LEVEL SETS, BUBBLES AND EXCURSIONS OF A BROWNIAN SHEET

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Abstract

This paper presents several recent results concerning level sets, bubbles and excursions of a Brownian sheet, along with the main methods of proof and directions of current research.

1 Introduction

This paper discusses several recent developments in the study of level sets, bubbles and excursions of the Brownian sheet. It intends to provide a review of much of the existing body of literature on this topic and to present some new results on level sets and excursions, along with key techniques in their proofs and some open problems.

The Brownian sheet is a real-valued, centered and continuous Gaussian process $(W(t),\ t \in \mathbb{R}^2_+)$ indexed by the positive quadrant in the plane, with covariance given by

$$E(W(s_1, s_2)W(t_1, t_2)) = (s_1 \wedge t_1)(s_2 \wedge t_2). \tag{1}$$

It is one of the natural extensions of Brownian motion to higher-dimensional time. No single multiparameter Gaussian process can play as central a role as standard Brownian motion plays in the theory of one-parameter processes. Indeed, one could consider the solutions to the heat equation driven by white noise, to the wave equation driven by white noise (essentially, the Brownian sheet [30]), and to an elliptic equation driven by white noise (such as the Whittle sheet [23]), to all be basic random fields of equal importance, and each with very different properties. There are also other important Gaussian random fields, such as Lévy's Brownian motion [14, 22, 26]. However, the Brownian sheet is probably the random field that arises in the widest number of contexts and has been the most studied.

Study of the Brownian sheet began with work of Kitagawa [17], Chentsov [4] and Yeh [28]. Important results on sample path properties were then obtained by Orey and Pruitt [21], Czörgö and Révèsz [5], Pyke [24] and Walsh [29], among many others. Motivation for the work reported here comes from results on level sets by Adler [1] and Kendall [16], followed by more recent work of Dalang and Walsh [11, 12].

2 Basic properties of the Brownian sheet

One immediately observes from Eq. (1) that the Brownian sheet vanishes on the coordinate axes, and that for fixed s_2 (resp. fixed s_1), the map $s_1 \mapsto W(s_1, s_2)$

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(resp. $s_2 \mapsto W(s_1, s_2)$ is a Brownian motion with speed s_2 (resp. s_1). Many other Gaussian processes are also embedded in the Brownian sheet. For instance, $u \mapsto W(u, 1-u)$ is a Brownian bridge, while $u \mapsto W(u, 1/u)$ is an Ornstein-Uhlenbeck process (see [30]).

The fact that $W(s_1, 0) = W(0, s_2) = 0$ is akin to an initial condition, and it is a common misunderstanding to assume that properties of the Brownian sheet are induced by this initial condition. In fact, the Brownian sheet can be equivalently defined as the solution of the stochastic hyperbolic partial differential equation

$$\frac{\partial^2}{\partial s_1 \partial s_2} W(s_1, s_2) = \dot{\eta}(s_1, s_2),$$

where $\dot{\eta}$ denotes two-parameter white noise, with the initial conditions $W(s_1, 0) = W(0, s_2) = 0$, for all $s_1 \geq 0$ and $s_2 \geq 0$. Most properties of the Brownian sheet that relate to horizontal and vertical lines, including the surprising "propagation of singularities" phenomenon discovered by Walsh in [29], are not induced by the coordinate axes but by the fact that the characteristic directions of the hyperbolic operator $\partial^2/(\partial s_1 \partial s_2)$ are the horizontal and vertical directions.

Let $\leq$ be the (partial) order on $\mathbb{R}^2$ defined by

$$s = (s_1, s_2) \leq t = (t_1, t_2) \quad \Longleftrightarrow \quad s_1 \leq t_1 \text{ and } s_2 \leq t_2.$$

Viewed along any monotone increasing curve (in this partial order), the Brownian sheet is a Brownian motion (after a deterministic time-change). Clearly, if one thinks of the Brownian sheet as a random function of two real variables, then the surface that it defines is extremely irregular and its restriction to any monotone curve looks like a sample path of ordinary Brownian motion.

3 Problems that lead to the Brownian sheet

The study of the Brownian sheet can be motivated by the many contexts in which it arises. We mention two of them. The first is a problem in bivariate statistics that makes use of the Brownian sheet.

