Moment bounds and asymptotics for the stochastic wave equation

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Abstract: We consider the stochastic wave equation on the real line driven by space-time white noise and with irregular initial data. We give bounds on higher moments and, for the hyperbolic Anderson model, explicit formulas for second moments. These bounds imply weak intermittency and allow us to obtain sharp bounds on growth indices for certain classes of initial conditions with unbounded support.

MSC 2010 subject classifications: Primary 60H15. Secondary 60G60, 35R60.

Keywords: nonlinear stochastic wave equation, hyperbolic Anderson model, intermittency, growth indices.

1 Introduction

In this paper, we will study the following stochastic wave equation:

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \kappa^2 \frac{\partial^2}{\partial x^2}\right) u(t,x) = \rho(u(t,x))\,\dot{W}(t,x), & x \in \mathbb{R},\ t \in \mathbb{R}_+^*, \\ u(0,\circ) = g(\circ), \quad \frac{\partial u}{\partial t}(0,\circ) = \mu(\circ), \end{cases} \tag{1.1}$$

where $\mathbb{R}_+^* = \,]0,\infty[$, $\dot{W}$ is space-time white noise, $\rho(u)$ is globally Lipschitz, $\kappa > 0$ is the speed of wave propagation, g and μ are the (deterministic) initial position and velocity, respectively, and $\circ$ denotes the spatial dummy variable. The linear case, $\rho(u) = \lambda u$, $\lambda \neq 0$, is called *the hyperbolic Anderson model* [19]. If $\rho(u) = \lambda\sqrt{\varsigma^2 + u^2}$, then we call (1.1) the *near-linear Anderson model.*

*Research partially supported by the Swiss National Foundation for Scientific Research.

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This equation has been extensively studied during last two decades by many authors: see e.g., [4, 6, 7, 32, 37] for some early work, [15, 16, 37] for an introduction, [19, 20] for asymptotic properties of moments, [12, 17, 18, 21, 26, 27, 28, 33, 34, 35] for the stochastic wave equation in the spatial domain $\mathbb{R}^d$, $d > 1$, [22, 36] for regularity of the solution, [2, 3] for the stochastic wave equation with values in Riemannian manifolds, and [11, 30, 31] for wave equations with polynomial nonlinearities.

In this paper, we consider initial data with very little regularity. In particular, we assume that the initial position g belongs to $L^2_{loc}(\mathbb{R})$, the set of locally square integrable Borel functions, and the initial velocity μ belongs to $\mathcal{M}(\mathbb{R})$, the set of locally finite Borel measures. Denote the solution to the homogeneous equation by

$$J_0(t,x) := \frac{1}{2}\left(g(x+\kappa t)+g(x-\kappa t)\right) + \left(\mu * G_\kappa(t,\circ)\right)(x) \,, \tag{1.2}$$

where

$$G_\kappa(t,x) = \frac{1}{2}H(t)1_{[-\kappa t,\kappa t]}(x)$$

is the wave kernel function. Here, $H(t)$ is the Heaviside function (i.e., $H(t) = 1$ if $t \ge 0$ and 0 otherwise), and "$*$" denotes convolution in the space variable. Regarding the stochastic pde (spde) (1.1), we interpret it in the integral (mild) form:

$$u(t,x) = J_0(t,x) + I(t,x),$$

where

$$I(t,x) := \iint_{[0,t]\times\mathbb{R}} G_\kappa\left(t-s,x-y\right)\rho\left(u\left(s,y\right)\right)W\left(\mathrm{d}s,\mathrm{d}y\right).$$

We call $I(t,x)$ the *stochastic integral part* of the random field solution. This stochastic integral is interpreted in the sense of Walsh [37].

The first contribution of this paper concerns estimates and exact formulas for moments of the random field solution to (1.1) (for the stochastic *heat* equation, this type of result has recently been obtained in [9]). Consider for instance the case where $\rho(u)^2 = \lambda^2(\varsigma^2 + u^2)$ for some λ and $\varsigma \in \mathbb{R}$, and let $I_n(\cdot)$ be the modified Bessel function of the first kind of order n, or simply the *hyperbolic Bessel function* ([29, 10.25.2, p. 249]):

$$I_n(x) := \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(x^2/4\right)^k}{k!\,\Gamma(n+k+1)} \,, \tag{1.3}$$

(see [25, 38] for its relation with the wave equation). Define two kernel functions $\mathcal{K}(t,x) := \mathcal{K}(t,x;\kappa,\lambda)$ and $\mathcal{H}(t) := \mathcal{H}(t;\kappa,\lambda)$ as follows:

$$\mathcal{K}\left(t,x;\,\kappa,\,\lambda\right) := \begin{cases} \dfrac{\lambda^2}{4} I_0\left(\sqrt{\dfrac{\lambda^2\left((\kappa t)^2-x^2\right)}{2\kappa}}\right) & \text{if } -\kappa t \le x \le \kappa t\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{1.4}$$

and

$$\mathcal{H}(t;\ \kappa,\lambda) := (1\star\mathcal{K})(t,x) = \cosh\left(|\lambda|\sqrt{\kappa/2}\,t\right) - 1\,, \tag{1.5}$$

where "$\star$" denotes the convolution in both space and time variables (note that the second equality in (1.5) is proved in Lemma 3.7 below). For $t'\ge t\ge 0$ and $x,x'\in\mathbb{R}$, define two functions

$$T_\kappa(t,t',x) := \left[\left(\frac{t+t'}{2}-\frac{|x|}{2\kappa}\right)\wedge t\right]\ 1_{\{|x|\le\kappa(t+t')\}}\,, \tag{1.6}$$

$$X_\kappa(x,x',t) := \begin{cases}[(x+x')/2-\kappa t/2]\vee x, & \text{if } x\le x',\\ [(x+x')/2+\kappa t/2]\wedge x, & \text{if } x>x',\end{cases} \tag{1.7}$$

where $x\vee y:=\max(x,y)$ and $x\wedge y:=\min(x,y)$. Clearly, $T_\kappa(t,t,0)=t$ and $X_\kappa(x,x,0)=x$. Our Theorem 2.3 yields in particular the exact formulas in the next two corollaries.

Corollary 1.1 (Constant initial data)**.** *Suppose that $\rho^2(x)=\lambda^2(\varsigma^2+x^2)$ with $\lambda\neq 0$. Let $\mathcal{H}(t)$ be defined as above. If $g(x)\equiv w$ and $\mu(\mathrm{d}x)=\widetilde{w}\,\mathrm{d}x$ with $w,\widetilde{w}\in\mathbb{R}$, then for all $t'\ge t\ge 0$ and x, $x'\in\mathbb{R}$, setting $T=T_\kappa(t,t',x-x')$,*

$$\begin{aligned}\mathbb{E}\left[u(t,x)u(t',x')\right] =& -\varsigma^2-\frac{4\kappa\widetilde{w}^2}{\lambda^2}+(w+\kappa\widetilde{w}t)(w+\kappa\widetilde{w}t')-(w+\kappa\widetilde{w}T)^2\\ &+\left(w^2+\varsigma^2+\frac{4\kappa\widetilde{w}^2}{\lambda^2}\right)\cosh\left(\frac{\sqrt{\kappa}|\lambda|}{\sqrt{2}}T\right)+\frac{2\sqrt{2\kappa}w\widetilde{w}}{|\lambda|}\sinh\left(\frac{\sqrt{\kappa}|\lambda|}{\sqrt{2}}T\right).\end{aligned} \tag{1.8}$$

Corollary 1.2 (Dirac delta initial velocity)**.** *Suppose that $\rho^2(x)=\lambda^2(\varsigma^2+x^2)$ with $\lambda\neq 0$. Let $\mathcal{H}(t)$ and $\mathcal{K}(t,x)$ be defined as above. If $g\equiv 0$ and $\mu=\delta_0$, then for all $t'\ge t\ge 0$ and x, $x'\in\mathbb{R}$, setting $T=T_\kappa(t,t',x-x')$ and $X=X_\kappa(x,x',t'-t)$,*

$$\mathbb{E}\left[u(t,x)u(t',x')\right]=\lambda^{-2}\,\mathcal{K}(T,X)+\varsigma^2\,\mathcal{H}(T)\,.$$

In particular, $||u(t,x)||_2^2=\lambda^{-2}\,\mathcal{K}(t,x)+\varsigma^2\,\mathcal{H}(t)$.

These two corollaries are proved in Section 3.4. With our moment formulas, it becomes possible to study very precisely two asymptotic properties of the stochastic wave equation. The first one is the mathematical *intermittency* property, which is defined, as in [5], via the moment Lyapunov exponents. Recall that the *upper and lower moment Lyapunov exponents* for constant initial data are defined as follows:

$$\overline{m}_p(x):=\limsup_{t\to+\infty}\frac{\log\mathbb{E}\left[|u(t,x)|^p\right]}{t},\qquad \underline{m}_p(x):=\liminf_{t\to+\infty}\frac{\log\mathbb{E}\left[|u(t,x)|^p\right]}{t}.$$

If the initial conditions are constants, then $\overline{m}_p(x)=:\overline{m}_p$ and $\underline{m}_p(x)=:\underline{m}_p$ do not depend on x. Mathematical intermittency is the property that $\underline{m}_p=\overline{m}_p=:m_p$ and $m_1<m_2/2<$

$\cdots < m_p/p < \cdots$. It is implied by the property that $\underline{m}_2 > 0$ and $m_1 = 0$ (see [5, Definition III.1.1, p. 55]), which is called *full intermittency*, while *weak intermittency*, defined in [24] and [13, Theorem 2.3] is the property $\overline{m}_2 > 0$ and $\overline{m}_p < +\infty$, for all $p \geq 2$.

Dalang and Mueller showed in [19] that for the wave equation in spatial domain $\mathbb{R}^3$ with spatially homogeneous colored noise, with $\rho(u) = u$ and constant initial position and velocity, $\overline{m}_p$ and $\underline{m}_p$ are both bounded, from above and below respectively, by some constant times $p^{4/3}$. For the stochastic wave equation in spatial dimension 1 with space-time white noise, Conus *et al* [13] show that if the initial position and velocity are bounded and measurable functions, then the moment Lyapunov exponents satisfy $\overline{m}_p \leq Cp^{3/2}$ for $p \geq 2$, and $\overline{m}_2 \geq c(\kappa/2)^{1/2}$ for positive initial data. The difference in the exponents—3/2 versus 4/3 in the three dimensional wave equation—reflects the distinct nature of the driving noises. Recently Conus and Balan [1] studied this problem when the noise is Gaussian, spatially homogeneous and behaves in time like a fractional Brownian motion with Hurst index $H > 1/2$.

As a direct consequence of our moment bounds, we recover the result $\overline{m}_p \leq Cp^{3/2}$ for $p \geq 2$ of [13] (see Theorem 2.7). We extend their lower bound on the *upper* Lyapunov exponent $\overline{m}_2$ to the *lower* Lyapunov exponent, by showing that $\underline{m}_2 \geq c(\kappa/2)^{1/2}$. In the case of the Anderson model $\rho(u) = \lambda u$, we show that $\overline{m}_2 = \underline{m}_2 = |\lambda| (\kappa/2)^{1/2}$.

The second application of our moment formulas in Theorem 2.3 is to the study of *growth indices*, defined by Conus and Khoshnevisan in [14] as follows

$$\underline{\lambda}(p) := \sup \left\{ \alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geq \alpha t} \log \mathbb{E}\left(|u(t,x)|^p\right) > 0 \right\}, \tag{1.9}$$

$$\overline{\lambda}(p) := \inf \left\{ \alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geq \alpha t} \log \mathbb{E}\left(|u(t,x)|^p\right) < 0 \right\}. \tag{1.10}$$

As discussed in [14], these growth indices give information about the location of exponentially large values of $u(t,x)$, and, in particular, how quickly they propagate away from the origin. In [14, Theorem 5.1], it was shown that if $\rho(0) = 0$, then for initial data with exponential decay at $\pm\infty$, $0 < \underline{\lambda}(p) \leq \overline{\lambda}(p) < +\infty$, for all $p \geq 2$. Since $G_\kappa(t,x)$ has support in the space-time cone $|x| \leq \kappa t$, it is clear that if the initial data have compact support and $\rho(0) = 0$, then any high peaks related to intermittency, must remain in a space-time cone. Hence $\underline{\lambda}(p) \leq \overline{\lambda}(p) \leq \kappa$. In [14, Theorem 5.1], it is shown that if the initial data consists of functions with compact support, then $\underline{\lambda}(p) = \overline{\lambda}(p) = \kappa$ for all $p \geq 2$. On the other hand, if the initial data is not compactly supported and does not decay at $\pm\infty$, for instance, if $g(\circ) \equiv 1$, then $\underline{\lambda}(p) = \overline{\lambda}(p) = +\infty$. We shall show that the rate of decay at $\pm\infty$ needed to have values of $\underline{\lambda}(p)$ and $\overline{\lambda}(p)$ in $]\kappa, +\infty[$ is exponential. In fact, our moment estimates allow us to show in particular (see the more precise statement in Theorem 2.9) that if the initial position and velocity are bounded below by $ce^{-\beta|x|}$ and above by $Ce^{-\tilde{\beta}|x|}$, with $\beta \geq \tilde{\beta}$, then

$$\kappa \left(1 + \frac{l^2}{8\kappa\beta^2}\right)^{\frac{1}{2}} \leq \underline{\lambda}(p) \leq \overline{\lambda}(p) \leq \kappa \left(1 + \frac{L^2}{8\kappa\tilde{\beta}^2}\right)^{\frac{1}{2}},$$

for certain explicit constants l and L. In the case of the Anderson model $\rho(u) = \lambda u$ and for $p = 2$ and $\beta = \tilde{\beta}$, we obtain

$$\underline{\lambda}(2) = \overline{\lambda}(2) = \kappa \left(1 + \frac{\lambda^2}{8\kappa\beta^2}\right)^{1/2}.$$

Since the growth indices of order two depend on the asymptotic behavior of $E(u(t,x)^2)$ as $t \to \infty$, this equality highlights, in a somewhat surprising way, how the initial data significantly affects (through the decay rate β) the behavior of the solution for all time, despite the presence of the driving noise.

This paper is organized as follows. We state our main results in Section 2. The proofs of the existence, uniqueness and moment bounds are given in Section 3, along with the proof of weak intermittency. Finally, we prove the results on the growth indices in Section 4.

Note. This paper corresponds mostly to Section 3 of the unpublished notes [10]. This material will not be published elsewhere.

2 Main results

Let $\{W_t(A) : A \in \mathcal{B}_b(\mathbb{R}),\, t \geq 0\}$ be a space-time white noise defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{B}_b(\mathbb{R})$ is the collection of Borel sets with finite Lebesgue measure. Let $(\mathcal{F}_t,\, t \geq 0)$ be the standard filtration generated by this space-time white noise, i.e., $\mathcal{F}_t = \sigma\left(W_s(A) : 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R})\right) \vee \mathcal{N}$, where $\mathcal{N}$ is the σ-field generated by all P-null sets in $\mathcal{F}$. We use $\|\cdot\|_p$ to denote the $L^p(\Omega)$-norm. A random field $Y(t,x)$, $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, is said to be *adapted* if for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $Y(t,x)$ is $\mathcal{F}_t$-measurable, and it is said to be *jointly measurable* if it is measurable with respect to $\mathcal{B}(\mathbb{R}_+^* \times \mathbb{R}) \times \mathcal{F}$.

Definition 2.1. A random field $(u(t,x),\, (t,x) \in \mathbb{R}_+ \times \mathbb{R})$, is called a *solution* to (1.1) if

(1) $u(t,x)$ is adapted and jointly measurable;

(2) for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $\left(G_\kappa^2(\cdot,\circ) \star \|\rho(u(\cdot,\circ))\|_2^2\right)(t,x) < +\infty$, where $\star$ denotes the simultaneous convolution in both space and time variables (and $\cdot$ denotes the time dummy variable);

(3) for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $u(t,x) = J_0(t,x) + I(t,x)$ a.s., where

$$I(t,x) = \iint_{\mathbb{R}_+ \times \mathbb{R}} G_\kappa\left(t-s, x-y\right) \rho\left(u\left(s,y\right)\right) W\left(\mathrm{d}s, \mathrm{d}y\right); \tag{2.1}$$

(4) the function $(t,x) \mapsto I(t,x)$ from $\mathbb{R}_+ \times \mathbb{R}$ into $L^2(\Omega)$ is continuous;

Remark 2.2. In the case of the stochastic *heat* equation, one often requires that $(t,x) \mapsto u(t,x)$ is L^2-continuous. However, this condition is not appropriate for the stochastic wave equation with irregular initial data. Indeed, consider the stochastic wave equation (1.1) with $g \in L^2_{loc}(\mathbb{R})$ and $\mu = 0$. In this case, $J_0(t,x) = 1/2\,(g(\kappa t + x) + g(\kappa t - x))$. Since the initial position g may not be defined for every x, the function $(t,x) \mapsto J_0(t,x)$ may not even be defined for certain (t,x). Therefore, for these (t,x), $u(t,x)$ may not be well-defined (see Example 2.5). Nevertheless, as we will show later, $I(t,x)$ is always well defined for each $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, and it has a continuous version under our assumptions. For the stochastic heat equation with deterministic initial conditions, this problem does not arise because in that equation, $(t,x) \mapsto J_0(t,x)$ is continuous over $\mathbb{R}_+^* \times \mathbb{R}$ thanks to the smoothing effect of the heat kernel.

2.1 Existence, uniqueness and moment bounds

Assume that $\rho : \mathbb{R} \mapsto \mathbb{R}$ is globally Lipschitz continuous with Lipschitz constant $\mathrm{Lip}_\rho > 0$. In particular, there will be constants $L_\rho > 0$ and $\overline{\varsigma} \geq 0$ such that

$$\rho(x)^2 \leq L_\rho^2 \left(\overline{\varsigma}^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}\,. \tag{2.2}$$

Note that $L_\rho \leq \sqrt{2}\,\mathrm{Lip}_\rho$ and the inequality may be strict. In cases where we want to bound the second moment from below, we will sometimes assume that for some constants $l_\rho > 0$ and $\underline{\varsigma} \geq 0$,

$$\rho(x)^2 \geq l_\rho^2 \left(\underline{\varsigma}^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}\,. \tag{2.3}$$

We shall also give particular attention to the Anderson model, which is a special case of the following near-linear growth condition: for some constants $\varsigma \geq 0$ and $\lambda \neq 0$,

$$\rho(x)^2 = \lambda^2 \left(\varsigma^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}\,. \tag{2.4}$$

Recall the definition of $\mathcal{K}(t,x;\kappa,\lambda)$ and $\mathcal{H}(t;\kappa,\lambda)$ in (1.4) and (1.5), respectively. In certain cases, we will replace λ by another value, and we use the following conventions:

$$\mathcal{K}(t,x) := \mathcal{K}(t,x;\ \kappa,\lambda)\,,\quad \overline{\mathcal{K}}(t,x) := \mathcal{K}(t,x;\ \kappa, L_\rho)\,,$$
$$\underline{\mathcal{K}}(t,x) := \mathcal{K}(t,x;\ \kappa, l_\rho)\,,\quad \widehat{\mathcal{K}}(t,x) := \mathcal{K}(t,x;\ \kappa, a_{p,\overline{\varsigma}}\, z_p\ L_\rho)\,,\ p \geq 2\,,$$

where the constant $a_{p,\overline{\varsigma}} (\leq 2)$ is defined by

$$a_{p,\overline{\varsigma}} := \begin{cases} 2^{(p-1)/p} & \overline{\varsigma} \neq 0,\ p > 2, \\ \sqrt{2} & \overline{\varsigma} = 0,\ p > 2, \\ 1 & p = 2, \end{cases} \tag{2.5}$$

and z_p is the optimal universal constant in the Burkholder-Davis-Gundy inequality (see [14, Theorem 1.4]) and so $z_2 = 1$ and $z_p \leq 2\sqrt{p}$ for all $p \geq 2$. Note that the kernel function

$\widehat{\mathcal{K}}(t,x)$ depends on the parameters p and $\bar{\varsigma}$, which is usually clear from the context. The same conventions will apply to $\mathcal{H}(t)$, $\overline{\mathcal{H}}(t)$, $\underline{\mathcal{H}}(t)$ and $\widehat{\mathcal{H}}(t)$.

In the next theorem, the existence and uniqueness results extend, in the spirit of [9], the classical existence results [6, 7, 37] as well as the more recent results of [13]. In fact, our assumptions on g and μ are essentially minimal. However, the main contribution concerns the bounds on moments of the solution, and, in particular, the explicit formulas (2.10) and (2.11). Recall that $\mathcal{M}(\mathbb{R})$ is the set of locally finite (signed) Borel measures over $\mathbb{R}$.

Theorem 2.3. *Suppose that $g \in L^2_{loc}(\mathbb{R})$, $\mu \in \mathcal{M}(\mathbb{R})$ and ρ is Lipschitz continuous with linear growth (2.2). Define $\overline{\mathcal{K}}$, $\overline{\mathcal{H}}$ as above, and T_κ, X_κ as in (1.6), (1.7). Then the stochastic wave equation (1.1) has a random field solution in the sense of Definition 2.1: $u(t,x) = J_0(t,x) + I(t,x)$ for $t > 0$ and $x \in \mathbb{R}$. Moreover,*
(1) $u(t,x)$ is unique (in the sense of versions);
(2) $(t,x) \mapsto I(t,x)$ is $L^p(\Omega)$-continuous for all integers $p \geq 2$.
Furthermore, for all $t' \geq t \geq 0$, x, $x' \in \mathbb{R}$, by denoting $T := T_\kappa(t,t',x-x')$ and $X := X_\kappa(x,x',t'-t)$, the following moment estimates hold:
(3) For all even integers $p \geq 2$,

$$||u(t,x)||_p^2 \leq \begin{cases} J_0^2(t,x) + \left(J_0^2 \star \overline{\mathcal{K}}\right)(t,x) + \bar{\varsigma}^2\, \overline{\mathcal{H}}(t) & \text{if } p=2, \\ 2J_0^2(t,x) + \left(2J_0^2 \star \widehat{\mathcal{K}}_p\right)(t,x) + \bar{\varsigma}^2\, \widehat{\mathcal{H}}_p(t) & \text{if } p>2, \end{cases} \tag{2.6}$$

and

$$\mathbb{E}\left[u(t,x)u(t',x')\right] \leq J_0(t,x)J_0(t',x') + \left(J_0^2 \star \overline{\mathcal{K}}\right)(T,X) + \bar{\varsigma}^2\, \mathcal{H}(T)\,; \tag{2.7}$$

(4) If ρ satisfies (2.3), then

$$||u(t,x)||_2^2 \geq J_0^2(t,x) + \left(J_0^2 \star \underline{\mathcal{K}}\right)(t,x) + \underline{\varsigma}^2\ \underline{\mathcal{H}}(t), \tag{2.8}$$

and

$$\mathbb{E}\left[u(t,x)u(t',x')\right] \geq J_0(t,x)J_0(t',x') + \left(J_0^2 \star \underline{\mathcal{K}}\right)(T,X) + \underline{\varsigma}^2\, \mathcal{H}(T)\,; \tag{2.9}$$

(5) In particular, if $\rho(u)^2 = \lambda^2\left(\varsigma^2 + u^2\right)$, then

$$||u(t,x)||_2^2 = J_0^2(t,x) + \left(J_0^2 \star \mathcal{K}\right)(t,x) + \varsigma^2\ \mathcal{H}(t), \tag{2.10}$$

and

$$\mathbb{E}\left[u(t,x)u(t',x')\right] = J_0(t,x)J_0(t',x') + \left(J_0^2 \star \mathcal{K}\right)(T,X) + \varsigma^2\, \mathcal{H}(T)\,. \tag{2.11}$$

Remark 2.4. We note that the structure of the formula (2.10) and of the bounds in (2.6) and (2.8) are similar to those in [9, Theorem 2.4]. In fact, this structure is generic and applies in principle to a wide class of spde's of the form $Lu(t,x) = \rho(u(t,x))\dot{W}(t,x)$, where L is a pde operator such as $L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ (heat), $L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ (wave) and so on. Of course, L must satisfy suitable conditions. The kernel functions $\mathcal{K}$ and $\mathcal{H}$ here have very different behaviors than those in [9], and this will lead to the different behavior of Lyapunov exponents and of growth indices in the stochastic heat and wave equations. Related formulas and bounds will be given for the space-fractional heat equation in a forthcoming paper.

The proofs of Theorem 2.3 and its two corollaries 1.1 and 1.2 are given at the end of Section 3. Notice that formula (2.10) shows that $||u(t,x)||_2$ depends in a monotone way on the function $J_0^2(\cdot,\circ)$.

Example 2.5. Let $g(x) = |x|^{-1/4}$ and $\mu \equiv 0$. Clearly, $g \in L^2_{loc}(\mathbb{R})$ and

$$J_0^2(t,x) = \frac{1}{4}\left(\frac{1}{|x+\kappa t|^{1/4}} + \frac{1}{|x-\kappa t|^{1/4}}\right)^2.$$

The function $J_0^2(t,x)$ equals $+\infty$ on the characteristic lines $x = \pm\kappa t$ that originate at $(0,0)$, where the singularity of g occurs. Nevertheless, the stochastic integral part $I(t,x)$ is well-defined for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$ and the random field solution $u(t,x)$ in the sense of Definition 2.1 does exist according to Theorem 2.3.

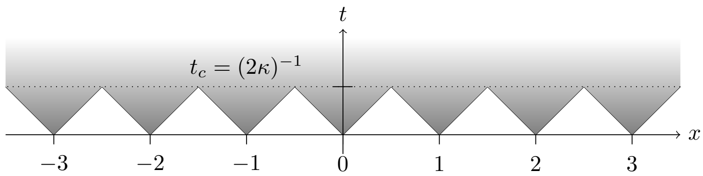


Figure 1: When $g(x) = \sum_{n\in\mathbb{N}} 2^{-n}\left(|x-n|^{-1/2} + |x+n|^{-1/2}\right)$ and $\mu \equiv 0$, the random field solution $u(t,x)$ is only defined in the unshaded regions and in particular not for $t > t_c = (2\kappa)^{-1}$.

Example 2.6. Let $g(x) = |x|^{-1/2}$ and $\mu \equiv 0$. Clearly, $g \notin L^2_{loc}(\mathbb{R})$. So Theorem 2.3 does not apply. In this case, the solution $u(t,x)$ is well-defined outside of the triangle $\kappa t \geq |x|$. But because

$$J_0^2(t,x) = \frac{1}{4}\left(\frac{1}{|x+\kappa t|^{1/2}} + \frac{1}{|x-\kappa t|^{1/2}}\right)^2,$$

and this function is not locally integrable over domains that intersect the characteristic lines $x = \pm\kappa t$, the random field solution exists only in the two "triangles" $\kappa t \leq |x|$. Another example is shown in Figure 1.

2.2 Weak intermittency

Recall that $u(t,x)$ is said to be fully intermittent if the Lyapunov exponent of order 1 vanishes and the lower Lyapunov exponent of order 2 is strictly positive: $m_1 = 0$ and $\underline{m}_2 > 0$. The solution is called *weakly intermittent* if $\overline{m}_2 > 0$ and $\overline{m}_p < +\infty$ for all $p \geq 2$.

Theorem 2.7. *Assuming (2.2), suppose that $g(x) \equiv w$ and $\mu(\mathrm{d}x) = \widetilde{w}\mathrm{d}x$ with $w, \widetilde{w} \in \mathbb{R}$. Then we have the following two properties:*

(1) *For all even integers* $p \geq 2$,

$$\overline{m}_p \leq \begin{cases} L_\rho \sqrt{2\kappa}\, p^{3/2} & \text{if } \overline{\varsigma} \neq 0 \text{ and } p > 2, \\ L_\rho \sqrt{\kappa}\, p^{3/2} & \text{if } \overline{\varsigma} = 0 \text{ and } p > 2, \\ L_\rho \sqrt{\kappa/2} & \text{if } p = 2. \end{cases} \tag{2.12}$$

(2) *If* (2.3) *holds for some* $l_\rho \neq 0$, *and if* $|\underline{\varsigma}| + |w| + |\widetilde{w}| \neq 0$ *with* $w\widetilde{w} \geq 0$, *then* $\underline{m}_2 \geq |\, l_\rho \,|\sqrt{\kappa/2}$ *and so* $u(t,x)$ *is weakly intermittent.*

(3) *If* $\rho(u)^2 = \lambda^2(\varsigma^2 + u^2)$, *with* $\lambda \neq 0$, *and if* $|\underline{\varsigma}| + |w| + |\widetilde{w}| \neq 0$, *then* $\underline{m}_2 = \overline{m}_2 = |\lambda|\sqrt{\kappa/2}$.

Remark 2.8. We do not know if a lower bound of the form $\underline{m}_p \geq Cp^{3/2}$ holds. For this kind of bound in the stochastic wave and heat equations in $\mathbb{R}_+ \times \mathbb{R}^3$ in the case where $\rho(u) = \lambda u$ and the driving noise is spatially colored, see [19].

2.3 Growth indices

Recall the definition of $\underline{\lambda}(p)$ and $\overline{\lambda}(p)$ in (1.9) and (1.10). Define

$$\mathcal{M}_G^\beta(\mathbb{R}) := \left\{ \mu \in \mathcal{M}(\mathbb{R}) : \int_{\mathbb{R}} e^{\beta|x|} |\mu|(\mathrm{d}x) < +\infty \right\}, \quad \beta \geq 0.$$

We use subscript "+" to denote the subset of non-negative measures. For example, $\mathcal{M}_+(\mathbb{R})$ is the set of non-negative Borel measures over $\mathbb{R}$ and $\mathcal{M}_{G,+}^\beta(\mathbb{R}) = \mathcal{M}_G^\beta(\mathbb{R}) \cap \mathcal{M}_+(\mathbb{R})$.

The next theorem improves the result of [14, Theorem 5.1] by giving sharp bounds on $\underline{\lambda}(p)$ and $\overline{\lambda}(p)$ in the case where $\rho(0) = 0$ and the initial data have exponential decay at $\pm\infty$.

Theorem 2.9. *We have the following:*

(1) *Suppose that* $|\rho(u)| \leq L_\rho |u|$ *with* $L_\rho \neq 0$ *and the initial data satisfy the following two conditions:*

  (a) *The initial position* $g(x)$ *is a Borel function such that* $|g(x)|$ *is bounded from above by some function* $ce^{-\beta_1|x|}$ *with* $c > 0$ *and* $\beta_1 > 0$ *for almost all* $x \in \mathbb{R}$*;*

  (b) *The initial velocity* $\mu \in \mathcal{M}_G^{\beta_2}(\mathbb{R})$ *for some* $\beta_2 > 0$.

  Then for all even integers $p \geq 2$,

$$\overline{\lambda}(p) \leq \begin{cases} \kappa \left(1 + \dfrac{a_{p,\overline{\varsigma}}^2\, z_p^2\, L_\rho^2}{8\kappa\,(\beta_1 \wedge \beta_2)^2}\right)^{1/2} & \text{if } p > 2, \\ \kappa \left(1 + \dfrac{L_\rho^2}{8\kappa\,(\beta_1 \wedge \beta_2)^2}\right)^{1/2} & \text{if } p = 2. \end{cases} \tag{2.13}$$

(2) Suppose that $|\rho(u)| \geq l_\rho |u|$ with $l_\rho \neq 0$ and the initial data satisfy the following two conditions:

(a') The initial position $g(x)$ is a non-negative Borel function bounded from below by some function $c_1 e^{-\beta_1'|x|}$ with $c_1 > 0$ and $\beta_1' > 0$ for almost all $x \in \mathbb{R}$;

(b') The initial velocity $\mu(\mathrm{d}x)$ has a density $\mu(x)$ that is a non-negative Borel function bounded from below by some function $c_2 e^{-\beta_2'|x|}$ with $c_2 > 0$ and $\beta_2' > 0$ for almost all $x \in \mathbb{R}$.

Then

$$\underline{\lambda}(p) \geq \kappa \left(1 + \frac{l_\rho^2}{8\kappa\left(\beta_1' \wedge \beta_2'\right)^2}\right)^{1/2}, \quad \textit{for all even integers } p \geq 2. \tag{2.14}$$

In particular, we have the following two special cases:

(3) For the hyperbolic Anderson model $\rho(u) = \lambda u$ with $\lambda \neq 0$, if the initial velocity μ satisfies all Conditions (a), (b), (a') and (b') with $\beta := \beta_1 \wedge \beta_2 = \beta_1' \wedge \beta_2'$, then

$$\underline{\lambda}(2) = \overline{\lambda}(2) = \kappa \left(1 + \frac{\lambda^2}{8\kappa\beta^2}\right)^{1/2}. \tag{2.15}$$

(4) If $l_\rho |u| \leq |\rho(u)| \leq L_\rho |u|$ with $l_\rho \neq 0$ and $L_\rho \neq 0$, and both $g(x)$ and $\mu(x)$ are non-negative Borel functions with compact support, then

$$\overline{\lambda}(p) = \underline{\lambda}(p) = \kappa, \quad \textit{for all even integers } p \geq 2.$$

Note that for Conclusion (3), clearly, $\beta_i' \geq \beta_i$, $i = 1, 2$. Hence, the condition $\beta_1 \wedge \beta_2 = \beta_1' \wedge \beta_2'$ has only two possible cases: $\beta_1' = \beta_1 \leq \beta_2 \leq \beta_2'$ and $\beta_2' = \beta_2 \leq \beta_1 \leq \beta_1'$.

Remark 2.10. As mentioned in the introduction, Theorem 2.9 shows that the initial data significantly affects the behavior of the solution for all time. This theorem also shows that in addition to depending on the rate of decay at $\pm\infty$ of the initial data, the behaviour of the growth indices also depends on the rate of growth of the nonlinearity of ρ. However, when the initial data are compactly supported, the rate of growth of the non-linearity ρ plays no role.

3 Proof of Theorem 2.3

In this section, we first give some key technical results, then we prove Theorem 2.3.

3.1 Computing the kernel function $\mathcal{K}(t,x)$

Define the backward space-time cone:

$$\Lambda(t,x)=\left\{(s,y)\in\mathbb{R}_+\times\mathbb{R}:\ 0\le s\le t,\ \ |y-x|\le\kappa(t-s)\right\}, \tag{3.1}$$

so the wave kernel function can be written $G_\kappa\left(t-s,x-y\right)=\frac{1}{2}1_{\{\Lambda(t,x)\}}\left(s,y\right)$. The change of variables $u=\kappa s-y$, $w=\kappa s+y$ will play an important role: see Figure 2.

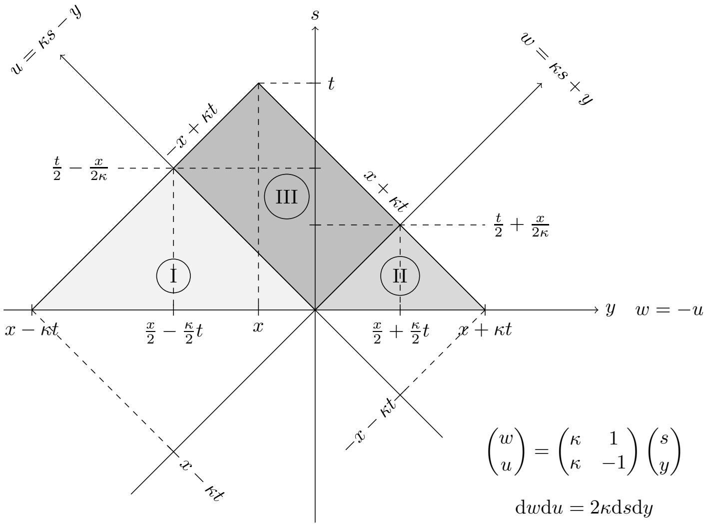


Figure 2: Change variables for the case where $|x|\le\kappa t$.

For all $n\in\mathbb{N}^*$ and $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$, define

$$\mathcal{L}_0(t,x):=\lambda^2G_\kappa^2(t,x)\quad\text{and}\quad\mathcal{L}_n(t,x)=(\mathcal{L}_0\star\cdots\star\mathcal{L}_0)(t,x),$$

where there are $n+1$ convolutions of $\mathcal{L}_0(\cdot,\circ)$ in the second equation.

Proposition 3.1. *For all $n\in\mathbb{N}$, and $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$,*

$$\mathcal{L}_n(t,x)=\begin{cases}\frac{\lambda^{2n+2}\left((\kappa t)^2-x^2\right)^n}{2^{3n+2}(n!)^2\kappa^n} & \text{if } -\kappa t\le x\le\kappa t,\\ 0 & \text{otherwise,}\end{cases} \tag{3.2}$$

$$\mathcal{K}(t,x)=\sum_{n=0}^{\infty}\mathcal{L}_n(t,x),\ \text{and} \tag{3.3}$$

$$\left(\mathcal{K}\star\mathcal{L}_0\right)(t,x)=\mathcal{K}(t,x)-\mathcal{L}_0(t,x)\,. \tag{3.4}$$

Moreover, there are non-negative functions $B_n(t)$ such that for all $n\in\mathbb{N}$, the function $B_n(t)$ is nondecreasing in t, and $\mathcal{L}_n(t,x)\le\mathcal{L}_0(t,x)B_n(t)$ for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$, and

$$\sum_{n=1}^{\infty}\left(B_n(t)\right)^{1/m}<+\infty,\quad\text{for all } m\in\mathbb{N}^*.$$

Proof. Formula (3.2) clearly holds for $n = 0$. By induction, suppose that it is true for n. Then we evaluate $\mathcal{L}_{n+1}(t,x)$ from the definition and a change of variables (see Figure 2):

$$\mathcal{L}_{n+1}(t,x) = (\mathcal{L}_0 \star \mathcal{L}_n)(t,x) = \frac{\lambda^{2n+4}}{2^{3n+4}(n!)^2\kappa^n}\,\frac{1}{2\kappa}\int_0^{x-\kappa t} \mathrm{d}u\; u^n \int_0^{x+\kappa t} \mathrm{d}w\; w^n$$
$$= \frac{\lambda^{2(n+1)+2}\left((\kappa t)^2 - x^2\right)^{n+1}}{2^{3(n+1)+2}((n+1)!)^2\kappa^{n+1}}$$

for $-\kappa t \le x \le \kappa t$, and $\mathcal{L}_{n+1}(t,x) = 0$ otherwise. This proves (3.2). The series in (3.3) converges to $\mathcal{K}(t,x;\kappa,\lambda)$ by (1.3) and (1.4). As a direct consequence, we have (3.4). Take $B_n(t) = \frac{\lambda^{2n}(\kappa t)^{2n}}{2^{3n}(n!)^2\kappa^n}$, which is non-negative and nondecreasing in t. Then clearly, $\mathcal{L}_n(t,x) \le \mathcal{L}_0(t,x)B_n(t)$. To show the convergence, by the ratio test, for all $m \in \mathbb{N}^*$, we have that

$$\frac{(B_n(t))^{1/m}}{(B_{n-1}(t))^{1/m}} = \left(\frac{\lambda\sqrt{\kappa}\,t}{2\sqrt{2}}\right)^{\frac{2}{m}}\left(\frac{1}{n}\right)^{\frac{2}{m}} \to 0,$$

as $n \to \infty$. This completes the proof of Proposition 3.1. ☐

3.2 A proposition used for $L^p(\Omega)$–continuity

We need some notation: for $\beta \in \,]0,1[\,$, $\tau > 0$, $\alpha > 0$ and $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, define

$$B_{t,x,\beta,\tau,\alpha} := \left\{(t',x') \in \mathbb{R}_+^* \times \mathbb{R} :\ \beta t \le t' \le t+\tau,\ |x - x'| \le \alpha\right\}\,. \tag{3.5}$$

Lemma 3.2. *Let $\tau = 1/2$ and $\alpha = \kappa/2$. Fix $\beta \in \,]0,1[\,$ and $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. Then for all $(t',x') \in B_{t,x,\beta,\tau,\alpha}$ and all $(s,y) \in [0,t'[\,\times\mathbb{R}$, $G_\kappa(t'-s,x'-y) \le G_\kappa(t+1-s,x-y)$.*

Proof. See Figure 3. The gray box is the set $B_{t,x,\beta,\tau,\alpha}$. Clearly, we need $\alpha/\kappa + \tau = 1$. Therefore, we can choose $\tau = 1/2$ and $\alpha = \kappa/2$. ☐

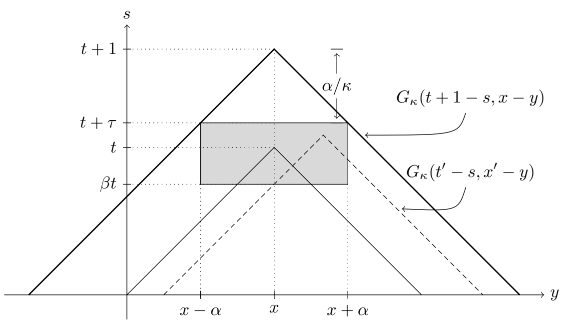


Figure 3: Illustration of the proof of Lemma 3.2.

For $p \ge 2$ and $X \in L^2(\mathbb{R}_+ \times \mathbb{R}, L^p(\Omega))$, set

$$||X||_{M,p}^2 := \iint_{\mathbb{R}_+^*\times\mathbb{R}} \mathrm{d}s\mathrm{d}y\, ||X(s,y)||_p^2 < +\infty\,.$$

The next proposition is useful in particular for checking $L^p(\Omega)$–continuity of the random field $I(t,x)$ in (2.1). We need some notation: For $X \in L^2\left(\mathbb{R}_+ \times \mathbb{R}, L^p(\Omega)\right)$ with $p \geq 2$, define

$$||X||_{M,p}^2 := \iint_{\mathbb{R}_+^* \times \mathbb{R}} ||X(s,y)||_p^2 \, \mathrm{d}s\mathrm{d}y < +\infty \,. \tag{3.6}$$

Let $\mathcal{P}_p$ denote the closure in $L^2\left(\mathbb{R}_+ \times \mathbb{R}, L^p(\Omega)\right)$ of *simple processes* (see [37]). According to Itô's isometry, $\iint X \mathrm{d}W$ is a well-defined Walsh integral for all elements of $\mathcal{P}_2$.

Proposition 3.3. *Suppose that for some even integer $p \in [2,+\infty[$, a random field $Y = \big(Y(t,x) : (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\big)$ has the following properties:*

(i) Y is adapted and jointly measurable;

(ii) for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $||Y(\cdot,\circ)G_\kappa(t-\cdot,x-\circ)||_{M,p}^2 < +\infty$.

Then for each $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $Y(\cdot,\circ)G_\kappa(t-\cdot,x-\circ) \in \mathcal{P}_2$, the following Walsh integral

$$w(t,x) = \iint_{]0,t[\times\mathbb{R}} G_\kappa\left(t-s,x-y\right) Y\left(s,y\right) W(\mathrm{d}s,\mathrm{d}y)$$

is well-defined and the resulting random field w is adapted. Moreover, w is $L^p(\Omega)$-continuous over $\mathbb{R}_+^* \times \mathbb{R}$.

Proof. The proof of this proposition is similar, but simpler, than that of [9, Proposition 3.4]. The main difference is the proof of the $L^p(\Omega)$-continuity statement. In particular, for two points $(t,x), (t',x') \in \mathbb{R}_+ \times \mathbb{R}$, denote

$$(\,t_*, x_*\,) = \begin{cases} (t',x') & \text{if } t' \leq t, \\ (t,x) & \text{if } t' > t, \end{cases} \qquad \text{and} \qquad \big(\hat{t},\hat{x}\big) = \begin{cases} (t,x) & \text{if } t' \leq t. \\ (t',x') & \text{if } t' > t. \end{cases}$$

Choose $\beta \in \,]0,1[$, $\tau = 1/2$ and $\alpha = \kappa/2$. Fix $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. Let $B := B_{t,x,\beta,\tau,\alpha}$ be the set defined in (3.5). Assume that $(t',x') \in B$. By [9, Lemma 3.3], we have that

$$\begin{aligned}
&||w(t,x) - w\left(t',x'\right)||_p^p \\
&\quad \leq 2^{p-1} z_p^p \left(\int_0^{t_*} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \, ||Y\left(s,y\right)||_p^2 \left(G_\kappa(t-s,x-y) - G_\kappa(t'-s,x'-y)\right)^2 \right)^{p/2} \\
&\qquad + 2^{p-1} z_p^p \left(\int_{t_*}^{\hat{t}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \; ||Y\left(s,y\right)||_p^2 \, G_\kappa^2\left(\hat{t}-s,\hat{x}-y\right) \right)^{p/2} \\
&\quad \leq 2^{p-1} z_p^p \left(L_1(t,t',x,x')\right)^{p/2} + 2^{p-1} z_p^p \left(L_2(t,t',x,x')\right)^{p/2} \;.
\end{aligned}$$

We first consider L_1. By Lemma 3.2,

$$\left(G_\kappa\left(t-s,x-y\right) - G_\kappa\left(t'-s,x'-y\right)\right)^2 \leq 4 G_\kappa^2\left(t+1-s,x-y\right),$$

and the left-hand side converges pointwise to 0 for almost all (t,x) as $(t',x') \to (t,x)$. Further,

$$\iint_{[0,t_*]\times\mathbb{R}} \mathrm{d}s\mathrm{d}y\, G_\kappa^2\left(t+1-s,x-y\right) ||Y\left(s,y\right)||_p^2 \le ||Y(\cdot,\circ)G_\kappa(t+1-\cdot,x-\circ)||_{M,p}^2\,,$$

which is finite by (ii). Hence, by the dominated convergence theorem,

$$\lim_{(t',x')\to(t,x)} L_1(t,t',x,x') = 0.$$

Similarly, for L_2, by Lemma 3.2,

$$G_\kappa^2\left(\hat{t}-s,\hat{x}-y\right) \le G_\kappa^2\left(t+1-s,x-y\right).$$

By the monotone convergence theorem, $\lim_{(t',x')\to(t,x)} L_2(t,t',x,x') = 0$, because

$$\iint_{\left[t_*,\hat{t}\right]\times\mathbb{R}} \mathrm{d}s\mathrm{d}y\, G_\kappa^2\left(t+1-s,x-y\right) ||Y(s,y)||_p^2 \le ||Y(\cdot,\circ)G_\kappa(t+1-\cdot,x-\circ)||_{M,p}^2$$

is finite by (ii). This completes the proof of Proposition 3.3. □

3.3 One lemma on the initial data

In a Picard iteration scheme, the initial data enters already into the very first step, and the next lemma will be needed. For $g \in L^2_{loc}(\mathbb{R})$ and $\mu \in \mathcal{M}(\mathbb{R})$, define two nondecreasing functions:

$$\Psi_g(x) = \int_{-x}^{x} \mathrm{d}y\, g^2(y), \quad \text{and} \quad \Psi_\mu^*(x) = \left(|\mu|\left([-x,x]\right)\right)^2\,, \quad \text{for all } x \ge 0. \tag{3.7}$$

Lemma 3.4. *If $g \in L^2_{loc}(\mathbb{R})$ and $\mu \in \mathcal{M}(\mathbb{R})$, then for all $v \in \mathbb{R}$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,*

$$\left(\left[v^2 + J_0^2\right] \star G_\kappa^2\right)(t,x) \le \frac{\kappa t^2}{4}\left(v^2 + 3\Psi_\mu^*\left(|x|+\kappa t\right)\right) + \frac{3}{16}\, t\, \Psi_g\left(|x|+\kappa t\right) < +\infty.$$

Moreover, for all $v \in \mathbb{R}$ and all compact sets $K \subseteq \mathbb{R}_+ \times \mathbb{R}$,

$$\sup_{(t,x)\in K} \left(\left[v^2 + J_0^2\right] \star G_\kappa^2\right)(t,x) < +\infty.$$

Proof. Suppose $t > 0$. Notice that $|(\mu * G_\kappa(s,\cdot))(y)| \le |\mu|\left([y-\kappa s, y+\kappa s]\right)$, and so, recalling (1.2),

$$\left(\left[v^2 + J_0^2\right] \star G_\kappa^2\right)(t,x) = \frac{1}{4}\left(v^2 \iint_{\Lambda(t,x)} \mathrm{d}s\mathrm{d}y + \iint_{\Lambda(t,x)} \mathrm{d}s\mathrm{d}y\, J_0^2\left(s,y\right)\right)$$

$$\leq \frac{1}{4}\Bigg(v^2\kappa t^2 + \frac{3}{4}\int_0^t \mathrm{d}s \int_{x-\kappa(t-s)}^{x+\kappa(t-s)} \mathrm{d}y\, \left(g^2(y+\kappa s) + g^2(y-\kappa s) \right.$$

$$\left. + 4|\mu|^2\left([y-\kappa s, y+\kappa s]\right)\right)\Bigg).$$

Clearly, for all $(s,y)\in\Lambda(t,x)$, by (3.7),

$$|\mu|^2\left([y-\kappa s, y+\kappa s]\right) \leq |\mu|^2\left([x-\kappa t, x+\kappa t]\right) \leq \Psi_\mu^*\left(|x|+\kappa t\right).$$

The integral for g^2 can be easily evaluated by the change of variables in Figure 2:

$$\begin{aligned}
\int_0^t \mathrm{d}s \int_{x-\kappa(t-s)}^{x+\kappa(t-s)} \left(g^2(y+\kappa s)+g^2(y-\kappa s)\right)\mathrm{d}y &= \frac{1}{2\kappa}\iint_{I\cup II\cup III}\left(g^2(u)+g^2(w)\right)\mathrm{d}u\mathrm{d}w \\
&\leq \frac{1}{2\kappa}\int_{x-\kappa t}^{x+\kappa t}\mathrm{d}w\int_{-x-\kappa t}^{-x+\kappa t}\mathrm{d}u\left(g^2(u)+g^2(w)\right) \\
&\leq t\,\Psi_g(|x|+\kappa t),
\end{aligned}$$

where I, II and III denote the three regions in Figure 2 and Ψ_g is defined in (3.7). Therefore,

$$\left(\left[v^2+J_0^2\right]\star G_\kappa^2\right)(t,x) \leq \frac{1}{4}\left(\left(v^2+3\Psi_\mu^*\left(|x|+\kappa t\right)\right)\kappa t^2 + \frac{3}{4}\,t\,\Psi_g\left(|x|+\kappa t\right)\right) < +\infty.$$

Finally, let $a = \sup\left\{|x|+\kappa t : (t,x)\in K\right\}$, which is finite because K is a compact set. Then,

$$\sup_{(t,x)\in K}\left(\left[v^2+J_0^2\right]\star G_\kappa^2\right)(t,x) \leq \frac{\kappa a^2}{4}\left(v^2+3\Psi_\mu^*(a)\right) + \frac{3}{16}\,a\,\Psi_g(a) < +\infty,$$

which completes the proof of Lemma 3.4. □

3.4 Proof of Theorems 2.3 and 2.7

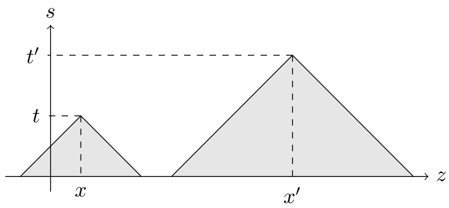
Lemma 3.5. *Recall the definitions of $T_\kappa(t,t',x)$ and $X_\kappa(x,x',t)$ in (1.6) and (1.7), respectively. For all $t'\geq t\geq 0$, and $x,x'\in\mathbb{R}$, by denoting $T := T_\kappa(t,t',x'-x)$ and $X := X_\kappa(x,x',t'-t)$, we have that*

$$G_\kappa(t-s,x-z)G_\kappa(t'-s,x'-z) = \frac{1}{2}G_\kappa(T-s,X-z), \tag{3.8}$$

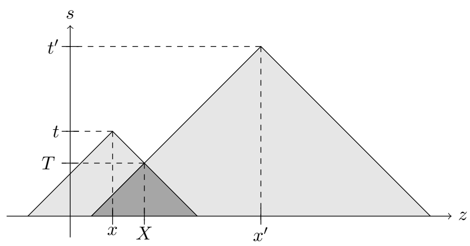
$$\int_{\mathbb{R}}\mathrm{d}z\, G_\kappa(t,x-z)G_\kappa(t',x'-z) = \frac{\kappa}{2}\,T, \tag{3.9}$$

$$\iint_{\mathbb{R}_+\times\mathbb{R}}\mathrm{d}s\mathrm{d}z\, G_\kappa(t-s,x-z)G_\kappa(t'-s,x'-z) = \frac{\kappa}{4}\,T^2. \tag{3.10}$$

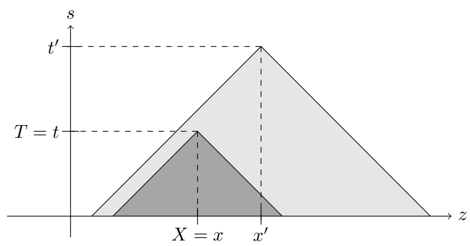
Proof. Recall that $\Lambda(t, x)$ is the space-time cone defined in (3.1). Since $G_\kappa(t-s, x-y) = \frac{1}{2}1_{\{\Lambda(t,x)\}}(s, y)$, by multiplying a factor 4 on both sides of (3.8), this equality reduces to a geometric property of the intersection of the two space-time cones; see Figure 4. We leave the elementary proof to the reader. The other two equalities (3.9) and (3.10) are direct consequences of (3.8). ☐



(a) the case where $|x - x'| \geq \kappa(t + t')$: no intersection between two cones.



(b) the case where $|x-x'| \leq \kappa(t+t')$ and two cones have partial intersection.



(c) the case where $|x-x'| \leq \kappa(t+t')$ and small cone is a subset of the big one.

Figure 4: The two lightly shaded regions denote the support of the functions $(s, z) \mapsto G_\kappa(t-s, x-z)$ and $(s, z) \mapsto G_\kappa(t'-s, y'-z)$, respectively.

Proof of Theorem 2.3. The proof follows the same six steps as those in the proof of [9, Theorem 2.4] with some minor changes:

(1) Both proofs rely on the computation of the kernel function $\mathcal{K}(t, x)$. Here, Proposition 3.1 plays the role of [9, Proposition 2.2].

(2) In the Picard iteration scheme (i.e., Steps 1–4 in the proof of [9, Theorem 2.4]), one needs to check the $L^p(\Omega)$-continuity of the stochastic integral, which then guarantees that at the next step, the integrand is again in $\mathcal{P}_2$, via [9, