Level Sets and Excursions of the Brownian Sheet

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1 Introduction

The objective of these notes is to present several recent results concerning level sets and excursions of what is one of the fundamental Gaussian random fields, namely the Brownian sheet. Recall that a real-valued, mean zero, continuous Gaussian process

$$W = (W(s_1, s_2),\ (s_1, s_2) \in \mathbb{R}_+^2)$$

is termed a *Brownian sheet* provided its covariance is given by the formula

$$E(W(s_1, s_2)W(t_1, t_2)) = (s_1 \wedge t_1)(s_2 \wedge t_2). \tag{1}$$

Early interest in this process came from limit theorems in multivariate statistics [35], of the following kind. Let $U_1, U_2, \dots$ be i.i.d. random points that are uniformly distributed in $[0,1]^2$, and, for $s = (s_1, s_2) \in [0,1]^2$, set $[0, s] = [0, s_1] \times [0, s_2]$. The empirical distribution function of the first n points is

$$F_n(s) = \frac{1}{n} \text{card } \{i \le n : U_i \in [0, s]\}.$$

Then, as $n \to \infty$, the normalized empirical distribution function

$$s \mapsto \sqrt{n}(F_n(s) - F(s))$$

converges weakly to the process known as the *pinned Brownian sheet*, which can be represented by $\tilde{W}(s) = W(s) - s_1 s_2 W(1, 1)$. This, and other, limit theorems provide statistical motivation for the study of the Brownian sheet. An early reference in this direction is Kitagawa [22]. Additional statistical motivation can be found in Adler [2] and the references therein.

Continuity properties of the Brownian sheet seem to have been first studied by Chentsov [5] and Yeh [39]. A more detailed study of sample path properties of the Brownian sheet was carried out by Orey and Pruitt [32], following some results of Zimmerman [44] and Pyke [34]. They derived the modulus of continuity of the Brownian sheet, and studied laws of the iterated logarithm, as well as recurrence properties of this process. Other surprising sample path properties of the Brownian sheet can be found in Walsh [41].

There was also much interest in Markov properties of this process (McKean [25], Pitt [33], Nualart [29], Rozanov [37]). In particular, there are a priori several natural ways of formulating Markov properties for random fields, such as the germ-field Markov property and the sharp Markov property. The result of Dalang and Russo [12, Theorem 3.12], which identified the monotone curves for which these two Markov properties are identical, opened the door to the essentially complete results of Dalang and Walsh [13] concerning the sharp Markov property of the Brownian sheet.

There has been much effort in understanding the level sets of the Brownian sheet. Recall that for $x \in \mathbb{R}$, the *level set of* W *at level* x is the random closed set

$$L(x) = \{(s_1, s_2) \in \mathbb{R}_+^2 : W(s_1, s_2) = x\}.$$

Adler [1] computed the Hausdorff dimension of this set (see Theorem 3), and Kendall [19] showed that this set is disconnected at “almost all” (but not all!) of its points. Dalang and Walsh [14] studied geometric properties of the level set and, in particular, the shape of this set in the neighborhood of certain of its points. Several further properties, concerning level sets, excursions and “bubbles” of the Brownian sheet have been examined by Dalang and Mountford [6]–[10]. The main objective of these notes is to present a substantial number of these recent results. Other kinds of properties of level sets and bubbles can be found in Ehm [18] and Khoshnevisan [20].

Many other properties of the Brownian sheet have been examined in the literature. We mention in particular results concerning the small ball problem (Kuelbs and Li [23], Tallagrand [38]) and potential theory of the Brownian sheet (Khoshnevisan and Shi [21]).

A separate set of ideas which also motivates the study of the Brownian sheet comes from the fact that this process is central to the theory of multi-parameter stochastic integrals [4] and is the basic example of a solution to a (hyperbolic) stochastic partial differential equation driven by white noise, of the form:

$$\frac{\partial^2 X(s_1, s_2)}{\partial s_1 \partial s_2} = \alpha(X(s_1, s_2))\dot{W}_{s_1,s_2} + \beta(X(s_1, s_2)), \tag{2}$$

with vanishing initial conditions $X(s_1, 0) = X(0, s_2) = 0$. In the case $\alpha \equiv 1$ and $\beta \equiv 0$, the solution X is a Brownian sheet [42]. Therefore, it is natural to expect that many properties of the Brownian sheet will carry over to the solutions of equations with non-constant coefficients. For potential-theoretic questions, this is indeed the case (see the results of E. Nualart [31]).

In yet another direction, the Brownian sheet is also connected to Malliavin calculus (see for instance Nualart [30]). Indeed, this process provides a simple representation of the Ornstein-Uhlenbeck process on Wiener space, which is a stationary process $s \mapsto U_s$ with values in the space $C(\mathbb{R}_+, \mathbb{R})$ whose law, at any fixed time, is Wiener measure. This process can be defined by

$$U_s(t) = e^{-s/2} W(t, e^s),$$

where W is a Brownian sheet [26], and many properties of the Ornstein-Uhlenbeck process on Wiener space can be deduced from this representation. In fact, we will see an example of this in Section 7.

The outline of these notes is as follows. In Section 2, we give the construction of the Brownian sheet, its relationship with white noise, and the basic properties of this process (independence of increments, time inversion, and modulus of continuity). In Section 3, we recall basic properties of excursions of Brownian motion and formulate analogous properties for the Brownian sheet, which are studied in the subsequent sections. In Section 4, we state and prove recent results concerning non-independence of excursions of the Brownian sheet (Theorems 4 and 5). In Section 5, we show that distinct excursion sets of the Brownian sheet can share a common boundary point (Theorem 6). Section 6 is the only section in which we prove a result concerning Brownian motion, namely the classical result of Dvoretsky, Erdős and Kakutani [17] concerning the absence of points of increase of Brownian motion. For this, we follow the proof of Burdzy [3]. This result contrasts with the property of the Brownian sheet that is examined in Section 7, namely the fact that the Brownian sheet admits points of increase along certain exceptional horizontal lines. This result also establishes a potential-theoretic property of the Ornstein-Uhlenbeck process on Wiener space. Finally, in Section 8, we study the process known as additive Brownian motion, which is simpler than, but closely related to, the Brownian sheet, and establish several results concerning level sets and excursions of this process. We recall that a two-parameter process $(X(u_1, u_2),\ (u_1, u_2) \in \mathbb{R}_+^2)$ is an *additive Brownian motion* if

$$X(u_1, u_2) = B_1(u_1) + B_2(u_2),$$

where $B_1(\cdot)$ and $B_2(\cdot)$ are independent Brownian motions.

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2 Construction of the Brownian sheet

In this section, we show how to construct white noise and the Brownian sheet, we establish the fundamental relationships between these two objects, and we give some of the basic properties of the Brownian sheet.

2.1 White noise

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $(E, \mathcal{E}, \nu)$ be a σ-finite measure space. *White noise on E based on ν* is a random set function

$$A \mapsto W(A),$$

defined for $A \in \mathcal{E}$ with $\nu(A) < \infty$, with values in $L^2(\Omega, \mathcal{F}, P)$, such that:

(a) $W(A)$ is a Gaussian random variable with mean 0 and variance $\nu(A)$;
(b) if A and B are disjoint, then $W(A)$ and $W(B)$ are independent and

$$W(A \cup B) = W(A) + W(B).$$

The covariance of $W(A)$ and $W(B)$ is easily computed from the definition:

$$\begin{aligned} E(W(A)W(B)) &= E((W(A \setminus B) + W(A \cap B))(W(B \setminus A) + W(B \cap A))) \\ &= E(W(A \cap B)^2) \\ &= \nu(A \cap B). \end{aligned}$$

2.2 Construction of white noise

A fundamental question is to construct a white noise, or to show that such an object exists. We will show that there is a Gaussian process

$$(W(A), \quad A \in \mathcal{E} \text{ with } \nu(A) < \infty)$$

with covariance function

$$c(A, B) = \nu(A \cap B).$$

This will clearly satisfy (a) and the independence statement in (b) above.

One checks the additivity property of W by checking that

$$E((W(A \cup B) - W(A) - W(B))^2) = 0 \ \text{ if } \ A \cap B = \emptyset.$$

Indeed, if A and B are disjoint, then the expectation is equal to

$$\nu(A \cup B) + \nu(A) + \nu(B) - 2\nu(A) - 2\nu(B) + 2\nu(A \cap B) = 0.$$

The existence of a Gaussian process with covariance $c(A, B)$ will follow from a general result on Gaussian processes [28, Neveu], provided $c(A, B)$ is non-negative definite. That is, for $a_i \in \mathbb{R}$ and $A_i \in \mathcal{E}$ with $\nu(A_i) < \infty$, $i = 1, \dots, n$,

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n a_i a_j c(A_i, A_j) &= \sum_{i=1}^n \sum_{j=1}^n a_i a_j \int_E 1_{A_i}(x) 1_{A_j}(x) \nu(dx) \\ &= \int_E \left(\sum_{i=1}^n a_i 1_{A_i}(x) \right)^2 \nu(dx) \\ &\geq 0. \end{aligned}$$

2.3 From white noise to the Brownian sheet

Having constructed white noise, we can use this object to construct a Brownian sheet. Set $E = \mathbb{R}_+^2$, let $\mathcal{E}$ be the Borel σ-field of $\mathbb{R}_+^2$, and let ν denote Lebesgue measure. For $t = (t_1, t_2)$, set $[0,t] = [0,t_1] \times [0,t_2]$ and

$$W(t_1, t_2) = W([0,t]).$$

This defines a mean-zero Gaussian process $W = (W(t_1,t_2),\ (t_1,t_2) \in \mathbb{R}_+^2)$ with covariance

$$E(W(s_1,s_2)W(t_1,t_2)) = \nu([0,s] \cap [0,t]) = (s_1 \wedge t_1)(s_2 \wedge t_2),$$

in agreement with (1). Note that it is not a priori clear that the sample paths of this process are continuous. This question will be examined in Subsection 2.7 below.

2.4 From the Brownian sheet to white noise

If a Brownian sheet W is given, then we can use it to construct a white noise, rather than the other way around. If $R = [s_1, t_1] \times [s_2, t_2]$, where $s_1 < t_1$ and $s_2 < t_2$, define the planar increment $\Delta_R W$ of W over R by

$$\Delta_R W = W(t_1,t_2) - W(s_1,t_2) - W(t_1,s_2) + W(s_1,s_2),$$

and define a set function W on rectangles by

$$W(R) = \Delta_R W.$$

Extend this set function to finite unions disjoint rectangles by additivity, then to finite unions of (not necessarily disjoint) rectangles by decomposing such a union into a union of disjoint rectangles, and then to Borel sets A with $\nu(A) < \infty$ by approximating A by finite unions A_n of rectangles so that

$$\lim_{n\to\infty} (\nu(A \setminus A_n) + \nu(A_n \setminus A)) = 0,$$

and set

$$W(A) = L^2\text{-}\lim_{n\to\infty} W(A_n).$$

The set function $A \mapsto W(A)$ satisfies

$$E((W(A) - W(A_n))^2) = \nu(A \setminus A_n) + \nu(A_n \setminus A) \longrightarrow 0 \qquad \text{as } n \to \infty,$$

and one checks immediately that this set function is a white noise.

A direct consequence of this construction is the following.

Proposition 1. *The Brownian sheet has independent planar increments, that is,* $\Delta_{R_1} W, \dots, \Delta_{R_n} W$ *are independent if the rectangles* R_i *are pairwise disjoint.*

2.5 Basic properties

Directly from the covariance of the Brownian sheet, we see that the sheet vanishes on the coordinate axes:

$$W(s_1, 0) = W(0, s_2) = 0 \qquad a.s.$$

This is analogous to an initial condition in the formulation of (2) and could be replaced by some other condition.

To get an idea of the behavior of the sample paths of the sheet, we can observe its restriction to various curves. For example, for fixed $s_2 > 0$,

$$s_1 \mapsto W(s_1, s_2)$$

is the restriction of the sheet to a horizontal line. From (1), the covariance of this one-parameter Gaussian process is that of Brownian motion with speed $\sqrt{s_2}$, so the restriction of the sheet to such lines is a Brownian motion. The same occurs if we fix $s_1 > 0$ and consider

$$s_2 \mapsto W(s_1, s_2),$$

which is the restriction of the sheet to a vertical line and is a Brownian motion with speed $\sqrt{s_1}$.

One can also consider

$$u \mapsto W(u, 1-u), \qquad 0 \leq u \leq 1,$$

which is the sheet restricted to the segment with extremities $(0, 1)$ and $(1, 0)$. This is a Gaussian process which vanishes at times 0 and 1, and the calculation of its covariance shows that it is a Brownian bridge [36, Chap. I, §3].

As a final example, consider

$$u \mapsto W(e^u, e^{-u}),$$

which is the restriction of the sheet to the hyperbola $s_1 s_2 = 1$. The calculation of the covariance shows that this process is stationary, with variance 1, and is in fact the Ornstein-Uhlenbeck process [36, Chap. I, §3].

