Hölder-continuity for the nonlinear stochastic heat equation with rough initial conditions

Le Chen[*] and **Robert C. Dalang**[*]

Institut de mathématiques

École Polytechnique Fédérale de Lausanne

Station 8,

CH-1015 Lausanne,

Switzerland

e-mails: le.chen@epfl.ch, robert.dalang@epfl.ch

Abstract: We study space-time regularity of the solution of the nonlinear stochastic heat equation in one spatial dimension driven by space-time white noise, with a rough initial condition. This initial condition is a locally finite measure μ with, possibly, exponentially growing tails. We show how this regularity depends, in a neighborhood of $t = 0$, on the regularity of the initial condition. On compact sets in which $t > 0$, the classical Hölder-continuity exponents $\frac{1}{4}-$ in time and $\frac{1}{2}-$ in space remain valid. However, on compact sets that include $t = 0$, the Hölder continuity of the solution is $\left(\frac{\alpha}{2} \wedge \frac{1}{4}\right)-$ in time and $\left(\alpha \wedge \frac{1}{2}\right)-$ in space, provided μ is absolutely continuous with an α-Hölder continuous density.

AMS 2010 subject classifications: Primary 60H15. Secondary 60G60, 35R60.

Keywords: nonlinear stochastic heat equation, rough initial data, sample path Hölder continuity, moments of increments.

1 Introduction

Over the last few years, there has been considerable interest in the stochastic heat equation with non-smooth initial data:

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right) u(t,x) = \rho(u(t,x))\, \dot{W}(t,x), & x \in \mathbb{R},\ t \in \mathbb{R}_+^*, \\ \quad u(0,\cdot) = \mu(\cdot)\ . \end{cases} \tag{1.1}$$

[*]Research partially supported by the Swiss National Foundation for Scientific Research.

In this equation, $\dot{W}$ is a space-time white noise, $\rho : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function and $\mathbb{R}_+^* =]0, \infty[$. The initial data μ is a signed Borel measure, which we assume belongs to the set

$$\mathcal{M}_H(\mathbb{R}) := \left\{ \text{signed Borel measures } \mu, \text{ s.t. } \int_{\mathbb{R}} e^{-ax^2} |\mu|(\mathrm{d}x) < +\infty, \text{ for all } a > 0 \right\}.$$

In this definition, $|\mu| := \mu_+ + \mu_-$, where $\mu = \mu_+ - \mu_-$ and $\mu_\pm$ are the two non-negative Borel measures with disjoint support that provide the Jordan decomposition of μ. The set $\mathcal{M}_H(\mathbb{R})$ can be equivalently characterized by the condition that

$$\left(|\mu| * G_\nu(t,\cdot)\right)(x) = \int_{\mathbb{R}} G_\nu(t, x-y) |\mu|(\mathrm{d}y) < +\infty \;, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}, \tag{1.2}$$

where $*$ denotes the convolution in the space variable and $G_\nu(t,x)$ is the one-dimensional heat kernel function

$$G_\nu(t,x) := \frac{1}{\sqrt{2\pi\nu t}} \exp\left\{ -\frac{x^2}{2\nu t} \right\}, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R} \,. \tag{1.3}$$

Therefore, $\mathcal{M}_H(\mathbb{R})$ is precisely the set of initial conditions for which the homogeneous heat equation has a solution for all time.

The use of non-smooth initial data is initially motivated by the *parabolic Anderson model* (in which $\rho(u) = u$) with initial condition given by the Dirac delta function $\mu = \delta_0$ (see [2], and more recently, [13, 7, 6]). These papers are mainly concerned with the study of the intermittency property, which is a property that concerns the behavior of moments of the solution $u(t,x)$. Some very precise moment estimates have also been recently obtained by the authors in [5].

In this paper, we are interested in space-time regularity of the sample paths $(t,x) \mapsto u(t,x)$, and, in particular, in how this regularity depends, in a neighborhood of $\{0\} \times \mathbb{R}$, on the regularity of the initial condition μ.

Given a subset $D \subseteq \mathbb{R}_+ \times \mathbb{R}$ and positive constants β_1, β_2, denote by $C_{\beta_1,\beta_2}(D)$ the set of functions $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with the following property: for each compact subset $\tilde{D} \subset D$, there is a finite constant c such that for all (t,x) and (s,y) in $\tilde{D}$,

$$|v(t,x) - v(s,y)| \le c \left(|t-s|^{\beta_1} + |x-y|^{\beta_2} \right).$$

Let

$$C_{\beta_1-,\beta_2-}(D) := \cap_{\alpha_1 \in \,]0,\beta_1[} \cap_{\alpha_2 \in \,]0,\beta_2[} C_{\alpha_1,\alpha_2}(D) \,.$$

When the measure μ has a bounded density f with respect to Lebesgue measure, then the initial condition is written $u(0,x) = f(x)$, for all $x \in \mathbb{R}$. When f is bounded, then the Hölder continuity of u was already studied in [28, Corollary 3.4, p.318]. In [2], it is stated, for the parabolic Anderson model, that if the initial data satisfies

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} \sqrt{t} \left(\mu * G_\nu(t,\circ) \right)(x) < \infty, \quad \text{for all } T > 0,$$

then $u \in C_{\frac{1}{4}-,\frac{1}{2}-}(\mathbb{R}_+^* \times \mathbb{R})$, a.s. In [21, 26], this result is extended to the case where the initial data is a continuous function with tails that grow at most exponentially at $\pm\infty$. Hölder continuity properties for more general parabolic problems, but mainly on bounded domains rather than $\mathbb{R}$, and with function-valued initial conditions, have aso been obtained using maximal inequalities and stochastic convolutions: see [3, 20, 25].

Sanz-Solé and Sarrà [24] considered the stochastic heat equation over $\mathbb{R}^d$ with spatially homogeneous colored noise which is white in time. Assuming that the spectral measure $\tilde{\mu}$ of the noise satisfies

$$\int_{\mathbb{R}^d} \frac{\tilde{\mu}(\mathrm{d}\xi)}{(1+|\xi|^2)^\eta} < +\infty, \quad \text{for some } \eta \in \,]0,1[, \tag{1.4}$$

they proved that if the initial data is a bounded ρ-Hölder continuous function for some $\rho \in \,]0,1[$, then

$$u \in C_{\frac{1}{2}(\rho\wedge(1-\eta))-,\rho\wedge(1-\eta)-}\left(\mathbb{R}_+ \times \mathbb{R}\right) \;, \quad \text{a.s.} \;,$$

where $a \wedge b := \min(a,b)$. For the case of space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$, the spectral measure $\tilde{\mu}$ is Lebesgue measure and hence the exponent η in (1.4) (with $d = 1$) can take the value $\frac{1}{2} - \epsilon$ for any $\epsilon > 0$. Their result ([23, Theorem 4.3]) implies that

$$u \in C_{\left(\frac{1}{4}\wedge\frac{\rho}{2}\right)-,\left(\frac{1}{2}\wedge\rho\right)-}\left(\mathbb{R}_+ \times \mathbb{R}\right) \;, \quad \text{a.s.}$$

More recently, in the paper [6, Lemma 9.3], assuming that the initial condition μ is a finite measure, Conus *et al* obtain tight upper bounds on moments of u and bounds on moments of spatial increments of u at fixed positive times: in particular, they show that u is Hölder continuous in x with exponent $\frac{1}{2} - \epsilon$.

Finally, in the papers [11, 12], Dalang, Khoshnevisan and Nualart considered a system of stochastic heat equations with vanishing initial conditions driven by space-time white noise, and proved that $u \in C_{\frac{1}{4}-,\frac{1}{2}-}\left(\mathbb{R}_+ \times \mathbb{R}\right)$.

The purpose of this paper is to extend the above results to the case where $\mu \in \mathcal{M}_H(\mathbb{R})$. In particular, we show that $u \in C_{\frac{1}{4}-,\frac{1}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}\right)$. Indeed, it is necessary to exclude the line $\{0\} \times \mathbb{R}$ unless the initial data μ has a density f that is sufficiently smooth (see part (2) of Theorem 3.1). Indeed, in this case, the regularity of u in the neighborhood of $t = 0$ can be no better than the regularity of f.

Recall that the rigorous interpretation of (1.1), used in [5], is the following integral equation:

$$\begin{aligned} u(t,x) &= J_0(t,x) + I(t,x), \\ I(t,x) &= \iint_{[0,t]\times\mathbb{R}} G_\nu\left(t-s, x-y\right) \rho\left(u\left(s,y\right)\right) W\left(\mathrm{d}s, \mathrm{d}y\right), \end{aligned} \tag{1.5}$$

where $J_0(t,x) := \left(\mu * G_\nu(t,\cdot)\right)(x)$, and the stochastic integral is interpreted in the sense of Walsh [28]. The regularity of $(t,x) \mapsto J_0(t,x)$ is classical (see Lemma 2.3), so the main effort is to understand the Hölder-regularity of $(t,x) \mapsto I(t,x)$ at $t = 0$. This is a delicate issue. In Theorem 3.1 and Proposition 4.3, we give sufficient conditions for sample path Hölder continuity of this function at $t = 0$. However, we have not resolved this question for all initial conditions μ. We do, however, show that for certain absolutely continuous μ with a locally unbounded density f, the function $t \mapsto u(t,x)$ from $[0,1]$ into $L^p(\Omega, \mathcal{F}, P)$, can have an optimal Hölder exponent that is arbitrarily close to 0 (see Proposition 3.5).

The difficulties for proving the Hölder continuity of u lie in part in the fact that for initial data satisfying (1.2), $\mathbb{E}\left[|u(t,x)|^p\right]$ need not be bounded over $x \in \mathbb{R}$, and mainly in the fact that the initial data is irregular. Indeed, standard techniques, which isolate the effects of initial data by using the $L^p(\Omega)$-boundedness of the solution, fail in our case (see Remark 3.2). Instead, the initial data play an active role in our proof. We also note that Fourier transform techniques are not directly applicable here because μ need not be a tempered measure.

Finally, it is natural to ask in what sense the measure μ is indeed the initial condition for the stochastic heat equation? We show in Proposition 3.4 that $u(t,\cdot)$ converges weakly (in the sense of distribution theory) to μ as $t \downarrow 0$, so that μ is the initial condition of (1.1) in the classical sense used for deterministic p.d.e.'s [14, Chapter 7, Section 1].

The paper is structured as follows. In Section 2, we recall the results of [5] that we need here. Our main results are stated in Section 3. The proofs are presented in Section 4. Finally, some technical lemmas are listed in the appendix.

2 Some preliminaries

Let $W = \{W_t(A),\ A \in \mathcal{B}_f(\mathbb{R}), t \geq 0\}$ be a space-time white noise defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{B}_f(\mathbb{R})$ is the collection of Borel sets with finite Lebesgue measure. Let

$$\mathcal{F}_t^0 = \sigma\left(W_s(A),\ 0 \leq s \leq t,\ A \in \mathcal{B}_f(\mathbb{R})\right) \vee \mathcal{N}, \quad t \geq 0,$$

be the natural filtration of W augmented by the σ-field $\mathcal{N}$ generated by all P-null sets in $\mathcal{F}$. For $t \geq 0$, define $\mathcal{F}_t := \mathcal{F}_{t+}^0 = \wedge_{s>t}\mathcal{F}_s^0$. In the following, we fix the filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, P\}$. We use $||\cdot||_p$ to denote the $L^p(\Omega)$-norm ($p \geq 1$). With this setup, W becomes a worthy martingale measure in the sense of Walsh [28], and $\iint_{[0,t]\times\mathbb{R}} X(s,y) W(\mathrm{d}s, \mathrm{d}y)$ is well-defined in this reference for a suitable class of random fields $\{X(s,y),\ (s,y) \in \mathbb{R}_+ \times \mathbb{R}\}$.

In this paper, we use $\star$ to denote the simultaneous convolution in both space and time variables.

Definition 2.1. A process $u = \left(u(t,x),\ (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\right)$ is called a *random field solution* to (1.5) if the following four conditions are satisfied:

(1) u is adapted, i.e., for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $u(t,x)$ is $\mathcal{F}_t$-measurable;

(2) u is jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+^* \times \mathbb{R}) \times \mathcal{F}$;

(3) $\left(G_\nu^2 \star ||\rho(u)||_2^2\right)(t,x) < +\infty$ for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, and the function $(t,x) \mapsto I(t,x)$ mapping $\mathbb{R}_+^* \times \mathbb{R}$ into $L^2(\Omega)$ is continuous;

(4) u satisfies (1.5) a.s., for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$.

Assume that $\rho : \mathbb{R} \mapsto \mathbb{R}$ is globally Lipschitz continuous with Lipschitz constant $\mathrm{Lip}_\rho > 0$. We consider the following growth conditions on ρ: for some constants $L_\rho > 0$ and $\overline{\varsigma} \geq 0$,

$$|\rho(x)|^2 \leq L_\rho^2 \left(\overline{\varsigma}^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R} . \tag{2.1}$$

Note that $L_\rho \leq \sqrt{2}\,\mathrm{Lip}_\rho$, and the inequality may be strict. Of particular importance is the linear case (the *parabolic Anderson model*): $\rho(u) = \lambda u$ with $\lambda \neq 0$, which is a special case of the following quasi-linear growth condition: for some constant $\varsigma \geq 0$,

$$|\rho(x)|^2 = \lambda^2 \left(\varsigma^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R} . \tag{2.2}$$

Define the kernel functions:

$$\mathcal{K}(t,x) = \mathcal{K}(t,x;\nu,\lambda) := G_{\frac{\nu}{2}}(t,x)\left(\frac{\lambda^2}{\sqrt{4\pi\nu t}} + \frac{\lambda^4}{2\nu}\, e^{\frac{\lambda^4 t}{4\nu}}\, \Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)\right), \tag{2.3}$$

$$\mathcal{H}(t) = \mathcal{H}(t;\nu,\lambda) := (1 \star \mathcal{K})(t,x) = 2e^{\frac{\lambda^4 t}{4\nu}}\, \Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right) - 1, \tag{2.4}$$

where $\Phi(x) = \int_{-\infty}^x (2\pi)^{-1/2} e^{-y^2/2} \mathrm{d}y$, and the formula on the right-hand side is explained in [5, (2.18)]. Some functions related to $\Phi(x)$ are the error functions $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} \mathrm{d}y$ and $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$. Clearly, $\Phi(x) = \left(1 + \mathrm{erf}(x/\sqrt{2})\right)/2$.

Let z_p be the universal constant in the Burkholder-Davis-Gundy inequality (see [8, Theorem 1.4], in particular, $z_2 = 1$) which satisfies $z_p \leq 2\sqrt{p}$ for all $p \geq 2$. Let $a_{p,\overline{\varsigma}}$ be the constant defined by

$$a_{p,\overline{\varsigma}} := \begin{cases} 2^{(p-1)/p} & \text{if } \overline{\varsigma} \neq 0,\ p > 2, \\ \sqrt{2} & \text{if } \overline{\varsigma} = 0,\ p > 2, \\ 1 & \text{if } p = 2. \end{cases}$$

Notice that $a_{p,\overline{\varsigma}} \in [1,2]$. Denote $\overline{\mathcal{K}}(t,x) := \mathcal{K}(t,x;\nu,L_\rho)$, $\widehat{\mathcal{K}}_p(t,x) = \mathcal{K}(t,x;\nu,a_{p,\overline{\varsigma}} z_p\, L_\rho)$ and $\overline{\mathcal{H}}(t) := \mathcal{H}(t;\nu,L_\rho)$, $\widehat{\mathcal{H}}_p(t) = \mathcal{H}(t;\nu,a_{p,\overline{\varsigma}} z_p\, L_\rho)$.

The following theorem is mostly taken from [5, Theorem 2.4], except that (2.7) comes from [5, Corollary 2.8].