Consider a sequence (X_n) of independent and identically disributed random variables that are uniformly distributed on the unit square $[0, 1]^2$. Let $F_n(t_1, t_2)$ denote the number of integers $i \in \{1, \ldots, n\}$ such that $X_i \in [0, t_1] \times [0, t_2]$. One can show (cf. [2]) that

$$\frac{F_n(t_1, t_2) - n t_1 t_2}{\sqrt{n}} \Rightarrow \tilde{W}(t_1, t_2),$$

where $\tilde{W}(t_1, t_2) = W(t_1, t_2) - t_1 t_2 W(1, 1)$. The process $(\tilde{W}(t),\ t \in [0, 1]^2)$ is termed the *pinned Brownian sheet* and is analogous to the one-parameter Brownian bridge. Its properties are clearly very closely related to those of the Brownian sheet. If one wants to statistically test whether or not the random variables X_n are indeed uniformly distributed, then one needs information about the pinned Brownian sheet. In order to build confidence intervals and to compute p-values, knowledge about the distribution of the maximum of the Brownian sheet is needed. However, even though there are many asymptotic formulas for tails of this distribution [3, 27], there is *no known exact formula for this distribution,* which may explain why questions regarding distributional properties of the Brownian sheet are generally very difficult to answer.

The Brownian sheet also arises in other areas of stochastic analysis. For instance, in the study of Wiener space and Malliavin calculus, the Brownian sheet provides a representation of the Ornstein-Uhlenbeck process on Wiener space. This process, sometimes called the Malliavin process on Wiener space, is the $C(\mathbb{R}_+, \mathbb{R})$-valued solution $(X_t,\ t \geq 0)$ of the infinite-dimensional stochastic differential equation

$$dX_t(\cdot) = -X_t(\cdot)\, dt + dW(t, \cdot).$$

The distribution of this process is stationary in time, and for a fixed time t, the law of X_t is that of a standard Brownian motion. It turns out that the Brownian sheet provides a very simple representation of this process ([18, 15]):

$$X_t(s) = e^{-t}\ W(s, e^{2t}), \qquad s \geq 0,\ -\infty \leq t \leq +\infty.$$

This representation has proven very effective in the study of the Malliavin process (see for instance [19]).

4 Level sets, bubbles and excursions

For $x \in \mathbb{R}$, the non-negative quadrant can be decomposed into three subsets as follows:

$$\begin{aligned} L(x) &= \{s \in \mathbb{R}^2_+ : W(s) = x\}, \\ L_+(x) &= \{s \in \mathbb{R}^2_+ : W(s) > x\}, \\ L_-(x) &= \{s \in \mathbb{R}^2_+ : W(s) < x\}. \end{aligned}$$

Because sample paths of the Brownian sheet are continuous, the first subset is closed, while the second and third are open sets: each of these is a countable union of *components,* that is, of connected open subsets. In addition, the level set $L(x)$ is the common boundary of these two sets.

The first set of questions of interest here concerns the Hausdorff dimension of the level set and related sets. Other questions include topological properties of the level set (cf. [16], [5]). For instance, Kendall [16] proved that *almost all* points $t \in L(x)$ are points at which the level set is totally disconnected, that is, the connected component of $L(x)$ that contains t consists of the single point t. The "almost all" is with respect to local time measure on $L(x)$. On the other hand, the level set $L(x)$ separates the two open subsets $L_+(x)$ and $L_-(x)$. As such, a deterministic result from planar topology (see [16]) implies that this set cannot be totally disconnected, so it must contain non-trivial connected subsets. However, there is currently no information on the nature of such subsets (e.g. their Hausdorff dimension, where they occur, what such subsets might look like, etc), though it is known that they cannot consist of Jordan arcs that are differentiable even at a single point [6].

Another set of questions of interest here concerns geometric properties of the decomposition

$$\mathbb{R}^2_+ = L_-(x) \cup L(x) \cup L_+(x).$$

Indeed, one would like to understand the nature of the contact between $L_-(x)$ and $L_+(x)$, and the nature of $L(x)$ in the neighborhood of boundary points of components of $L_-(x)$. Several results in this direction are described in the papers of Dalang and Walsh [11, 12].

Much recent progress in understanding the nature of the contact between $L_-(x)$ and $L_+(x)$ comes from the study of points of increase. This will be described in Section 6 below.

Finally, just as there has been for several years a very good understanding of excursions of ordinary Brownian motion, one would like to understand the behavior of excursions of the Brownian sheet, where an excursion is defined as follows. First, we term a *Brownian bubble* a single connected component of $L_\pm(x)$. An *excursion* of the Brownian sheet is then the restriction of the Brownian sheet to any fixed bubble (for instance, the bubble that contains the point $t = (1, 1)$). The excursion is *positive* (resp. *negative*) if the restriction of the Brownian sheet to this bubble is positive (resp. negative). We shall describe below a surprising recent result concerning the lack of independence between different excursions of the Brownian sheet.