2.6 Inversion properties

It is useful to examine properties of the Brownian sheet under inversions of one or both of its coordinates. Direct computation of covariances shows that the three processes W_1, W_2 and W_3 defined by

$$W_1(s_1, s_2) = s_1 W_{\frac{1}{s_1}, s_2}, \quad W_2(s_1, s_2) = s_2 W_{s_1, \frac{1}{s_2}}, \quad W_3(s_1, s_2) = s_1 s_2 W_{\frac{1}{s_1}, \frac{1}{s_2}},$$

are again Brownian sheets.

These inversion properties are useful for translating results regarding behavior of W for s_1 or s_2 near 0 to its behavior for s_1 or s_2 near ∞, and vice-versa.

2.7 Continuity of the Brownian sheet

Recall that a process $(\tilde{X}_t)$ is a *modification* of (X_t) if for all t, $\tilde{X}_t = X_t$ a.s. With this definition, we can state the following extension of Kolmogorov's classical Continuity Theorem. In this theorem, we use the notation

$$|t - s| \stackrel{\text{def}}{=} |s_1 - t_1| + |s_2 - t_2|$$

if $s = (s_1, s_2)$ and $t = (t_1, t_2)$.

Theorem 1. *Fix $d \in \mathbb{N}^*$ and let $(X_t,\ t \in [0,1]^d)$ be a process with values in a separable Banach space. Suppose there exist $\gamma > 0$, $c > 0$, and $\varepsilon > 0$ such that*

$$E(|X_t - X_s|^\gamma) \leq c|t - s|^{d+\varepsilon}.$$

Then there is a modification $\tilde{X}$ of X such that

$$E\left(\left(\sup_{s \neq t} \frac{|\tilde{X}_s - \tilde{X}_t|}{|s - t|^\alpha}\right)^\gamma\right) < \infty, \qquad \textit{for all } \alpha \in [0, \frac{\varepsilon}{\gamma}[\,.$$

In particular, for $\alpha \in [0, \frac{\varepsilon}{\gamma}[$, $s \mapsto \tilde{X}_s(\omega)$ is Hölder-continuous with exponent α.

A proof of this theorem can be found in [36, Chap. I, Theorem (2.1)]. With this theorem, it it straightforward to check that a Brownian sheet $(W_t,\ t \in [0,1]^2)$ has a continuous modification. Indeed,

$$E(|W_s - W_t|^2) \leq |s_1 - t_1| + |s_2 - t_2| = |s - t|.$$

Because $W_s - W_t$ is a Gaussian random variable, this implies that for $p \geq 1$, there is a constant c_p such that

$$E(|W_s - W_t|^p) \leq c_p\, |s - t|^{p/2}.$$

Take $p > 4$, $\varepsilon = \frac{p}{2} - 2 > 0$, and $\gamma = p$. Then

$$\frac{\varepsilon}{\gamma} = \frac{p-4}{2p} \uparrow \frac{1}{2} \quad \text{as } p \to \infty.$$

Therefore, Theorem 1 allows us to conclude the following.

Corollary 1. *(W_t) has a modification which is Hölder-continuous with exponent α, for $0 < \alpha < \frac{1}{2}$.*

We will always assume that (W_t) *is* this modification.

3 Excursions, level sets and bubbles

We shall first recall some properties of excursions of Brownian motion, and then consider analogous properties for the Brownian sheet. A fundamental reference for properties of Brownian motion is the monograph [36].

3.1 Excursions of Brownian motion

For a (standard) Brownian motion $B = (B(u), u \geq 0)$ and $x \in \mathbb{R}$, an *excursion interval* of B away from x is a (random) interval $[u_1, u_2] \subset \mathbb{R}_+$ such that

$$B(u_1) = B(u_2) = x, \text{ and } B(u) \neq x \text{ for } u_1 < u < u_2.$$

We shall state three well-known properties of excursions of Brownian motion. For a detailed study of excursions of this process, we refer the reader to [36, Chapter XII].

Fact 1. Distinct excursion intervals of Brownian motion cannot share an endpoint.

This property is essentially a consequence of the strong Markov property of Brownian motion. Indeed, at the first hit T of x after hitting $x+\varepsilon$, $B(T+\cdot)$ is again a Brownian motion, so

$$\inf_{T<u<T+\delta} B(u) < y < \sup_{T<u<T+\delta} B(u), \qquad \text{for all } \delta > 0.$$

This means that T, which is the right endpoint of an excursion of B above x, is not the left endpoint of an excursion, and one deduces that there is no right endpoint of one excursion that is also the left endpoint of some other excursion, which is Fact 1.

Fact 2. Given $(|B(u)|,\ u \geq 0)$, the signs of the excursions of $(B(u),\ u \geq 0)$ away from 0 are i.i.d. random variables, independent of $(|B(u)|,\ u \geq 0)$ and positive with probability $\frac{1}{2}$.

This is a classical result. See for instance [36, Chap. XII, Ex.(2.16)].

Fact 3. With probability one, Brownian motion has no points of increase.

The formal statement of this third result, due to Dvoretzky, Erdős and Kakutani [17], is the following.

Theorem 2.

$$P\{\exists\varepsilon > 0, \exists t > \varepsilon, \quad B(t-h) < B(t) < B(t+h), \quad \forall h \in]0,\varepsilon]\} = 0.$$

This result extends Fact 1 to a statement that concerns all levels simultaneously.

3.2 Level sets, bubbles and excursions of the Brownian sheet

For $x \in \mathbb{R}$, the *level set of* W *at level* x is the random set

$$L(x) = \{(s_1, s_2) \in \mathbb{R}_+^2 : W(s_1, s_2) = x\}.$$

Because sample paths of the Brownian sheet are continuous, this is a closed set. This set has some geometric and topological complexity, as is attested by the following result, due to R. Alder [1].

Theorem 3. *A.s., for all* $x \in \mathbb{R}$*, the Hausdorff dimension of* $L(x)$ *is* $3/2$*. For the d-parameter Brownian sheet, the Hausdorff dimension of* $L(x)$ *is* $d - \frac{1}{2}$*.*

For $x \in \mathbb{R}$, an *x-bubble,* or *excursion set* of the Brownian sheet W away from x, is one connected component of the random set

$$\{(s_1, s_2) \in \mathbb{R}_+^2 : W(s_1, s_2) \neq x\}.$$

Note that this set is the disjoint union of the two sets

$$L_+(x) = \{(s_1, s_2) \in \mathbb{R}_+^2 : W(s_1, s_2) > x\}$$

and

$$L_-(x) = \{(s_1, s_2) \in \mathbb{R}_+^2 : W(s_1, s_2) < x\}.$$

A bubble contained in $L_+(x)$ (respectively $L_-(x)$) is an *upwards* (respectively *downwards*) bubble. When $x = 0$, we prefer to say a *positive* (respectively *negative*) bubble.

We now ask three questions concerning properties of the Brownian sheet that are analogues of the properties of Brownian motion stated in Facts 1, 2 and 3 in the previous subsection.

Analogue of Fact 1. Can distinct bubbles share a boundary point ?

Analogue of Fact 2. Given $|W| = (|W(s_1, s_2)|,\ (s_1, s_2) \in \mathbb{R}_+^2)$, are the signs of the 0-bubbles of W i.i.d. and independent of $|W|$? By definition, the sign of a 0-bubble is *positive* (that is, equal to $+1$) if $W > 0$ in this bubble, and is *negative* (that is, equal to -1) otherwise.

Analogue of Fact 3. Recall that for each fixed s_2, the process $s_1 \mapsto W(s_1, s_2)$ is a Brownian motion. Therefore, for all $s_2 > 0$,

$$P\{s_1 \mapsto W(s_1, s_2) \text{ has a point of increase}\} = 0.$$

This observation leaves open the answer to the following question: What is the value of

$$P\{\exists s_2 > 0 : s_1 \mapsto W(s_1, s_2) \text{ has a point of increase}\} \ ?$$

The questions above will be addressed respectively in Sections 5, 4 and 7. Their answers are summarized in Table 1.

Table 1. Comparison of properties of Brownian motion and the Brownian sheet

	Brownian motion	Brownian sheet
Fact 1:	No	Yes
Fact 2:	Yes	No
Fact 3:	0	1

4 Non-independence of excursions of the Brownian sheet

In this section, we address the analogue for the Brownian sheet of Fact 2 in Section 3. The following striking result is due to John B. Walsh [43] (note that all the σ-fields that we consider below are completed with P-null sets).

Theorem 4. *The two σ-fields $\sigma(|W(s_1, s_2)|, (s_1, s_2) \in \mathbb{R}_+^2) \vee \sigma(\text{sign } W(1, 1))$ and $\sigma(W(s_1, s_2), (s_1, s_2) \in \mathbb{R}_+^2)$ are equal.*

This theorem states that given $|W|$ and the sign of the sheet at a single point, one can determine the sign of the sheet everywhere. It was inspired by the deeper result in [10], in which one is given the level set of the sheet at level 0, rather than the absolute value of the sheet. The latter result will be discussed at the end of this section.

4.1 A property of correlated Brownian motions

The proof of Theorem 4 is based on the following lemma for correlated Brownian motions.

Lemma 1. *Fix $a > 0$ and $b > 0$. Let B_1 and B_2 be independent Brownian motions. For $u \geq 0$, set*

$$X_1(u) = a\, B_1(u), \qquad X_2(u) = aB_1(u) + b\, B_2(u),$$

and define the two σ-fields

$$\mathcal{G}_1 = \sigma(|X_1(u)|, |X_2(u)|,\ u \geq 0) \vee \sigma(\text{sign } X_1(1)),$$
$$\mathcal{G}_2 = \sigma(|X_1(u)|, |X_2(u)|,\ u \geq 0) \vee \sigma(\text{sign } X_1(1), \text{sign } X_2(1)).$$

Then $\mathcal{G}_1 = \mathcal{G}_2$.

This lemma states that for the correlated Brownian motions X_1 and X_2, the observation over time of their absolute values and the sign of one of them at time 1 determines the sign of the other at the same time.

Proof. Recall [36, Chapter 4] that the *quadratic variation* $\langle X\rangle$ of a diffusion $(X(u),\ u \geq 0)$ is

$$\langle X\rangle_u = \lim_{n\to\infty} \sum_{k=1}^{[2^n u]} (X(k2^{-n}) - X((k-1)2^{-n}))^2 \quad \text{a.s.}$$

The diffusion X can be replaced by $|X|$ to define $\langle |X|\rangle$, and it is well-known [36, Chapter 6] that

$$\langle |X|\rangle_u = \langle X\rangle_u, \qquad \text{for all } u \geq 0, \quad \text{a.s.}$$

In particular, $\langle X\rangle_u$ is $\sigma(|X|)$-measurable.

Clearly, $\mathcal{G}_1 \subset \mathcal{G}_2$. For the converse inclusion, we show that sign $(X_2(1))$ is $\mathcal{G}_1$-measurable. Set $Y(u) = |X_1(u)| + |X_2(u)|$, so that the process $(Y(u),\ u \geq 0)$ is $\mathcal{G}_1$-measurable. Clearly,

$$Y(u) = \begin{cases} |2aB_1(u) + b\,B_2(u)| & \text{if } X_1(u)\cdot X_2(u) > 0, \\ |bB_2(u)| & \text{if } X_1(u)\cdot X_2(u) < 0, \end{cases}$$

so

$$\frac{d\langle Y\rangle_u}{du} = \begin{cases} 4a^2 + b^2 & \text{if } X_1(u)\cdot X_2(u) > 0, \\ b^2 & \text{if } X_1(u)\cdot X_2(u) < 0. \end{cases}$$

Therefore, writting equality between sets that differ only by a null set,

$$\begin{aligned} \{X_2(1) > 0\} = & \left\{ \left.\frac{d\langle Y\rangle_u}{du}\right|_{u=1} = 4a^2 + b^2, \quad X_1(1) > 0 \right\} \\ & \cup \left\{ \left.\frac{d\langle Y\rangle_u}{du}\right|_{u=1} = b^2, \quad X_1(1) < 0 \right\} \\ & \in \mathcal{G}_1. \end{aligned}$$

The lemma is proved. ◊

4.2 Non-independence given the absolute value of the sheet

We are now ready to prove Theorem 4.

Proof of Theorem 4. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the two σ-fields in the statement of the theorem. Clearly, $\mathcal{H}_1 \subset \mathcal{H}_2$. In order to show that $\mathcal{H}_2 \subset \mathcal{H}_1$, it suffices to show that

$$\text{sign } W(s_1, s_2) \text{ is } \mathcal{H}_1\text{-mesurable}, \qquad \text{for all } (s_1, s_2) \in \mathbb{R}_+^2.$$

Fix $u_2 \in \mathbb{Q}$ such that $u_2 > 1$, and set

$$\begin{aligned} X_1(u) &= W(u,1), \\ X_2(u) &= W(u,u_2) = B_1(u) + \sqrt{u_2 - 1}\ B_2(u), \end{aligned}$$

where B_1 and B_2 are the independent Brownian motions defined by

$$B_1(u) = W(u,1), \qquad B_2(u) = \frac{1}{\sqrt{u_2 - 1}}(W(u,u_2) - W(u,1)).$$

By Lemma 1, sign $(W(1,u_2))$ is $\mathcal{H}_1$-measurable.

