\|<2\|x\|} \rho(x-y)\rho(y)\,dy, \\ T_2 &:= \int_{2\|x\|\leq\|y\|\leq 10N_0} \rho(x-y)\rho(y)\,dy, \\ T_3 &:= \int_{\|y\|>10N_0} \rho(x-y)\rho(y)\,dy. \end{aligned} \tag{B.9}$$

We estimate each T_i separately.

It will turn out that the main contribution to $\kappa(x)$ comes from T_2. Therefore, we begin by bounding that quantity: If $2\|x\| \leq \|y\|$, then $\|x-y\| \leq \frac{3}{2}\|y\|$; thus,

$$\begin{aligned} T_2 &\geq \left(\frac{2}{3}\right)^{d/2} \int_{2\|x\|\leq\|y\|\leq 10N_0} \frac{e^{-3\|y\|/2}}{\|y\|^{d/2}}\rho(y)\,dy \\ &\geq C_1 \int_{2\|x\|\leq\|y\|\leq 10N_0} \frac{dy}{\|y\|^d}. \end{aligned} \tag{B.10}$$

We integrate this in polar coordinates to find that $T_2 \geq C_2(\ln N_0 + \ln(1/\|x\|))$. Because $T_1, T_3 \geq 0$, it follows that $\kappa(x)$ is bounded below by a constant multiple of $\ln(N_0/\|x\|)$. This proves half of the lemma.

For the other half, we note that if $2\|x\| \leq \|y\|$, then $\|x-y\| \geq \|y\|/2$. Therefore, we can use an argument, similar to the one we used to bound T_2 from below, in order to prove that

$$T_2 \leq C_3(\ln(10N_0) + \ln(1/\|x\|)), \tag{B.11}$$

and since $\|x\| \leq N_0/2$, the right-hand side is bounded above by $C_4(\ln N_0 + \ln(1/\|x\|))$, provided C_4 is chosen large enough.

Next we bound T_3. Note that if $\|y\| > 10N_0$, then $\|x-y\| \geq 9N_0$. Consequently, $\rho(x-y)$ is bounded from above, and hence $T_3 \leq C_4 \int_{\mathbb{R}^d} \rho(y)\,dy < \infty$.

Finally, we estimate T_1 by first writing it as

$$T_1 = T_{11} + T_{12}, \tag{B.12}$$

where

$$\begin{aligned} T_{11} &:= \int_{\substack{\|y\|\leq 2\|x\| \\ \|y-x\|\geq\|x\|/2}} \rho(x-y)\rho(y)\,dy, \\ T_{12} &:= \int_{\substack{\|y\|\leq 2\|x\| \\ \|y-x\|<\|x\|/2}} \rho(x-y)\rho(y)\,dy. \end{aligned} \tag{B.13}$$

If $\|y-x\| \geq \|x\|/2$, then $\rho(x-y) \leq 2^{d/2}\|x\|^{-d/2}$, and thus,

$$\begin{aligned} T_{11} &\leq \frac{2^{d/2}}{\|x\|^{d/2}} \int_{\substack{\|y\|\leq 2\|x\| \\ \|y-x\|\geq \|x\|/2}} \frac{\exp(-\|y\|)}{\|y\|^{d/2}}\, dy \\ &\leq \frac{2^{d/2}}{\|x\|^{d/2}} \int_{\|y\|\leq 2\|x\|} \frac{dy}{\|y\|^{d/2}} \\ &\leq C_5. \end{aligned} \tag{B.14}$$

The last line follows from integrating in polar coordinates.

In order to estimate the remaining term T_{12}, we note that if $\|y\| \leq 2\|x\|$ and $\|y-x\| < \|x\|/2$, then $\|y\| \geq \|x\|/2$, and hence $\rho(y) \leq 2^{d/2}\|x\|^{-d/2}$. Consequently,

$$\begin{aligned} T_{12} &\leq \frac{2^{d/2}}{\|x\|^{d/2}} \int_{\|y-x\|\leq \|x\|/2} \rho(x-y)\, dy \\ &\leq \frac{2^{d/2}}{\|x\|^{d/2}} \int_{\|z\|\leq \|x\|/2} \frac{dz}{\|z\|^{d/2}} \\ &\leq C_5, \end{aligned} \tag{B.15}$$

for the same constant C_5 as before. These remarks together prove the lemma. □

Now we prove Theorem B.2.

