Hitting probabilities for systems of stochastic PDEs: an overview

Robert C. Dalang

Abstract We consider a d-dimensional random field that solves a possibly nonlinear system of stochastic partial differential equations, such as stochastic heat or wave equations. We present results, obtained in joint works with Davar Khoshnevisan and Eulalia Nualart, and with Marta Sanz-Solé, on upper and lower bounds on the probabilities that the random field visits a deterministic subset of $\mathbb{R}^d$, in terms, respectively, of Hausdorff measure and Newtonian capacity of the subset. These bounds determine the critical dimension above which points are polar, but do not, in general, determine whether points are polar in the critical dimension. For linear SPDEs, we discuss, based on joint work with Carl Mueller and Yimin Xiao, how the issue of polarity of points can be resolved in the critical dimension.

2010 *Mathematics Subject Classification*: Primary: 60H15; secondary: 60J45, 60G15, 60H07, 60G60.

Key words: Hitting probabilities, systems of stochastics PDEs, Malliavin calculus, stochastic heat equation, stochastic wave equation, spatially homogeneous Gaussian noise, capacity, Hausdorff measure.

1 Introduction

A basic problem in probabilistic potential theory, according to [14, Chapter 10], is the following. Let $U = (U(x),\ x \in \mathbb{R}^k)$ be an $\mathbb{R}^d$-valued continuous stochastic process defined on a probability space $(\Omega, \mathscr{F}, P)$. Let $I \subset \mathbb{R}^k$ be a fixed compact

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The research of the author is partly supported by the Swiss National Foundation for Scientific Research

set with positive Lebesgue measure. The range of U over I is the random compact subset $U(I)$ of $\mathbb{R}^d$ consisting of all points visited by U: for $\omega \in \Omega$,

$$U(I)(\omega) = \{U(x)(\omega),\ x \in I\}.$$

Given a compact subset $A \subset \mathbb{R}^d$, is $P\{U(I) \cap A \neq \emptyset\}$ *positive?* If the answer is "no," then we say that A is *polar* for U, otherwise, A is *non-polar.* If $A = \{z\}$ consists of a single point $z \in \mathbb{R}^d$, then $\{U(I) \cap A \neq \emptyset\} = \{\exists x \in I : U(x) = z\}$, and we say that z is *polar* for U if the singleton $\{z\}$ is polar for U.

A related question concerns *hitting probabilities.* Namely, for a compact subset $A \subset \mathbb{R}^d$, what are bounds on

$$P\{U(I) \cap A \neq \emptyset\}\,?$$

In particular, one would like upper and lower bounds that are sufficient to determine whether or not points are polar for U.

In the case where the d components of U are i.i.d., there is typically a *critical value* $Q(k)$ such that:

- if $d > Q(k)$, then there is lots of room to move around in the value space and points are polar;
- if $1 \leq d < Q(k)$, then points are *non-polar*;
- at the critical value $d = Q(k)$, either situation may occur and it is usually more difficult to decide whether or not points are polar.

The oldest results in this direction concern standard Brownian motion and are due to Paul Lévy [18] (see also [15, Theorem 2.2]), where it is shown that points are polar for Brownian motion in all dimensions $d \geq 2$ (and they are non-polar in dimension $d = 1$). This kind of result was extended to classical Markov processes (see [3, 13, 22]). Extensions to multiparameter processes came later and a discussion of these extensions can be found in [14].

In this paper, we shall mainly discuss results on hitting probabilities for systems of stochastic partial differential equations (SPDEs), though we shall begin with the results on Gaussian random fields that motivated this study and that also serve as benchmarks for research on non-Gaussian random fields and solutions of nonlinear systems of SPDEs.

2 Benchmark results for Gaussian random fields

In view of the many special properties of Gaussian random fields, it is natural to begin the study of hitting probabilities with such random fields. We start with a very particular random field, namely the Brownian sheet. Indeed, the results obtained by Khoshnevisan and Shi [16] were the starting point for much of the later research on this topic.

2.1 First example: the Brownian sheet

Let $(W(x),\ x \in \mathbb{R}_+^k)$ denote an k-parameter $\mathbb{R}^d$-valued *Brownian sheet*, that is, a centered continuous Gaussian random field

$$W(x) = (W_1(x), \dots, W_d(x))$$

with covariance

$$E[W_i(x)W_j(y)] = \delta_{i,j} \prod_{\ell=1}^{k} \min(x_\ell, y_\ell), \qquad i, j \in \{1, \dots, d\},$$

where $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$, and $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. It is clear that the Brownian sheet is a multi-parameter extension of standard Brownian motion, which corresponds to the case $k = 1$.

In order to state the first theorem, we introduce some notation concerning potential theory. For all Borel sets $F \subset \mathbb{R}^d$, let $\mathscr{P}(F)$ denote the set of all probability measures with compact support in F. For all $\alpha \in \mathbb{R}$ and $\mu \in \mathscr{P}(\mathbb{R}^k)$, we let $I_\alpha(\mu)$ denote the *α-dimensional energy* of μ, that is,

$$I_\alpha(\mu) := \iint \mathrm{K}_\alpha(\|x-y\|)\, \mu(dx)\, \mu(dy),$$

where, for $r > 0$,

$$\mathrm{K}_\alpha(r) := \begin{cases} r^{-\alpha} & \text{if } \alpha > 0, \\ \max(\log(1/r), 1) & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha < 0. \end{cases} \tag{1}$$

For all $\alpha \in \mathbb{R}$ and Borel sets $F \subset \mathbb{R}^k$, $\mathrm{Cap}_\alpha(F)$ denotes the *α-dimensional Bessel-Riesz capacity of F*, that is,

$$\mathrm{Cap}_\alpha(F) := \left[\inf_{\mu \in \mathscr{P}(F)} I_\alpha(\mu) \right]^{-1},$$

where, by definition, $1/\infty := 0$.

Finally, for $M > 0$, we let $B(0, M)$ denote the open ball in $\mathbb{R}^d$ of radius M centered at the origin.

Theorem 1. [16] *Fix $M > 0$. Let I be a* box, *that is, $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$, where $0 < a_\ell < b_\ell < \infty$, $\ell = 1, \dots, k$. There exists $0 < C < \infty$ (C depends only on M, k, d, $\min_{\ell=1,\dots,k} a_\ell$, $\max_{\ell=1,\dots,k} b_\ell$) such that for all compact sets $A \subset B(0, M)$ ($\subset \mathbb{R}^d$),*

$$\frac{1}{C}\, \mathrm{Cap}_{d-2k}(A) \le P\{W(I) \cap A \neq \emptyset\} \le C\, \mathrm{Cap}_{d-2k}(A). \tag{2}$$

The statement and proof of this theorem can be found in [16, Theorem 1.1]. Notice that the constraint $A \subset B(0,M)$ is needed for the lower bound. Indeed, imagine translating the set A off to infinity. Then the probability that the random field W hits A would become less and less likely, simply because W has continuous sample paths and the parameter set I is compact.

Example 1. What does Theorem 1 tell us about polarity of points? In the case where $A = \{z\}$, it is not difficult to check that

$$\mathrm{Cap}_{d-2k}(\{z\}) = \begin{cases} 1 \text{ if } d < 2k, \\ 0 \text{ if } d \geq 2k. \end{cases}$$

Therefore, the upper bound in (2) tells us that *points are polar for W in all dimensions $d \geq 2k$,* and the lower bound in (2) tells us that *points are non-polar for W in dimensions* $1 \leq d < 2k$. In particular, $d = 2k$ is the critical dimension and points are polar in this critical dimension.

The result of Theorem 1 is essentially the optimal result to aim for. Indeed, the upper and lower bounds in (2) are identical up to a constant. As we will see, bounds as good as (2) are not available for wider classes of Gaussian random fields.

2.2 Anisotropic Gaussian random fields

We consider here a wider class of Gaussian random fields, studied in particular in [2, 27]. These random fields will typically have different behaviors in different directions, hence the name "anisotropic."

Let $(V(x),\ x \in \mathbb{R}^k)$ be a centered continuous Gaussian random field with values in $\mathbb{R}^d$. We write $V(x) = (V_1(x), \ldots, V_d(x))$ and we assume that the components $V_i = (V_i(x),\ x \in \mathbb{R}^k)$ are i.i.d. real-valued random fields. The canonical metric associated with these random fields is

$$\Delta(x,y) = \|V_1(x) - V_1(y)\|_{L^2}.$$

Let I be a box as in Theorem 1. Assume the following condition:

(C) There exists $0 < c < \infty$ and $H_1, \ldots, H_k \in]0,1[$ such that for all $x \in I$,

$$c^{-1} \leq \Delta(0,x) \leq c,$$

$x \mapsto \Delta(0,x)$ is differentiable on I, and for all $x,y \in I$,

$$c^{-1} \sum_{j=1}^{k} |x_j - y_j|^{H_j} \leq \Delta(x,y) \leq c \sum_{j=1}^{k} |x_j - y_j|^{H_j}.$$

Given $\alpha \geq 0$, recall that the α-dimensional *Hausdorff measure* of F is defined by

$$\mathscr{H}_\alpha(F) = \lim_{\varepsilon\to 0^+} \inf \left\{ \sum_{i=1}^{\infty} (2r_i)^\alpha : F \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ \sup_{i\geq 1} r_i \leq \varepsilon \right\},$$

where $B(x,r)$ denotes the open (Euclidean) ball of radius $r>0$ centered at $x\in\mathbb{R}^d$. When $\alpha<0$, we define $\mathscr{H}_\alpha(F)$ to be infinite.

Theorem 2. [2] *Fix $M>0$. Set $Q=\sum_{j=1}^k \frac{1}{H_j}$. Then there is $0<C<\infty$ such that for every compact set $A\subset B(0,M)$ ($\subset\mathbb{R}^d$),*

$$C^{-1}\,\mathrm{Cap}_{d-Q}(A) \leq P\{V(I)\cap A\neq\emptyset\} \leq C\mathscr{H}_{d-Q}(A). \tag{3}$$

Compared to Theorem 1, Theorem 2 applies to a wide class of Gaussian random fields, but there is a major difference: Hausdorff measure appears on the right-hand side of (3), instead of capacity, as in (2). To see why this is significant, we again consider the issue of polarity of points.

Example 2. Suppose that $A=\{z\}$. Then

$$\mathrm{Cap}_{d-Q}(\{z\}) = \begin{cases} 1 \text{ if } d<Q,\\ 0 \text{ if } d=Q,\\ 0 \text{ if } d>Q, \end{cases} \qquad \mathscr{H}_{d-Q}(\{z\}) = \begin{cases} \infty \text{ if } d<Q,\\ 1 \ \text{ if } d=Q,\\ 0 \ \text{ if } d>Q. \end{cases}$$

In particular, points are polar for V when $d>Q$, and are non-polar when $q<Q$. Therefore, if Q is an integer, then $d=Q$ is the critical dimension for hitting points. However, if $d=Q$, then the statement of Theorem 2 reduces essentially to $0\leq P\{\exists x\in I: V(x)=z\}\leq 1$, which is not particularly informative, and the issue of polarity of points in this critical dimension is not answered by Theorem 2.

2.3 Funaki's random string

There is one further important result on Gaussian random fields, which brings us closer to SPDEs. This concerns the solution to a system of stochastic heat equations driven by space-time white noise, also known as Funaki's random string.

Let $(u(t,x),\ (t,x)\in\mathbb{R}_+\times\mathbb{R})$ be an $\mathbb{R}^d$-valued random field such that

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \dot{W}(t,x), \qquad x\in\mathbb{R},\ t>0, \tag{4}$$

where $u(0,\cdot):\mathbb{R}\to\mathbb{R}^d$ is a given deterministic and smooth function, and $\dot{W}(t,x)$ is an $\mathbb{R}^d$-valued space-time white noise (that is, each of the d components of $\dot{W}$ is a real-valued space-time white noise, and these components are independent random fields).

Theorem 3. [19] *The critical dimension for hitting points is $d = 6$ and points are polar in this dimension.*

The proof of Mueller and Tribe [19] uses a rather clever extension of the argument used by Paul Lévy for Brownian motion. Namely, since the solution of (4) is not stationary in time, they reduce the problem to the study of polarity of points for a "stationary pinned string,", which solves the stochastic heat equation with a random initial condition and has stationary spatial increments, then they use a scaling property and a time reversal argument, as was used by Paul Lévy. We note that [19] also treats the issue of double points for the solution of (4).

The method of Mueller and Tribe is quite specific to the stochastic heat equation. For instance, it does not apply to the stochastic wave equation, nor even to the stochastic heat equation with deterministic non-constant coefficients: if $(t,x) \mapsto \sigma(t,x)$ is a smooth but non-constant function, then the method of [19] does not apply to the solution $(u(t,x))$ of the SPDE.

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \sigma(t,x)\dot{W}(t,x), \qquad x \in \mathbb{R},\ t > 0.$$

3 Hitting probabilities for non-Gaussian random fields

The results presented in the previous section, concerning Gaussian random fields, tell us what inequalities it is reasonable to aim for in the case of non-Gaussian random fields. We now present the results that have been obtained in this direction.

3.1 Systems of nonlinear wave equations in spatial dimension 1

In the paper [11], E. Nualart and the author considered the reduced stochastic wave equation in two parameters, which is obtained from the classical stochastic wave equation in one spatial dimension by a rotation of coordinates. We state however their results here for the classical stochastic wave equation.

Let $u = (u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R})$ be an $\mathbb{R}^d$-valued random field such that

$$\frac{\partial^2}{\partial t^2}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \qquad x \in \mathbb{R},\ t > 0, \tag{5}$$

where $u(0,\cdot)$ and $\frac{\partial}{\partial t}u(0,\cdot)$ are given smooth functions from $\mathbb{R}$ into $\mathbb{R}^d$, $\dot{W}(t,x)$ is an $\mathbb{R}^d$-valued space-time white noise, and $v \mapsto \sigma(v)$ is a matrix-valued function $\sigma(v) = (\sigma_{i,j}(v),\ i,j = 1,\dots,d)$, where each $v \mapsto \sigma_{i,j}(v)$ infinitely differentiable with bounded partial derivatives. Assume also that σ is *strongly elliptic,* that is, there is $\rho > 0$ such that for all $v \in \mathbb{R}^d$ and $z \in \mathbb{R}^d$ with $\|z\| = 1$,

$$\|\sigma(v)z\|^2 = \sum_{i=1}^{d} \left(\sum_{j=1}^{d} \sigma_{i,j}(v) z_j \right)^2 \geq \rho^2.$$

The next theorem was obtained in [11, Theorem 5.1] for the reduced stochastic wave equation. In the case where the SPDE (5) also contains a nonlinear drift term, a slightly weaker result in given in [11, Corollary 5.3].

Theorem 4. *Let u be the solution of* (5). *Let $I = [t_0,t_1] \times [x_0,x_1]$ be a rectangle, where $0 < t_0 < t_1$ and $x_0 < x_1$ and let $M > 0$. Under the assumptions just stated, there exists a finite positive constant C such that, for all compacts sets $A \subset B(0,M)$ ($\subset \mathbb{R}^d$),*

$$\frac{1}{C} \operatorname{Cap}_{d-4}(A) \leq P\{u(I) \cap A \neq \emptyset\} \leq C \operatorname{Cap}_{d-4}(A).$$

It follows from this theorem (and the properties of capacity mentioned in Example 1) that $d = 4$ is the critical dimension for hitting points and points are polar in this critical dimension.

The proof of Theorem 4 uses Malliavin calculus, and, for the upper bound, Cairoli's maximal inequality for multiparameter martingales (see [14, Chapter 7]). This last property, which was already used by Khoshnevisan and Shi [16] for Theorem 1, will not be available for the other SPDEs that we will consider in this paper.

3.2 Other non-linear systems of SPDEs

In this subsection, we consider a wide class of systems of SPDEs, that includes in particular systems of heat and wave equations.

Let L be a partial differential operator. For instance,

$$L = \frac{\partial}{\partial t} - \Delta \qquad \text{or} \qquad L = \frac{\partial^2}{\partial t^2} - \Delta$$

in the case of the heat (respectively wave) operator, where Δ is the Laplacian in the spatial variables,

Let $u = (u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^k)$, where

$$u(t,x) = (u^1(t,x), \ldots, u^d(t,x)) \in \mathbb{R}^d$$

is the solution of

$$\begin{cases} Lu^1(t,x) = b^1(u(t,x)) + \sum_{j=1}^d \sigma_{1,j}(u(t,x))\dot{W}_j(t,x), \\ \quad \vdots \\ Lu^d(t,x) = b^d(u(t,x)) + \sum_{j=1}^d \sigma_{d,j}(u(t,x))\dot{W}_j(t,x), \end{cases} \tag{6}$$

where $t \in]0,T]$, $x \in \mathbb{R}^k$, with suitable initial conditions. The functions $b^i : \mathbb{R}^d \to \mathbb{R}$ and $\sigma_{i,j} : \mathbb{R}^d \to \mathbb{R}$, $i,j = 1,\ldots,d$, are assumed to be C^∞-functions with bounded derivatives of all positive orders. The matrix $\sigma = (\sigma_{i,j})$ is assumed to be strongly elliptic.

The choice of initial conditions should ensure that the system (6) is well-posed. For instance, in the case of the heat operator, the initial value $u(0,\cdot)$ is given, whereas for the wave operator, the initial velocity $\frac{\partial}{\partial t}u(0,\cdot)$ should also be given.

The noise process $\dot{W}(t,x) = (\dot{W}_1(t,x),\dots,\dot{W}_d(t,x))$ is assumed to be Gaussian, white in time, and possibly correlated in space. We will only consider the spatially homogeneous case, with a spatial correlation given by a Riesz kernel. More precisely, we fix $k \geq 1$ and suppose $\beta \in]0,k\wedge 2[$ or $k = 1 = \beta$. We suppose that the covariance of the noise is

$$E(\dot{W}_\ell(t,x)\dot{W}_j(s,y)) = \delta(t-s)\|x-y\|^{-\beta}\delta_{\ell,j}, \tag{7}$$

unless $k = 1 = \beta$, in which $\dot{W}$ is an $\mathbb{R}^d$-valued space-time white noise.

For many choices of the operator L, the system (6) has a unique solution, and the optimal Hölder exponents for the solution can be determined. Often, the Hölder regularity is different in the time variable and in the spatial variable (or could even be different in each of the spatial variables). We assume here that we have, for all $p \geq 2$, the bounds

$$c(p)\Delta(t,x;s,y) \leq \|u(t,x)-u(s,y)\|_{L^p} \leq C(p)\Delta(t,x;s,y), \tag{8}$$

where

$$\Delta(t,x;s,y) = |t-s|^{H_1} + \|x-y\|^{H_2} \tag{9}$$

and $H_1, H_2 \in]0,1]$. Define

$$Q = \frac{1}{H_1} + \frac{k}{H_2}. \tag{10}$$

The type of result that has been obtained in many cases, which we will summarize in Section 3.5, takes the following form.

Typical result 5 *Fix $\eta > 0$ and $M > 0$. Let $I = [t_0,t_1]$, with $0 < t_0 < t_1$, and let J be a box in $\mathbb{R}^k$. Then there are positive constants c_η and C_η such that, for all compact sets $A \subset B(0,M)$ ($\subset \mathbb{R}^d$),*

$$c_\eta \, \mathrm{Cap}_{d-Q+\eta}(A) \leq P\{u(I\times J)\cap A \neq \emptyset\} \leq C_\eta \mathscr{H}_{d-Q-\eta}(A). \tag{11}$$

The bounds in (11) are similar to those in Theorem 2, with Hausdorff measure on the right-hand side. They are slightly less good, since there is an additional term $+\eta$ (resp. $-\eta$) in the dimension of the capacity (resp. the Hausdorff measure).

The inequalities in (11) imply that the critical dimension for hitting points is $d = Q$, with Q defined in (10) (assuming that Q is an integer). However, in the critical dimension $d = Q$, the issue of polarity of points is not answered.

The bounds in (11) and the methods used to obtain these bounds also lead to information about the Hausdorff dimensions of level sets of u and of the range of u, as well as to bounds on the probability that a level set of u meets a given subset of

$\mathbb{R}_+ \times \mathbb{R}^k$: see for instance [6, Theorems 2.4 and 3.2] as well as [7, Corollary 1.5 and Theorem 1.6].

In the next two sections, we explain the main methods that have been developed to prove this kind of result. Then we will indicate specific PDE operators L for which this program has been carried out.

3.3 Proving the upper bound

Beginning with [6, Theorem 3.3], various sufficient conditions for obtaining the upper bound in (11) have been identified (see also [8, Section 2.3]). These apply in principle to any continuous random field $u = \{u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^k\}$.

Theorem 6. *Let $D \subset \mathbb{R}^d$. In addition to knowing the Hölder exponents H_1 and H_2 that appear in the upper bound in* (8)*, assume that for any $x \in \mathbb{R}^k$, $u(t,x)$ has a probability density function $p_{t,x}$, and*

$$\sup_{z\in D}\ \sup_{(t,x)\in I\times J} p_{t,x}(z) \leq C < \infty. \tag{12}$$

Then for any $\eta > 0$, for every Borel set $A \subset D$,

$$P\{u(I)\cap A \neq \emptyset\} \leq C\mathscr{H}_{d-Q-\eta}(A). \tag{13}$$

We note that the conclusion (13) only uses the upper bound in (8) (the lower bound is not needed). In the case of a Gaussian random field, condition (12) is usually easy to verify. In the non-Gaussian case, the existence of a probability density function and the uniform bound in (12) can often be obtained by using Malliavin calculus.

3.4 Proving the lower bound

Sufficient conditions for obtaining the lower bound in (11) have also been identified in [6]. These conditions were later weakened in [12, Remark 3.9]. Let $u = \{u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ be an $\mathbb{R}^d$-valued continuous process.

Theorem 7. *Let $\Delta(s,y;t,x)$ be defined as in* (9)*. Assume that:*

(a) for all $(t,x) \in I \times J$, the probability density function of $u(t,x)$ exists, is continuous and strictly positive;

(b) for any $(t,x) \neq (s,y)$ in $I \times J$, the (two-point) probability density function $p_{s,y;t,x}$ of $(u(s,y),u(t,x))$ exists, and there are $c > 0$, $\gamma \geq \frac{1}{H_1} + \frac{k}{H_2}$ and $p > d(\gamma - \frac{1}{H_1} - \frac{k}{H_2})$ such that for all $z_1, z_2 \in [-N,N]^d$,

$$p_{s,y;t,x}(z_1,z_2) \le c[\Delta(s,y;t,x)]^{-\gamma} \left[\frac{(\Delta(s,y;t,x))^2}{\|z_1-z_2\|^2} \wedge 1 \right]^{p/2d}. \tag{14}$$

Then

$$P\{u(I\times J)\cap A \ne \emptyset\} \ge c\,\mathrm{Cap}_{\gamma-Q}(A),$$

where $Q = \frac{1}{H_1} + \frac{k}{H_2}$.

Usually, the best possible choice of γ is $\gamma = d$. Both properties (a) and (b) can be obtained by using Malliavin calculus. For instance, an early result on the positivity of densities was obtained by Kohatsu-Higa [17] (see also [20]). The upper bound in (b) is often difficult to obtain. In some cases [7], a stronger upper bound has been obtained.

Since Theorem 7 is not stated as such anywhere in the literature, we shall sketch its proof here. The main technical ingredient is the following anisotropic extension of [12, Theorem 3.8] (see also [12, Remark 3.9]). Recall the definition of the function $K_\alpha(r)$ in (1).

Lemma 8 *Suppose that condition (b) of Theorem 7 holds. For any I, K compact subsets of* $[0,T]$ *and* $\mathbb{R}^k$*, respectively, both with diameter* ≤ 1*, there exists a constant* $C = C(H_1,H_2,\gamma,d,k,N)$ *such that, for every* $z_1,z_2 \in \mathbb{R}^d$ *with* $0 \le \|z_1-z_2\| \le N$*,*

$$\mathscr{I} := \int_{I\times K} dt dx \int_{I\times K} ds dy\, p_{s,y;t,x}(z_1,z_2) \le C K_{\gamma - \frac{1}{H_1} - \frac{k}{H_2}}(\|z_1-z_2\|).$$

Proof. Define $\eta = \|z_1-z_2\|$, and suppose that $\rho_0 > 0$ is such that $I\times K \subset \{(s,y)\in \mathbb{R}_+\times\mathbb{R}^k : |s|^{H_1} + \|y\|^{H_2} \le \rho_0\}$. Clearly, $\mathscr{I} = \mathscr{I}_1 + \mathscr{I}_2$, where

$$\mathscr{I}_1 = \int_{I\times K} dt dx \int_{I\times K} ds dy\, 1_{\{\Delta(s,y;t,x)\le \frac{\rho_0\eta}{N}\}}\, p_{s,y;t,x}(z_1,z_2),$$
$$\mathscr{I}_2 = \int_{I\times K} dt dx \int_{I\times K} ds dy\, 1_{\{\Delta(s,y;t,x) > \frac{\rho_0\eta}{N}\}}\, p_{s,y;t,x}(z_1,z_2).$$

We bound each term separately. Observe that by (14) and since $p>0$,

$$\begin{aligned}
\mathscr{I}_1 &\le c\int_{I\times K} dt dx \int_{I\times K} ds dy\, 1_{\{\Delta(s,y;t,x)\le \frac{\rho_0\eta}{N}\}} [\Delta(s,y;t,x)]^{-\gamma} \left[\frac{(\Delta(s,y;t,x))^2}{\|z_1-z_2\|^2}\right]^{p/2d} \\
&= c\int_{I\times K} dt dx \int_{I\times K} ds dy\, 1_{\{\Delta(s,y;t,x)\le \frac{\rho_0\eta}{N}\}} [\Delta(s,y;t,x)]^{-\gamma+\frac{p}{d}}\, \eta^{-\frac{p}{d}} \\
&\le \tilde c\, \eta^{-\frac{p}{d}} \int_0^{\rho_0^{H_1^{-1}}} dr \int_0^{\rho_0^{H_2^{-1}}} du\, u^{k-1}\, 1_{\{r^{H_1}+u^{H_2}\le \frac{\rho_0\eta}{N}\}} (r^{H_1}+u^{H_2})^{-\gamma+\frac{p}{d}}.
\end{aligned}$$

Use the change of variables $w = u^{H_2/H_1}$ to see that

$$\mathscr{I}_1 \le \tilde c'\, \eta^{-\frac{p}{d}} \int_0^{\rho_0^{H_1^{-1}}} dr \int_0^{\rho_0^{H_1^{-1}}} dw\, w^{\frac{H_1}{H_2}-1} w^{(k-1)\frac{H_1}{H_2}}\, 1_{\{r^{H_1}+w^{H_1}\le \frac{\rho\eta}{N}\}} (r^{H_1}+w^{H_1})^{-\gamma+\frac{p}{d}}.$$

Pass to polar coordinates in the variables (r,w) to see that

$$\mathscr{I}_1 \leq \tilde{c}'' \eta^{-\frac{p}{d}} \int_0^{2(\frac{\rho\eta}{N})^{H_1^{-1}}} d\rho\, \rho\, \rho^{k\frac{H_1}{H_2}-1} \rho^{-\gamma H_1 + \frac{pH_1}{d}}.$$

Since $k\frac{H_1}{H_2} - \gamma H_1 + \frac{pH_1}{d} > -1$ because $p > d(\gamma - \frac{1}{H_1} - \frac{k}{H_2})$ by hypothesis, the integral is finite and we obtain

$$\mathscr{I}_1 \leq \hat{c}'' \eta^{-\frac{p}{d}} \eta^{\frac{k}{H_2} - \gamma + \frac{p}{d} + \frac{1}{H_1}} = \hat{c}'' \eta^{-(\gamma - \frac{1}{H_1} - \frac{k}{H_2})}. \tag{15}$$

Now observe that by (14) and since $p > 0$,

$$\begin{aligned}
\mathscr{I}_2 &\leq c \int_{I\times K} dt dx \int_{I\times K} ds dy\, 1_{\{\Delta(s,y;t,x) > \frac{\rho_0\eta}{N}\}} [\Delta(s,y;t,x)]^{-\gamma} \\
&\leq \tilde{c} \int_0^{\rho_0^{H_1^{-1}}} dr \int_0^{\rho_0^{H_2^{-1}}} du\, 1_{\{r^{H_1} + u^{H_2} > \frac{\rho_0\eta}{N}\}} u^{k-1} (r^{H_1} + u^{H_2})^{-\gamma} \\
&\leq \tilde{c} (\mathscr{I}_{2,1} + \mathscr{I}_{2,2} + \mathscr{I}_{2,3}),
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{I}_{2,1} &= \int_0^{(\frac{\rho_0\eta}{2N})^{H_2^{-1}}} du \int_{(\frac{\rho_0\eta}{2N})^{H_1^{-1}}}^{\rho_0^{H_1^{-1}}} dr\, u^{k-1} (r^{H_1} + u^{H_2})^{-\gamma}, \\
\mathscr{I}_{2,2} &= \int_{(\frac{\rho_0\eta}{2N})^{H_2^{-1}}}^{\rho_0^{H_2^{-1}}} du \int_0^{(\frac{\rho_0\eta}{2N})^{H_1^{-1}}} dr\, u^{k-1} (r^{H_1} + u^{H_2})^{-\gamma}, \\
\mathscr{I}_{2,3} &= \int_{(\frac{\rho_0\eta}{2N})^{H_1^{-1}}}^{\rho_0^{H_1^{-1}}} dr \int_{(\frac{\rho_0\eta}{2N})^{H_2^{-1}}}^{\rho_0^{H_2^{-1}}} du\, u^{k-1} (r^{H_1} + u^{H_2})^{-\gamma}.
\end{aligned}$$

Clearly, since $\gamma > H_1^{-1}$,

$$\begin{aligned}
\mathscr{I}_{2,1} &\leq c \int_0^{(\frac{\rho_0\eta}{2N})^{H_2^{-1}}} du\, u^{k-1} \int_{(\frac{\rho_0\eta}{2N})^{H_1^{-1}}}^{\rho_0^{H_1^{-1}}} dr\, r^{-\gamma H_1} \leq c' \eta^{kH_2^{-1}} \eta^{-\gamma + H_1^{-1}} \\
&= c' \eta^{-(\gamma - \frac{1}{H_1} - \frac{k}{H_2})}.
\end{aligned} \tag{16}$$

Similarly, since $\gamma > kH_2^{-1}$,

$$\begin{aligned}
\mathscr{I}_{2,2} &\leq c \int_0^{(\frac{\rho_0\eta}{2N})^{H_1^{-1}}} dr \int_{(\frac{\rho_0\eta}{2N})^{H_2^{-1}}}^{\rho_0^{H_2^{-1}}} du\, u^{k-1-\gamma H_2} \leq c' \eta^{H_1^{-1}} \eta^{kH_2^{-1} - \gamma} \\
&\leq c' \eta^{-(\gamma - \frac{1}{H_1} - \frac{k}{H_2})}.
\end{aligned} \tag{17}$$

Finally,

$$\mathscr{I}_{2,3} \leq c \int_{(\frac{\rho_0\eta}{2N})^{H_1^{-1}}}^{\rho_0^{H_1^{-1}}} dr \int_{(\frac{\rho_0\eta}{2N})^{H_2^{-1}}}^{\rho_0^{H_2^{-1}}} du \, \frac{u^{k-1}}{r^{\gamma H_1} + u^{\gamma H_2}}.$$

Use the change of variables $w = u^{H_2/H_1}$ to see that

$$\mathscr{I}_{2,3} \leq \tilde{c} \int_{(\frac{\rho_0\eta}{2N})^{H_1^{-1}}}^{\rho_0^{H_1^{-1}}} dr \int_{(\frac{\rho_0\eta}{2N})^{H_1^{-1}}}^{\rho_0^{H_1^{-1}}} dw \, \frac{w^{\frac{H_1}{H_2}-1} w^{(k-1)\frac{H_1}{H_2}}}{r^{\gamma H_1} + w^{\gamma H_1}}.$$

We bound the integrand above by $cw^{k\frac{H_1}{H_2}-1}(r+w)^{-\gamma H_1}$, then pass to polar coordinates in the variables (r,w) to see that

$$\mathscr{I}_{2,3} \leq c' \int_{(\frac{\rho_0\eta}{2N})^{H_1^{-1}}}^{2\rho_0^{H_1^{-1}}} d\rho \, \rho \, \rho^{k\frac{H_1}{H_2}-1} \rho^{-\gamma H_1}. \tag{18}$$

If $\gamma > \frac{1}{H_1} + \frac{k}{H_2}$, then we replace $2\rho^{H_1^{-1}}$ by $+\infty$ in the upper bound, to get

$$\mathscr{I}_{2,3} \leq \tilde{c}' \, \eta^{-(\gamma - \frac{1}{H_1} - \frac{k}{H_2})}. \tag{19}$$

Putting together (15)–(17) and (19) proves the lemma when $\gamma > \frac{1}{H_1} + \frac{k}{H_2}$.

If $\gamma = \frac{1}{H_1} + \frac{k}{H_2}$, then from (18),

$$\mathscr{I}_{2,3} \leq c \int_{(\frac{\rho_0\eta}{2N})^{H_1^{-1}}}^{2\rho_0^{H_1^{-1}}} d\rho \, \rho^{-1} = c \left[\log(2\rho_0^{H_1^{-1}}) + H_1^{-1} \log\left(\frac{2N}{\rho_0 \eta}\right) \right], \tag{20}$$

and this is bounded above by $cK_0(\eta)$. Putting together (15)–(17) and (20) proves the lemma when $\gamma = \frac{1}{H_1} + \frac{k}{H_2}$. □

With Lemma 8, the proof of Theorem 7 follows as in [12, Section 3.2].

3.5 Results for systems of non-linear equations

We now list the specific operators for which the methods outlined in Section 3.2–3.4 have been completely carried out.

Heat equation, $k = 1$, space-time white noise [7]. In this case,

$$Lu^i(t,x) = \frac{\partial u^i}{\partial t}(t,x) - \frac{\partial^2 u^i}{\partial x^2}(t,x)$$

and $\dot{W}_j(t,x)$ is space-time white noise. For any fixed $\eta > 0$, the bound (14) has been proved with $\gamma = d + \eta$. It is well-known that $H_1 = \frac{1}{4}$, $H_2 = \frac{1}{2}$, so the bounds on hitting probabilites are

$$c_\eta \text{ Cap}_{d+\eta-6}(A) \le P\{u(I\times J)\cap A \neq \emptyset\} \le C_\eta \mathscr{H}_{d-\eta-6}(A)$$

(see [7, Theorem 1.2]). These bounds are consistent with the result of Theorem 3.

Heat equation, $k \ge 1$, spatially homogeneous noise [8]. In this case,

$$Lu^i(t,x) = \frac{\partial u^i}{\partial t}(t,x) - \Delta u^i(t,x)$$

and $\dot W(t,x)$ is spatially homogeneous noise with covariance given in (7).

It is shown in [23, Theorem 2.1] that for any $H_1 < \frac{2-\beta}{4}$ and $H_2 < \frac{2-\beta}{2}$, the upper bound of (8) holds. For any $\eta > 0$, the inequality (14) has been obtained for $\gamma = d+\eta$. Setting $Q = \frac{4+6k}{2-\beta}$, the bounds on hitting probabilities are

$$c_\eta \text{ Cap}_{d+\eta-Q}(A) \le P\{u(I\times J)\cap A \neq \emptyset\} \le C_\eta \mathscr{H}_{d-\eta-Q}(A)$$

(see [8, Theorem 1.2]).

Wave equation, $k \in \{1,2,3\}$, spatially homogeneous noise [12]. In this case,

$$Lu^i(t,x) = \frac{\partial^2 u^i}{\partial t^2}(t,x) - \Delta u^i(t,x)$$

and $\dot W_j(t,x)$ is spatially homogeneous noise with covariance given in (7).

It is shown in [12, Proposition 2.2] that the upper bound of (8) holds with

$$H_1 = H_2 = \frac{2-\beta}{2}.$$

For any $\eta > 0$, the inequality (14) has been obtained for $\gamma = d+\eta+\frac{4d^2}{2-\beta}$ ($\eta > 0$). Defining Q as in (10), the bounds on hitting probabilities are

$$c_\eta \text{ Cap}_{d+\eta-Q+\frac{4d^2}{2-\beta}}(A) \le P\{u(I\times J)\cap A \neq \emptyset\} \le C_\eta \mathscr{H}_{d-\eta-Q}(A) \qquad (21)$$

(see [12, Theorems 2.1 and 4.3]). It is expected that the optimal value of γ should be no more than $\gamma = d+\eta$, and this would considerably improve the lower bound in (21).

4 Polarity of points in critical dimensions

As we have mentioned, the results in Sections 3.2–3.5 do not answer the question of polarity of points in critical dimensions. Building on works of Talagrand [25, 26] for fractional Brownian motion, we have obtained results in this direction for a wide class of Gaussian random fields.

Let $v = (v(t,x),\ t \geq 0, x \in \mathbb{R}^k)$ be an $\mathbb{R}^d$-valued Gaussian random field with i.i.d. components. Fix $I \times J$ as in (11). Suppose that there are $C > 0$ and $H_1, H_2 \in]0,1[$ such that for all $(t,x), (s,y) \in I \times J$,

$$\|v(t,x) - v(s,y)\|_{L^2} \leq C\Delta(t,x;s,y) := |t-s|^{H_1} + \|x-y\|^{H_2}. \tag{22}$$

We consider the following assumption on v.

Assumption 9 *There is a random field* $(V(A,t,x),\ A \in \mathscr{B}(\mathbb{R}_+), t \geq 0, x \in \mathbb{R}^k)$ *and* $\varepsilon_0 > 0$ *such that:*

(a) for fixed $(t,x) \in (I \times J)^{(\varepsilon_0)}$ *(this denotes an* ε_0*-enlargement of* $I \times J$*),* $A \mapsto V(A,t,x)$ *is an* $\mathbb{R}^d$*-valued Gaussian white noise;*

(b) when $A \cap B = \emptyset$*,* $V(A,\cdot,\cdot)$ *and* $V(B,\cdot,\cdot)$ *are independent;*

(c) there are constants $c_0 \in \mathbb{R}_+$*,* $a_0 \in \mathbb{R}$ *and* $\gamma_1 > 0$*,* $\gamma_2 > 0$ *such that for all* $a_0 \leq a \leq b \leq +\infty$*,* (t,x)*,* $(s,y) \in (I \times J)^{(\varepsilon_0)}$*:*

$$\begin{aligned} &\|v(t,x) - v(s,y) - (V([a,b[,t,x) - V([a,b[,s,y))\|_{L^2} \\ &\qquad \leq c\left[a^{\gamma_1}|t-s| + a^{\gamma_2}\|x-y\| + b^{-1}\right] \end{aligned} \tag{23}$$

and

$$\|V([0,a_0],t,x) - V([0,a_0],s,y)\|_{L^2} \leq c_0(|t-s| + \|x-y\|). \tag{24}$$

(d) There is a constant $\tilde{c} > 0$ *such that for all* $(t,x) \in (I \times J)^{(\varepsilon_0)}$*, and* $i = 1,\ldots,d$*, we have* $\|v_i(t,x)\|_{L^2} \geq \tilde{c}$*;*

(e) There is $\rho > 0$ *with the following property. For* $(t,x) \in I \times J$*, there are* $(t',x') \in (I \times J)^{(\varepsilon_0)}$*,* $\delta_j \in]\alpha_j, 1]$*,* $j = 1,2$*, and* $C > 0$ *such that for all* $i = 1,\ldots,d$*,* $(s,y), (\bar{s},\bar{y}) \in (I \times J)^{(\varepsilon_0)}$ *with* $\Delta(s,y;t,x) \leq 2\rho$ *and* $\Delta(\bar{s},\bar{y};t,x) \leq 2\rho$*,*

$$\left|E[(v_i(s,y) - v_i(\bar{s},\bar{y}))v_i(t',x')]\right| \leq C(|s-\bar{s}|^{\delta_1} + \|y-\bar{y}\|^{\delta_2}).$$

Remark 1. (1) If there exist exponents γ_j such that (23) holds, then it can be checked that a possible choice for the Hölder exponents in (22) is that they satisfy

$$\gamma_j = \tfrac{1}{H_j} - 1 \tag{25}$$

(see [10, Proposition 2.2]).

(2) Condition (c) states that if $|t-s| \sim 2^{-n/H_1}$ and $\|x-y\| \sim 2^{-n/H_2}$, then the increment $\|v(t,x) - v(s,y)\|$ is well-approximated by the increment $\|V([2^n, 2^{n+1}[,t,x) - V([2^n, 2^{n+1}[,s,y)\|$. If we are considering several increments over boxes of different sizes, then this approximation is useful because V has lots of independence built into it.

(3) Property (d) is a non-degeneracy assumption, while property (e) states that covariances of v_i are smoother than sample paths of v (since $\delta_j > \alpha_j$).

Theorem 10. [10, Theorem 2.6] *Under Assumption 9, if $d = Q := \frac{1}{H_1} + \frac{k}{H_2}$, then for all $z \in \mathbb{R}^d$,*

$$P\{\exists (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^k : v(t,x) = z\} = 0,$$

that is, points are polar for v.

It turns out that Theorem 10 is widely applicable, since Assumption 9 is satisfied by the solutions to many systems of linear SPDEs. In addition, in the case where $\Delta(t,x;s,y)$ not only satisfies the upper bound (22) but also the lower bound (8), we know from Section 3.5 that Q defined in Theorem 10 is the critical dimension for the SPDEs considered there. So the next proposition establishes polarity of points in the critical dimension for several types of SPDEs.

Proposition 11 [10, Sections 7–9] *Let $v = (v(t,x))$ be the solution of a system of linear SPDEs. Assumption 9 is satisfied in the following cases:*

(a) systems of linear heat *equations in spatial dimension $k = 1$, driven by space-time white noise, with possibly non-constant coefficients;*

(b) systems of linear wave *equations in spatial dimension $k = 1$, driven by space-time white noise;*

(c) systems of linear heat *equations in spatial dimension $k \geq 1$, driven by spatially homogeneous Gaussian noise with covariance given in* (7)*;*

(d) systems of linear wave *equations in spatial dimension $k \geq 1$, driven by spatially homogeneous Gaussian noise with covariance given in* (7).

In particular, in each of these four cases, points are polar for v in the critical dimension.

Via Case (a) of Proposition 11, one recovers the results of Theorem 3. As as example of how one uses Theorem 10 to establish the claims of Proposition 11, we consider the case of *wave* equations in spatial dimension 1 driven by space-time white noise $\hat{W}$. Let $v = (v(t,x), t \in \mathbb{R}_+, x \in \mathbb{R})$ solve

$$\begin{cases} \frac{\partial^2}{\partial t^2} v_j(t,x) = \frac{\partial^2}{\partial x^2} v_j(t,x) + \dot{\hat{W}}_j(t,x), & j = 1,\dots,d, \\ v(0,x) = 0, \qquad \frac{\partial}{\partial t} v(0,x) = 0, & x \in \mathbb{R}. \end{cases} \tag{26}$$

Corollary 12 [10, Theorem 9.1] *Suppose $d = 4$ (critical dimension). Then points are polar for v.*

The main point, in order to apply Theorem 10, is to determine the random field V of Assumption 9. Letting $G(t,x)$ denote the fundamental solution of the wave equation, it is well-known that the solution of (26) has the representation

$$v(t,x) = \int_0^t \int_{\mathbb{R}} G(t-s, x-y)\, \hat{W}(ds,dy). \tag{27}$$

Taking space-time Fourier transforms of G and $\hat{W}$ leads to the representation

$$v(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-i\xi \cdot x - i\tau t}}{2|\xi|} \left[\frac{1-e^{it(\tau+|\xi|)}}{\tau+|\xi|} - \frac{1-e^{it(\tau-|\xi|)}}{\tau-|\xi|} \right] W(d\tau, d\xi), \qquad (28)$$

where W is another space-time white noise. Note that this representation already appears in [1]. We call it a *harmonizable* representation of v.

With the representation (28), it is shown in [10, Lemmas 9.3 and 9.6] that Assumption 9 is satisfied by setting

$$V(A,t,x) := \iint_{\{\max(|\tau|^{\frac{1}{2}}, |\xi|^{\frac{1}{2}}) \in A\}} \frac{e^{-i\xi \cdot x - i\tau t}}{2|\xi|} \left[\frac{1-e^{it(\tau+|\xi|)}}{\tau+|\xi|} - \frac{1-e^{it(\tau-|\xi|)}}{\tau-|\xi|} \right] W(d\tau, d\xi),$$

where W is again a space-time white noise.

For condition (c) of Assumption 9, the formula (25) applied to the Hölder exponent $\frac{1}{2}$ gives $\gamma := (\frac{1}{2})^{-1} - 1 = 1$, so it is necessary to check that

$$\begin{aligned} &\|v(t,x) - v(s,y) - (V([a,b[,t,x) - V([a,b[,s,y))\|_{L^2} \\ &\quad \le c_0 \left[a^1 |t-s| + a^1 |x-y| + b^{-1} \right]. \end{aligned}$$

Proving this inequality requires estimating some double integrals.

It turns out that for the other examples mentioned in Proposition 11, the same Fourier transform method applied to the standard representation (27) applies, and then the candidate process V is easily obtained as above from the harmonizable representation of v.

In future work, we plan to extend the above results to the solution of systems of nonlinear SPDEs, and to study the issue of existence/nonexistence of multiple point in critical dimensions, extending the recent result of [9].

Finally, we also mention the papers [4, 5, 21, 24], which are also concerned with hitting probabilities for SPDEs.

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