Fix $u_1 \in \mathbb{Q}$ such that $u_1 > 1$, and set

$$\begin{aligned} \tilde{X}_1(v) &= W(1,v), \\ \tilde{X}_2(v) &= W(u_1,v) = \tilde{B}_1(v) + \sqrt{u_1 - 1}\ \tilde{B}_2(v), \end{aligned}$$

where $\tilde{B}_1$ and $\tilde{B}_2$ are the independent Brownian motions defined by

$$\tilde{B}_1(v) = W(1,v), \qquad \tilde{B}_2(v) = \frac{1}{\sqrt{u_1 - 1}}\ (W(u_1,v) - W(1,v)).$$

By the first part of the proof, sign $W(1,v)$ is $\mathcal{H}_1$-measurable, so by Lemma 1, sign $W(u_1,v)$ is too.

To show that sign $W(u_1,v)$ is $\mathcal{H}_1$-measurable for (u_1,v) in the other three quadrants relative to $(1,1)$, one can use the inversion properties of the Brownian sheet or continue with arguments similar to those above. This proves Theorem 4. ♢

4.3 Non-independence given a level set

In Theorem 4, one is given the absolute value of the Brownian sheet at every point in the non-negative quadrant. A related question is whether or not the sign of an excursion can be determined from the shape of the level set and the sign of other excursions. This question is addressed in [10] as follows.

For a closed set $A \subset \mathbb{R}^2_+$, let $d(t,A)$ be the infimum over $s \in A$ of $|s-t|$. Set

$$\begin{aligned} \mathcal{G} &= \sigma\{d(t,L(0)),\ t \text{ with rational coordinates}\} = \sigma\{d(t,L(0)),\ t \in \mathbb{R}^2_+\} \\ &= \sigma\{I_{L(0)\cap D \neq \emptyset},\ D \text{ a square in } \mathbb{R}^2_+\}. \end{aligned}$$

Clearly, the information that is contained in this σ-field is the position of the level set $L(0)$.

Let $E_1, E_2, \ldots$ be an enumeration of the 0-bubbles that is measurable with respect to the σ-field $\mathcal{G}$. Such an enumeration can be obtained as follows. Let $(q_i,\ i \in \mathbb{N})$ be an enumeration of the points in $]0,\infty[^2$ with rational coordinates. Let E_1 be the bubble containing q_1, and for $i > 1$, let E_i be the bubble containing q_{J_i}, where

$$J_i = \inf\{k \geq 1 : q_k \notin \cup_{\ell=1}^{i-1} E_\ell \text{ and } W(q_k) \neq 0\}.$$

Given such an enumeration of bubbles, define random variables S_i with values in $\{-1, 1\}$ such that

$$S_i W(t) > 0 \qquad \text{for } t \in E_i,$$

that is, S_i is the sign of W on E_i.

The following result is established in [10].

Theorem 5. *For $n \in \mathbb{N}$, let $\mathcal{F}_n = \mathcal{G} \vee \sigma(S_i,\ i > n)$. Then for all $n \geq 0$,*

$$\mathcal{F}_n = \mathcal{F}_0.$$

In other words, given the level set $L(0)$ and the sign of all but finitely many bubbles, one can determine the signs of the remaining bubbles.

The proof of Theorem 5 is quite technical, and requires a rather deep study of properties of the level set $L(0)$, for which the reader is referred to [10]. It is natural to conjecture that in fact, $\mathcal{F}_0 = \mathcal{G} \vee \sigma(S_1)$, that is, given the level set $L(0)$ and the sign of a single excursion, then the signs of all remaining excursions are determined. This is however an open problem.

5 Bubbles with common boundary points

We now address the issue of the analogue for the Brownian sheet of Fact 1 of Section 4. The following result, due to [9], shows that distinct bubbles of opposite sign can have a common boundary point.

5.1 The main result

Theorem 6. *Fix $q \in \mathbb{R}$ and $h > 0$. With positive probability, there exist uncountably many points $(t_1, t_2) \in [2,3]^2$ such that*

$$W(t_1, t_2 + u) < q < W(t_1 + u, t_2), \qquad \text{for all } u \in]0, h]. \tag{3}$$

(and, of course, $W(t_1, t_2) = q$ by continuity).

The rectangle $[2,3]^2$ in the statement of the theorem could be replaced by any other rectangle: it will simply be convenient to have a bounded rectangle that is sufficiently far away from the coordinate axes. It is shown in [9, Theorem 9] that the Hausdorff dimension of $\{(t_1, t_2) \in \mathbb{R}_+^2 : (3) \text{ holds}\}$ is $1/2$, a.s. Here, we shall only prove the following proposition.

Proposition 2. *Fix $q \in \mathbb{R}$. With positive probability, there exists $(t_1, t_2) \in [2,3]^2$ such that for all $u \in]0, 1]$,*

$$W(t_1, t_2 + u) < q < W(t_1 + u, t_2) \text{ and } W(t_1 - u, t_2) > q. \tag{4}$$

Remark 1. (a) One can prove, though this is non-trivial, that for (t_1, t_2) satisfying (4), the segment $[t_1 - 1, t_1[\times\{t_2\}$ and $]t_1, t_1 + 1] \times \{t_2\}$ belong to the same upwards q-bubble.

(b) Even though the property (4) is stronger than (3), Proposition 2 is (much) easier to prove than Theorem 6, in part because there cannot be more than one point $(t_1, t_2) \in [2, 3]^2$ that satisfies (4). Indeed, if one tries to put two such points in $[2, 3]^2$, then one immediately notices that the inequalities required by property (4) at each of the two points are incompatible.

5.2 An approximating event

In this subsection, we begin the proof of Proposition 2, following [9, Section 5]. For $t \in \mathbb{R}_+^2$ and $n \in \mathbb{N}$, set

$$\begin{aligned} W_R^{t,n}(u) &= W(t_1 + 2^{-2n} + u, t_2) - W(t_1 + 2^{-2n}, t_2), \\ W_U^{t,n}(v) &= W(t_1, t_2 + 2^{-2n} + v) - W(t_1, t_2 + 2^{-2n}), \\ W_L^{t,n}(u) &= W(t_1 - 2^{-2n} - u, t_2) - W(t_1 - 2^{-2n}, t_2). \end{aligned}$$

Let $g(u) = u^{3/4}$, and

$$\begin{aligned} F_R(t, n) &= \{W_R^{t,n}(u) \geq g(u) - 2^{-n},\ 0 \leq u \leq 1\}, \\ F_U(t, n) &= \{W_U^{t,n}(v) \leq -g(v) + 2^{-n},\ 0 \leq v \leq 1\}, \\ F_L(t, n) &= \{W_L^{t,n}(u) \geq g(u) - 2^{-n},\ 0 \leq u \leq 1\}. \end{aligned}$$

The interpretation of these events is that on $F_U(t, n)$, for instance, the process $v \mapsto W_U^{t,n}(v)$ stays below the graph of $-g(\cdot) + 2^{-n}$. Similarly, on the events $F_R(t, n)$ and $F_L(t, n)$, the sheet increases as we move away from t, at a rate guaranteed by $g(\cdot)$.

We now recall a property of the Brownian motions $W_i^{t,n}$, $i \in \{R, U, L\}$.

Lemma 2. *There exists $k > 0$ such that for all large n and for all $t \in [2, 3]^2$,*

$$P(F_i(t, n)) \geq K 2^{-n}, \qquad i \in \{R, U, L\}. \tag{5}$$

For the proof of this lemma, the reader is referred to [9, Lemma 12]. In addition to the increase or decrease of W as we move away from t, we also will need to specify the value of W near t. Set

$$\begin{aligned} F_0(t, n) = \{q + 2^{-n} \leq &\left\{ \begin{matrix} W(t_1 + 2^{-2n}, t_2) \\ W(t_1 - 2^{-2n}, t_2) \\ W(t_1, t_2 - 2^{-2n}) \end{matrix} \right\} \leq q + 2^{-n+1}, \\ &q - 2^{-n+1} \leq \left\{ \begin{matrix} W(t_1, t_2 + 2^{-2n}) \\ W(t_1, t_2) \end{matrix} \right\} \leq q - 2^{-n}\}. \end{aligned}$$

The probability $P(F_0(t,n))$ can be estimated as follows. First, because $W(t_1,t_2)$ is a Gaussian random variable with mean 0 and varaince between 4 and 9, there is $c_0 > 0$ such that for all $t \in [2,3]^2$,

$$P\{q - 2^{-n+1} \leq W(t_1,t_2) \leq q - 2^{-n}\} \geq c_0 2^{-n}.$$

Next, set $Z_1 = W(t_1,t_2)$ and $Z_2 = 2^n t_1^{-1}(W(t_1,t_2+2^{-2n}) - W(t_1,t_2))$. Then Z_1 and Z_2 are independent, Z_2 is $N(0,1)$, and

$$\begin{aligned} &P\{q - 2^{-n+1} \leq W(t_1,t_2+2^{-2n}) \leq q - 2^{-n} \\ &\qquad\qquad\qquad \mid q - 2^{-n+1} \leq W(t_1,t_2) \leq q - 2^{-n}\} \\ &\quad = P\{q - 2^{-n+1} \leq Z_1 + t_1 2^{-n} Z_2 \leq q - 2^{-n} \\ &\qquad\qquad\qquad \mid q - 2^{-n+1} \leq Z_1 \leq q - 2^{-n}\} \\ &\quad \geq c_1, \end{aligned}$$

where c_1 is a constant that does not depend on n or $t \in [2,3]^2$. In particular, given that $q - 2^{-n+1} \leq W(t_1,t_2) \leq q - 2^{-n}$, all four other inequalities in the definition of $F_0(t,n)$ hold with positive probability, bounded below by a constant that does not depend on n or $t \in [2,3]^2$. Therefore,

$$P(F_0(t,n) \geq c\, 2^{-n}. \tag{6}$$

Let

$$F(t,n) = F_0(t,n) \cap F_L(t,n) \cap F_U(t,n) \cap F_R(t,n).$$

Notice that by the independent increments property of the Brownian sheet, $F_U(t,n)$ and $F_R(t,n)$ are independent, and independent of $F_0(t,n)$ and $F_L(t,n)$. Further, the event $F(t,n)$ is "approximately" the event described in the statement of Proposition 2.

5.3 A lower bound

We now estimate the probability of $F(t,n)$. By the independence properties just mentionned and (5),

$$\begin{aligned} P(F(t,n)) &= P(F_0(t,n) \cap F_L(t,n)) P(F_U(t,n)) P(F_R(t,n)) \\ &\geq K^2 2^{-2n} P(F_0(t,n) \cap F_L(t,n)). \end{aligned}$$

The events $F_0(t,n)$ and $F_L(t,n)$ are *not* independent. However, given F_0, which is essentially the event $\{q - 2^{-n+1} \leq W(t_1,t_2) \leq q - 2^{-n}\}$, the process $u \mapsto W(t_1 - 2^{-2n} - u, t_2) - W(t_1 - 2^{-2n}, t_2)$ is a Brownian motion run backwards in time, which is nothing but a Brownian bridge run for at least two units of time. Therefore, its behavior for $0 \leq u \leq 1$ is well-approximated by that of a Brownian motion. In particular, it is possible to check that (5) remains valid in the form

$$P(F_L(t,n) \mid F_0(t,n)) \geq K2^{-n}. \tag{7}$$

For details, the reader is referred to [9, Lemma 16]. By (6) and (7),

$$\begin{aligned} P(F_0(t,n) \cap F_L(t,n)) &= CP(F_0(t,n))P(F_L(t,n) \mid F_0(t,n)) \\ &\geq c2^{-2n}, \end{aligned}$$

and therefore, there is $c > 0$ such that for all large n and all $t \in [2,3]^2$,

$$P(F(t,n)) \geq c2^{-4n}.$$

Let $\mathbb{D}_{2n}$ be the set of points in $[2,3]^2$ with coordinates that are dyadic of order $2n$, so that card $\mathbb{D}_{2n} = 2^{4n}$, and let X_n be the number of $t \in \mathbb{D}_{2n}$ such that $F(t,n)$ occurs. Then

$$P\{X_n > 0\} = P(\cup_{t\in\mathbb{D}_{2n}} F(t,n)). \tag{8}$$

As noted in Remark 1(b) and in view of the definition of $F_0(t,n)$, it is easy to check that, the events in this union are disjoint, so this is equal to

$$\sum_{t\in\mathbb{D}_{2n}} P(F(t,n)) \geq 2^{4n} c\, 2^{-4n} = c > 0. \tag{9}$$

Set $G = \limsup_{n\to\infty} P\{X_n > 0\}$. By Fatou's lemma,

$$P(G) \geq \limsup_{n\to\infty} P\{X_n > 0\} > c > 0.$$

5.4 Proof of the main result

We now show that for $\omega \in G$, the desired behavior (4) of W occurs. By the definition of G, there exists a sequence $n_k \uparrow \infty$ such that $X_{n_k}(\omega) > 0$, so there exists a sequence of points $(t^{(k)}) \subset [2,3]^2$ such that $\omega \in F(t^{(k)}, n_k)$, for all k.