Theorem 2.2 (Existence, uniqueness and moments)**.** *Suppose that the function ρ is Lipschitz continuous and satisfies (2.1), and $\mu \in \mathcal{M}_H(\mathbb{R})$. Then the stochastic integral equation (1.5) has a random field solution $u = \{u(t,x),\ (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\}$. Moreover:*
(1) u is unique (in the sense of versions).

(2) $(t,x)\mapsto u(t,x)$ *is* $L^p(\Omega)$*-continuous for all integers* $p\geq 2$.
(3) For all even integers $p\geq 2$, *all* $t>0$ *and* $x,y\in\mathbb{R}$,

$$||u(t,x)||_p^2 \leq \begin{cases} J_0^2(t,x)+\left(J_0^2\star\overline{\mathcal{K}}\right)(t,x)+\overline{\varsigma}^2\,\overline{\mathcal{H}}(t), & \text{if } p=2,\\ 2J_0^2(t,x)+\left(2J_0^2\star\widehat{\mathcal{K}}_p\right)(t,x)+\overline{\varsigma}^2\,\widehat{\mathcal{H}}_p(t), & \text{if } p>2.\end{cases} \tag{2.5}$$

(4) In particular, if $|\rho(u)|^2=\lambda^2\left(\varsigma^2+u^2\right)$, *then for all* $t>0$ *and* $x,y\in\mathbb{R}$,

$$||u(t,x)||_2^2 = J_0^2(t,x)+\left(J_0^2\star\mathcal{K}\right)(t,x)+\varsigma^2\ \mathcal{H}(t). \tag{2.6}$$

Moreover, if $\mu=\delta_0$ *(the Dirac delta function), then*

$$||u(t,x)||_2^2=\frac{1}{\lambda^2}\mathcal{K}(t,x)+\varsigma^2\,\mathcal{H}(t). \tag{2.7}$$

The next lemma is classical. A proof can be found in [5, Lemma 3.8].

Lemma 2.3. *The function* $(t,x)\mapsto J_0(t,x)=\left(\mu * G_\nu(t,\cdot)\right)(x)$ *with* $\mu\in\mathcal{M}_H(\mathbb{R})$ *is smooth for* $t>0$: $J_0(t,x)\in C^\infty\left(\mathbb{R}_+^*\times\mathbb{R}\right)$. *If, in addition,* $\mu(\mathrm{d}x)=f(x)\mathrm{d}x$ *where* f *is continuous, then* J_0 *is continuous up to* $t=0$: $J_0\in C^\infty\left(\mathbb{R}_+^*\times\mathbb{R}\right)\ \cap\ C\left(\mathbb{R}_+\times\mathbb{R}\right)$, *and if* f *is* α*-Hölder continuous, then* $J_0\in C^\infty\left(\mathbb{R}_+^*\times\mathbb{R}\right)\ \cap\ C_{\alpha/2,\alpha}\left(\mathbb{R}_+\times\mathbb{R}\right)$.

For $p\geq 2$ and $X\in L^2\left(\mathbb{R}_+\times\mathbb{R},L^p(\Omega)\right)$, set

$$||X||_{M,p}^2 := \iint_{\mathbb{R}_+^*\times\mathbb{R}} ||X\left(s,y\right)||_p^2\,\mathrm{d}s\mathrm{d}y<+\infty\ .$$

When $p=2$, we write $||X||_M$ instead of $||X||_{M,2}$. In [28], $\iint X\mathrm{d}W$ is defined for *predictable* X such that $||X||_M<+\infty$. Let $\mathcal{P}_p$ denote the closure in $L^2\left(\mathbb{R}_+\times\mathbb{R},L^p(\Omega)\right)$ of simple processes. Clearly, $\mathcal{P}_2\supseteq\mathcal{P}_p\supseteq\mathcal{P}_q$ for $2\leq p\leq q<+\infty$, and according to Itô's isometry, $\iint X\mathrm{d}W$ is well-defined for all elements of $\mathcal{P}_2$. The next lemma, taken from [5, Lemma 3.3], gives easily verifiable conditions for checking that $X\in\mathcal{P}_2$. In the following, we will use $\cdot$ and $\circ$ to denote the time and space dummy variables respectively.

Lemma 2.4. *Let* $\mathcal{G}(s,y)$ *be a deterministic measurable function from* $\mathbb{R}_+^*\times\mathbb{R}$ *to* $\mathbb{R}$ *and let* $Z=\left(Z\left(s,y\right),\ \left(s,y\right)\in\ \mathbb{R}_+^*\ \times\mathbb{R}\right)$ *be a process with the following properties:*

(1) Z *is adapted and jointly measurable with respect to* $\mathcal{B}(\mathbb{R}^2)\times\mathcal{F}$;

(2) $\mathbb{E}\left[\iint_{[0,t]\times\mathbb{R}}\mathcal{G}^2\left(t-s,x-y\right)Z^2\left(s,y\right)\mathrm{d}s\mathrm{d}y\right]<\infty$, *for all* $(t,x)\in\mathbb{R}_+\times\mathbb{R}$.

Then for each $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, *the random field* $(s,y)\in\,]0,t[\,\times\mathbb{R}\mapsto\mathcal{G}\left(t-s,x-y\right)Z\left(s,y\right)$ *belongs to* $\mathcal{P}_2$ *and so the stochastic convolution*

$$(\mathcal{G}\star Z\dot{W})(t,x):=\iint_{[0,t]\times\mathbb{R}}\mathcal{G}\left(t-s,x-y\right)Z\left(s,y\right)W\left(\mathrm{d}s,\mathrm{d}y\right)$$

is a well-defined Walsh integral and the random field $\mathcal{G}\star Z\dot{W}$ *is adapted. Moreover, for all even integers* $p\geq 2$ *and* $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$\left|\left|(\mathcal{G}\star Z\dot{W})(t,x)\right|\right|_p^2\leq z_p^2\,||\mathcal{G}(t-\cdot,x-\circ)Z(\cdot,\circ)||_{M,p}^2\,.$$

3 Main results

If the initial data is of the form $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$, where f is a bounded function, then it is well-known (see [28]) that the solution u is bounded in $L^p(\Omega)$ for all $p \geq 2$. In addition, by the moment formula (2.5),

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} ||u(t,x)||_p^2 \leq 2\,C^2 + \left(2\,C^2 + \overline{\varsigma}^2\right)\widehat{\mathcal{H}}_p\,(T) < +\infty, \quad \text{for all } T > 0, \tag{3.1}$$

where $C = \sup_{x\in\mathbb{R}}|f(x)| = \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} J_0(t,x)$. From this bound, one can easily derive that that $u \in C_{1/4-,1/2-}\left(\mathbb{R}_+^* \times \mathbb{R}\right)$, a.s.: see Remark 4.6 below. We will extend this classical result to the case where μ can be unbounded either locally, such as $\mu = \delta_0$, or at $\pm\infty$, such as $\mu(\mathrm{d}x) = e^{|x|^a}\mathrm{d}x$, $a \in \,]1,2[$, or both. However, for irregular initial conditions, Hölder continuity of u will be obtained only on $\mathbb{R}_+^* \times \mathbb{R}$, and this continuity extends to $\mathbb{R}_+ \times \mathbb{R}$ when the initial condition is continuous.

We need a set of initial data defined as follows:

$$\mathcal{M}_H^*(\mathbb{R}) := \left\{\mu(\mathrm{d}x) = f(x)\mathrm{d}x, \text{ s.t. } \exists a \in \,]1,2[\,,\ \sup_{x\in\mathbb{R}}|f(x)|e^{-|x|^a} < +\infty\right\}.$$

Clearly, $\mathcal{M}_H^*(\mathbb{R}) \subset \mathcal{M}_H(\mathbb{R})$, and $\mathcal{M}_H^*(\mathbb{R})$ includes all absolutely continuous measures whose density functions are bounded by functions of the type $c_1 e^{c_2|x|^a}$ with $c_1, c_2 > 0$ and $a \in \,]1,2[$ (see Lemma 5.1).

Theorem 3.1. *Suppose that ρ is Lipschitz continuous. Then the solution $u(t,x) = J_0(t,x) + I(t,x)$ to (1.5) has the following sample path regularity:*

(1) If $\mu \in \mathcal{M}_H(\mathbb{R})$, then $I \in C_{\frac{1}{4}-,\frac{1}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}\right)$ a.s. Therefore,

$$u = J_0 + I \in C_{\frac{1}{4}-,\frac{1}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}\right), \quad a.s.$$

(2) If $\mu \in \mathcal{M}_H^*(\mathbb{R})$, then $I \in C_{\frac{1}{4}-,\frac{1}{2}-}\left(\mathbb{R}_+ \times \mathbb{R}\right)$, a.s. If, in addition, $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$, where f is a continuous function, then

$$u \in C\left(\mathbb{R}_+ \times \mathbb{R}\right) \cap C_{\frac{1}{4}-,\frac{1}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}\right), \quad a.s.$$

If $\mu \in \mathcal{M}_H^*(\mathbb{R})$ and, in addition, $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$, where f is an α-Hölder continuous function, then

$$u \in C_{\left(\frac{\alpha}{2}\wedge\frac{1}{4}\right)-,\left(\alpha\wedge\frac{1}{2}\right)-}\left(\mathbb{R}_+ \times \mathbb{R}\right) \cap C_{\frac{1}{4}-,\frac{1}{2}-}\left(\mathbb{R}_+^* \times \mathbb{R}\right), \quad a.s.$$

This theorem will be proved in Section 4.2.

Remark 3.2. The standard approach (e.g., that is used in [10, p.54 –55], [24], [26] and [28]) for proving Hölder continuity cannot be used to establish the above theorem. For instance, consider the case where $\rho(u) = u$ and $\mu = \delta_0$. The classical argument, as presented in [26, p.432] (see also the proof of Proposition 1.5 in [1] and the proof of Corollary 3.4 in [28]), uses Burkholder's inequality for $p > 1$ and Hölder's inequality with $q = p/(p-1)$ to obtain

$$
\begin{aligned}
||I(t,x) - I(t',x')||_{2p}^{2p} \leq & C_{p,T} \left(\int_0^{t\vee t'}\!\!\int_{\mathbb{R}} \mathrm{d}s\mathrm{d}y \left(G_\nu(t-s,x-y) - G(t'-s,x'-y')\right)^2 \right)^{p/q} \\
& \times \int_0^{t\vee t'}\!\!\int_{\mathbb{R}} \mathrm{d}s\mathrm{d}y \left(G_\nu(t-s,x-y) - G(t'-s,x'-y')\right)^2 \left(1 + ||u(s,y)||_{2p}^{2p}\right).
\end{aligned}
$$

However, by Hölder's inequality, (2.7) and (2.3),

$$
||u(s,y)||_{2p}^2 \geq ||u(s,y)||_2^2 \geq G_{\nu/2}(s,y) \frac{1}{\sqrt{4\pi\nu s}}\,.
$$

Therefore, $||u(s,y)||_{2p}^{2p} \geq C G_{\nu/(2p)}(s,y) s^{1/2-p}$. The second term in the above bound is not $\mathrm{d}s$–integrable in a neighborhood of $\{0\}\times\mathbb{R}$ unless $p < 3/2$. Therefore, this classical argument does not apply in the presence of an irregular initial condition such as δ_0.

Example 3.3 (Dirac delta initial data)**.** Suppose $\rho(u) = \lambda u$ with $\lambda \neq 0$. If $\mu = \delta_0$, then neither $x \mapsto J_0(0,x)$ nor $x \mapsto \lim_{t\to 0_+} ||I(t,x)||_2$ is a continuous function. Indeed, this is clear for $J_0(0,x) = \delta_0(x)$. For $\lim_{t\to 0_+} ||I(t,x)||_2$, by (2.7),

$$
||I(t,x)||_2^2 = ||u(t,x)||_2^2 - J_0^2(t,x) = \frac{\lambda^2}{2\nu} e^{\frac{\lambda^4 t}{4\nu}}\, \Phi\left(\lambda^2 \sqrt{\frac{t}{2\nu}}\right) G_{\nu/2}(t,x)\,.
$$

Therefore, $\lim_{t\to 0_+} ||I(t,x)||_2^2$ equals 0 if $x \neq 0$, and $+\infty$ if $x = 0$. (We note that $I(0,x) \equiv 0$ by definition.)

Example 3.3 suggests that $||I(t,x)||_2^2$ tends to $\frac{\lambda^2}{4\nu}\delta_0(x)$ as $t \to 0_+$ in the weak sense, i.e.,

$$
\lim_{t\to 0_+} \left\langle ||I(t,\cdot)||_2^2\,, \phi(\cdot) \right\rangle = \frac{\lambda^2}{4\nu}\phi(0), \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}),
$$

where $C_c^\infty(\mathbb{R})$ denotes smooth functions with compact support. Furthermore, the following proposition shows that the random field solution of (1.5) satisfies the initial condition $u(0,\circ) = \mu$ in a weak sense.

Proposition 3.4. *For all $\phi \in C_c^\infty(\mathbb{R})$ and $\mu \in \mathcal{M}_H(\mathbb{R})$,*

$$
\lim_{t\to 0_+} \int_{\mathbb{R}} \mathrm{d}x\, u(t,x)\phi(x) = \int_{\mathbb{R}} \mu(\mathrm{d}x)\, \phi(x) \quad \text{in } L^2(\Omega).
$$

The proof of this proposition is presented in Section 4.5. In the next proposition, rather than considering sample path continuity, we shows that the map $t \mapsto I(t,x)$, from $[0,1]$ into $L^p(\Omega, \mathcal{F}, P)$, may be quite far from $\frac{1}{4}$–Hölder continuous at the origin, and in fact, the Hölder-exponent may be arbitrarily near 0.

Proposition 3.5. *Suppose $\rho(u) = \lambda u$ with $\lambda \neq 0$ and $\mu(\mathrm{d}x) = |x|^{-a}\,\mathrm{d}x$ with $0 < a \le 1$, so that $J_0(0,x) = |x|^{-a}$ is not locally bounded. Fix $p \ge 2$. Then:*

(1) If $a < 1/2$, then for all $x \in \mathbb{R}$, $\lim_{t\to 0_+} ||I(t,x)||_p \equiv 0$.

(2) There is $c > 0$ such that for all $t > 0$, $||I(t,0)||_p \ge c\, t^{\frac{1-2a}{4}}$.

In particular, when $\frac{1}{2} < a < 1$, $\lim_{t\to 0_+} ||I(t,0)||_p = +\infty$, and when $0 < a < \frac{1}{2}$, $t \mapsto I(t,0)$ from $\mathbb{R}_+$ to $L^p(\Omega)$ cannot be smoother than $\frac{1-2a}{4}$–Hölder continuous (in this case $\frac{1-2a}{4} \in\ ]0, 1/4[$).