Proof of Theorem B.2. Thanks to Lemma B.5,

$$I_0(g*\mu) \leq \frac{1}{c_1} \iint \kappa(x-y)\, \nu(dy)\, \nu(dx), \tag{B.16}$$

where $\nu(dx) := (g*\mu)(x)\, dx$. Because $|\hat{\nu}(\xi)| = |\hat{g}(\xi)\hat{\mu}(\xi)| \leq |\hat{\mu}(\xi)|$, Lemma B.4 and Proposition B.3 together imply that

$$\begin{aligned} I_0(g*\mu) &\leq \frac{1}{c_1(2\pi)^d} \iint |\hat{\mu}(\xi)|^2\, \hat{\kappa}(\xi)\, d\xi \\ &= \frac{1}{c_1} \iint \kappa(x-y)\, \mu(dx)\, \mu(dy). \end{aligned} \tag{B.17}$$

Another application of Lemma B.5 shows that the latter term is at most $(c_2/c_1)I_0(\mu)$, whence follows the theorem with $c := c_2/c_1$. □

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References

L. Arnold and O. Imkeller. Stratonovich calculus with spatial parameters and anticipative problems in multiplicative ergodic theory. *Stoch. Proc. Appl.* **62** (1), 19–54 (1996).

V. Bally, A. Millet and M. Sanz-Solé. Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equations. *Ann. Probab.* **23**, 178–222 (1995).

V. Bally and E. Pardoux. Malliavin calculus for white noise driven parabolic spdes. *Potential Anal.* **9**, 27–64 (1998).

R. M. Blumenthal and R. K. Getoor. *Markov Processes and Potential Theory*. Academic Press, New York (1968).

S. M. Buckley and P. Koskela. Sobolev–Poincaré implies John. *Math. Res. Lett.* **2** (1995).

S. M. Buckley and P. Koskela. Criteria for imbeddings of Sobolev-Poincaré type. *Internat. Math. Res. Notices* **18**, 881–901 (1996).

R. C. Dalang, D. Khoshnevisan and E. Nualart. Hitting probabilities for systems of non-linear heat equations with multiplicative noise. *Ann. Probab.* (2007). To appear.

R. C. Dalang, C. Mueller and L. Zambotti. Hitting properties of s.p.d.e.'s with reflection. *Ann. Probab.* **34**, 1423–1450 (2006).

R. C. Dalang and E. Nualart. Potential theory for hyperbolic spdes. *Ann. Probab.* **32**, 2099–2148 (2004).

C. Dellacherie and P. A. Meyer. *Probabilités et Potentiels*. Hermann, Paris (1975).

J. Doob. *Classical Potential Theory and Its Probabilistic Counterpart*. Springer-Verlag, Berlin, New York (1984).

A. M. Garsia. Continuity properties of gaussian processes with multidimensional time parameter. In *Proc. Sixth Berkeley Symp. Math. Statis. and Probab. Vol. II: Probability Theory*, pages 369–274 (1972).

B. Heinkel. Mesures majorantes et régularité des fonctions aléatoires. In *Statistical and Physical Aspects of Gaussian Processes (Saint-Flour, 1980), Colloq. Internat. CNRS*, volume 307, pages 407–434. Paris (1981).

J.-P. Kahane. *Some Random Series of Functions*. D. C. Heath, Lexington, Mass. (1968).

M. Kassmann. A note on integral inequalities and embeddings of Besov spaces. *J. Inequal. Pure Appl. Math.* **3** (5), Article 107 (2003).

D. Khoshnevisan. *Multiparameter Processes*. Springer-Verlag, New York (2002).

H. Kunita. *Stochastic Flows and Stochastic Differential Equations*. Cambridge University Press, Cambridge (1991).

P.-L. Morien. The Hölder and the Besov regularity of the density for the solution of a parabolic stochastic partial differential equation. *Bernoulli* **5**, 275–298 (1998).

C. Mueller and R. Tribe. Hitting properties of the random string. *Electron. J. Probab.* **7**, 1–29 (2002). With corrections.

D. Nualart and E. Pardoux. Markov field properties of solutions of white noise driven quasi-linear parabolic PDEs. *Stochastics Stoch. Rep.* **48**, 17–44 (1994).

S. C. Port and C. J. Stone. *Brownian Motion and Classical Potential Theory*. Academic Press, New York (1978).

E. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton (1970).

J. B. Walsh. An Introduction to Stochastic Partial Differential Equations. In *Ecole d'Eté de Probabilités de Saint-Flour XIV, Lect. Notes in Math.*, volume 1180, pages 266–437. Springer-Verlag (1986).

Donghsheng Wu and Yimin Xiao. Fractal properties of the random string processes. In *High Dimensional Probability*, volume 1180, pages 128–147. IMS Lecture Notes–Monograph Series (2007).

L. Zambotti. Integration by parts on convex sets of paths and applications to s.p.d.e.'s with reflection. *Prob. Theory and Rel. Fields* **123** (4), 579–600 (2002).

L. Zambotti. Integration by parts on δ-bessel bridges, $\delta > 3$, and related s.p.d.e.'s. *Ann. Probab.* **31** (1), 323–348 (2003).