5 Hausdorff dimensions

The main result on the Hausdorff dimension of level sets of the Brownian sheet is the following theorem, due to R. Adler [1].

Theorem 1 *Almost surely, for all $x \in \mathbb{R}$, the Hausdorff dimension of $L(x)$ is $\frac{3}{2}$.*

In fact, Adler's result was proved for the d-parameter Brownian sheet, in which case $\frac{3}{2} = 2 - \frac{1}{2}$ should be replaced by $d - \frac{1}{2}$. Adler's result implies that $L(x)$ is a very complicated closed set. A picture of this closed set, obtained by simulation, can be found in [11].

A much more recent result, due to T. Mountford [20], establishes the following surprising property.

Theorem 2 *Almost surely, the Hausdorff dimension of the boundary of any fixed bubble is in the interval* $[1.25,\ 1.5[$.

Notice that the interval $[1.25,\ 1.5[$ is open on the right: the Hausdorff dimension of any fixed bubble is strictly less than $\frac{1}{2}$, which means that "most of $L(x)$ is in between the boundaries of individual bubbles."

The following remains an open problem.

Problem 1. Do the boundaries of all bubbles have the same Hausdorff dimension?

Recent work of the authors and D. Khoshnevisan motivates the following conjecture: the answer to the question in Problem 1 is yes, and the Hausdorff dimension of all bubbles should be

$$\frac{3}{2} - \frac{1}{4}\left(5 - \sqrt{13 + 4\sqrt{5}}\right).$$

6 Points of increase

A *point of increase* of a Brownian motion $(B(u),\ u \geq 0)$ is a (possibly random) time $u > 0$ such that for some $\varepsilon > 0$,

$$B(u-h) < B(u) < B(u+h), \qquad \text{for } 0 < h < \varepsilon.$$

Dvoretsky, Erdös and Kakutani [13] have established the following celebrated result.

Theorem 3 *Almost surely, Brownian motion has no point of increase.*

For the Brownian sheet, the question of existence of a point of increase can become either of the following : (a) do there exist bubbles of opposite signs that share a boundary point? or (b) Do there exist curves, or even monotone curves, along which the Brownian sheet has a point of increase? Formally, does there exist a (possibly random) function $\gamma : [0,1] \rightarrow]0,\infty[$, $r \in]0,1[$ and $\varepsilon > 0$ such that for $0 < h < \varepsilon$,

$$W(\gamma(r-h)) < W(\gamma(r)) < W(\gamma(r+h))? \tag{2}$$

If so, we say that W has a *point of increase* (at the random level $W(\gamma(r))$). If in fact, given $x \in I\!R$, there exists a (possibly random) function $\gamma : [0,1] \rightarrow]0,\infty[$ and $h > 0$ such that for $0 < h < \varepsilon$,

$$W(\gamma(r-h)) < x < W(\gamma(r+h)), \tag{3}$$

then we say that W has a *point of increase at the fixed level* x. Question (a) was answered affirmatively in [20], but since a boundary point of an open set in the plane may not be the endpoint of a curve that is otherwise contained in the open set, answering question (b) gives more information about the contact between bubbles.

Theorem 3, together with the basic properties mentioned in Section 2, show that the Brownian sheet cannot have a point of increase on any given deterministic monotone curve γ, as it behaves like a Brownian motion along such a curve. Therefore, any curve γ with property (2) or (3) must in fact be random.

The following result is due to Dalang and Mountford [7].

Theorem 4 *For all $x \in I\!R$, with positive probability, there exists a monotone curve contained in the square $[1,2]^2$ along which the Brownian sheet has a point of increase at level x.*

The conclusion can be made valid with probability one if the point of increase is allowed to occur anywhere in the positive quadrant.

The result of Theorem 4 implies that a bubble in $L_-(x)$ can share a common boundary point with a bubble of $L_+(x)$. On the other hand, there is no control over the regularity of the curve γ in (3).

The proof of Theorem 4 in [7] is quite demanding. On the other hand, it is natural to ask, as did John B. Walsh, whether or not the conclusion of Theorem 4 might not be true for any continuous function of two variables. It turns out that this is not the case, as shown in the following result of Dalang and Mountford [8].

Theorem 5 *Let A be the subset of $C(\mathbb{R}_+^2, \mathbb{R})$ that consists of functions f with the property that there exists a monotone curve along which f has a point of increase. Then A has first Baire category.*

Of course, the set $C(\mathbb{R}_+^2, \mathbb{R})$ is not of first Baire category, by the Baire Category Theorem, so this result implies that sample paths of the Brownian sheet are contained in a topologically small subset of $C(\mathbb{R}_+^2, \mathbb{R})$. In this regard, the behavior of the Brownian sheet differs from that of standard Brownian motion, whose sample paths are outside the (topologically small) set of functions which admit a point of increase.

In view of Theorem 5, it is natural to ask whether points of increase might exist along some particular type of curve, say even along horizontal lines. The following is currently an open problem.

Problem 2. Given $x \in \mathbb{R}$, do there exist horizontal lines along which the Brownian sheet has a point of increase at level x? More precisely, given $x \in \mathbb{R}$, is there a point $(T_1, T_2) \in]0, \infty[^2$ and $\varepsilon > 0$ such that

$$W((T_1 - h) \vee 0, T_2) < x < W(T_1 + h, T_2), \qquad \text{for } 0 < h < \varepsilon?$$

We note that if such horizontal lines do exist, then they must be random, because as mentioned in Section 2, the Brownian sheet behaves like a Brownian motion along any deterministic horizontal line, and hence does not have any point of increase.

While Problem 2 remains open, Dalang and Mountford [9] have established the following result.

Theorem 6 *With positive probability, there are horizontal lines along which the Brownian sheet has a point of increase at random levels, that is, there exist random points $(T_1, T_2) \in [1, 2]^2$ such that*

$$W(T_1 - h, T_2) < W(T_1, T_2) < W(T_1 + h, T_2), \qquad \textit{for } \ 0 < h < \frac{1}{2}.$$

Again, the conclusion can be made valid with probability one if the point (T_1, T_2) is not constrained to belong to $[1, 2]^2$, but is allowed to be anywhere in $\mathbb{R}_+^2$.

Theorem 6 has the following implication with regard to properties of the Ornstein-Uhlenbeck process on Wiener space.

Corollary 7 *The Ornstein-Uhlenbeck process on Wiener space hits with probability one the set of paths in $C(\mathbb{R}_+, \mathbb{R})$ that have points of increase.*

Since Theorem 6 is a statement concerning points of increase at random levels, it says nothing about the level set $L(x)$ at a fixed level x. One would like to modify the statement in some way so as to obtain a related result valid at fixed levels.

In order to do this, it is useful to understand how results such as those in Theorems 4, 6 or even 2 are proved. They all make effective use of the following local decomposition of the Brownian sheet, which was already used in [16] and [11, 12]. For fixed $t = (t_1, t_2) \in [1, 2]^2$, one can write

$$W(t_1 + u_1, t_2 + u_2) = W(t_1, t_2) + X_1(u_1) + X_2(u_2) + \eta_{t_1,t_2}(u_1, u_2). \tag{4}$$

The processes X_1 and X_2 are defined for u_1 and u_2 both positive and negative, and are independent. The specific distribution depends on whether u_i is positive or negative: $(X_1(u_1),\ u_1 \geq 0)$ is a Brownian motion (with speed u_2), while $(X_1(-u_1),\ 0 \leq u_1 \leq t_1)$ is a Brownian bridge with value 0 at times 0 and t_1, and similar statements hold for X_2. However, locally near 0, X_1 behaves like a Brownian motion, whether u_1 runs forwards or backwards from 0. In addition, the last term $\eta_{t_1,t_2}(u_1, u_2)$ is small relative to X_1 and X_2: it is of order $\sqrt{u_1 u_2}$, while $X_i(u_i)$ is of order $\sqrt{u_i}$, for small u_1 and u_2. This means that the behavior of the Brownian sheet W in the neighborhood of a point t is essentially the same as that of *additive Brownian motion* X defined by

$$X(u_1, u_2) = X_1(u_1) + X_2(u_2), \qquad u_1 \in \mathbb{R},\ u_2 \in \mathbb{R},$$

where X_1 and X_2 are independent, and are Brownian motions for both positive and negative time.

Theorems 4, 6 and 2 are all based on this local decomposition, and an accounting of the error term η, which can be quite involved. But intuition and conjectures about the properties of the Brownian sheet can be obtained from corresponding properties of additive Brownian motion. Of course, this must be done with care: for instance, while the conclusions of Theorems 2 and 4 are valid for additive Brownian motion, with a similar (and in fact, simpler) proof, the conclusion of Theorem 6 does not hold for additive Brownian motion, because for any T_2, the path $t_1 \mapsto X(t_1, T_2)$ is that of a Brownian motion with initial value $X_2(T_2)$, and this cannot have a point of increase by Theorem 3.

Substantial information about bubbles of additive Brownian motion was obtained by Dalang and Walsh in [12]. By examining how bubbles of this process come into contact with each other, one is led to the following definition: we term a *broken line* with corner at (t_1, t_2) the union of two segments of the form $\{t_1\} \times [t_2, t_2 + h]$ and $[t_1, t_1 + h] \times \{t_2\}$. The following theorem is established in [9].

Theorem 8 *With probability one, there exist broken lines along which the Brownian sheet has a point of increase at fixed levels, that is, for all $x \in \mathbb{R}$ and $h > 0$, there exists $(T_1, T_2) \in \mathbb{R}^2_+$ such that*

$$W(T_1, T_2 + u) < x < W(T_1 + u, T_2), \qquad \textit{for } \ 0 < u < h. \tag{5}$$

In addition, it is shown in [9] that the set of all (T_1, T_2) with the property (5) has Hausdorff dimension $\frac{1}{2}$. In particular, this set is uncountable and cannot consist only of isolated points.

PROOF OF THEOREM 8. We shall define below an event $A(t,n)$, which describes the fact that "W has an approximate point of increase of order n along a broken line with corner at t."

Let X_n be the number of points t with dyadic coordinates of order $2n$ contained in $[2,3]^2$ for which $A(t,n)$ occurs. We will use the estimates in (8) and (9) below to check that there is $\varepsilon > 0$ such that for all large n,

$$\frac{E(X_n)^2}{E(X_n^2)} \geq \varepsilon. \tag{6}$$

It follows from this and the Cauchy-Schwartz inequality that

$$P\{X_n > 0\} \geq \frac{E(X_n)^2}{E(X_n^2)} \geq \varepsilon > 0,$$

and from Fatou's lemma that

$$P\left(\limsup_{n\to\infty} \{X_n > 0\}\right) \geq \varepsilon.$$

Therefore, with positive probability, there is a sequence (t^n) of elements of $[2,3]^2$ such that $A(t_n, n)$ occurs for all large n. In view of the definition of the event $A(t_n, n)$ given below, the limit of a convergent subsequence of (t^n) will have the property requested in the conclusion of Theorem 8.

In order to complete the proof, it now suffices to define the event $A(t,n)$ and to establish the inequality (6). Set

$$A(t,n) = A_U(t,n) \cap A_0(t,n) \cap A_R(t,n),$$

where

$$\begin{aligned}
A_0(t,n) &= \{|W(t) - x| \leq 2^{-n}\}, \\
A_U(t,n) &= \{W(t_1, t_2 + v) - W(t_1, t_2) < -2^{-n},\ 2^{-2n} \leq v \leq 1\}, \\
A_R(t,n) &= \{W(t_1 + u, t_2) - W(t_1, t_2) > 2^{-n},\ 2^{-2n} \leq u \leq 1\}.
\end{aligned}$$

Then $A_0(t,n)$ describes the fact that $W(t)$ is approximately equal to x, $A_U(t,n)$ occurs if W stays below $W(t)$ along a vertical segment with lower extremity at t, and $A_R(t,n)$ occurs if W stays above $W(t)$ along a horizontal segment with left extremity at t. So $A(t,n)$ does indeed occur if and only if W has an approximate point of increase of order n along a broken line with corner at t.

From the definition of X_n,

$$E(X_n) = \sum_t P(A(t,n)) \qquad \text{and} \qquad E(X_n^2) = \sum_s \sum_t P(A(s,n) \cap A(t,n)), \tag{7}$$

where each sum is over the 2^{4n} points in $[2,3]^2$ with coordinates that are dyadic of order $2n$. It is shown in [9] that there is $c > 0$ such that for all large n,

$$P(A(t,n)) \geq c\, 2^{-3n}, \tag{8}$$

and there is a constant $C < \infty$ such that for all large n,

$$P(A(s,n) \cap A(t,n)) \leq C\, 2^{-3n} \frac{2^{-n}}{\sqrt{|s_1 - t_1|}} \frac{2^{-n}}{\sqrt{|s_2 - t_2|}} \frac{2^{-n}}{\sqrt{|s_1 - t_1| \vee |s_2 - t_2|}}. \tag{9}$$

From (7) and (9), one can deduce as in [9] that there are constants $k > 0$ and $K < \infty$ such that

$$E(X_n) \geq k\, 2^{-n} \qquad \text{and} \qquad E(X_n^2) \leq K\, 2^{2n}.$$

This proves (6) and completes the proof of Theorem 8. $\diamondsuit$

7 Dependence between excursions

One of the features of excursions of standard Brownian motion away from a fixed level is that these excursions are independent of each other, and there is a beautiful point process description of excursions [25, Chapter VIII]. Can one develop such a theory for the Brownian sheet? It turns out that it will not be possible to describe excursions of the Brownian sheet independently of each other.

Indeed, one feature which makes possible the independence of excursions of Brownian motion is the fact that distinct excursion intervals never have a common endpoint, but are separated by infinitely many other excursions. If a positive excursion interval could share an endpoint with a negative excursion interval, then knowing that one of the excursions is positive would indicate that the other is negative (given that a.s., 0 is not a local minimum or maximum of Brownian motion.

Theorems 4 and 8 above show that distinct bubbles of the Brownian sheet can share boundary points, and given this, it is natural to suspect that excursions of the Brownian sheet in distinct bubbles are *not* independent. This intuition is formalized in the following theorem of Dalang and Mountford [10].

Theorem 9 *Given the level set $L(0)$ and the sign of all but finitely many excursions of the Brownian sheet away from 0, the sign of the remaining excursions is fully determined.*

PROOF. The complete proof of this theorem is highly technical and will be given in [10]. However, the main ideas are as follows. Suppose that we are given $L(0)$, and the sign of all excursions of W except the one that contains $(1,1)$: call this excursion $C(1,1)$. We are going to show how to determine the sign of $C(1,1)$.

Suppose for a minute that $W(1,1) > 0$. Define

$$\tau = \inf\{s_1 > 1 : W(s_1, 1) = 0\},$$

and set $B(v) = W(\tau, 1 + v/\tau)$. Then $B = (B(v),\ v \geq 0)$ is a standard Brownian motion.

We now focus on excursions of B near $v = 0$. On average, B "has as many" positive excursions as negative excursions near 0. Each negative excursion of B corresponds to a negative bubble of W, which is certainly distinct from $C(1,1)$ since we have assumed that $W > 0$ on $C(1,1)$.

On the other hand, a positive excursion of B corresponds to a positive bubble of W, and this positive bubble may or may not coincide with $C(1,1)$. In fact, think of a positive excursion of B near $v = 0$ that is comparatively long and goes up to a relatively high level, and reaches its maximum at $v = \varepsilon$, say. It is quite likely that the point $(\tau, 1 + \varepsilon/\tau)$ will be part of $C(1,1)$, since there is a good chance that W will be positive along some path that connects the segment $[1, \tau[\times\{1\}$ to the point $(\tau, 1+\varepsilon)$.

Therefore, positive excursions of B that correspond to bubbles of W distinct from $C(1,1)$ are fewer than negative excursions of B that correspond to such distinct bubbles.

The key observation is now that the occurence of such excursions is determined by the level set $L(0)$ and the sign of all excursions except that of $C(1,1)$. Formally, let F_n^+ (resp. F_n^-) be the event "there is a positive (resp. negative) excursion of B originating in $[2^{-(n+1)}, 2^{-n}]$ of length 2^{-n} that corresponds to an excursion of W *distinct* from $C(1,1)$."

One can show that if $W(1,1) > 0$, then

$$P\left(\limsup_{n\to\infty} F_n^+\right) = 0 \quad \text{and} \quad P\left(\limsup_{n\to\infty} F_n^-\right) = 1,$$

while if $W(1,1) < 0$, then

$$P\left(\limsup_{n\to\infty} F_n^+\right) = 1 \quad \text{and} \quad P\left(\limsup_{n\to\infty} F_n^-\right) = 0.$$

Clearly, the events $\limsup_{n\to\infty} F_n^+$ and $\limsup_{n\to\infty} F_n^-$ are determined by $L(0)$ and the sign of all but finitely many excursions of W.

The main technical difficulty is to quantify the likelihood that a positive excursion of W corresponds to an excursion of W distinct from $C(1,1)$. Details of this can be found in [10]. $\diamondsuit$

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