Proof. (1) By the moment bounds formulas (2.5) and (2.6), it suffices to consider second moment and show that $\lim_{t\to 0_+} ||I(t,x)||_2 \equiv 0$. For some constant $C_a > 0$, the Fourier transform of μ is $C_a|x|^{-1+a}$ (see [27, Lemma 2 (a), p. 117]), which is non-negative. Hence Bochner's theorem (see, e.g., [15, Theorem 1, p.152]) implies that μ, and therefore $x \mapsto J_0(t,x)$, is non-negative definite. Such functions achieves their maximum at the origin (see, e.g., [15, Theorem 1, p. 152]), and so

$$0 < J_0(t,x) \le J_0(t,0) = \int_{\mathbb{R}} \mathrm{d}y\, \frac{1}{|y|^a} G_\nu(t,y) = 2\int_0^{+\infty} \mathrm{d}y\, \frac{e^{-y^2/(2\nu t)}}{y^a\sqrt{2\pi\nu t}}\,.$$

Then by a change of variable and using Euler's integral (see [19, 5.2.1, p.136]),

$$J_0(t,0) = 2\int_0^{+\infty} \mathrm{d}u\, \frac{e^{-u}}{(2\nu t u)^{a/2}\sqrt{2\pi\nu t}}\, \frac{\sqrt{2\nu t}}{2\sqrt{u}} = \frac{\Gamma\left(\frac{1-a}{2}\right)}{\sqrt{\pi}(2\nu t)^{a/2}}\,, \tag{3.2}$$

where $\Gamma(x)$ is Euler's Gamma function [19]. By (2.6) and the above bound,

$$||I(t,x)||_2^2 = \left(J_0^2 \star \mathcal{K}\right)(t,x) \le \int_0^t \mathrm{d}s\, \left(\frac{\lambda^2}{\sqrt{4\pi\nu(t-s)}} + \frac{\lambda^4}{2\nu} e^{\frac{\lambda^4(t-s)}{4\nu}}\right) \frac{C}{s^a}.$$

The integral converges if and only if $a < 1$. Finally, using the Beta integral (see [19, (5.12.1), p.142])

$$\int_0^t \mathrm{d}s\, s^{\mu-1}(t-s)^{\nu-1} = t^{\mu+\nu-1}\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}, \quad \text{for } t > 0,\ \mu > 0 \text{ and } \nu > 0, \tag{3.3}$$

we see that $||I(t,x)||_2^2 \le C_1\, t^{1/2-a} + C_2\, t^{1-a}$, so $\lim_{t\to 0_+} ||I(t,x)||_2^2 = 0$ when $a < 1/2$.

(2) Now consider the function $t \mapsto I(t,0)$ from $\mathbb{R}_+$ to $L^p(\Omega)$. Since $(x-y)^2 \le 2(x^2+y^2)$, as in (3.2), we see that

$$J_0(t,x) = \int_{\mathbb{R}} \mathrm{d}y\, \frac{1}{|y|^a} G_\nu(t, x-y) \ge \frac{1}{\sqrt{2}} \exp\left(-\frac{x^2}{\nu t}\right) \frac{\Gamma\left(\frac{1-a}{2}\right)}{\sqrt{\pi}} \frac{1}{(\nu t)^{a/2}}.$$

Hence,

$$J_0^2(t,x) \geq C G_{\nu/2}\left(\frac{t}{2},x\right) t^{1/2-a}.$$

Since $\mathcal{K}(t,x) \geq G_{\nu/2}(t,x)\frac{\lambda^2}{\sqrt{4\pi\nu t}}$ by (2.3),

$$||I(t,x)||_2^2 \geq \frac{C' \exp\left(-\frac{2x^2}{\nu t}\right)}{t} \int_0^t \mathrm{d}s\, s^{1/2-a} = C'' \exp\left(-\frac{2x^2}{\nu t}\right) t^{\frac{1-2a}{2}}.$$

If $x=0$, then for all integers $p \geq 2$, since $I(0,x) \equiv 0$,

$$||I(t,0) - I(0,0)||_p^2 \geq ||I(t,0)||_2^2 \geq C''\, t^{\frac{1-2a}{2}}.$$

When $0 < a < 1/2$, the function $t \mapsto I(t,0)$ from $\mathbb{R}_+$ to $L^p(\Omega)$ cannot be smoother than η-Hölder continuous at $t=0$ with $\eta = \frac{1-2a}{4} \in \,]0,1/4[\,$. $\square$

4 Proofs of the main results

Establishing Hölder continuity relies on Kolmogorov's continuity theorem. We present a formulation of this result that is suitable for our purposes.

4.1 Kolmogorov's continuity theorem

For $x=(x_1,\dots,x_N)$ and $y=(y_1,\dots,y_N)$, define

$$\tau_{\alpha_1,\dots,\alpha_N}(x,y) := \sum_{i=1}^N |x_i - y_i|^{\alpha_i} \quad , \qquad \text{with } \alpha_1,\dots,\alpha_N \in \,]0,1]. \tag{4.1}$$

This defines a metric on $\mathbb{R}^N$ that is not induced by a norm except when $\alpha_i = 1$ for $i = 1,\dots,N$. We refer the interested readers to [16, Theorem 4.3] or [22, Theorem 2.1, on p. 62] for the isotropic case ($\alpha_1 = \cdots = \alpha_N$). For the anisotropic case (where the α_i are not identical), see [17, Theorem 1.4.1, p. 31] and [11, Corollary A.3, p. 34]. We state a version (Proposition 4.2 below), which is a consequence of these references and is convenient for our purposes.

Definition 4.1. (Hölder continuity) A function $f: D \mapsto \mathbb{R}$ with $D \subseteq \mathbb{R}^N$ is said to be *locally (and uniformly) Hölder continuous* with indices $(\alpha_1,\dots,\alpha_N)$ if for all compact sets $K \subseteq D$, there exists a constant A_K such that for all $x,y \in K$, $|f(x)-f(y)| \leq A_K \sum_{i=1}^N |x_i - y_i|^{\alpha_i}$.

Proposition 4.2. *Let $\{X(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ be a random field indexed by $\mathbb{R}_+ \times \mathbb{R}^d$. Suppose that there exist $d+1$ constants $\alpha_i \in \,]0,1]$, $i = 0,1,\dots,d$, and $p > \sum_{i=0}^d \alpha_i^{-1}$ such that, for all $n > 1$, there is a constant $C_{p,n}$ such that*

$$||X(t,x) - X(s,y)||_p \leq C_{p,n} \tau_{\alpha_0,\dots,\alpha_d}\left((t,x),(s,y)\right),$$

for all (t,x) and (s,y) in $K_n := [1/n, n] \times [-n, n]^d$, where the metric $\tau_{\alpha_0,\dots,\alpha_d}$ is defined in (4.1) with $N = d+1$. Then X has a modification which is locally Hölder continuous in $\mathbb{R}_+^* \times \mathbb{R}^d$ with indices $(\beta\alpha_0, \dots, \beta\alpha_d)$, for all $\beta \in \;]0, \beta_p[$, where $\beta_p = 1 - p^{-1}\sum_{i=0}^d \alpha_i^{-1}$. In addition, for all $0 \le \beta < \beta_p$,

$$E\left[\left(\sup_{\substack{(t,x),(s,y)\in K_n\\(t,x)\neq(s,y)}} \frac{|X(t,x)-X(s,y)|}{[\tau_{\alpha_0,\dots,\alpha_d}((t,x),(s,y))]^\beta}\right)^p\right] < +\infty.$$

If the compact sets K_n can be taken to be $[0,n] \times [-n,n]^d$, then the same local Hölder continuity of X extends to $\mathbb{R}_+ \times \mathbb{R}^d$ and the moment bound on increments of X applies with this new K_n.

4.2 Moment estimates

The main moment estimate that is needed for this proof is the following.

Proposition 4.3. *Fix $\varsigma \in \mathbb{R}$ and $\mu \in \mathcal{M}_H(\mathbb{R})$.*

(1) *For all $p \ge 2$ and $n > 1$, there is a constant $C_{n,p}$ such that for all $t, t' \in [1/n, n]$ and $x, x' \in [-n, n]$,*

$$\|I(t,x) - I(t',x')\|_p \le C_{n,p}\left(|t-t'|^{\frac{1}{4}} + |x-x'|^{\frac{1}{2}}\right). \tag{4.2}$$

(2) *If, in addition, $\mu \in \mathcal{M}_H^*(\mathbb{R})$, then there exists a constant $C_{n,p}^*$ such that for all (t,x), $(t',x') \in [0,n] \times [-n,n]$, (4.2) holds with $C_{n,p}$ replaced by $C_{n,p}^*$.*

The proof of this proposition will be given at the end of this section. We note that by Proposition 3.5, the conclusion in part (2) above is not valid for all $\mu \in \mathcal{M}_H(\mathbb{R})$.

Assuming Proposition 4.3, we now prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 2.3, we only need to establish the Hölder-continuity statements for I instead of u. Part (1) (respectively (2)) follows from Proposition 4.3(1) (respectively Proposition 4.3(2)) and Proposition 4.2. This proves Theorem 3.1. □

The next two propositions are needed to establish Proposition 4.3.

Proposition 4.4. *Given $\varsigma \in \mathbb{R}$ and $\mu \in \mathcal{M}_H(\mathbb{R})$, let $J_0^*(t,x) = (|\mu| * G_\nu(t,\cdot))(x)$ and $h(t,x) = \varsigma^2 + 2\left[J_0^*(t,x)\right]^2$. Then we have:*
(1) *For all $n > 1$, there exist constants $C_{n,i}$, $i = 1,3,5$, such that for all $t, t' \in [1/n, n]$, with $t < t'$, and $x, x' \in [-n, n]$,*

$$\iint_{[0,t]\times\mathbb{R}} \mathrm{d}s\mathrm{d}y\, h(s,y)\left(G_\nu(t-s,x-y) - G_\nu(t'-s,x-y)\right)^2 \le C_{n,1}\sqrt{t'-t}, \tag{4.3}$$

$$\iint_{[0,t]\times\mathbb{R}} \mathrm{d}s\mathrm{d}y\, h(s,y)\left(G_\nu(t-s,x-y) - G_\nu(t-s,x'-y)\right)^2 \le C_{n,3}\left|x-x'\right|, \tag{4.4}$$

$$\iint_{[t,t']\times\mathbb{R}} \mathrm{d}s\mathrm{d}y\, h(s,y) G_\nu^2(t'-s,x'-y) \le C_{n,5}\sqrt{t'-t}\,. \tag{4.5}$$

(2) *If, in addition, $\mu \in \mathcal{M}_H^*(\mathbb{R})$, then there exist constants $C_{n,i}^*$, $i=1,3,5$, such that for all $(t,x),(t',x') \in [0,n]\times[-n,n]$, (4.3)–(4.5) hold with $C_{n,i}$ replaced by $C_{n,i}^*$, $i=1,3,5$.*

Proposition 4.5. *Given $\bar\varsigma \in \mathbb{R}$ and $\mu \in \mathcal{M}_H(\mathbb{R})$, let $J_0^*(t,x) = (|\mu| * G_\nu(t,\cdot))(x)$. Then:*
(1) For all $n>1$, there exist three constants

$$C_{n,2} = \frac{\sqrt{\pi n}}{\sqrt{4\nu}} C_{n,1}, \quad C_{n,4} = \frac{\sqrt{\pi n}}{\sqrt{4\nu}} C_{n,3}, \quad \text{and} \quad C_{n,6} = \frac{\sqrt{\pi n}}{\sqrt{4\nu}} C_{n,5}, \tag{4.6}$$

such that for all $t,t' \in [1/n,n]$ with $t<t'$ and $x,x' \in [-n,n]$,

$$\left|\left(\left(\bar\varsigma^2 + 2\,|J_0^*|^2\right) \star G_\nu^2 \star (G_\nu(\cdot,\circ) - G_\nu(\cdot + t'-t,\circ))^2\right)(t,x)\right| \le C_{n,2}\sqrt{t'-t}, \tag{4.7}$$

$$\left|\left(\left(\bar\varsigma^2 + 2\,|J_0^*|^2\right) \star G_\nu^2 \star (G_\nu(\cdot,\circ) - G_\nu(\cdot,\circ + x'-x))^2\right)(t,x)\right| \le C_{n,4}|x'-x|, \tag{4.8}$$

$$\iint_{[t,t']\times\mathbb{R}} \mathrm{d}s\mathrm{d}y \left(\left(\bar\varsigma^2 + 2\,|J_0^*|^2\right) \star G_\nu^2\right)(s,y)\, G_\nu^2(t'-s,x'-y) \le C_{n,6}\sqrt{t'-t}. \tag{4.9}$$

(2) If, in addition, $\mu \in \mathcal{M}_H^*(\mathbb{R})$, then there exist constants

$$C_{n,2}^* = \frac{\sqrt{n}}{\sqrt{\pi\nu}} C_{n,1}^*, \quad C_{n,4}^* = \frac{\sqrt{n}}{\sqrt{\pi\nu}} C_{n,3}^*, \quad \text{and} \quad C_{n,6}^* = \frac{\sqrt{n}}{\sqrt{\pi\nu}} C_{n,5}^*,$$

such that for all $(t,x),(t',x') \in [0,n]\times[-n,n]$, (4.7)–(4.9) hold with $C_{n,i}$ replaced by $C_{n,i}^*$, $i=2,4,6$.

The proofs of these two propositions are given in the Sections 4.3 and 4.4. Assuming Propositions 4.4 and 4.5, we now prove Proposition 4.3.

Proof of Proposition 4.3. We first prove part (1). Without loss of generality, assume that $\mu \ge 0$. Otherwise, we simply replace μ in the following arguments by $|\mu|$. Fix $n>1$. By parts (1) of Propositions 4.4 and 4.5, there exist $C_{n,i}>0$ for $i=1,\dots,6$ such that for all (t,x) and $(t',x') \in [1/n,n]\times[-n,n]$ with $t'>t$, the six inequalities in Propositions 4.4 and 4.5 hold. By (2.1) and Lemma 2.4, for all even integers $p\ge 2$,

$$||I(t,x) - I(t',x')||_p^p \le 2^{p-1} z_p^p\, L_\rho^p\, I_1(t,t',x,x')^{p/2} + 2^{p-1} z_p^p\, L_\rho^p\, I_2(t,t'\,;\,x')^{p/2} \;,$$

where

$$I_1(t,t',x,x') = \iint_{[0,t]\times\mathbb{R}} \mathrm{d}s\mathrm{d}y\; (G_\nu(t-s,x-y) - G_\nu(t'-s,x'-y))^2 \left[\bar\varsigma^2 + ||u(s,y)||_p^2\right], \tag{4.10}$$

$$I_2(t,t'\,;\,x') = \iint_{[t,t']\times\mathbb{R}} \mathrm{d}s\mathrm{d}y\; G_\nu^2(t'-s,x'-y)\left(\bar\varsigma^2 + ||u(s,y)||_p^2\right). \tag{4.11}$$

By the subadditivity of $x \mapsto |x|^{2/p}$ and since $2^{2(p-1)/p} \leq 4$,

$$||I(t,x) - I(t',x')||_p^2 \leq 4 z_p^2 \, L_\rho^2 \left[I_1(t,t',x,x') + I_2(t,t' \,;\, x') \right].$$

Notice that

$$\mathcal{K}(t,x;\nu,\lambda) = \Upsilon(t;\nu,\lambda) \; G_\nu^2(t,x), \quad \text{with } \Upsilon(t;\nu,\lambda) = \lambda^2 + \lambda^4 \sqrt{\frac{\pi t}{\nu}} e^{\frac{\lambda^4 t}{4\nu}} \Phi\left(\lambda^2 \sqrt{\frac{t}{2\nu}}\right).$$

Denote $\Upsilon_*(t) := \Upsilon(t\,;\,\nu,\, a_{p,\bar{\varsigma}} z_p\, L_\rho) < +\infty$, for all $t \in \mathbb{R}_+$. Clearly, $\Upsilon_*(t) \leq \Upsilon_*(n)$ for $t \leq n$. Hence, it follows from (2.5) and (2.4) that

$$||u(s,y)||_p^2 \leq 2\, J_0^2(s,y) + \Upsilon_*(n) \left(\left(\bar{\varsigma}^2 + 2\, J_0^2 \right) \star G_\nu^2 \right)(s,y), \quad \text{for } s \leq t \leq n. \tag{4.12}$$

We shall use this bound in order to estimate I_1 and I_2.

We first consider the case where $x = x'$. Set $h = t' - t$. Then

$$\begin{aligned} I_1(t,t',x,x) \leq & \left(\left(\bar{\varsigma}^2 + 2J_0^2 \right) \star \left(G_\nu(\cdot,\circ) - G_\nu(\cdot + h,\circ) \right)^2 \right)(t,x) \\ & + \Upsilon_*(n) \left(\left(\bar{\varsigma}^2 + 2J_0^2 \right) \star G_\nu^2 \star \left(G_\nu(\cdot,\circ) - G_\nu(\cdot + h,\circ) \right)^2 \right)(t,x). \end{aligned}$$

By parts (1) of Propositions 4.4 and 4.5,

$$I_1(t,t',x,x) \leq \left(C_{n,1} + \Upsilon_*(n) C_{n,2} \right) \; |h|^{1/2}.$$

Similarly, we have that

$$I_2(t,t' \,;\, x') \leq \left(C_{n,5} + \Upsilon_*(n) C_{n,6} \right) \; |h|^{1/2}.$$

Hence, for all $x \in [-n,n]$ and $1/n \leq t < t' \leq n$,

$$||I(t,x) - I(t',x)||_p^2 \leq 4 z_p^2 \, L_\rho^2 \left(C_{n,1} + C_{n,5} + \Upsilon_*(n) \left(C_{n,2} + C_{n,6} \right) \right) \, |t' - t|^{1/2} \;. \tag{4.13}$$

Now consider the case where $t = t' \geq 1/n$. Denote $\zeta = x' - x$. In this case, $I_2 = 0$. By (4.12) above and parts (1) of Propositions 4.4 and 4.5,

$$||I(t,x) - I(t,x')||_p^2 \leq 4 z_p^2 \, L_\rho^2 \; \left[C_{n,3} + \Upsilon_*(n) C_{n,4} \right] \; |\zeta|.$$

Combining this with (4.13), we see that

$$||I(t,x) - I(t',x')||_p^2 \leq \widetilde{C}_{p,n} \left(|t' - t|^{1/2} + |x' - x| \right),$$

for all $1/n \leq t < t' \leq n$, $x, x' \in [-n,n]$, where $\widetilde{C}_{p,n}$ is a finite constant. This proves (1).

The conclusion in part (2) can be proved in the same way by applying parts (2) of Propositions 4.4 and 4.5 below instead of parts (1). We simply replace all $C_{n,i}$ above by $C_{n,i}^*$ for $i = 1, \ldots, 6$. The remaining statements follow immediately. This completes the proof of Proposition 4.3. $\square$

Remark 4.6 (Case of bounded initial data)**.** In the case where the initial data is bounded: $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$, where f is a bounded function such that $|f(x)| \le C$, the conclusions of Proposition 4.3 follow from the following standard (and much simpler) argument: By (3.1), for $0 \le t \le t' \le T$, and $x, x' \in \mathbb{R}$

$$I_1(t,t',x,x') \le A_T \iint_{[0,t']\times\mathbb{R}} \mathrm{d}s\mathrm{d}y \ \left(G_\nu(t-s,x-y) - G_\nu(t'-s,x'-y)\right)^2,$$

where $I_1(t,t',x,x')$ is defined in (4.10) and A_T is a finite constant. Then by Proposition 5.2, for some constant $C' > 0$ depending only on ν,

$$I_1\left(t,t',x,x'\right) \le A_T C' \left(|x-x'| + \sqrt{|t'-t|}\right) \ .$$

Similarly, $I_2\left(t,t',x,x'\right)$, defined in (4.11), is bounded by $A_T C'\sqrt{|t'-t|}$ with the same constants A_T and C'. Therefore,

$$||I(t,x) - I\left(t',x'\right)||_p^2 \le 4z_p^2 A_T C' \left(|x-x'| + |t-t'|^{1/2}\right) \ ,$$

for all $0 \le t \le t \le T$ and $x, x' \in \mathbb{R}$. The Hölder continuity follows from Proposition 4.2.

4.3 Proofs of part (1) of the Propositions 4.4 and 4.5

Lemma 4.7. *For all $L > 0$, $\beta \in \,]0,1[\,$, $t > 0$, $x \in \mathbb{R}$, $\nu > 0$, and h with $|h| \le \beta L$, we have that*

$$\begin{aligned}&|G_\nu(t,x+h) - G_\nu(t,x)| \\ &\qquad \le |h| \left(\frac{C}{\sqrt{2\nu t}} + \frac{1}{(1-\beta)L}\right) \left[G_\nu(t,x) + e^{\frac{3L^2}{2\nu t}} \left\{G_\nu\left(t,x-2L\ \right) + G_\nu\left(t,x+2L\ \right)\right\}\right]\end{aligned}$$

and

$$\begin{aligned}&|G_\nu(t,x+h) + G_\nu(t,x-h) - 2G_\nu(t,x)| \\ &\qquad \le 2|h| \left(\frac{C}{\sqrt{2\nu t}} + \frac{1}{(1-\beta)L}\right) \left[G_\nu(t,x) + e^{\frac{3L^2}{2\nu t}} \left\{G_\nu\left(t,x-2L\ \right) + G_\nu\left(t,x+2L\ \right)\right\}\right],\end{aligned}$$

where $C := \sup_{x\in\mathbb{R}} \frac{1}{|x|}|e^{-x^2/2} - 1| \approx 0.451256$.

Proof. Fix $L > 0$ and $\beta \in \,]0,1[$. Assume that $|h| \le \beta L$. Define

$$f(t,x,h) = G_\nu(t,x+h) + G_\nu(t,x-h) - 2G_\nu(t,x),$$

$$I(t,x,h) = \begin{cases} h^{-1}\, G_\nu^{-1}(t,x-L)\Big[G_\nu(t,x+h) - G_\nu(t,x)\Big] & \text{if } x \ge 0, \\ h^{-1}\, G_\nu^{-1}(t,x+L)\Big[G_\nu(t,x+h) - G_\nu(t,x)\Big] & \text{if } x \le 0. \end{cases}$$

Clearly,

$$\left|\frac{f(t,x,h)}{h\ \left(G_\nu(t,x+L)+G_\nu(t,x-L)\right)}\right| \le |I(t,x,h)| + |I(t,x,-h)|\ . \tag{4.14}$$

We will bound $|I(t,x,h)|$ for $-\beta L \le h \le \beta L$. If $x \ge 0$, then

$$I(t,x,h) = \frac{1}{h}\left(\ e^{-\frac{(x+h)^2}{2\nu t}+\frac{(x-L)^2}{2\nu t}} - e^{-\frac{x^2}{2\nu t}+\frac{(x-L)^2}{2\nu t}}\ \right),$$

and so

$$\frac{\partial}{\partial x} I(t,x,h) = -\frac{1}{\nu t} e^{-\frac{(x+h)^2}{2\nu t}+\frac{(x-L)^2}{2\nu t}} - \frac{L}{\nu t}\, I(t,x,h).$$

Hence,

$$|I(t,x,h)| \le \int_0^x (\nu t)^{-1} e^{-\frac{(y+h)^2}{2\nu t}+\frac{(y-L)^2}{2\nu t}} \mathrm{d}y + \frac{L}{\nu t}\int_0^x |I(t,y,h)|\,\mathrm{d}y + |I(t,0,h)|\,.$$

Let C be the constant defined in the proposition. Then

$$|I(t,0,h)| \le \frac{C}{\sqrt{2\nu t}} e^{\frac{L^2}{2\nu t}}, \quad \text{for all } h \in \mathbb{R}.$$

Since $|h| \le \beta L$,

$$\int_0^x \frac{1}{\nu t} e^{-\frac{(y+h)^2}{2\nu t}+\frac{(y-L)^2}{2\nu t}} dy \le \int_0^\infty \frac{1}{\nu t} e^{-\frac{(y+h)^2}{2\nu t}+\frac{(y-L)^2}{2\nu t}} dy = \frac{e^{\frac{L^2-h^2}{2\nu t}}}{L+h} \le \frac{e^{\frac{L^2}{2\nu t}}}{(1-\beta)L}.$$

Therefore,

$$|I(t,x,h)| \le C_{t,L,\beta} + \frac{L}{\nu t}\int_0^x |I(t,y,h)|\,\mathrm{d}y, \quad \text{with } C_{t,L,\beta} := \left(\frac{C}{\sqrt{2\nu t}} + \frac{1}{(1-\beta)L}\right) e^{\frac{L^2}{2\nu t}}.$$

Apply Bellman-Gronwall's lemma (see [18, Lemma 12.2.2]) to get

$$|I(t,x,h)| \le C_{t,L,\beta}\, e^{\frac{Lx}{\nu t}} = C_{t,L,\beta}\, e^{\frac{L|x|}{\nu t}},$$

and so, by definition of $I(t,x,h)$,

$$|G_\nu(t,x+h) - G_\nu(t,x)| \le C_{t,L,\beta}|h|\left(G_\nu(t,x+L)+G_\nu(t,x-L)\right) e^{\frac{L|x|}{\nu t}}. \tag{4.15}$$

By symmetry, for $x \le 0$, we get the same bound for $|I(t,x,h)|$. Hence, from (4.14),

$$|f(t,x,h)| \le 2C_{t,L,\beta}|h|\left(G_\nu(t,x+L)+G_\nu(t,x-L)\right)\exp\left(\frac{L|x|}{\nu t}\right). \tag{4.16}$$

Finally, some calculations show that

$$\begin{aligned}
&\Big(G_\nu(t,x+L)+G_\nu(t,x-L)\Big)e^{\frac{L|x|}{\nu t}}\\
&\qquad\qquad =G_\nu(t,x)e^{-\frac{L^2}{2\nu t}}+G_\nu(t,x-2L)e^{\frac{3L^2}{2\nu t}}1_{\{x\geq 0\}}+G_\nu(t,x+2L)e^{\frac{3L^2}{2\nu t}}1_{\{x\leq 0\}}\\
&\qquad\qquad \leq G_\nu(t,x)e^{-\frac{L^2}{2\nu t}}+\Big(G_\nu(t,x-2L)+G_\nu(t,x+2L)\Big)e^{\frac{3L^2}{2\nu t}}\,.
\end{aligned}$$

The desired conclusions now follow from (4.15) and (4.16). □

Proof of Proposition 4.4 (1). Assume that $\overline{\varsigma}=0$. Set $\bar{z}=(z_1+z_2)/2$. Set

$$I(t,x;t',x')=\iint_{[0,t]\times\mathbb{R}}\mathrm{d}s\mathrm{d}y\,\left[J_0^*(s,y)\right]^2\left(G_\nu\left(t-s,x-y\right)-G_\nu(t'-s,x'-y)\right)^2.$$

Write $\left[J_0^*(s,y)\right]^2$ as a double integral and then use Lemma 5.3 to get

$$\begin{aligned}
I(t,x;t',x')=&\int_0^t\mathrm{d}s\iint_{\mathbb{R}^2}|\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\,G_{2\nu}(s,z_1-z_2)\\
&\times\int_{\mathbb{R}}\mathrm{d}y\,G_{\nu/2}\left(s,y-\bar{z}\right)\left(G_\nu\left(t-s,x-y\right)-G_\nu(t'-s,x'-y)\right)^2.
\end{aligned}\tag{4.17}$$

In the following, we use $\int\mathrm{d}yG(G-G)^2$ to denote the $\mathrm{d}y$–integral in (4.17). Expand $(G-G)^2=G^2-2GG+G^2$ and apply Lemma 5.3 to each term:

$$\begin{aligned}
&\left(G_\nu\left(t-s,x-y\right)-G_\nu(t'-s,x'-y)\right)^2\\
&=\frac{1}{\sqrt{4\pi\nu(t-s)}}G_{\nu/2}\left(t-s,x-y\right)+\frac{1}{\sqrt{4\pi\nu(t'-s)}}G_{\nu/2}\left(t'-s,x'-y\right)\\
&\quad-2G_{2\nu}\left(\frac{t+t'}{2}-s,x-x'\right)G_{\nu/2}\left(\frac{2(t-s)(t'-s)}{t+t'-2s},y-\frac{(t-s)x'+(t'-s)x}{t+t'-2s}\right).
\end{aligned}$$

Then integrate over y using the semigroup property of the heat kernel:

$$\begin{aligned}
&\int_{\mathbb{R}}\mathrm{d}y\,G_{\nu/2}(s,y-\bar{z})\left(G_\nu(t-s,x-y)-G_\nu(t'-s,x'-y)\right)^2\\
&=\frac{1}{\sqrt{4\pi\nu(t-s)}}G_{\nu/2}\left(t,x-\bar{z}\right)+\frac{1}{\sqrt{4\pi\nu(t'-s)}}G_{\nu/2}\left(t',x'-\bar{z}\right)\\
&\quad-2\,G_{2\nu}\left(\frac{t+t'}{2}-s,x-x'\right)G_{\nu/2}\left(\frac{2(t-s)(t'-s)}{t+t'-2s}+s,\frac{(t-s)x'+(t'-s)x}{t+t'-2s}-\bar{z}\right).
\end{aligned}\tag{4.18}$$

Property (4.3). Set $x=x'$ in (4.17) and let $h=t'-t$. Then $\frac{2(t-s)(t'-s)}{t+t'-2s}+s=t+\frac{(t-s)h}{2(t-s)+h}$ and (4.18) becomes

$$\int\mathrm{d}y\,G(G-G)^2=\left[\frac{1}{\left(4\pi\nu(t-s)\right)^{\frac{1}{2}}}+\frac{1}{\left(4\pi\nu(t'-s)\right)^{\frac{1}{2}}}-\frac{1}{\left(\pi\nu\left(\frac{t+t'}{2}-s\right)\right)^{\frac{1}{2}}}\right]G_{\nu/2}\left(t,x-\bar{z}\right)$$

$$
\begin{aligned}
&+ \frac{1}{\sqrt{4\pi\nu(t'-s)}} \left(\frac{G_{\nu/2}\left(t', x-\bar{z}\right)}{G_{\nu/2}\left(t, x-\bar{z}\right)} - 1 \right) G_{\nu/2}\left(t, x-\bar{z}\right) \\
&- \frac{1}{\sqrt{\pi\nu\left(\frac{t+t'}{2} - s\right)}} \left(\frac{G_{\nu/2}\left(t + \frac{(t-s)h}{2(t-s)+h}, x-\bar{z}\right)}{G_{\nu/2}\left(t, x-\bar{z}\right)} - 1 \right) G_{\nu/2}\left(t, x-\bar{z}\right) \\
:= {}& I_1 + I_2 - I_3 \,.
\end{aligned}
$$

We first consider I_2. Because $1/n \le t \le t' \le n$, we have that $h \in [0, n^2 t]$, so by Lemma 5.7, we find after simplification that

$$
|I_2| \le \frac{3\sqrt{1+n^2}}{4\sqrt{\pi\nu t(t'-s)}} G_{\frac{\nu(1+n^2)}{2}}\left(t, x-\bar{z}\right)\sqrt{h},
$$

and so

$$
\int_0^t \mathrm{d}s\, G_{2\nu}(s, z_1 - z_2)|I_2| \le \sqrt{h} \int_0^t \mathrm{d}s\, \frac{3\sqrt{1+n^2}}{4\sqrt{\pi\nu t(t'-s)}} G_{\nu(1+n^2)/2}\left(t, x-\bar{z}\right) G_{2\nu}(s, z_1 - z_2).
$$

Apply Lemma 5.4 to $G_{\nu(1+n^2)/2}\left(\cdots\right) G_{2\nu}(\cdots)$ and integrate over $\mathrm{d}z_1\mathrm{d}z_2$ to get

$$
\iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2) \int_0^t \mathrm{d}s\, G_{2\nu}|I_2| \le \frac{3\left(1+n^2\right)\sqrt{h}}{2\sqrt{\pi\nu}} \left(|\mu| * G_{2\nu(1+n^2)}(t,\cdot)\right)^2(x) \int_0^t \frac{\mathrm{d}s}{\sqrt{s(t'-s)}}.
$$

By the Beta integral (see (3.3)), the $\mathrm{d}s$–integral is less than or equal to π. So

$$
\iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2) \int_0^t \mathrm{d}s\, G_{2\nu}(\cdots)|I_2| \le \frac{3\left(1+n^2\right)\sqrt{\pi}}{2\sqrt{\nu}} \left(|\mu| * G_{2\nu(1+n^2)}(t,\cdot)\right)^2(x)\sqrt{h}. \tag{4.19}
$$

As for I_3, notice that since $s \in [0,t]$, $\frac{(t-s)h}{2(t-s)+h} \le \frac{th}{h} \le n^2 t$ for all $h \ge 0$. Apply Lemma 5.7 with $r = \frac{(t-s)h}{2(t-s)+h}$ to obtain that

$$
\left| \frac{G_{\nu/2}\left(t + \frac{(t-s)h}{2(t-s)+h}, x-\bar{z}\right)}{G_{\nu/2}\left(t, x-\bar{z}\right)} - 1 \right| \le \frac{3}{2\sqrt{2}} \exp\left(\frac{n^2(x-\bar{z})^2}{\nu t\left(1+n^2\right)}\right) \frac{\sqrt{h}}{\sqrt{t}}, \quad \text{for all } h \ge 0,
$$

where we have used the inequality $\frac{(t-s)h}{2(t-s)+h} \le \frac{h}{2}$. Multiplying out the exponentials, we obtain

$$
|I_3| \le \frac{3\sqrt{1+n^2}}{2\sqrt{2\pi\nu t(t-s)}} G_{\nu(1+n^2)/2}\left(t, x-\bar{z}\right)\, \sqrt{h}.
$$

Then by the same arguments as for I_2, we have that

$$\iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2) \int_0^t \mathrm{d}s\, G_{2\nu}|I_3| = \frac{3\left(1+n^2\right)\sqrt{\pi}}{\sqrt{2\nu}} \left(|\mu| * G_{2\nu(1+n^2)}(t,\cdot)\right)^2(x)\sqrt{h}.$$

Now let us consider I_1. Apply Lemma 5.4 to $G_{2\nu}\left(s, z_1 - z_2\right) G_{\nu/2}\left(t, x - \bar{z}\right)$ to get

$$\begin{aligned}\int_0^t \mathrm{d}s\, G_{2\nu}(s, z_1 - z_2)|I_1| \leq & \frac{\sqrt{t}}{\sqrt{\pi\nu}} G_{2\nu}(t, x - z_1) G_{2\nu}(t, x - z_2) \\ & \times \int_0^t \mathrm{d}s \left| (s(t-s))^{-\frac{1}{2}} + (s(t'-s))^{-\frac{1}{2}} - 2\left(s\left[\frac{t+t'}{2} - s\right]\right)^{-\frac{1}{2}} \right|.\end{aligned}$$

The integrand is bounded by

$$\left|(s(t-s))^{-\frac{1}{2}} - \left(s\left[\frac{t+t'}{2} - s\right]\right)^{-\frac{1}{2}}\right| + \left|(s(t'-s))^{-\frac{1}{2}} - \left(s\left[\frac{t+t'}{2} - s\right]\right)^{-\frac{1}{2}}\right|.$$

Taking into account the signs of the increment, this is equal to $[s(t-s)]^{-1/2} - [s(t'-s)]^{-1/2}$. Integrate the r.h.s. of the above inequality using the formula $\int_0^t \frac{\mathrm{d}s}{\sqrt{s(t'-s)}} = 2\arctan\left(\frac{\sqrt{t}}{\sqrt{t'-t}}\right)$ for all $t' > t \geq 0$ to find that

$$\int_0^t \mathrm{d}s \left| (s(t-s))^{-\frac{1}{2}} + (s(t'-s))^{-\frac{1}{2}} - 2\left(s\left[\frac{t+t'}{2} - s\right]\right)^{-\frac{1}{2}} \right| \leq \pi - 2\arctan\left(\sqrt{t/h}\right).$$

It is an elementary calculus exercise to show that the function $f(x) := x\left(\pi - 2\arctan\left(x\right)\right)$ for $x \geq 0$ is non-negative and bounded from above, and $f(x) \leq \lim_{x\to+\infty} f(x) = 2$. Hence, $\pi - 2\arctan\left(\sqrt{t/h}\right) \leq 2\sqrt{h/t}$. Therefore,

$$\iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2) \int_0^t \mathrm{d}s\, G_{2\nu}(s, z_1 - z_2)|I_1| \leq \frac{2\sqrt{h}}{\sqrt{\pi\nu}} \left(|\mu| * G_{2\nu}(t,\cdot)\right)^2(x). \tag{4.20}$$

We conclude from (4.19)–(4.20) that for all $(t,x), (t',x) \in [1/n, n] \times [-n, n]$ with $t' > t$,

$$I(t,x;t',x) \leq \left(C_\nu^\star \left(|\mu| * G_{2\nu}(t,\cdot)\right)^2(x) + C_{n,\nu}^* \left(|\mu| * G_{2\nu(1+n^2)}(t,\cdot)\right)^2(x)\right)\sqrt{h},$$

where

$$C_\nu^\star = \frac{2}{\sqrt{\pi\nu}}, \quad \text{and} \quad C_{n,\nu}^* := \frac{3\sqrt{\pi}\left(1+\sqrt{2}\right)\left(1+n^2\right)}{2\sqrt{\nu}}.$$

As for the contribution of the constant ς, it corresponds to the initial data $\mu(\mathrm{d}x) \equiv \varsigma\,\mathrm{d}x$ and we apply Proposition 5.2. Finally, by the smoothing effect of the heat kernel (Lemma 2.3), we can choose the following constant

$$C_{n,1} = \varsigma^2 \frac{\sqrt{2}-1}{\sqrt{\pi\nu}} + \sup 2\left(C_\nu^\star \left(|\mu| * G_{2\nu}(s,\cdot)\right)^2(y) + C_{n,\nu}^* \left(|\mu| * G_{2\nu(1+n^2)}(s,\cdot)\right)^2(y)\right),$$

for (4.3), where the supremum is over $(s,y) \in [1/n, n] \times [-n, n]$. This proves (4.3).

Property (4.4). Set $t = t'$ in (4.17) and $\bar{x} = \frac{x+x'}{2}$. Consider the integral in (4.17)

$$\int_0^t \mathrm{d}s\, G_{2\nu}(s, z_1 - z_2) \int \mathrm{d}y\, G(G-G)^2,$$

which is denoted by $\int \mathrm{d}s\, G \int \mathrm{d}y\, G(G-G)^2$ for convenience. By (4.18),

$$\begin{aligned} \int \mathrm{d}y\, G_{\nu/2}(s, y - \bar{z}) \left(G_\nu(t-s, x-y) - G_\nu(t-s, x'-y)\right)^2 & \\ = \frac{1}{\sqrt{4\pi\nu(t-s)}} \Big[G_{\nu/2}\left(t, x - \bar{z}\right) + G_{\nu/2}\left(t, x' - \bar{z}\right)\Big] & \\ - 2\, G_{2\nu}\left(t-s, x-x'\right) G_{\nu/2}\left(t, \bar{x} - \bar{z}\right). & \end{aligned} \tag{4.21}$$

Then apply Lemma 5.5 to integrate over s:

$$\begin{aligned} \int \mathrm{d}s\, G \int \mathrm{d}y\, G(G-G)^2 = & \frac{1}{4\nu}\left(G_{\nu/2}\left(t, x-\bar{z}\right) + G_{\nu/2}\left(t, x'-\bar{z}\right)\right) \operatorname{erfc}\left(\frac{|z_1 - z_2|}{\sqrt{4\nu t}}\right) \\ & - \frac{1}{2\nu} G_{\nu/2}\left(t, \bar{x} - \bar{z}\right) \operatorname{erfc}\left(\frac{1}{\sqrt{2t}}\left[\frac{|z_1 - z_2|}{\sqrt{2\nu}} + \frac{|x - x'|}{\sqrt{2\nu}}\right]\right). \end{aligned}$$

It follows from the definition of $\operatorname{erfc}(x)$ that $\operatorname{erfc}\left(|x| + h\right) \ge \operatorname{erfc}\left(|x|\right) - \frac{2e^{-x^2}}{\sqrt{\pi}} h$ for $h \ge 0$ and we apply this inequality to the last factor to obtain,

$$\begin{aligned} \int \mathrm{d}s\, G \int & \mathrm{d}y\, G(G-G)^2 \\ & \le \frac{1}{\nu}\, G_{\nu/2}\left(t, \bar{x} - \bar{z}\right) \frac{|x - x'|}{\sqrt{4\pi\nu t}} \exp\left(-\frac{(z_1 - z_2)^2}{4\nu t}\right) \\ & \quad + \frac{1}{4\nu}\left[G_{\nu/2}\left(t, x - \bar{z}\right) + G_{\nu/2}\left(t, x' - \bar{z}\right) - 2G_{\nu/2}\left(t, \bar{x} - \bar{z}\right)\right] \operatorname{erfc}\left(\frac{|z_1 - z_2|}{\sqrt{4\nu t}}\right). \end{aligned}$$

Now apply Lemma 4.7 with $h = \frac{x'-x}{2}$, $L = 2n$ and $\beta = 1/2$: there are two constants

$$\begin{aligned} C_n' &= \sup_{s \in [1/n, n]} C_{2n, 1/2, \nu s} = \frac{C\sqrt{n}}{\sqrt{2\nu}} + \frac{1}{n}, \qquad C \approx 0.451256\,, \\ C_n'' &= \sup_{s \in [1/n, n]} C''_{2n, 1/2, \nu s} = C_n' \exp\left(\frac{6\, n^3}{\nu}\right), \end{aligned}$$

where $C'_{L,\beta,\nu s}$ and $C''_{L,\beta,\nu s}$ are defined in Lemma 4.7, such that for $\left|\frac{x-x'}{2}\right| \le \beta L = n$,

$$
\begin{aligned}
&\left| G_{\nu/2}\left(t,x-\bar z\right)+G_{\nu/2}\left(t,x'-\bar z\right)-2G_{\nu/2}\left(t,\bar x-\bar z\right)\right| \\
&\qquad \le \left\{ C_n''\left[G_{\nu/2}\left(t,\bar x-\bar z-2L\right)+G_{\nu/2}\left(t,\bar x-\bar z+2L\right)\right] + C_n'\, G_{\nu/2}\left(t,\bar x-\bar z\right)\right\} |x-x'| \,.
\end{aligned}
$$

Note that $t\ge 1/n$ is essential for this inequality to be valid. By Lemma 5.5, we have that $\operatorname{erfc}\left(\frac{|z_1-z_2|}{\sqrt{4\nu t}}\right)\le \sqrt{4\pi\nu t}\, G_{2\nu}\left(t,z_1-z_2\right)$, and so

$$
\begin{aligned}
\left|\int \mathrm{d}s\, G\int \mathrm{d}y\, G(G-G)^2\right| \le{}& \left(\frac{1}{\nu}+\frac{\sqrt{\pi t}}{\sqrt{4\nu}}C_n'\right)|x-x'|\; G_{\nu/2}\left(t,\bar x-\bar z\right)G_{2\nu}\left(t,z_1-z_2\right) \\
&+\frac{\sqrt{\pi t}\; C_n''}{\sqrt{4\nu}}\,|x-x'|\; G_{\nu/2}\left(t,\bar x-\bar z-2L\right)G_{2\nu}\left(t,z_1-z_2\right) \\
&+\frac{\sqrt{\pi t}\; C_n''}{\sqrt{4\nu}}\,|x-x'|\; G_{\nu/2}\left(t,\bar x-\bar z+2L\right)G_{2\nu}\left(t,z_1-z_2\right)\,.
\end{aligned}
$$

Now apply Lemma 5.4:

$$
\begin{aligned}
\left|\int \mathrm{d}s\, G\int \mathrm{d}y\, G(G-G)^2\right| \le{}& \left(\frac{1}{\nu}+\frac{\sqrt{\pi n}}{\sqrt{4\nu}}C_n'\right)|x-x'|\, G_{2\nu}\left(t,\widetilde{x}_1-z_1\right)G_{2\nu}\left(t,\widetilde{x}_1-z_2\right) \\
&+\frac{\sqrt{\pi n}\; C_n''}{\sqrt{4\nu}}\,|x-x'|\; G_{2\nu}\left(t,\widetilde{x}_2-z_1\right)G_{2\nu}\left(t,\widetilde{x}_2-z_2\right) \\
&+\frac{\sqrt{\pi n}\; C_n''}{\sqrt{4\nu}}\,|x-x'|\; G_{2\nu}\left(t,\widetilde{x}_3-z_1\right)G_{2\nu}\left(t,\widetilde{x}_3-z_2\right),
\end{aligned}
$$

where $\widetilde{x}_1=\bar x$, $\widetilde{x}_2=\bar x-2L$ and $\widetilde{x}_3=\bar x+2L$. Clearly, $\widetilde{x}_i\in[-5n,5n]$ for all $i=1,2,3$. Finally, after integrating over $|\mu|(\mathrm{d}z_1)$ and $|\mu|(\mathrm{d}z_2)$, we see that

$$
I(t,x;t,x')\le C_{n,3}'\,|x-x'|
$$

for all $t\in[1/n,n]$, and $x,x'\in[-n,n]$, where the constant is equal to

$$
C_{n,3}'=\left(\frac{1}{\nu}+\frac{\sqrt{\pi n}}{\sqrt{4\nu}}\left(C_n'+2C_n''\right)\right)\sup_{(s,y)\in[1/n,n]\times[-5n,5n]}\left(|\mu|*G_{2\nu}(s,\cdot)\right)^2(y)\,.
$$

As for the contribution of the constant ς, it corresponds to the initial data $|\mu|(\mathrm{d}x)\equiv\varsigma\,\mathrm{d}x$ and we apply Proposition 5.2. Finally, one can choose, for (4.4),

$$
C_{n,3}=\frac{\varsigma^2}{\nu}+\left(\frac{2}{\nu}+\frac{\sqrt{\pi n}}{\sqrt{\nu}}\left(C_n'+2C_n''\right)\right)\sup_{(s,y)\in[1/n,n]\times[-5n,5n]}\left(|\mu|*G_{2\nu}(s,\cdot)\right)^2(y)\,.
$$

This constant $C_{n,3}$ is clearly finite. This completes the proof of (4.4).

Property (4.5). We first consider the contribution of $J_0^*(t,x)$. As before, let

$$
I\left(t,x;t',x'\right)=\iint_{[t,t']\times\mathbb{R}}\mathrm{d}s\mathrm{d}y\;\left[J_0^*\left(s,y\right)\right]^2G_\nu^2(t'-s,x'-y).
$$

Set $\bar{z} = (z_1 + z_2)/2$. Similar to the arguments leading to (4.17), we have

$$\begin{aligned} I\left(t,x;t',x'\right) = \int_t^{t'} \mathrm{d}s \iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\, G_{2\nu}(s,z_1-z_2) \\ \times \int_{\mathbb{R}} \mathrm{d}y\, G_{\nu/2}\left(s,y-\bar{z}\right) G_\nu^2\left(t'-s,x'-y\right). \end{aligned} \tag{4.22}$$

Apply Lemma 5.3 to $G_\nu^2\left(t'-s,x'-y\right)$ and then integrate over y,

$$I\left(t,x;t',x'\right) = \int_t^{t'} \mathrm{d}s \iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2) \frac{1}{\sqrt{4\pi\nu(t'-s)}} G_{2\nu}(s,z_1-z_2) G_{\nu/2}\left(t',x'-\bar{z}\right).$$

Now apply Lemma 5.4 to $G_{2\nu}(s,z_1-z_2)G_{\nu/2}\left(t',x'-\bar{z}\right)$. Then by Lemma 5.8 and the fact that $\arcsin(x) \le \pi x/2$ for $x \in [0,1]$, we see that

$$I\left(t,x;t',x'\right) \le \left|J_0^*\left(2t',x'\right)\right|^2 \frac{2\sqrt{t'}}{\sqrt{\pi\nu}} \arcsin\left(\sqrt{\frac{t'-t}{t'}}\right) \le \left|J_0^*\left(2t',x'\right)\right|^2 \sqrt{\frac{\pi}{\nu}}\sqrt{t'-t}.$$

Therefore,

$$I\left(t,x;t',x'\right) \le C_{n,5}'\sqrt{t'-t}, \quad \text{with } C_{n,5}' = \sqrt{\frac{\pi}{\nu}} \sup_{(s,y)\in[1/n,n]\times[-n,n]} |J_0^*(2s,y)|^2.$$

As for the contribution of $\bar{\varsigma}$, it corresponds to the initial data $|\mu|(\mathrm{d}x) \equiv \bar{\varsigma}\,\mathrm{d}x$ and we apply Proposition 5.2. Finally, we can choose

$$C_{n,5} = \frac{\bar{\varsigma}^2}{\sqrt{\pi\nu}} + 2\sqrt{\frac{\pi}{\nu}} \sup_{(s,y)\in[1/n,n]\times[-n,n]} |J_0^*(2s,y)|^2 \tag{4.23}$$

for (4.5). This completes the proof of (4.5) and therefore part (1) of Proposition 4.4. □

Proof of Proposition 4.5 (1). We first prove (4.7) and (4.8). Denote

$$I(t,x;t',x') = \iint_{[0,t]\times\mathbb{R}} \mathrm{d}s\mathrm{d}y \left(|J_0^*|^2 \star G_\nu^2\right)(s,y) \left(G_\nu\left(t-s,x-y\right) - G_\nu(t'-s,x'-y)\right)^2.$$

Let $\bar{z} = (z_1+z_2)/2$. As in (4.17), replace $|J_0^*(u,z)|^2$ by the double integral. By Lemma 5.3,

$$\begin{aligned} I\left(t,x;t',x'\right) = \iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2) \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, \frac{1}{\sqrt{4\nu\pi(s-u)}} G_{2\nu}(u,z_1-z_2) \\ \times \iint_{\mathbb{R}^2} \mathrm{d}y\mathrm{d}z\, G_{\nu/2}\left(u,z-\bar{z}\right) G_{\nu/2}(s-u,y-z) \\ \times \left(G_\nu\left(t-s,x-y\right) - G_\nu\left(t'-s,x'-y\right)\right)^2. \end{aligned}$$

We first integrate over $\mathrm{d}z$ using the semigroup property and then integrate over $\mathrm{d}u$ by using Lemma 5.5 and use the fact that $s \le t \le n$ to obtain

$$
\begin{aligned}
I\left(t,x;t',x'\right) \le & \frac{\sqrt{\pi n}}{\sqrt{4\nu}} \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2)\, G_{2\nu}(s,z_1-z_2) \\
& \times \int_{\mathbb{R}} \mathrm{d}y\, G_{\nu/2}\left(s,y-\bar{z}\right)\left(G_\nu\left(t-s,x-y\right)-G_\nu\left(t'-s,x'-y\right)\right)^2 .
\end{aligned} \tag{4.24}
$$

Comparing this upper bound with (4.17), we can apply Proposition 4.4 to conclude that (4.7) and (4.8) are true with the constants $C_{n,2}$ and $C_{n,4}$ given in (4.6). As for (4.9), let

$$
I\left(t,x;t',x'\right) = \iint_{[t,t']\times\mathbb{R}} \mathrm{d}s\mathrm{d}y \left(|J_0^*|^2 \star G_\nu^2\right)(s,y)\, G_\nu^2(t'-s,x'-y).
$$

By arguments similar to those leading to (4.24), we have that

$$
\begin{aligned}
I\left(t,x;t',x'\right) \le & \frac{\sqrt{\pi n}}{4\nu} \int_t^{t'} \mathrm{d}s \iint_{\mathbb{R}^2} |\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2) G_{2\nu}(s,z_1-z_2) \\
& \times \int_{\mathbb{R}} \mathrm{d}y\, G_{\nu/2}\left(s,y-\bar{z}\right) G_\nu^2(t'-s,x'-y).
\end{aligned}
$$

Comparing this upper bound with (4.22), we can apply Proposition 4.4 to conclude that (4.9) is true with the corresponding constant $C_{n,6}$ given in (4.6). This completes the proof of part (1) of Proposition 4.5. $\square$

4.4 Proofs of part (2) of the Propositions 4.4 and 4.5

Lemma 4.8. *For $a \ge 1$ and $b \ge (a\,e)^{-1}$, we have that $|x| \le e^{b|x|^a}$ for all $x \in \mathbb{R}$.*

Proof. The case where $x = 0$ is clearly true. We only need to consider the case where $x > 0$. Equivalently, we need to solve the critical case where the graphs of the two functions $\log x$ and $b\,x^a$ intersect exactly once ($x > 0$), that is,

$$
\log x = b\,x^a, \quad \text{and} \quad \frac{1}{x} = a\,b\,x^{a-1},
$$

which implies $x = e^{1/a}$ and $b = (a e)^{-1}$. When b is bigger than this critical value, the function $b|x|^a$ will dominate $\log x$ for all $x > 0$. $\square$

Lemma 4.9. *Let $g(x) = e^{c|x|^a}$ with $c > 0$ and $a > 1$. For all $n > 0$, the following properties hold:*
(1) For all $x, z \in \mathbb{R}$, $0 \le t \le t' \le n$,

$$
\left|g\left(x-\sqrt{t}\,z\right) - g\left(x-\sqrt{t'}\,z\right)\right| \le a\,c\,\exp\left(c_1|x|^a + c_2|z|^a\right)\left|t'-t\right|^{1/2},
$$

where the two constants $c_1 := c_1(a,c)$ and $c_2 := c_2(n,a,c)$ can be chosen as follows:

$$
c_1(a,c) = \left(c + \frac{a-1}{a\,e}\right)2^{a-1}, \quad \textit{and} \quad c_2(n,a,c) = c_1(a,c)\,n^{a/2} + \frac{1}{a\,e}.
$$

(2) For all $x, x' \in [-n,n]$, $z \in \mathbb{R}$ and $t \in [0,n]$,

$$\left|g\left(x-\sqrt{t}\,z\right)-g\left(x'-\sqrt{t}\,z\right)\right| \le c_3\,\exp\left(c_4|z|^a\right)\,|x'-x|$$

where the two constants $c_3 := c_3(n,a,c)$ and $c_4 := c_4(n,a,c)$ can be chosen as follows:

$$c_3(n,a,c) := a\,c\,e^{c_1\,n^a}, \quad \textit{and} \quad c_4(n,a,c) = c_1\,n^{a/2}.$$

Proof. (1) Because $a > 1$, the function g belongs to $C^1(\mathbb{R})$, is convex and $g'(x) \ge 0$ for $x \ge 0$. Hence,

$$\left|g\left(x-\sqrt{t}\,z\right)-g\left(x-\sqrt{t'}\,z\right)\right| \le \left|g'\left(|x|+\sqrt{n}\,|z|\right)\right| \cdot \left|\sqrt{t'}\,z-\sqrt{t}\,z\right|.$$

Let $b = (a\,e)^{-1}$. By Lemma 4.8, $|g'(x)| = a\,c\,|x|^{a-1}e^{c|x|^a} \le a\,c\,e^{(c+(a-1)b)|x|^a}$. Thus

$$\left|g'\left(|x|+\sqrt{n}\,|z|\right)\right| \le a\,c\,e^{(c+(a-1)b)\left(|x|+\sqrt{n}\,|z|\right)^a} \le a\,c\,e^{c_1|x|^a+c_1n^{a/2}|z|^a}, \tag{4.25}$$

where we have applied the inequality $(x+y)^a \le 2^{a-1}(x^a+y^a)$ for all $x, y \ge 0$. Clearly,

$$|\sqrt{t'}-\sqrt{t}| \le \sqrt{t'-t}. \tag{4.26}$$

Finally, apply Lemma 4.8 to $|z|$, and combining all the above bounds proves (1).

(2) Similarly to (1),

$$\left|g\left(x-\sqrt{t}\,z\right)-g\left(x'-\sqrt{t}\,z\right)\right| \le \left|g'\left(|n|+\sqrt{n}\,|z|\right)\right| \cdot |x-x'|\,,$$

and by (4.25), $|g'\left(|n|+\sqrt{n}\,|z|\right)| \le a\,c\,e^{c_1n^a+c_1n^{a/2}|z|^a}$. This proves (2). ☐

For $c > 0$ and $a \in [0,2[$, define the constant

$$K_{a,c}(\nu t) := \left(e^{c|\cdot|^a} * G_\nu(t,\cdot)\right)(0)\,.$$

For $0 \le t \le n$, we have that

$$K_{a,c}(\nu t) = \int_{\mathbb{R}} \mathrm{d}y\; e^{c\left(\sqrt{t}\,|y|\right)^a} G_\nu(1,y) \le \int_{\mathbb{R}} \mathrm{d}y\; e^{c\left(\sqrt{n}\,|y|\right)^a} G_\nu(1,y) = K_{a,c}(\nu n). \tag{4.27}$$

Proof of Proposition 4.4 (2). Because $\mu \in \mathcal{M}_H^*(\mathbb{R})$, there are a function $f(x)$ and two constants $a \in [1,2[$ and $c > 0$ such that $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ and $c = \sup_{x\in\mathbb{R}} |f(x)|e^{-|x|^a} < +\infty$. In the following, we assume that $x, x' \in [-n,n]$, and $t, t' \in [0,n]$. Set $g(x) = e^{2^a|x|^a}$ and assume that $\varsigma = 0$. From (4.17),

$$I(t,x;t',x') \le c^2 \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \mathrm{d}z_1\mathrm{d}z_2\; e^{|z_1|^a+|z_2|^a} G_{2\nu}(s,z_1-z_2)$$

$$\times \int_{\mathbb{R}} \mathrm{d}y\, G_{\nu/2}\left(s, y-\bar{z}\right)\left(G_{\nu}\left(t-s, x-y\right)-G_{\nu}(t'-s, x'-y)\right)^{2}.$$

We shall apply the change of variables $z=\bar{z}$ and $w=\Delta z$: since

$$|z_1|^a+|z_2|^a=\left|z+\frac{w}{2}\right|^a+\left|z-\frac{w}{2}\right|^a\le 2^{a-1}\left(|z|^a+\left|\frac{w}{2}\right|^a\right)\times 2=2^a|z|^a+|w|^a,$$

we see that

$$e^{|z_1|^a+|z_2|^a}\le e^{2^a|z|^a+|w|^a}=e^{|w|^a}g(z),$$

and it follows that

$$\begin{aligned}I(t,x;t',x')\le c^2\int_0^t \mathrm{d}s\int_{\mathbb{R}}\mathrm{d}z\left(e^{|\cdot|^a}*G_{2\nu}(s,\cdot)\right)(0)g(z)&\\ \times\int_{\mathbb{R}}\mathrm{d}y\, G_{\nu/2}(s,y-z)\left(G_{\nu}(t-s,x-y)-G_{\nu}(t'-s,x'-y)\right)^2&\\ \le c^2\, K_{a,1}(2\nu n)\int_0^t\mathrm{d}s\int_{\mathbb{R}}\mathrm{d}z\, g(z)&\\ \times\int_{\mathbb{R}}\mathrm{d}y\, G_{\nu/2}(s,y-z)\left(G_{\nu}(t-s,x-y)-G_{\nu}(t'-s,x'-y)\right)^2,&\end{aligned}\tag{4.28}$$

where the second inequality is due to (4.27).

Property (4.3)**.** For the moment, we continue to assume that $\varsigma=0$. Set $x=x'$. Apply (4.18) with $x=x'$ and $\bar{z}$ replaced by z, integrate over $\mathrm{d}z$, and use (4.28) to see that,

$$\begin{aligned}I(t,x;t',x)\le c^2\, K_{a,1}(2\nu n)\int_0^t\mathrm{d}s\Bigg(&\frac{1}{\sqrt{4\pi\nu(t-s)}}\left(g*G_{\nu/2}(t,\cdot)\right)(x)\\ &+\frac{1}{\sqrt{4\pi\nu(t'-s)}}\left(g*G_{\nu/2}(t',\cdot)\right)(x)\\ &-\frac{2}{\sqrt{4\pi\nu\left(\frac{t+t'}{2}-s\right)}}\left(g*G_{\nu/2}\left(t+\frac{(t-s)h}{2(t-s)+h},\cdot\right)\right)(x)\Bigg)\\ \le c^2\, K_{a,1}(2\nu n)\int_0^t\mathrm{d}s\left(I_1+I_2+I_3\right),&\end{aligned}$$

where, letting $h=t'-t$,

$$\begin{aligned}I_1&=\left[(4\pi\nu(t-s))^{-\frac{1}{2}}+(4\pi\nu(t'-s))^{-\frac{1}{2}}-\left(\pi\nu\left(\frac{t+t'}{2}-s\right)\right)^{-\frac{1}{2}}\right]\left(g*G_{\nu/2}(t,\cdot)\right)(x),\\ I_2&=\frac{1}{\sqrt{4\pi\nu(t'-s)}}\left[\left(g*G_{\nu/2}(t',\cdot)\right)(x)-\left(g*G_{\nu/2}(t,\cdot)\right)(x)\right],\end{aligned}$$

$$I_3 = \frac{2}{\sqrt{4\pi\nu\left(\frac{t+t'}{2}-s\right)}}\left[\left(g * G_{\nu/2}\left(t+\frac{(t-s)h}{2(t-s)+h},\cdot\right)\right)(x) - \left(g * G_{\nu/2}(t,\cdot)\right)(x)\right].$$

Set $\bar{t} = \frac{t+t'}{2}$. By (4.26),

$$\begin{aligned}
\int_0^t I_1 \mathrm{d}s &= \frac{1}{\sqrt{\pi\nu}}\left(\sqrt{t}+\sqrt{t'}-\sqrt{h}-2\sqrt{\bar{t}}+2\sqrt{\bar{t}-t}\right)\left(g * G_{\nu/2}(t,\cdot)\right)(x) \\
&\le \frac{1}{\sqrt{\pi\nu}}\left(\left|\sqrt{t}-\sqrt{\bar{t}}\right|+\left|\sqrt{t'}-\sqrt{\bar{t}}\right|-\sqrt{h}+2\sqrt{\frac{h}{2}}\right)\left(g * G_{\nu/2}(t,\cdot)\right)(x) \\
&\le \frac{1}{\sqrt{\pi\nu}}\left(4\sqrt{\frac{h}{2}}-\sqrt{h}\right)\left(g * G_{\nu/2}(t,\cdot)\right)(x).
\end{aligned}$$

By Lemma 4.9, for some constants $c_i > 0$, $i = 1, 2$,

$$\begin{aligned}
|I_2| &\le \frac{1}{\sqrt{4\pi\nu(t'-s)}}\int_{\mathbb{R}} \mathrm{d}z \left|g\left(x-\sqrt{t}\,z\right)-g\left(x-\sqrt{t'}\,z\right)\right| G_{\nu/2}(1,z) \\
&\le \frac{1}{\sqrt{4\pi\nu(t-s)}} a\, 2^a e^{c_1|x|^a}\left(e^{c_2|\cdot|^a} * G_{\nu/2}(1,\cdot)\right)(0)\sqrt{h}\,.
\end{aligned}$$

Hence, for all $0 \le t \le t' \le n$ and $x \in [-n, n]$,

$$\int_0^t \mathrm{d}s\, |I_2| \le \frac{a\, 2^a\sqrt{n}}{\sqrt{\pi\nu}} e^{c_1|n|^a} K_{a,c_2}\left(\frac{\nu}{2}\right)\sqrt{h}\,.$$

Similarly, because $\frac{(t-s)h}{2(t-s)+h} \le \frac{h}{2}$, for all $0 \le s \le t \le t' \le n$ and $x \in [-n, n]$,

$$\int_0^t \mathrm{d}s\, |I_3| \le \frac{a\, 2^a\sqrt{n}}{\sqrt{2\pi\nu}} e^{c_1|n|^a} K_{a,c_2}\left(\frac{\nu}{2}\right)\sqrt{h}\,.$$

Therefore, for all $0 \le t \le t' \le n$ and $x \in [-n, n]$, $I(t, x; t', x) \le \widetilde{C}^*_{n,1}\sqrt{t'-t}$ with

$$\begin{aligned}
\widetilde{C}^*_{n,1} = \frac{c^2\, K_{a,1}(2\nu n)}{\sqrt{2\pi\nu}}\Bigg[&\left(\sqrt{2}+1\right) a\, 2^a\sqrt{n}\, e^{c_1|n|^a} K_{a,c_2}\left(\frac{\nu}{2}\right) \\
&+\left(4-\sqrt{2}\right)\sup_{(s,y)\in[0,n]\times[-n,n]}\left(e^{2^a|\cdot|^a} * G_{\nu/2}(s,\cdot)\right)(y)\Bigg].
\end{aligned}$$

Finally, as for (4.3), the contribution of the constant $\overline{\varsigma}$ can be calculated by using Proposition 5.2. Therefore, one can choose

$$C^*_{n,1} = \overline{\varsigma}^2\,\frac{\sqrt{2}-1}{\sqrt{\pi\nu}} + 2\,\widetilde{C}^*_{n,1}.$$

Property (4.4). Assume again that $\varsigma = 0$. Set $t = t'$ and $\bar{x} = \frac{x+x'}{2}$. Recalling (4.21), we see that the inequality (4.28) reduces to

$$
\begin{aligned}
I(t,x;t,x') \le c^2\, K_{a,1}(2\nu n) \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}z\, g(z) \Bigg\{ & \frac{1}{\sqrt{4\pi\nu(t-s)}} \left[G_{\nu/2}(t,x-z) + G_{\nu/2}(t,x'-z) \right] \\
& - 2\, G_{2\nu}(t-s,x-x')\, G_{\nu/2}(t,\bar{x}-z) \Bigg\}.
\end{aligned}
$$

Then integrate over $\mathrm{d}s$ using Lemma 5.6:

$$
\begin{aligned}
I(t,x;t,x') \le c^2\, K_{a,1}(2\nu n) \int_{\mathbb{R}} \mathrm{d}z\, g(z) \Bigg\{ & \frac{\sqrt{t}}{\sqrt{\pi\nu}} \left[G_{\nu/2}(t,x-z) + G_{\nu/2}(t,x'-z) \right] \\
& - 2 \left[2t\, G_{2\nu}(t,x-x') - \frac{1}{2\nu} |x-x'| \operatorname{erfc}\left(\frac{|x-x'|}{\sqrt{4\nu t}} \right) \right] G_{\nu/2}(t,\bar{x}-z) \Bigg\}.
\end{aligned}
$$

Denote $F(x) = \left(g * G_{\nu/2}(t,\cdot) \right)(x)$. Then integrating over $\mathrm{d}z$ gives

$$
\begin{aligned}
I(t,x;t,x') \le\; & c^2\, K_{a,1}(2\nu n) \Bigg\{ \frac{\sqrt{t}}{\sqrt{\pi\nu}} \left[F(x) + F(x') \right] \\
& - 2 \left[\frac{\sqrt{t}}{\sqrt{\pi\nu}} e^{-\frac{(x-x')^2}{4\nu t}} - \frac{1}{2\nu} |x-x'| \operatorname{erfc}\left(\frac{|x-x'|}{\sqrt{4\nu t}} \right) \right] F(\bar{x}) \Bigg\} \\
\le\; & c^2\, K_{a,1}(2\nu n) \Bigg\{ \frac{\sqrt{t}}{\sqrt{\pi\nu}} |F(x) - F(\bar{x})| + \frac{\sqrt{t}}{\sqrt{\pi\nu}} |F(x') - F(\bar{x})| \\
& + \frac{2\sqrt{t}}{\sqrt{\pi\nu}} \left(1 - e^{-\frac{|x-x'|^2}{4\nu t}} \right) F(\bar{x}) + \frac{1}{\nu} |x-x'|\, F(\bar{x}) \Bigg\}.
\end{aligned}
$$

Notice that $0 \le 1 - e^{-x^2/2} \le \widetilde{C}\, |x|$, where the universal constant $\widetilde{C}$ is given in Lemma 4.7. By part (2) of Lemma 4.9, for some constants c_i, $i = 3,4$,

$$
\begin{aligned}
|F(x) - F(\bar{x})| &\le \int_{\mathbb{R}} \mathrm{d}z \left| g\left(x - \sqrt{t}\, z\right) - g\left(\bar{x} - \sqrt{t}\, z\right) \right| G_{\nu/2}(1,z) \\
&\le c_3 \left(e^{c_4|\cdot|^a} * G_{\nu/2}(1,\cdot) \right)(0)\ |x - \bar{x}| \\
&= \frac{c_3}{2} K_{a,c_4}\left(\frac{\nu}{2} \right)\ |x - x'|.
\end{aligned}
$$

Similarly, $|F(x') - F(\bar{x})| \le \frac{c_3}{2} K_{a,c_4}\left(\frac{\nu}{2}\right)\ |x - x'|$. Hence,

$$
I(t,x;t,x') \le c^2\, K_{a,1}(2\nu n) \left\{ \frac{c_3\sqrt{n}}{\sqrt{\pi\nu}} K_{a,c_4}\left(\frac{\nu}{2} \right) + \left(\frac{\widetilde{C}\sqrt{2}}{\nu\sqrt{\pi}} + \frac{1}{\nu} \right) F(\bar{x}) \right\} |x - x'|\,.
$$

Therefore, for all $0 \le t \le n$ and $x, x' \in [-n, n]$, $I(t, x; t, x') \le \widetilde{C}^*_{n,3} |x - x'|$ with

$$\widetilde{C}^*_{n,3} = c^2\, K_{a,1}(2\nu n) \left\{ \frac{c_3 \sqrt{n}}{\sqrt{\pi\nu}} K_{a,c_4}\left(\frac{\nu}{2}\right) + \left(\frac{\widetilde{C}\,\sqrt{2}}{\nu\sqrt{\pi}} + \frac{1}{\nu}\right) \sup_{(s,y)\in[0,n]\times[-n,n]} \left(g * G_{\nu/2}(s,\cdot)\right)(y) \right\},$$

and $\widetilde{C}^*_{n,3} < +\infty$ by definition of g. Finally, the contribution of the constant $\overline{\varsigma}$ in (4.4) is given in Proposition 5.2. Therefore, one can choose

$$C^*_{n,3} = \frac{\overline{\varsigma}^2}{\nu} + 2\, \widetilde{C}^*_{n,3}\,.$$

Property (4.5)**.** As for (4.5), notice that $J^*_0(t, x) \le c \left(e^{|\cdot|^a} * G_\nu(t, \cdot)\right)(x)$. By checking the proof of part (1) (see (4.23)), one can choose,

$$C^*_{n,5} = \frac{\overline{\varsigma}^2}{\sqrt{\pi\nu}} + 2\, c^2 \sqrt{\pi/\nu} \sup_{(s,y)\in[0,n]\times[-n,n]} \left(e^{|\cdot|^a} * G_\nu(2s, \cdot)\right)^2 (y).$$

This completes the proof of part (2) of Proposition 4.4. $\square$

Proof of Proposition 4.5 (2). If $\mu \in \mathcal{M}^*_H(\mathbb{R})$, then by Proposition 4.4 (2), the l.h.s. of (4.7) is bounded by

$$C^*_{n,1}\sqrt{t'-t}\left(1 \star G^2_\nu\right)(t, x) = C^*_{n,1}\frac{\sqrt{t}}{\sqrt{\pi\nu}}\sqrt{t'-t} \le C^*_{n,1}\frac{\sqrt{n}}{\sqrt{\pi\nu}}\sqrt{t'-t}.$$

Hence, $C^*_{n,2} = \frac{\sqrt{n}}{\sqrt{\pi\nu}} C^*_{n,1}$. The same arguments apply to the other two constants $C^*_{n,4}$ and $C^*_{n,6}$, i.e., (4.8) and (4.9). Note that it was not possible to use the above argument in the proof of part (1) of Proposition 4.4. This completes the proof of Proposition 4.5 (2). $\square$

4.5 Checking the initial condition

Proof of Proposition 3.4. Because $u(t, x) = J_0(t, x) + I(t, x)$, and because it is standard that (see [14, Chapter 7, Section 6] and also [4, Lemma 2.6.14, p.89]),

$$\lim_{t\to 0_+} \int_{\mathbb{R}} \mathrm{d}x\, J_0(t, x)\phi(x) = \int_{\mathbb{R}} \mu(\mathrm{d}x)\, \phi(x),$$

we only need to prove that

$$\lim_{t\to 0_+} \int_{\mathbb{R}} \mathrm{d}x\, I(t, x)\phi(x) = 0 \quad \text{in } L^2(\Omega).$$

Recall that the Lipschitz continuity of ρ implies the linear growth condition (2.1). Fix $\phi \in C_c^\infty(\mathbb{R})$. Denote $L(t) := \int_{\mathbb{R}} I(t,x)\phi(x)\mathrm{d}x$. By the stochastic Fubini theorem (see [28, Theorem 2.6, p. 296]), whose assumptions are easily checked,

$$L(t) = \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathrm{d}x \, G_\nu(t-s, x-y)\phi(x) \right) \rho(u(s,y)) W(\mathrm{d}s, \mathrm{d}y).$$

Hence, by (2.1),

$$\mathbb{E}\left[L(t)^2\right] \le L_\rho^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \left(\int_{\mathbb{R}} \mathrm{d}x \, G_\nu(t-s,x-y)\phi(x) \right)^2 \left(\bar{\varsigma}^2 + ||u(s,y)||_2^2 \right).$$

By the moment formula (2.5), we can write the above upper bound as

$$\mathbb{E}\left[L(t)^2\right] \le L_\rho^2 \left[L_1(t) + L_2(t) + L_3(t) + L_4(t)\right],$$

where

$$\begin{aligned}
L_1(t) &= \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \left(\int_{\mathbb{R}} \mathrm{d}x \, G_\nu(t-s,x-y)\phi(x) \right)^2 J_0^2(s,y), \\
L_2(t) &= \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \left(\int_{\mathbb{R}} \mathrm{d}x \, G_\nu(t-s,x-y)\phi(x) \right)^2 \left(J_0^2 \star \overline{\mathcal{K}} \right)(s,y), \\
L_3(t) &= \bar{\varsigma}^2 \int_0^t \mathrm{d}s \, \overline{\mathcal{H}}(s) \int_{\mathbb{R}} \mathrm{d}y \left(\int_{\mathbb{R}} \mathrm{d}x \, G_\nu(t-s,x-y)\phi(x) \right)^2, \\
L_4(t) &= \bar{\varsigma}^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \left(\int_{\mathbb{R}} \mathrm{d}x \, G_\nu(t-s,x-y)\phi(x) \right)^2.
\end{aligned}$$

From now on, we may assume that $\mu \in \mathcal{M}_{H,+}(\mathbb{R})$, because otherwise, one can simply replace the above $J_0(s,y)$ by $J_0^*(s,y) = \left(|\mu| * G_\nu(s,\circ)\right)(y)$.

(1) Consider $L_1(t)$ first. Write out both $J_0^2(s,y)$ and $\left(\int_{\mathbb{R}} \mathrm{d}x \, G_\nu(t-s,x-y)\phi(x)\right)^2$ in the forms of double integrals, and apply Lemma 5.3, to see that

$$\begin{aligned}
L_1(t) = \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y & \left(\iint_{\mathbb{R}^2} \mathrm{d}x_1 \mathrm{d}x_2 \, G_{\nu/2}(t-s, \bar{x}-y) G_{2\nu}(t-s, \Delta x) \phi(x_1)\phi(x_2) \right) \\
& \times \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2) \, G_{\nu/2}(s, \bar{z}-y) G_{2\nu}(s, \Delta z),
\end{aligned} \tag{4.29}$$

where $\bar{x} = \frac{x_1+x_2}{2}$, $\Delta x = x_1 - x_2$ and similarly for $\bar{z}$ and Δz. Integrate over $\mathrm{d}y$ first using the semigroup property of the heat kernel and then integrate over $\mathrm{d}s$ by using Lemma 5.5, we see that

$$L_1(t) = \iint_{\mathbb{R}^2} \mathrm{d}x_1 \mathrm{d}x_2 \, \phi(x_1)\phi(x_2) \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2) \, G_{\nu/2}(t, \bar{x}-\bar{z}) \frac{1}{4\nu} \mathrm{erfc}\left(\frac{1}{\sqrt{4\nu t}} \left[|\Delta x| + |\Delta z|\right] \right).$$

By (5.2),

$$\mathrm{erfc}\left(\frac{1}{\sqrt{4\nu t}} \left[|\Delta x| + |\Delta z|\right] \right) \le e^{-\frac{(|\Delta x| + |\Delta z|)^2}{4\nu t}} \le e^{-\frac{|\Delta x|^2}{4\nu t}} e^{-\frac{|\Delta z|^2}{4\nu t}} = 4\pi\nu\sqrt{t} \, G_{2\nu}\left(1, \frac{\Delta x}{\sqrt{t}}\right) G_{2\nu}(t, \Delta z).$$

By the change of variables $y = (x_1 + x_2)/2$ and $w = (x_1 - x_2)/\sqrt{t}$,

$$L_1(t) \leq \pi\, t \iint_{\mathbb{R}^2} \mathrm{d}y\mathrm{d}w\; G_{2\nu}(1,w)\phi\left(y+\frac{\sqrt{t}}{2}w\right)\phi\left(y-\frac{\sqrt{t}}{2}w\right)$$
$$\times \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\; G_{\nu/2}(t, y-\bar{z})G_{2\nu}(t,\Delta z).$$

By Lemma 5.4,

$$\iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\; G_{\nu/2}(t, y-\bar{z})G_{2\nu}(t,\Delta z) \leq 2\left(\mu * G_{2\nu}(t,\cdot)\right)^2(y) = 2J_0^2(2t,y).$$

For some constants a and $c \geq 0$, $|\phi(x)| \leq c\, 1_{[-a,a]}(x)$. If $|y| > a$, then the two sets $\left\{w\in\mathbb{R} : \left|\frac{\sqrt{t}}{2}w \pm y\right| \leq a\right\}$ have empty intersection. Hence,

$$L_1(t) \leq 2c^2\pi \int_{|y|\leq a} \mathrm{d}y\, t\, J_0^2(2t,y) \int_{\mathbb{R}} \mathrm{d}w\; G_{2\nu}(1,w) = 2c^2\pi \int_{|y|\leq a} \mathrm{d}y\, t\, J_0^2(2t,y).$$

Clearly, by assuming that $t \leq 1$,

$$\sqrt{t}\, J_0(2t,y) = \int_{\mathbb{R}} \mu(\mathrm{d}x)\frac{1}{\sqrt{4\pi\nu}} e^{-\frac{(y-x)^2}{4\nu t}} \leq \int_{\mathbb{R}} \mu(\mathrm{d}x)\frac{1}{\sqrt{4\pi\nu}} e^{-\frac{(y-x)^2}{4\nu}} = J_0(2,y).$$

Hence, Lebesgue's dominated convergence theorem implies that

$$\lim_{t\to 0} \sqrt{t}\, J_0(2t,y) = 0.$$

Because $\int_{|y|\leq a} \mathrm{d}y J_0^2(2,y) < +\infty$, by another application of Lebesgue's dominated convergence theorem, we see that $\lim_{t\to 0} L_1(t) = 0$.

(2) As for $L_2(t)$, because $\overline{\mathcal{K}}(t,x) \leq G_{\nu/2}(t,x)\frac{1}{\sqrt{t}}h(t)$, where $h(t) := L_\rho^2\left(\frac{1}{\sqrt{4\pi\nu}} + \frac{L_\rho^2\sqrt{t}}{2\nu}e^{\frac{L_\rho^4 t}{4\nu}}\right)$ is a nondecreasing function in t, we see that as in (4.29),

$$L_2(t) \leq \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \iint_{\mathbb{R}} \mathrm{d}x_1\mathrm{d}x_2\; G_{\nu/2}(t-s, \bar{x}-y)G_{2\nu}(t-s,\Delta x)\phi(x_1)\phi(x_2)$$
$$\times \int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}w\; G_{\nu/2}(s-r, y-w)\frac{1}{\sqrt{s-r}}h(t)$$
$$\times \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\; G_{\nu/2}(r, \bar{z}-w)G_{2\nu}(r,\Delta z).$$

Integrate first over $\mathrm{d}w$ using the semigroup property of the heat kernel, and then integrate over $\mathrm{d}r$ using (5.1), to find that

$$L_2(t) \leq \pi h(t)\sqrt{t} \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \iint_{\mathbb{R}} \mathrm{d}x_1\mathrm{d}x_2\; G_{\nu/2}(t-s,\bar{x}-y)\; G_{2\nu}(t-s,\Delta x)\phi(x_1)\phi(x_2)$$

$$\times \iint_{\mathbb{R}^2} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2)\, G_{\nu/2}(s,\bar{z}-y)G_{2\nu}(s,\Delta z).$$

Comparing the above bound with (4.29), we see that

$$L_2(t) \le \pi\sqrt{t}\, h(t) L_1(t) \to 0, \quad \text{as } t \to 0.$$

(3) Notice that $L_3(t) \le \overline{\mathcal{H}}(t)\, L_4(t)$, so we only need to consider $L_4(t)$, which is a special case of $L_1(t)$ with $\mu(\mathrm{d}x) = \varsigma\,\mathrm{d}x$. Since this μ belongs to $\mathcal{M}_H(\mathbb{R})$, $\lim_{t\to 0_+} L_4(t) = 0$ by part (1). This completes the proof of Proposition 3.4. $\square$

5 Appendix

Lemma 5.1. *If $|f(x)| \le c_1 e^{c_2|x|^a}$ for all $x \in \mathbb{R}$ with $c_1, c_2 > 0$ and $a \in]1,2[$, then there is $c_3 < +\infty$ such that for all $b \in]a,2[$, $|f(x)| \le c_3 e^{|x|^b}$ for all $x \in \mathbb{R}$.*

Proof. Notice that $c_2|x|^a \ge |x|^b$ if and only if $|x| \le c_2^{\frac{1}{b-a}}$. Hence, $c_2|x|^a - |x|^b \le c_2\, c_2^{\frac{a}{b-a}} - 0 = c_2^{\frac{b}{b-a}}$. Therefore, $c_1 \exp\left(c_2|x|^a - |x|^b\right) \le c_1 \exp(c_2^{\frac{b}{b-a}}) =: c_3$. $\square$

Proposition 5.2 (Proposition 3.5 of [5])**.** *There are three universal and optimal constants $C_1 = 1$, $C_2 = \frac{\sqrt{2}-1}{\sqrt{\pi}}$, and $C_3 = \frac{1}{\sqrt{\pi}}$, such that for all s, t with $0 \le s \le t$ and $x \in \mathbb{R}$,*

$$\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \left[G_\nu(t-r,x-z) - G_\nu(t-r,y-z)\right]^2 \le \frac{C_1}{\nu}|x-y|\ ,$$

$$\int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \left[G_\nu(t-r,x-z) - G_\nu(s-r,x-z)\right]^2 \le \frac{C_2}{\sqrt{\nu}}\sqrt{t-s}\ ,$$

$$\int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \left[G_\nu(t-r,x-z)\right]^2 \le \frac{C_3}{\sqrt{\nu}}\sqrt{t-s}\ ,$$

$$\iint_{\mathbb{R}_+\times\mathbb{R}} \left(G_\nu(t-r,x-z) - G_\nu(s-r,y-z)\right)^2 \mathrm{d}r\mathrm{d}z \le 2C_1\left(\frac{|x-y|}{\nu} + \frac{\sqrt{|t-s|}}{\sqrt{\nu}}\right)\ ,$$

where we use the convention that $G_\nu(t,\cdot) \equiv 0$ if $t \le 0$.

Lemma 5.3 (Lemma 5.4 of [5])**.** *For all t, $s > 0$ and x, $y \in \mathbb{R}$, we have that $G_\nu^2(t,x) = \frac{1}{\sqrt{4\pi\nu t}} G_{\nu/2}(t,x)$ and $G_\nu(t,x)G_\nu(s,y) = G_\nu\left(\frac{ts}{t+s}, \frac{sx+ty}{t+s}\right) G_\nu(t+s, x-y)$.*

Lemma 5.4 (Lemma 5.5 of [5])**.** *For all x, z_1 $z_2 \in \mathbb{R}$ and $t, s > 0$, denote $\bar{z} = \frac{z_1+z_2}{2}$, $\Delta z = z_1 - z_2$. Then $G_1(t, x-\bar{z})\, G_1(s,\Delta z) \le \frac{(4t)\vee s}{\sqrt{ts}} G_1((4t)\vee s, x-z_1)\, G_1((4t)\vee s, x-z_2)$, where $a \vee b := \max(a,b)$.*

Lemma 5.5 (Lemma 5.10 of [5])**.** *For $0 \le s \le t$ and x, $y \in \mathbb{R}$, we have that*

$$\int_0^t \mathrm{d}s\, G_\nu(s,x) G_\sigma(t-s,y) = \frac{1}{2\sqrt{\nu\sigma}} \mathrm{erfc}\left(\frac{1}{\sqrt{2t}}\left(\frac{|x|}{\sqrt{\nu}} + \frac{|y|}{\sqrt{\sigma}}\right)\right),$$

where ν and σ are strictly positive. In particular, by letting $x = 0$, we have that

$$\int_0^t \mathrm{d}s\, \frac{G_\sigma(t-s,y)}{\sqrt{2\pi\nu s}} = \frac{1}{2\sqrt{\nu\sigma}} \mathrm{erfc}\left(\frac{|y|}{\sqrt{2\sigma t}}\right) \le \frac{\sqrt{\pi t}}{\sqrt{2\nu}} G_\sigma(t,y)\,. \tag{5.1}$$

Note that the inequality in (5.1) is because by [19, (7.7.1), p.162],

$$\mathrm{erfc}(x) = \frac{2}{\pi} e^{-x^2} \int_0^\infty \mathrm{d}t\, \frac{e^{-x^2t^2}}{1+t^2} \le \frac{2}{\pi} e^{-x^2} \int_0^\infty \mathrm{d}t\, \frac{1}{1+t^2} = e^{-x^2}\,, \tag{5.2}$$

Lemma 5.6. *For $t > 0$, $\nu > 0$ and $x \in \mathbb{R}$, we have that*

$$\int_0^t \mathrm{d}s\, G_\nu(s,x) = 2t\, G_\nu(t,x) - \frac{|x|}{\nu} \mathrm{erfc}\left(\frac{|x|}{\sqrt{2\nu t}}\right).$$

Proof. The case where $x = 0$ can be easily verified. Assume that $x \neq 0$. By change of variables $y = |x|/\sqrt{2\nu s}$ and integration by parts, we have that

$$\int_0^t \mathrm{d}s\, G_\nu(s,x) = \int_{\frac{|x|}{\sqrt{2\nu t}}}^{+\infty} \mathrm{d}y\, \frac{|x|}{\sqrt{\pi}\,\nu\, y^2}\, e^{-y^2} = \frac{|x|}{\sqrt{\pi}\,\nu\, y}\, e^{-y^2}\Big|_{+\infty}^{\frac{|x|}{\sqrt{2\nu t}}} - \frac{|x|}{\nu} \int_{\frac{|x|}{\sqrt{2\nu t}}}^{+\infty} \mathrm{d}y\, \frac{2}{\sqrt{\pi}}\, e^{-y^2}\,.$$

Therefore, the conclusion follows from the definition of the function $\mathrm{erfc}(\cdot)$. ☐

Lemma 5.7. *If $\nu > 0$, $t > 0$, $n > 1$ and $x \in \mathbb{R}$, then for $r \in [0, n^2 t]$,*

$$\left|\frac{G_{\nu/2}(t+r,x)}{G_{\nu/2}(t,x)} - 1\right| \le \frac{3r}{t+r} \exp\left(\frac{n^2x^2}{\nu t\,(1+n^2)}\right) \le \frac{3}{2}\frac{\sqrt{r(1+n^2)}}{\sqrt{t}} G_{\frac{\nu}{2}}^{-1}(t,x)\, G_{\frac{\nu(1+n^2)}{2}}(t,x).$$

Proof. Fix $\nu > 0$, $t > 0$, $n > 1$, and $x \in \mathbb{R}$. For $r \in [0, n^2t]$, define

$$g_{t,x}(r) = \frac{G_{\nu/2}(t+r,x)}{G_{\nu/2}(t,x)} - 1 = \frac{\sqrt{t}}{\sqrt{t+r}} \exp\left(\frac{x^2}{\nu t}\frac{r}{t+r}\right) - 1\,.$$

Clearly $g_{t,x}(0) = 0$. Notice that

$$|g_{t,x}(r)| \le \left|\exp\left(\frac{x^2}{\nu t}\ \frac{r}{t+r}\right) - 1\right| + \exp\left(\frac{x^2}{\nu t}\ \frac{r}{t+r}\right)\left|\frac{\sqrt{t}}{\sqrt{t+r}} - 1\right|.$$

The second term on the right-hand side is bounded by $\exp\left(\frac{n^2x^2}{\nu(1+n^2)t}\right)\frac{r}{t+r}$ for all $r \in [0, n^2t]$, because $\frac{r}{r+t} \in \left[0, \frac{n^2}{1+n^2}\right]$ for $r \in [0, n^2t]$. To bound the first term, we use the fact that for fixed $a > 0$ and $b > 0$, $0 \le e^{ah} - 1 \le e^{ab}\frac{h}{b}$ for all $h \in [0, b]$. Apply this fact to $\exp\left(\frac{x^2}{\nu t}\ \frac{r}{t+r}\right) - 1$ with $a = \frac{x^2}{\nu t}$, $h = \frac{r}{r+t}$ and $b = \frac{n^2}{1+n^2}$: the first term is bounded by $2\exp\left(\frac{n^2x^2}{\nu t(1+n^2)}\right)\frac{r}{r+t}$ for all $r \in [0, n^2t]$. Adding these two bounds proves the first inequality. The second one follows from the inequality $t + r \ge 2\sqrt{tr}$. ☐

Lemma 5.8. $\int_t^{t'} \frac{1}{\sqrt{s(t'-s)}} \mathrm{d}s = 2 \arcsin\left(\sqrt{\frac{t'-t}{t'}}\right)$ *for all* $t' > 0$ *with* $t' \geq t \geq 0$.

Proof. For $t = 0$, the l.h.s. reduces to the Beta integral (see, e.g., (3.3). If $t \in]0, t']$, differentiate with respect to t on both sides. $\square$

References

[1] V. Bally, A. Millet and M. Sanz-Solé. Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equations. *Ann. Probab.*, 23(1):178–222, 1995.

[2] L. Bertini and N. Cancrini. The stochastic heat equation: Feynman-Kac formula and intermittence. *J. Statist. Phys.*, 78(5-6):1377–1401, 1995.

[3] Z. Brzeźniak. On stochastic convolution in Banach spaces and applications. *Stochastics Stochastic Rep.* 61(3-4):245–295, 1997.

[4] L. Chen. *Moments, intermittency, and growth indices for nonlinear stochastic PDE's with rough initial conditions.* PhD thesis, No. 5712, École Polytechnique Fédérale de Lausanne, 2013.

[5] L. Chen and R. C. Dalang. Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *Preprint, arXiv:1307.0600*, 2013.

[6] D. Conus, M. Joseph, D. Khoshnevisan, and S.-Y. Shiu. Initial measures for the stochastic heat equation. *Ann. Inst. Henri Poincaré Probab. Stat.*, to appear, 2013.

[7] D. Conus and D. Khoshnevisan. Weak nonmild solutions to some SPDEs. *Illinois Journal of Mathematics*, Vol. 54, No. 4, 2010.

[8] D. Conus and D. Khoshnevisan. On the existence and position of the farthest peaks of a family of stochastic heat and wave equations. *Probab. Theory Related Fields*, 152(3-4):681–701, 2012.

[9] R. C. Dalang, D. Khoshnevisan, C. Mueller, D. Nualart, and Y. Xiao. *A minicourse on stochastic partial differential equations.* Edited by D. Khoshnevisan and F. Rassoul-Agha, Springer-Verlag, Berlin, 2009.

[10] R. C. Dalang. The stochastic wave equation. In: *A minicourse on stochastic partial differential equations*, pp. 39–71. Springer, Berlin, 2009.

[11] R. C. Dalang, D. Khoshnevisan, and E. Nualart. Hitting probabilities for systems of non-linear stochastic heat equations with additive noise. *ALEA Lat. Am. J. Probab. Math. Stat.*, 3:231–271, 2007.

[12] R. C. Dalang, D. Khoshnevisan, and E. Nualart. Hitting probabilities for systems for non-linear stochastic heat equations with multiplicative noise. *Probab. Theory Related Fields*, 144(3-4):371–427, 2009.

[13] M. Foondun and D. Khoshnevisan. Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron. J. Probab.*, 14:no. 21, 548–568, 2009.

[14] A. Friedman. *Generalized functions and partial differential equations* Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

[15] I. M. Gel'fand and N. Y. Vilenkin. *Generalized functions. Vol. 4: Applications of harmonic analysis.* Academic Press, New York, 1964.

[16] D. Khoshnevisan. A primer on stochastic partial differential equations. In: *A minicourse on stochastic partial differential equations*, pp. 5–42. Springer, Berlin, 2009.

[17] H. Kunita. *Stochastic flows and stochastic differential equations* . Cambridge University Press, Cambridge, 1990.

[18] H.-H. Kuo. *Introduction to stochastic integration.* Springer, New York, 2006.

[19] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds). *NIST Handbook of mathematical functions.* U.S. Department of Commerce National Institute of Standards and Technology, Washington, DC, 2010.

[20] S. Peszat and J. Seidler. Maximal inequalities and space-time regularity of stochastic convolutions. *Mathematica Bohemica* 123(1): 7-32, 1998.

[21] J. Pospíšil and R. Tribe. Parameter estimates and exact variations for stochastic heat equations driven by space-time white noise. *Stoch. Anal. Appl.*, 25(3):593–611, 2007.

[22] D. Revuz and M. Yor. *Continuous martingales and Brownian motion* (3^{nd} ed.). Springer-Verlag, Berlin, 1999.

[23] M. Sanz-Solé and M. Sarrà. Path properties of a class of Gaussian processes with applications to spde's. In: *Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999)* , pp. 303–316. Amer. Math. Soc., Providence, RI, 2000.

[24] M. Sanz-Solé and M. Sarrà. Hölder continuity for the stochastic heat equation with spatially correlated noise. In: *Seminar on Stochastic Analysis, Random Fields and Applications, III* , pp. 259–268. Birkhäuser, Basel, 2002. (R. C. Dalang, M. Dozzi and F. Russo, eds).

[25] J. Seidler. Da Prato-Zabczyk's maximal inequality revisited. I. *Mathematica Bohemica* 118(1):67–106, 1993.

[26] T. Shiga. Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. *Canad. J. Math.*, 46(2):415–437, 1994.

[27] E. M. Stein. *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton, N.J., 1970.

[28] J. B. Walsh. An introduction to stochastic partial differential equations. In: *École d'été de probabilités de Saint-Flour, XIV—1984* , pp. 265–439. Springer, Berlin, 1986.