Because $[2,3]^2$ is compact, we can pass to a convergent subsequence, which we again denote $(t^{(k)})$, so there is $t = (t_1,t_2) \in [2,3]^2$ such that $t^{(k)} \to t$ as $k \to \infty$. Then, for $0 < u \leq 1$,

$$\begin{aligned} W(t_1+u,t_2) &= \lim_{k\to\infty} W(t_1^{(k)}+u, t_2^{(k)}) \\ &\geq \lim_{k\to\infty} (q+2^{-n_k}) + g(u-2^{-n_k}) - 2^{-n_k} \\ &= q + g(u) \\ &> q. \end{aligned}$$

Similarly,

$$W(t_1, t_2+u) < q \quad \text{and} \quad W(t_1-u, t_1) > q.$$

Therefore, t satisfies (4) and Proposition 2 is proved. ◊

Remark 2. Similar ideas are used in [9] to prove Theorem 6. However, a major difference occurs because there will typically be infinitely many points in $[2,3]^3$ that satisfy property (3), so that when definining approximating events analogous to the $F(t,n)$, these will no longer be disjoint. Therefore, the probability of the union of events in (8) will no longer be the sum of the probabilities of the events, and a different approach, known as the "second moment argument", is needed to obtain a lower bound as in (9). We will see this approach further on, in Subsection 7.1.

6 Absence of points of increase for Brownian motion

This section is devoted to proving the property of Brownian motion stated as Fact 3 in Section 3, namely, Brownian motion has no points of increase. We shall follow the approach of Burdzy [3].

6.1 The result of Dvoretsky, Erdős and Kakutani

Let f be a continuous real-valued function defined on $\mathbb{R}$. The function f is said to have a *point of increase* at $u \in \mathbb{R}$ if for some $\varepsilon > 0$,

$$f(u-h) < f(u) < f(u+h), \qquad \text{for all } 0 < h < \varepsilon.$$

The following famous property of Brownian motion was proved by Dvoretsky, Erdős and Kakutani [17].

Theorem 7. *With probability* 1*, sample paths of Brownian motion have no points of increase.*

Remark 3. Let W be a Brownian sheet. For fixed t_2, $t_1 \mapsto W(t_1,t_2)$ is a Brownian motion, so by Theorem 7, it has *a.s.* no points of increase. However, one can ask if there are random T_2 such that $t_1 \mapsto W(t_1,T_2)$ does have points of increase. We will see in Section 7 that the answer to this question is positive.

6.2 Reduction to a special case

We begin the proof of Theorem 7 by reducing the problem to a simpler one, using a sequence of elementary properties of Brownian motion. Throughout the remainder of this section, s and t will denote real numbers, rather than elements of $\mathbb{R}^2_+$.

Let $B = (B(t),\ t \geq 0)$ be a Brownian motion. For $\varepsilon > 0$, set

$$C_\varepsilon = \{\exists t > \varepsilon : B(t-h) < B(t) < B(t+h), \quad \forall h \in]0,\varepsilon[\}.$$

We need to show that

$$P(\cup_{\varepsilon \in \mathbb{Q}^*_+} C_\varepsilon) = 0,$$

or, equivalently, that $P(C_\varepsilon) = 0$, for all $\varepsilon > 0$. Set

$$\begin{aligned} A(y,\varepsilon,r) = \{\exists t > r+\varepsilon : & B(s) < B(t), && r \le s < t, \\ & B(t) < B(t+h), && 0 < h < \varepsilon, \\ & B(t+\varepsilon) > B(t) + y\}. \end{aligned}$$

Then

$$C_\varepsilon = \cup_{y \in \mathbb{Q}_+^*} \cup_{\varepsilon \in \mathbb{Q}_+^*} \cup_{r \in \mathbb{Q}_+^*} A(y,\varepsilon,r),$$

so it suffices to show that $P(A(y,\varepsilon,r)) = 0$ for fixed y, ε and r. We now express the event $A(y,\varepsilon,r)$ using increments of B after time r:

$$\begin{aligned} B(s) - B(r) &< B(t) - B(r), && r \le s < t, \\ B(t) - B(r) &< B(t+h) - B(r), && 0 < h < \varepsilon, \\ B(t+\varepsilon) - B(r) &> B(t) - B(r) + y. \end{aligned}$$

Use the independence of increments B and the fact that increments after r form a Brownian motion to see that the equality $P(A(y,\varepsilon,r)) = 0$ is equivalent to

$$P_0(A(y,\varepsilon,0)) = 0,$$

where under P_0, $B_0 = 0$ a.s. The properties that define $A(y,\varepsilon,0)$ are

$$\begin{aligned} \exists t > \varepsilon : \quad & B(s) < B(t), && 0 \le s < t, \\ & B(t) < B(s), && t < s < t+\varepsilon, \\ & B(t+\varepsilon) > B(t) + y. \end{aligned}$$

Clearly,

$$A(y,\varepsilon,0) = \cup_{M>1} \tilde{A}(y,\varepsilon,M),$$

where $\tilde{A}(y,\varepsilon,M)$ has the additional inequality $B(t) \le M$.

Set $\tilde{B}(s) = \frac{1}{M} B(M^2 s)$, which, by the scaling property of B, is again a Brownian motion. Let

$$\begin{aligned} A_0'(y,M) = \{\exists t > 0,\ \exists u > t : & \tilde{B}(s) < \tilde{B}(t), && 0 \le s < t, \\ & \tilde{B}(t) < \tilde{B}(s), && t < s \le u, \\ & \tilde{B}(t) \le 1, \\ & \tilde{B}(u) > \tilde{B}(t) + y/M\}. \end{aligned}$$

Then $P_0(A_0'(y,M)) \ge P_0(\tilde{A}(y,\varepsilon,M))$.

Define

$$\begin{aligned} A_0 = \{\exists v > 0,\ \exists t \in]0,v[: & B(s) < B(t), && 0 \le s < t, \\ & B(t) < B(s), && t < s \le v, \\ & B(t) \le 1, \\ & B(v) > B(t) + 2\}. \end{aligned} \tag{10}$$

We claim that it suffices to show that $P_0(A_0) = 0$.

Indeed, if $y/M \geq 2$, then $P(A_0'(y,M)) \leq P(A_0)$, so $P_0(A_0) = 0$ implies $P_0(A_0'(y,M)) = 0$. If $y/M < 2$, then $P(A_0) \leq P(A_0'(y,M))$, but

$$P(A_0) \geq P\left(\cup_{y,M:y/M<2}(A_0'(y,M) \cap \{B(u+\cdot) \text{ hits 2 before 0}\})\right),$$

so, for all $y > 0$ and $M > 0$ with $y/M < 2$,

$$P_0(A_0) \geq P_0(A_0'(y,M))P_{y/M}\{B \text{ hits 2 before 0}\}.$$

The last factor is positive, so $P_0(A_0) = 0$ implies $P_0(A_0'(y,M)) = 0$ also in this case.

In summary, in order to show that $P(C_\varepsilon) = 0$, it suffices to show that $P_0(A_0) = 0$.

6.3 Proof of the special case

In this subsection, we prove that $P_0(A_0) = 0$, where A_0 is defined in (10). As shown in the previous subsection, this will be sufficient to complete the proof of Theorem 7.

Fix $\varepsilon \in]0,1[$, set $M_0 = 0$, $U_0 = 0$, and for $k \geq 0$,

$$\begin{aligned} T_k &= \inf\{t > U_k : B(t) = M_k - \varepsilon \text{ or } B_t = M_k + 2\}, \\ M_{k+1} &= \max_{0 \leq t \leq T_k} B_t, \\ U_{k+1} &= \inf\{t > T_k : B(t) = M_{k+1}\}. \end{aligned}$$

Finally, for $k \geq 1$, let $X_k = M_k - M_{k-1}$.

It is not difficult to see that if $X_{k+1} = 2$, then "an approximate point of increase" occured at level M_k and time U_k. Indeed, $B_{U_k} = M_k$, $B(t) < M_k$ for $t \in]T_{k-1}, U_k[$, and when $X_{k+1} = 2$, $B_{T_{k+1}} = M_k + 2$, and $B(t) > M_k - \varepsilon$ for $t \in [U_k, T_{k+1}]$. In other words, during the time interval $[U_k, T_{k+1}]$, $B(\cdot)$ rises from level M_k to level $M_k + 2$ without going below level $M_k - \varepsilon$.

Set

$$A_\varepsilon = \{\exists k \geq 1 : M_k \leq 1 \text{ and } X_{k+1} = 2\}.$$

Then $A_0 \subset A_\varepsilon$, for all $\varepsilon > 0$, so it suffices to show that

$$\lim_{\varepsilon \downarrow 0} P_0(A_\varepsilon) = 0.$$

Notice that

$$\begin{aligned} P_0\{X_k > x\} &= P_0\{B \text{ hits } x \text{ before } -\varepsilon \text{ or } 2\} \\ &= \begin{cases} 0 & \text{if } x \geq 2, \\ \frac{\varepsilon}{x+\varepsilon} & \text{if } 0 < x < 2, \end{cases} \end{aligned} \tag{11}$$

so

$$E_0(X_k) = \varepsilon(\log(2+\varepsilon) - \log\varepsilon) \geq \varepsilon \log\frac{1}{\varepsilon}.$$

Let

$$N = \sup\{k \geq 0 : M_k \leq 1\} \qquad (\leq +\infty).$$

Then

$$P_0(A_\varepsilon) = P_0(\cup_{k\geq 1}\{M_k \leq 1 \text{ and } X_{k+1} = 2\}).$$

The events in the union are disjoint, so this is equal to

$$\sum_{k=0}^{\infty} P_0\{M_k \leq 1,\ X_{k+1} = 2\}.$$

By the strong Markov property at T_k, this equals

$$\sum_{k=0}^{\infty} P_0\{M_k \leq 1\} P_0\{X_{k+1} = 2\}.$$

By (11), this can be written

$$\frac{\varepsilon}{2+\varepsilon}\, E_0\left(\sum_{k=0}^{\infty} 1_{\{M_k \leq 1\}}\right).$$

Since $M_{k-1} \leq M_k$, for all k, this is equal to

$$\frac{\varepsilon}{2+\varepsilon}\, E_0(N+1). \tag{12}$$

To estimate $E_0(N+1)$, notice that $M_k = \sum_{j=1}^{k} X_j$, and the X_j are i.i.d. and positive. Apply Wald's equation [16, Section 3.1] to the stopping time

$$N+1 = \inf\{k \geq 0 : M_k \geq 2\},$$

which has finite expectation [16, Section 3.4, Theorem (4.1)], to get

$$E_0(M_{N+1}) = E_0(N+1)\, E_0(X_1),$$

so

$$E_0(N+1) = \frac{E_0(M_{N+1})}{E_0(X_1)} \leq \frac{3}{\varepsilon \log\frac{1}{\varepsilon}}\,.$$

By (12),

$$P_0(A_\varepsilon) \leq \frac{\varepsilon}{2+\varepsilon} \cdot \frac{3}{\varepsilon\log\frac{1}{\varepsilon}} = \frac{3}{2+\varepsilon}\ \frac{1}{\log\frac{1}{\varepsilon}}\,.$$

The right-hand side converges to 0 as $\varepsilon \downarrow 0$, so this completes the proof of Theorem 7. ♢

7 Points of increase of the Brownian sheet along horizontal lines

We now examine the analogue for the Brownian sheet of Fact 3 in Section 3. The following result, established in [9], shows that there are exceptional (random) horizontal lines on which the Brownian sheet does have a point of increase.

Theorem 8. *Fix $h > 0$. With positive probability, there exists $(t_1, t_2) \in [2, 3]^2$ such that*

$$W(t_1 - u, t_2) < W(t_1, t_2) < W(t_1 + u, t_2), \qquad \forall u \in]0, h]. \tag{13}$$

We remark that in order to get a statement that is valid with probability 1, it suffices to allow (t_1, t_2) to be anywhere in $\mathbb{R}_+^2$, rather than to require that $(t_1, t_2) \in [2, 3]^2$.

An equivalent way of stating this result is the following. Consider the Ornstein-Uhlenbeck process on Wiener space described in Section 1. At each fixed time, this process is a Brownian motion. However, Theorem 8 implies that at random times, this process visits the set of paths that have a point of increase. Other results of this type, in which the Brownian sheet along random horizontal lines has an unusual behavior compared with deterministic lines, will appear in a paper in preparation by Dalang and Khoshnevisan.

7.1 The second moment argument

We now begin the proof of Theorem 8, following [9, Section 2]. Let $g(u) = u^{3/4}$ and define $W_R^{t,n}(\cdot)$ and $W_L^{t,n}(\cdot)$ as in the beginning of the proof of Proposition 2. Set

$$F_R(t, n) = \{W_R^{t,n}(u) \geq g(u) - 2^{-n}, \quad 0 \leq u \leq 1\},$$
$$F_L(t, n) = \{W_L^{t,n}(u) \leq -g(u) + 2^{-n}, \quad 0 \leq u \leq 1\},$$

and

$$F^H(t, n) = F_L(t, n) \cap F_R(t, n).$$

Notice that this definition of $F_R(t, n)$ is the same as in the proof of Proposition 2 (see Subsection 5.2), while the definition of $F_L(t, n)$ is not.

If $F^H(t, n)$ occurs, then W nearly has a point of increase along the horizontal line through t. As in the proof of Proposition 2 (see Subsection 5.3), let X_n be the number of $t \in \mathbb{D}_{2n}$ such that $F^H(t, n)$ occurs. We shall need a lower bound on $P\{X_n > 0\}$. Use the Cauchy-Schwarz inequality to see that

$$(E(X_n))^2 \leq E(X_n^2) E(1_{\{X_n > 0\}}),$$

and therefore,

$$P\{X_n > 0\} \geq \frac{(E(X_n))^2}{E(X_n^2)}. \tag{14}$$

We will have the desired lower bound if we can show that there is $c < \infty$ such that for all large n,

$$E(X_n^2) \leq c\,(E(X_n))^2. \tag{15}$$

Proving this inequality is the objective of the next subsection. The argument just presented is known as the "second moment argument."

7.2 Estimating the first and second moments

To show (15), we first derive a lower bound on $E(X_n)$, and then an upper bound on $E(X_n^2)$. Clearly,

$$\begin{aligned} E(X_n) &= E\left(\sum_{t \in \mathbb{D}_{2n}} 1_{F^H(t,n)}\right) \\ &= \sum_{t \in \mathbb{D}_{2n}} P(F^H(t,n)) \\ &= \sum_{t \in \mathbb{D}_{2n}} P(F_L(t,n) \cap F_R(t,n)). \end{aligned}$$

The two events are independent, so this is equal to

$$\sum_{t \in \mathbb{D}_{2n}} P(F_L(t,n))\ P(F_R(t,n)) \geq 2^{4n}\ K^2\ 2^{-2n} = K^2\ 2^{2n},$$

by Lemma 2. We conclude therefore that

$$E(X_n) \geq K^2\ 2^{2n}. \tag{16}$$

On the other hand,

$$E(X_n^2) = E\left(\left(\sum_{t \in \mathbb{D}_{2n}} 1_{F^H(t,n)}\right)^2\right) = \sum_{s,t \in \mathbb{D}_{2n}} P(F^H(s,n) \cap F^H(t,n)).$$

We need an upper bound on the probability of the intersection of the two events $F^H(s,n)$ and $F^H(t,n)$. Clearly,

$$P(F^H(s,n) \cap F^H(t,n)) = P(F_L(s,n) \cap F_R(s,n) \cap F_L(t,n) \cap F_R(t,n)).$$

If these four last events were independent, this would be $\leq C2^{-4n}$, so we would get

$$E(X_n^2) \leq 2^{4n} 2^{4n} C 2^{-4n} = C\, 2^{4n} \leq \frac{C}{K^4} (E(X_n))^2,$$

which is what we would like. Unfortunately, $F_L(s,n)$, $F_L(t,n)$, $F_R(s,n)$ and $F_R(t,n)$ are clearly *not* independent.

It is not difficult to see that if $F_R(s,n)$ occurs, then this favors the occurence of $F_R(t,n)$. Lemma 3 below makes this statement precise.

Assuming Lemma 3 for the moment, or the equivalent statement in Lemma 4, we now check that the bound in Lemma 4 is sufficient to obtain the desired inequality

$$E(X_n^2) \leq C\ 2^{4n}. \tag{17}$$

Indeed, by Lemma 4 and the bound on the cardinality of the set $E_{i,j}$ defined just above Lemma 4,

$$\begin{aligned} E(X_n^2) &= \sum_{s,t \in \mathbb{D}_{2n}} P(F^H(s,n) \cap F^H(t,n)) \\ &\leq \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{(s,t) \in E_{i,j}} C\, 2^{-(2n+2i)} (\exp(-c(j-i) 2^{-(n-j)}) \wedge 1) \\ &\leq C\, 2^{2n} \sum_{i=0}^{n} \sum_{j=0}^{n} 2^{2j} (\exp(-c(j-i) 2^{-(n-j)}) \wedge 1). \end{aligned}$$

We split the sum into two parts, according as $i \geq j$ or $i < j$. When $i \geq j$, the last factor equals 1 and the first part of the sum becomes

$$\sum_{i=0}^{n} \sum_{j=0}^{i} 2^{2j} \leq 2 \sum_{i=0}^{n} 2^{2i} \leq 4\, 2^{2n}.$$

When $i < j$, the exponential plays a crucial role and the second part of the sum becomes

$$\begin{aligned} \sum_{i=0}^{n} \sum_{j=i+1}^{n} 2^{2j} \exp(-c(j-i) 2^{-(n-j)}) &= \sum_{j=1}^{n} \sum_{i=0}^{j-1} 2^{2j} \exp(-c(j-i) 2^{-(n-j)}) \\ &= \sum_{j=1}^{n} 2^{2j} \sum_{i=1}^{j} \exp(-c\, i\, 2^{-(n-j)}). \end{aligned}$$

The sum over i is geometric, equal to

$$\frac{\exp(-c 2^{-(n-j)}) - \exp(-c(j+1) 2^{-(n-j)})}{1 - \exp(-c\, 2^{-(n-j)})}.$$

Because the numerator is ≤ 1 and the denominator is $\geq c\ 2^{-(n-j)-1}$, we conclude that

$$E(X_n^2) \le C\ 2^{2n}\left(42^{2n} + c\sum_{j=1}^{n} 2^{2j}\cdot 2^{n-j+1}\right) \le C(4+4c)2^{4n}.$$

This proves (17).

Together, the inequalities (16) and (17) yield (15). By (14),

$$P\{X_n > 0\} \ge \frac{1}{c} > 0.$$

We now proceed as in the last lines of the proof of Proposition 2 (see Subsection 5.4). Set

$$G = \limsup_{n\to\infty} \{X_n > 0\}.$$

By Fatou's Lemma,

$$P(G) \ge \limsup_{n\to\infty} P\{X_n > 0\} \ge \frac{1}{c} > 0.$$

For $\omega \in G$, there exists a subsequence $n_k \uparrow \infty$ such that $X_{n_k}(\omega) > 0$, so there exists a sequence $(t^{(k)}) \subset [2,3]^2$ such that $\omega \in F^H(t^{(k)}, n_k)$. Pass to a convergent subsequence, which we again denote $(t^{(k)})$, so that $t^{(k)} \to t$. Then this t has the desired property (13), and Theorem 8 is proved. ♢

7.3 The key estimate

The proof of Theorem 8 made use of the following estimate in order to establish (17).

Lemma 3. *There is $c > 0$ and $C < \infty$ such that*

$$P(F^H(s,n) \cap F^H(t,n)) \le C\frac{2^{-4n}}{|s_1 - t_1| \vee 2^{-2n}}\left(\frac{|s_1-t_1|}{|s_2-t_2|}\right)^{c\sqrt{|t_2-s_2|}}.$$

As $s_1 - t_1$ decreases from 1 to 0, the denominator in the first ratio above decreases from 1 to 2^{-2n}, which means that the first ratio increases from 2^{-4n} to 2^{-2n}. For a fixed non-zero value of $s_1 - t_1$, the expression in parentheses increases to 1 as $|s_2 - t_2|$ decreases to 0. Therefore, the closer t is to s, the more the occurence of $F_R(s,n)$ favors the occurence of $F_R(t,n)$.

The aim of this subsection is to give the main ideas in the proof of Lemma 3. For $i, j \in \{0,\dots,n\}$, let $E_{i,j}$ be the set of couples (s,t) of elements of $\mathbb{D}_{2n}$ such that

$$2^{-2(n-i+1)} \le |s_1 - t_1| \le 2^{-2(n-i)} \quad \text{and} \quad 2^{-2(n-j+1)} \le |s_2 - t_2| \le 2^{-2(n-j)}.$$

Observe that card $E_{i,j} \le 2^{4n+2i+2j}$.

We shall use the (partial) order $\leq$ on $\mathbb{R}^2_+$ defined by

$$s = (s_1, s_2) \leq t = (t_1, t_2) \qquad \Longleftrightarrow \qquad s_1 \leq t_1 \text{ and } s_2 \leq t_2.$$

We also write $s \ll t$ if $s_1 < t_1$ and $s_2 < t_2$.

It is not difficult to see that Lemma 3 can be restated equivalently as follows.

Lemma 4. *There exists a finite constant $C > 0$ such that for all large $n \in \mathbb{N}$ and $(s,t) \in E_{i,j}$,*

$$P(F^H(s,n) \cap F^H(t,n)) \leq C\, 2^{-(2n+2i)} (\exp(-c(j-i)2^{-(n-j)}) \wedge 1).$$

The proof of Lemma 8 will make use of the following two lemmas.

Lemma 5. *Fix $K_i > 0$, $n_i \in \mathbb{N}^*$, $c > 0$ and let X_i, $i = 1, 2$, be independent random variables with density bounded by $K_i(|x|^{n_i} \vee 1) \exp(-x^2/c)$. Then there is a constant K such that the density of $X_1 + X_2$ is bounded by $K(|x|^{n_1+n_2} \vee 1) \exp(-x^2/(2c))$.*

Lemma 6. *Let B be a standard Brownian motion. There is $K > 0$ such that, for all $0 < \varepsilon < 1$, the conditional density of $B(1)$ given that B has not hit $-\varepsilon$ during the time interval $[0,1]$ is bounded by $K(|x| \vee 1) \exp(-x^2/2)$.*

For the proof of these two lemmas, the reader is referred to Lemmas 14 and 15 in [9].

Proof of Lemma 4. Fix $(s,t) \in E_{i,j}$ such that $s \leq t$ (the other relative positions of s and t are bounded similarly). Define

$$\hat{F}_R(s,t,n) = \left\{ W_R^{s,n}(u) \geq g(u) - 2^{-n},\ 0 \leq u \leq \frac{t_1 - s_1}{2} \right\},$$

$$\hat{F}_L(t,s,n) = \left\{ W_L^{t,n}(u) \leq -g(u) + 2^{-n},\ 0 \leq u \leq \frac{t_1 - s_1}{2} \right\},$$

$$\begin{aligned} G_R(t,s,n) = \{ & W_R^{t,n}(\cdot) \text{ hits 1 before } -2^{-n}, \\ & W_R^{s,n}(t_1 - s_1 + \cdot) \text{ hits 1 before } -2^{-n} \}. \end{aligned}$$

Observe that

$$F^H(s,n) \cap F^H(t,n) \subset F_L(s,n) \cap \hat{F}_R(s,t,n) \cap \hat{F}_L(t,s,n) \cap G_R(t,s,n),$$

and that the first three events on the right-hand side are mutually independent. Using Lemma 2 and the scaling property of Brownian motion, we see that

$$P(F_L(s,n)) \le C\,2^{-n}, \quad P(\hat{F}_R(s,t,n)) \le C\,2^{-i}, \quad P(\hat{F}_L(t,s,n)) \le C\,2^{-i}.$$

Let

$$H(s,t,n) = F_L(s,n) \cap \hat{F}_R(s,t,n) \cap \hat{F}_L(t,s,n) \cap \{W_R^{s,n}(t_1 - s_1) > -2^{-n}\}.$$

Then $P(H(s,t,n)) \le C\,2^{-n-2i}$, and therefore the conclusion will follow if we prove that

$$P(G_R(t,s,n)|H(s,t,n)) \le K\,2^{-n}(\exp(-c(j-i)2^{-(n-j)}) \wedge 1). \tag{18}$$

When $j \le i$, the right-hand side is equal to 2^{-n}. Because

$$G_R(t,s,n) \subset \{W_R^{t,n}(\cdot) \text{ hits } 1 \text{ before } -2^{-n}\},$$

the event on the right-hand side is independent of $H(s,t,n)$ and its probability is $\le 2^{-n}$, the inequality (18) is satisfied in this case.

Assume now that $i < j$. Set

$$Y = W(t_1, s_2) - W\left(\frac{s_1+t_1}{2} + 2^{-2n}, s_2\right),$$

$$\hat{Y} = E\left(Y \middle| W_L^{t,n}(u),\ 0 \le u \le \frac{t_1+s_1}{2}\right) = \frac{s_2}{t_2}\, W_L^{t,n}\left(\frac{s_1+t_1}{2} - 2^{-2n}\right),$$

$$\begin{aligned} Z &= W_R^{s,n}(t_1 - s_1) \\ &= W_R^{s,n}\left(\frac{s_1+t_1}{2}\right) + \hat{Y} + (Y - \hat{Y}) + (W(t_1 + 2^{-2n}, s_2) - W(t_1, s_2)). \end{aligned}$$

Then

$$G_R(t,s,n) = \{W_R^{t,n} \text{ hits } 1 \text{ before } -2^{-n},\ W_R^{(t_1,s_2),n}(\cdot) + Z \text{ hits } 1 \text{ before } -2^{-n}\}.$$

Notice that $(W_R^{t,n}(\cdot), W_R^{(t_1,s_2),n}(\cdot))$ is independent of $\sigma(H(s,t,n)) \vee \sigma(Z)$, and therefore, for each z, the event

$$\begin{aligned} G(t,s,n;z) = \{&W_R^{t,n}(\cdot) \text{ hits } 1 \text{ before } -2^{-n}, \\ &W_R^{(t_1,s_2),n}(\cdot) \text{ hits } 1-z \text{ before } -z-2^{-n}\} \end{aligned}$$

is independent of $\sigma(H(s,t,n)) \vee \sigma(Z)$. It follows that

$$P(G_R(t,s,n)|H(s,t,n)) = \int_{-2^{-n}}^{+\infty} P(G(t,s,n;z))\, f_{Z|H(s,t,n)}(z)\, dz, \tag{19}$$

where $f_{Z|H(s,t,n)}$ denotes the conditional density of Z given $H(s,t,n)$. Let $G'(t,s,n;z)$ be defined as $G(t,s,n;z)$, but with $1-z$ replaced by $1/2$. For $z \le 1/2$, $G(t,s,n;z) \subset G'(t,s,n;z)$, and $P(G'(t,s,n;z))$ is a non-decreasing function of z, which is therefore bounded by $P(G'(t,s,n;z \vee 2^{-n}))$.

Given $H(s,t,n)$, the law of $W_R^{s,n}((s_1+t_1)/2)$ is that of a standard Brownian motion at time $s_2^{1/2}(t_1-s_1)/2$ conditioned not to have hit -2^{-n}, and the law of Y is also, but at time $t_2^{1/2}(t_1-s_1)/2$. Because $Y-\hat{Y}$ is independent of $H(s,t,n)$, its law given $H(s,t,n)$ is still normal, with mean 0 and variance $s_2(1-s_2/t_2)(t_1-s_1)/2$. It follows therefore from Lemmas 5 and 6 that the conditional density of $2^{n-i}Z$ given $H(s,t,n)$ is bounded by

$$\psi(x) = K(|x|^3 \vee 1)e^{-x^2/2},$$

where K is a constant that does not depend on n, i or j, and therefore the conditional probability in (19) is no greater than

$$\int_{-2^{-i}}^{+\infty} P(G'(t,s,n;(2^{-(n-i)}x)\vee 2^{-n}))\,\psi(x)\,dx. \tag{20}$$

Let $k_0 = \sup_{x\geq 0} x^{1/2}\,2^{-x}$. Then the integral in (20) can be split into two integrals, the first over $x \leq (n-i)^{1/2}/(16k_0)$, the second over $x > (n-i)^{1/2}/(16k_0)$. By Lemma 7 below, the first integral is bounded by

$$\begin{aligned}
&\int_{-2^{-i}}^{+\infty} 2^{-n}\left(\frac{(2^{-(n-i)}x)\vee 2^{-n}}{2^{-(n-j+1)}}\right)^{c2^{-(n-j)}/\sqrt{s_2}} \psi(x)\,dx \\
&\quad = 2^{-n}(2^{i-j})^{c2^{-(n-j)}/\sqrt{s_2}} \int_{-2^{-i}}^{+\infty} (x\vee 2^{-i})^{c2^{-(n-j)}/\sqrt{s_2}}\psi(x)\,dx \\
&\quad \leq K\;2^{-n}\,2^{-c(j-i)2^{-(n-j)}/\sqrt{s_2}},
\end{aligned} \tag{21}$$

while the second integral is bounded by

$$\int_{(n-i)^{1/2}/(16k_0)}^{+\infty} P\{W_R^{t,n}(\cdot) \text{ hits 1 before } -2^{-n}\}\,\psi(x)\,dx \leq K2^{-n}e^{-(n-i)/C}.$$

For $c < 1/C$, the right-hand side is

$$\leq K\;2^{-n}\exp(-c(n-i)2^{-(n-j)}). \tag{22}$$

We observe that

$$2^{-c(n-i)2^{-(n-j)}} \leq 2^{-c(j-i)2^{-(n-j)}},$$

and, from (20), (21) and (22), we conclude that (18) holds with $i<j$. This completes the proof of Lemma 4. ◇

7.4 A property of planar Brownian motion

The estimate (21) above relied on the following fact.

Lemma 7. *There are $K > 0$ and $c > 0$ such that: for all $s, t \in [2,3]^2$ with $s_1 \le t_1$, $s_2 < t_2$ and $t_1 - s_1 \le \frac{1}{2}$, for all large n and $x \in [2^{-n}, \frac{1}{16}]$,*

$$P\{W_R^{t,n}(\cdot) \text{ hits } 1 \text{ before } -2^{-n},\ W_R^{(t_1,s_2),n}(\cdot) \text{ hits } 1 \text{ before } -x-2^{-n}\}$$

$$\le K\, 2^{-n} \left(\frac{x}{\sqrt{t_2 - s_2}} \right)^{c\sqrt{t_2-s_2}/\sqrt{s_2}}.$$

Because $(W_R^{t,n}(\cdot), W_R^{(t_1,s_2),n}(\cdot))$ has the same law as

$$(\sqrt{s_2}B_1, \sqrt{s_2}B_1 + \sqrt{t_2 - s_2}B_2),$$

where (B_1, B_2) is a standard planar Brownian motion started at the origin, Lemma 7 states a property of planar Brownian motion, concerning the manner in which such a process exits a certain parallelogram. For the proof of Lemma 7, the reader is referred to [9, Lemma 13].

8 Additive Brownian motion

As mentioned in the introduction, a two-parameter process $(X(u_1, u_2), (u_1, u_2) \in \mathbb{R}_+^2)$ is an *additive Brownian motion* if

$$X(u_1, u_2) = B_1(u_1) + B_2(u_2),$$

where $B_1(\cdot)$ and $B_2(\cdot)$ are independent Brownian motions. It is often convenient to replace $B_2(\cdot)$ by $-B_2(\cdot)$ in the definition of X: this does not affect the distribution of X, since $-B_2$ is again a Brownian motion.

In this section, we shall show that this process is closely related to the Brownian sheet, and examine fine properties of its level sets and excursions. Obtaining for the Brownian sheet analogous results to those presented in this section for additive Brownian motion remains a challenge for the futur.

8.1 Local relationship with the Brownian sheet

The local behavior of a Brownian sheet in the neighborhood of a single point is well-approximated by an additive Brownian motion. Indeed, it was observed in [14] that for fixed (t_1, t_2),

$$W(t_1 + u_1, t_2 + u_2) = W(t_1, t_2) + B_1(u_1) + B_2(u_2) + \varepsilon_{t_1,t_2}(u_1, u_2),$$

where

$$B_1(u_1) = W(t_1 + u_1, t_2) - W(t_1, t_2), \quad B_2(u_2) = W(t_1, t_2 + u_2) - W(t_1, t_2),$$

and

$$\varepsilon_{t_1,t_2}(u_1,u_2) = \Delta_{[t_1,t_1+u_1]\times[t_2,t_2+u_2]}W.$$

Therefore, $(B_1(u_1),\ u_1 \geq 0)$ and $(B_2(u_2),\ u_2 \geq 0)$ are independent Brownian motions (with respective speeds $\sqrt{t_2}$ and $\sqrt{t_1}$) that are independent of ε_{t_1,t_2}. Clearly, for small u_1 and u_2,

$$B_1(u_1) \approx \sqrt{u_1}, \qquad B_2(u_2) \approx \sqrt{u_2}, \qquad \varepsilon_{t_1,t_2}(u_1,u_2) \approx \sqrt{u_1 u_2},$$

so $\varepsilon_{t_1,t_2}(u_1,u_2)$ is of smaller order than $X(u_1,u_2) = B_1(u_1) + B_2(u_2)$. The local behavior of $W(\cdot)$ near (t_1,t_2) is therefore well-approximated by the behavior of $X(\cdot)$.

8.2 Global relationship with the Brownian sheet

Consider an $\mathbb{R}^d$-valued Brownian sheet and an $\mathbb{R}^d$-valued additive Brownian motion. While the previous result gives a local relationship between these two processes, there are also global relationships. For instance, Khoshnevisan and Shi [21] have shown that these two processes are *intersection-equivalent,* that is, for all compact subsets E of $\mathbb{R}^d$,

$$P\{X(\mathbb{R}^2_+) \cap E \neq \emptyset\} > 0 \Longleftrightarrow P\{W(\mathbb{R}^2_+) \cap E \neq \emptyset\} > 0.$$

8.3 Level sets and excursions of additive Brownian motion

Additive Brownian motion X is clearly a simpler process than the Brownian sheet W. One property which highlights this is that for all rectangles R, one immediately checks from the definition of X that $\Delta_R X \equiv 0$, whereas $\Delta_R W = W(R)$.

It is therefore natural to study level sets, bubbles and excursions of additive Brownian motion. In the remainder of this section, we use the notation

$$\begin{aligned} L(x) &= \{(u_1,u_2) : X(u_1,u_2) = x\},\\ L_+(x) &= \{ \quad " \qquad " \qquad > x\},\\ L_-(x) &= \{ \quad " \qquad " \qquad < x\}. \end{aligned}$$

By the local approximation property, small bubbles of the Brownian sheet should not be very different from bubbles of additive Brownian motion. A result that indicates some similarity between level sets of additive Brownian motion and those of the Brownian sheet is the following.

Proposition 3. *For additive Brownian motion, a.s., for all* $x \in \mathbb{R}$, $\dim\, L(x) = \frac{3}{2}$.

According to this proposition and Theorem 3, the Hausdorff dimension of the level sets of additive Brownian motion and of the level sets of the Brownian sheet are the same. We point out that most of the proofs of Theorem 3 for the Brownian sheet apply without any change to additive Brownian motion.

In the next subsections, we shall present some results concerning bubbles and level sets of additive Brownian motion for which corresponding results for the Brownian sheet are not available.

8.4 Structure of bubbles of additive Brownian motion

For the remainder of this section, we set

$$X(u_1, u_2) = B_1(u_1) - B_2(u_2),$$

so that

$$X(u_1, u_2) > 0 \Longleftrightarrow B_1(u_1) > B_2(u_2).$$

We are interested in describing individual components of $L_+(0)$. For this, we will term a *cross* a pair C of line segments

$$\{]s_1, t_1[\times\{u_2\},\ \{u_1\}\times]s_2, t_2[\}$$

such that $u_1 \in]s_1, t_1[$ and $u_2 \in]s_2, t_2[$ (that is, the two open line segments intersect). The point (u_1, u_2) is the *center* of the cross. Given a cross C as above, we set $R(C) = [s_1, t_1] \times [s_2, t_2]$, and term $R(C)$ the *rectangle generated by* C.

The following result is due to [15, Section 2].

Theorem 9. *Let E be a connected component of $L_+(0)$. Then there exists a (random)* distinguished cross

$$C(E) = \{]\sigma_1, \tau_1[\times\{S_2\},\ \{S_1\}\times]\sigma_2, \tau_2[\}$$

such that $C(E) \subset E \subset R(C(E))$, with the following additional properties. The interval $]\sigma_1, \tau_1[$ is an excursion interval for B_1 from some (random) value m up to some (random) maximum value $M > m$, and $B_1(S_1) = M$, $B_1(\sigma_1) = B_1(\tau_1) = m$, while the interval $]\sigma_2, \tau_2[$ is an excursion interval for B_2 from value M down to the minimum value m, and $B_2(S_2) = m$, $B_2(\sigma_2) = B_2(\tau_2) = M$. The positive value $M - m$ at the point (S_1, S_2) is the unique maximum of X on $R(C(E))$.

Remark 4. We note that E is the component of $L_+(0)$ which contains (S_1, S_2). It is not difficult to check that $X \leq 0$ on $\partial R(C(E))$, and in fact, all four corners of $R(C(E))$ belong to the same negative bubble. Therefore, E is surrounded by a negative bubble E_- (see the proof of [15, Lemma 5.3]).

Theorem 9 leaves open the question of determining which points in $R(C(E))$ do indeed belong to E. This question was addressed in [15], using the following algorithm.

ALGORITHM A. Fix $r = (r_1, r_2) \in R(C(E))$. If $X(r_1, r_2) \leq 0$, then output NO and stop. Otherwise, set $T_1^0 = r_1$, $T_2^0 = r_2$, and $H_0 = X(r)$. The algorithm proceeds in stages, beginning with $n = 1$.

Stage $2n-1$. Let

$$\begin{aligned} U_n &= \sup\{u < T_1^{n-1} : X(u, T_2^{n-1}) = 0\}, \\ U'_n &= \inf\{u > T_1^{n-1} : X(u, T_2^{n-1}) = 0\}, \\ H_{2n-1} &= \sup_{U_n < u < U'_n} X(u, T_2^{n-1}), \end{aligned}$$

and let T_1^n be the unique time point in $[U_n, U'_n]$ such that

$$X(T_1^n, T_2^{n-1}) = H_{2n-1}.$$

(a) If $T_1^n = S_1$, then output YES and stop.
(b) If $H_{2n-1} = H_{2n-2}$ (or, equivalently, $T_1^n = T_1^{n-1}$), then output NO and stop.
(c) Otherwise, proceed to Stage $2n$.

Stage $2n$. Let

$$\begin{aligned} V_n &= \sup\{v < T_2^{n-1} : X(T_1^n, v) = 0\}, \\ V'_n &= \inf\{v > T_2^{n-1} : X(T_1^n, v) = 0\}, \\ H_{2n} &= \sup_{V_n < v < V'_n} X(T_1^n, v), \end{aligned}$$

and let T_2^n be the unique time point in $[V_n, V'_n]$ such that

$$X(T_1^n, T_2^n) = H_{2n}.$$

(a) If $T_2^n = S_2$, then output YES and stop.
(b) If $H_{2n} = H_{2n-1}$ (or, equivalently, $T_2^n = T_2^{n-1}$), then output NO and stop.
(c) Otherwise, proceed to Stage $2n+1$.

The following result is established in [15, Proposition 2.2].

Theorem 10. *Consider E, σ_1, τ_1, S_1, σ_2, τ_2, and S_2 as in Theorem 9. For each $r \in R(C(E))$, Algorithm A terminates after a finite number of stages, and the algorithm outputs YES if and only if $r \in E$.*

For the proof of this theorem, the reader is referred to [15]. This algorithm can for instance be used to compute the expected area of a bubble given its height. Other uses of Algorithm A will appear in [11].

8.5 A Jordan curve in the boundary of a bubble

Let E be a connected component of $L_+(0)$ and consider the negative bubble E_-, mentioned in Remark 4, that contains the four corners of $R(C(E))$. In this subsection, we address the following question.

Question How is E separated from E_- ?

The answer to this question is provided by the following result [8].

Theorem 11. *Let E, m, M, σ_1, τ_1, σ_2, τ_2, S_1, and S_2 be in Theorem 9. Then there is a unique Jordan arc contained in $\partial E \subset L(0)$ with extremities (σ_1, S_2) and (S_1, τ_2), and even a unique closed Jordan curve J contained in ∂E that passes through the four points (σ_1, S_2), (S_1, τ_2), (τ_1, S_2) and (S_1, σ_2).*

Before proving Theorem 11, we mention some properties of the Jordan curve J mentioned in this theorem.

- J is nowhere differentiable [6].
- The Hausdorff dimension of J is ≥ 1 (in fact, this is true for any Jordan curve).
- The Hausdorff dimension of J is $\leq 3/2$ (since this is dim $L(0)$). In fact, according to a result of Mountford [27], dim $J \leq$ dim $\partial E < 3/2$.
- In a forthcoming paper [11], it will be shown that

$$\dim \partial E = \frac{3}{2} - \frac{1}{4}\left(5 - \sqrt{13 + 4\sqrt{5}}\right) < 1.422,$$

and therefore this expression is also an upper bound for dim J.

8.6 Construction of the Jordan curve

In this subsection, we follow [8, Section 3]. The second statement in Theorem 11 is in fact a consequence of the first, because the four sub-rectangles

$$\begin{aligned} &[\sigma_1, S_1] \times [S_2, \tau_2], \quad [S_1, \tau_1] \times [S_2, \tau_2], \\ &[\sigma_1, S_1] \times [\sigma_2, S_2], \quad [S_1, \tau_1] \times [\sigma_2, S_2], \end{aligned} \tag{23}$$

all play similar roles, and so the closed Jordan curve is just the union of the four Jordan arcs contained in $L(0) \cap \partial E$ that link (σ_1, S_2) to (S_1, τ_2), (S_1, τ_2) to (τ_1, S_2), (τ_1, S_2) to (S_1, σ_2), and finally (S_1, σ_2) to (σ_1, S_2). Therefore, in the proof of the theorem, we will focus on the first statement.

We are first going to prove this statement for fixed sample paths $B_1(\cdot\,;\omega)$ and $B_2(\cdot\,;\omega)$. Since we focus on the first of the four rectangles listed in (23), we replace the sample paths of B_1 and B_2 respectively by deterministic continuous functions f_1 and f_2. These functions will need to have certain properties, which will appear during the construction, and we will check later on that sample paths of Brownian motions satisfy these properties.

In order to simplify the notation, we put ourselves in the following situation. Fix real numbers $m < M$, $\sigma_1 < \tau_1$, and $\sigma_2 < \tau_2$. Assume that f_1 and f_2 are deterministic continuous functions such that

$$f_1(\sigma_1) = f_2(\sigma_2) \qquad \text{and} \qquad f_1(\tau_1) = f_2(\tau_2).$$

We are interested in constructing a Jordan arc contained in $\{(s_1, s_2) : f_1(s_1) = f_2(s_2)\}$ with extremities (σ_1, σ_2) and (τ_1, τ_2). Notice that the point (σ_1, σ_2) (resp. (τ_1, τ_2)) plays the role of (σ_1, S_2) (resp. (S_1, τ_2)) in Theorem 11.

Define the (partial) orders $\leq$ and $\triangle$ on $\mathbb{R}^2$ by

$$(s_1, s_2) \leq (t_1, t_2) \quad \Longleftrightarrow \quad s_1 \leq t_1 \text{ and } s_2 \leq t_2,$$
$$(s_1, s_2) \triangle (t_1, t_2) \quad \Longleftrightarrow \quad s_1 \leq t_1 \text{ and } s_2 \geq t_2.$$

We begin by examining a special case, which never occurs for sample paths of Brownian motion but does occur for the past minimum process or the future minimum process of a Brownian motion, and will be used further on. This special case will be a key ingredient for the general case.

The monotone case. The case where f_1 and f_2 are monotone is particularly simple. In order to state the result, we shall need the following definition, which also appears in [24].

Definition. Two monotone functions f_1 and f_2 have *a common flat level* if the inverse functions f_1^{-1} and f_2^{-1} have a common point of discontinuity.

Lemma 8. *Assume $m < M$, $\sigma_1 < \tau_1$ and $\sigma_2 < \tau_2$. For $i = 1, 2$, let f_i be a continuous monotone function with domain $[\sigma_i, \tau_i]$ and range $[m, M]$. Assume that $m < f_i(s) < M$ for $\sigma_i < s < \tau_i$, $i = 1, 2$, and f_1 and f_2 have no common flat levels. Then the set*

$$C(f_1, f_2) = \{(s_1, s_2) : f_1(s_1) = f_2(s_2)\}$$

is a monotone curve with endpoints (σ_1, σ_2) and (τ_1, τ_2). If f_1 and f_2 are non-decreasing and $a < b$, then the function $\psi : C(f_1, f_2) \to [a, b]$ defined by $\psi(s_1, s_2) = a + (s_1 + s_2 - \sigma_1 - \sigma_2)(b - a)/(\tau_1 + \tau_2 - \sigma_1 - \sigma_2)$ is continuous and one-to-one, or equivalently, ψ^{-1} is a continuous one-to-one parametrization of $C(f_1, f_2)$.

Proof. We only consider the case where f_1 and f_2 are non-decreasing. Let $D = \{(s_1, s_2) : f_1(s_1) \geq f_2(s_2)\}$. This set satisfies the conditions of [40, Theorem 2.7], and therefore its upper-left boundary Λ is a monotone curve and $\psi : \Lambda \to \mathbb{R}$ defined by $\psi(s_1, s_2) = s_1 + s_2$ is continuous and one-to-one (according to Walsh's proof). So we only need to show that $C(f_1, f_2) = \Lambda$.

By our assumptions, $(\sigma_1, s_2) \notin D$ for $\sigma_2 < s_2 < \tau_2$. For any such s_2 and $(s_1, s_2) \in \Lambda$ such that $\sigma_1 < s_1 < \tau_1$ and any large n, $(s_1 - \frac{1}{n}, s_2 + \frac{1}{n}) \notin D$, $s_1 - \frac{1}{n} \in [\sigma_1, \tau_1]$, $s_2 + \frac{1}{n} \in [\sigma_2, \tau_2]$, and therefore $f_1(s_1 - \frac{1}{n}) < f_2(s_2 + \frac{1}{n})$.

Letting $n \to \infty$, it follows by continuity that $f_1(s_1) \leq f_2(s_2)$. But $f_1(s_1) \geq f_2(s_2)$ since $(s_1, s_2) \in \Lambda \subset D$, so $f_1(s_1) = f_2(s_2)$ and $(s_1, s_2) \in C(f_1, f_2)$. Therefore $\Lambda \subset C(f_1, f_2)$.

We now establish the converse inclusion. Fix $(s_1, s_2) \in C(f_1, f_2)$. Assume $(s_1, s_2) \notin \Lambda$, that is, there is $(t_1, t_2) \in D$ such that $t_1 < s_1$ and $t_2 > s_2$. Then since $t_1 < s_1$, $(t_1, t_2) \in D$ and $s_2 < t_2$, we see that

$$f_1(s_1) \geq f_1(t_1) \geq f_2(t_2) \geq f_2(s_2),$$

so all of these inequalities are equalities because $f_1(s_1) = f_2(s_2)$. But then f_1 and f_2 have a common flat level, which contradicts the hypothesis. Therefore $C(f_1, f_2) \subset \Lambda$. ♢

Remark 5. (a) There is a one-to-one correspondence between horizontal (resp. vertical) segments of $C(f_1, f_2)$ and intervals on which f_1 (resp. f_2) is constant.

(b) If f_1 is non-increasing and f_2 is non-decreasing, then one should replace $s_1 + s_2$ by $s_1 - s_2$ in the definition of $\psi(s_1, s_2)$. If both f_1 and f_2 are non-decreasing, then no change is necessary.

The non-monotone case. We no longer assume that f_1 and f_2 are monotone, but we will assume that Hypothesis 1 below is satisfied. Fix

$$m < M, \qquad \sigma_1 < \tau_1, \qquad \sigma_2 < \tau_2 \tag{24}$$

as in Lemma 8. We assume now that f_i is a continuous function with domain Dom $f_i = [\sigma_i, \tau_i]$ and range Range $f_i = [m, M]$, satisfying the following hypothesis.

Hypothesis 1.

(a) $f_i(\text{int Dom } f_i) \subset \text{int Range } f_i$;

(b) the values of f_1 (resp. f_2) at distinct local extrema are distinct (in particular, there is no non-degenerate interval on which f_1 (resp. f_2) is constant);

(c) the functions $\underline{f}_1$ and $\underline{f}_2$ have no common flat levels, where $\underline{f}_1$ and $\underline{f}_2$ are defined as follows :

$$\underline{f}_i(s_i) = \begin{cases} \min\limits_{s_i \leq t_i \leq \tau_i} f_i(t_i), \text{ if } f_i(\sigma_i) < f_i(\tau_i), \\ \min\limits_{\sigma_i \leq t_i \leq s_i} f_i(t_i), \text{ if } f_i(\sigma_i) > f_i(\tau_i). \end{cases}$$

When this hypothesis is satisfied, the functions $\underline{f}_1$ and $\underline{f}_2$ satisfy the assumptions of Lemma 8, and we can consider the set $C(\underline{f}_1, \underline{f}_2)$.

The set $\{s_1 : f_1(s_1) > \underline{f}_1(s_1)\}$ is an open set, therefore a countable union of open intervals, each of which corresponds to an excursion of f_1 above $\underline{f}_1$, and also to an open horizontal segment of $C(\underline{f}_1, \underline{f}_2)$ ($\underline{f}_1$ is constant on each of these intervals). Similar statements are true of the set $\{s_2 : f_2(s_2) > \underline{f}_2(s_2)\}$. Let $S(f_1, f_2)$ be the union of all of these open segments, and set

$$L(f_1, f_2) = C(\underline{f}_1, \underline{f}_2) \setminus S(f_1, f_2).$$

This defines a closed set which is totally ordered (for $\leq$ or for $\underline{\triangle}$), but is not connected. Notice that each horizontal segment of $C(\underline{f}_1, \underline{f}_2)$ (on which $f_1 > \underline{f}_1$) corresponds to a *horizontal gap* in $L(f_1, f_2)$, while each vertical segment of $C(\underline{f}_1, \underline{f}_2)$ (on which $f_2 > \underline{f}_2$) corresponds to a *vertical gap* in $L(f_1, f_2)$. In addition, for each $(s_1, s_2) \in L(f_1, f_2)$,

$$f_1(s_1) = \underline{f}_1(s_1) = \underline{f}_2(s_2) = f_2(s_2),$$

so

$$L(f_1, f_2) \subset \{(s_1, s_2) : f_1(s_1) = f_2(s_2)\}.$$

Also, notice that each horizontal segment of $C(f_1, f_2)$ is of the form $[s_1, t_1] \times \{s_2\}$, where $[s_1, t_1]$ is an excursion interval of f_1 above $\underline{f}_1$, and similar statement is true for vertical segments of $C(f_1, f_2)$.

Finally, if I is any non-degenerate closed interval, then $L(f_1, f_2)$ can be parametrized by a continuous and one-to-one function $\varphi(f_1, f_2, I)$ defined on a closed subset of I, by defining ψ as in Lemma 8 using $\underline{f}_1$ and $\underline{f}_2$ and with $[a, b] = I$ and setting

$$\varphi(f_1, f_2, I) = \psi^{-1}|_{\psi(L(f_1,f_2))}. \tag{25}$$

As the range of φ is contained in $\mathbb{R}^2$, we use the notation $\varphi = (\varphi_1, \varphi_2)$. This function is continuous on its domain, even though the domain is not an interval but a closed set.

Given the domains of f_1 and f_2, we can define the rectangle

$$R(f_1, f_2) = (\text{Dom } f_1) \times (\text{Dom } f_2).$$

Note that $L(f_1, f_2) \subset R(f_1, f_2)$, and except for two points on the extremities of one of the diagonals of $R(f_1, f_2)$, $\partial R(f_1, f_2)$ is contained in $\{(s_1, s_2) : f_1(s_1) \neq f_2(s_2)\}$.

The set $L(f_1, f_2)$ will be part of the Jordan curve that we shall construct. Since this set is not connected, we must add additional points to create a Jordan curve. We shall do this by a recursive procedure, taking the union and the closure of the sets that are constructed.

I. *Filling in a horizontal gap*

Suppose $[s_1, t_1] \times \{s_2\}$ is a horizontal gap in $L(f_1, f_2)$. In particular

$$f_1(s_1) = \underline{f}_1(s_1) = \underline{f}_1(t_1) = f_1(t_1) = f_2(s_2) = \underline{f}_2(s_2)$$

and $f_1(u) > \underline{f}_1(u)$ for $s_1 < u < t_1$. We assume for simplicity that $f_1(\sigma_1) = m = f_2(\sigma_2)$, because the procedure in the other cases is similar.

Let $\underline{s}_1 \in [s_1, t_1]$ be an absolute maximum of f_1 on $[s_1, t_1]$; by Hypothesis 1(b), there is a single such maximum. Set

$$t_2 = \inf\{v \geq s_2 : f_2(v) = f_1(\underline{s}_1)\}.$$

Note that $s_2 < t_2 < \tau_2$, and since $\underline{f}_1$ and $\underline{f}_2$ have no common flat levels by Hypothesis 1(c), $f_2(s_2) < f_2(v) < f_2(t_2)$ for $s_2 < v < t_2$.

We now construct two pairs of functions (g_1, g_2) and (h_1, h_2) which satisfy (a) and (b) of Hypothesis 1. The function g_1 has domain $[s_1, \underline{s}_1]$ and is equal to f_1 on this interval. The function h_1 has domain $[\underline{s}_1, t_1]$ and is equal to f_1 on this interval. The two functions g_2 and h_2 are equal and have domain $[s_2, t_2]$ and are equal to f_2 on this interval.

We can now consider the sets $L(g_1, g_2)$ and $L(h_1, h_2)$. Notice that

$$\begin{aligned}
L(g_1, g_2) \cap L(h_1, h_2) &= \{(\underline{s}_1, t_2)\}, \\
R(g_1, g_2) \cap L(f_1, f_2) &= \{(s_1, s_2)\}, \\
R(h_1, h_2) \cap L(f_1, f_2) &= \{(t_1, s_2)\}, \\
R(g_1, g_2) \cap R(h_1, h_2) &= \{\underline{s}_1\} \times [s_2, t_2], \\
R(g_1, g_2) \cup R(h_1, h_2) &\subset R(f_1, f_2).
\end{aligned}$$

We call $R(g_1, g_2) \cup R(h_1, h_2) = [s_1, t_1] \times [s_2, t_2]$ the rectangle *associated* with the gap $[s_1, t_1] \times \{s_2\}$.

II. *Filling in a vertical gap*

This is similar to filling in a horizontal gap. Suppose $\{u_1\} \times [u_2, v_2]$ is a vertical gap in $L(f_1, f_2)$. In particular,

$$f_2(u_2) = \underline{f}_2(u_2) = \underline{f}_2(v_2) = f_2(v_2) = f_1(u_1) = \underline{f}_1(u_1),$$

and $f_2(u) > \underline{f}_2(u_2)$ for $u_2 < u < v_2$. We assume for simplicity that $f_1(\sigma_1) = m = f_2(\sigma_2)$, because the procedure in the other cases is similar.

Let $\underline{v}_2 \in [u_2, v_2]$ be an absolute maximum of f_2 on $[u_2, v_2]$; by Hypothesis 1(b), there is a single such maximum. Set

$$v_1 = \inf\{u \geq u_1 : f_1(u) = f_2(\underline{v}_2)\}.$$

Note that $u_1 < v_1 < \tau_1$, and, since $\underline{f}_1$ and $\underline{f}_2$ have no common flat levels, $f_1(u_1) < f_1(u) < f_1(v_1)$ for $u_1 < u < v_1$.

We now construct two pairs of functions (g_1, g_2) and (h_1, h_2), which satisfy (a) and (b) of Hypothesis 1. The functions g_1 and h_1 are equal, both have domain $[u_1, v_1]$ and are equal to f_1 on this interval. The function g_2 (resp. h_2) has domain $[u_2, \underline{v}_2]$ (resp. $[\underline{v}_2, v_2]$) and is equal to f_2 on its domain.

We can now consider the sets $L(g_1, g_2)$ and $L(h_1, h_2)$. Notice that

$$\begin{aligned}
L(g_1, g_2) \cap L(h_1, h_2) &= \{(v_1, \underline{v}_2)\}, \\
R(g_1, g_2) \cap L(f_1, f_2) &= \{(u_1, u_2)\}, \\
R(h_1, h_2) \cap L(f_1, f_2) &= \{(u_1, v_2)\}, \\
R(g_1, g_2) \cap R(h_1, h_2) &= [u_1, v_1] \times \{\underline{t}_2\}, \\
R(g_1, g_2) \cup R(h_1, h_2) &\subset R(f_1, f_2).
\end{aligned}$$

Again, we call $R(g_1,g_2)\cup R(h_1,h_2)=[u_1,v_1]\times[u_2,v_2]$ the rectangle *associated* with the gap $\{u_1\}\times[u_2,v_2]$.

III. *Parametrizing filled in gaps*

Suppose $[s_1,t_1]\times\{s_2\}$ is a horizontal gap in $L(f_1,f_2)$, which corresponds to an interval $]\alpha,\beta[\subset I$ in the complement of the domain of $\varphi(f_1,f_2,I)$.

In order to parameterize $L(g_1,g_2)$ and $L(h_1,h_2)$, we use respectively the intervals

$$I_1=\left[\alpha,\frac{\alpha+\beta}{2}\right]\qquad\text{and}\qquad I_2=\left[\frac{\alpha+\beta}{2},\beta\right].$$

Notice that $I_1\cup I_2=[\alpha,\beta]$ and $I_1\cap I_2$ is a singleton. Moreover, neither of these intervals overlaps with the domain of $\varphi(f_1,f_2,I)$, except at one endpoint, where both parameterizations agree. Further, distinct gaps in $L(f_1,f_2)$ lead to disjoint intervals.

For vertical gaps, one proceeds similarly. It is clear from I and II that rectangles associated with distinct horizontal (resp. vertical) gaps are disjoint. One can show that in fact, a rectangle associated with a horizontal gap will be disjoint from any rectangle associated with a vertical gap (see [8]).

Constructing the Jordan arc. We begin with m, M, σ_1, τ_1, σ_2, τ_2 as in (24). For $i=1,2$, we assume that f_i is continuous with domain $[\sigma_i,\tau_i]$ and range $[m,M]$ and satisfies Hypothesis 1.

We shall inductively define a sequence $(\mathcal{L}_k,\ k\geq 0)$ of sets of pairs of functions. A pair in $\mathcal{L}_k$ will be referred to as a *level* k pair. We assume that the following occurs.

Hypothesis 2. Hypothesis 1 is satisfied by all the pairs of functions that arise in the construction of the sequence $(\mathcal{L}_k,\ k\geq 0)$.

By definition, there is a single level 0 pair $(f_1^{(0)},f_2^{(0)})$, equal to (f_1,f_2), with parameterization interval $I^{(0)}=I=[0,1]$ and parameterization defined as in (25).

Once the set $\mathcal{L}_k$ of all level k pairs of functions has been constructed, we construct the set $\mathcal{L}_{k+1}$ of level $k+1$ pairs as follows. For each level k pair $(f_1^{(k)},f_2^{(k)})$, we consider $L(f_1^{(k)},f_2^{(k)})$, with its parameterization set, and construct all functions which arise while filling in horizontal or vertical gaps in this set (two new pairs for each gap), together with their parameterizations, following the procedures described in I, II and III above. The parameterization of $L(f_1^{(k)},f_2^{(k)})$ will be denoted $\varphi(f_1^{(k)},f_2^{(k)})$. The domain of this parameterization is determined as described in III.

All level $k+1$ pairs are therefore obtained from filling in gaps of level k pairs. All pairs are the restriction of (f_1,f_2) to some pair of intervals.

Consider the set

$$L = \bigcup_{k=0}^{\infty} \bigcup_{(g_1,g_2)\in\mathcal{L}_k} L(g_1, g_2),$$

parametrized by φ defined as follows. The domain of φ is

$$\bigcup_{k=0}^{\infty} \bigcup_{(g_1,g_2)\in\mathcal{L}_k} \mathrm{Dom}\ \varphi(g_1, g_2),$$

and for $x \in \mathrm{Dom}\ \varphi$,

$$\varphi(x) = \varphi(g_1, g_2)(x), \text{ if } x \in \mathrm{Dom}\ \varphi(g_1, g_2).$$

This definition is coherent, since if x belongs to more than one such domain, all corresponding parametrizations coincide at x.

The interest of this construction resides in the following theorem.

Theorem 12. *The closure $\overline{L}$ of L is a Jordan arc (with extremities (σ_1, σ_2) and (τ_1, τ_2) in the case where $f_1(\sigma_1) = m = f_2(\sigma_2)$).*

For the proof of this theorem, the reader is referred to [8, Theorem 6]. The following result concerns uniqueness of the Jordan arc.

Proposition 4. *The Jordan arc $\overline{L}$ constructed in Theorem 12 is the unique (Jordan) arc contained in $F = \{(s_1, s_2) : f_1(s_1) = f_2(s_2)\} \cap ([\sigma_1, \tau_1] \times [\sigma_2, \tau_2])$ with extremities (σ_1, σ_2) and (τ_1, τ_2). Further, this arc is contained in the boundary of a component of $\{(s_1, s_2) : f_1(s_1) \neq f_2(s_2)\}$.*

We shall now use the deterministic results described above to prove Theorem 11.

Proof of Theorem 11. By Theorem 12, it suffices to show that if B_1 and B_2 are independent diffusions, and $m < M$, $\sigma_1 < S_1 < \tau_1$ and $\sigma_2 < S_2 < \tau_2$ are (random) numbers as in Theorems 9 and 11, then for almost all $\omega \in \Omega$, the functions $f_1(\cdot) = B_1(\cdot\,; \omega)$ (resp. $f_2(\cdot) = B_2(\cdot\,; \omega)$) defined on $[\sigma_1(\omega), S_1(\omega)]$ (resp. $[S_2(\omega), \tau_2(\omega)]$) satisfy Hypotheses 1 and 2.

Hypothesis 1(a) holds because $[\sigma_i(\omega), \tau_i(\omega)]$ is an excursion interval of $B_i(\omega)$, $i = 1, 2$. Hypothesis 1(b) is a well-known property of diffusions. The proof that Hypothesis 1(c) holds is found in [24, Proposition 5].

As for Hypothesis 2, notice that there are only countably many pairs that arise in the construction of the sequence $(\mathcal{L}_k,\ k \geq 0)$, and each pair consists of two independent diffusions, each defined on one of its own excursion intervals. Therefore, Hypothesis 1 holds for this pair, and it follows that Hypothesis 2 holds. This completes the proof of Theorem 11. $\diamondsuit$

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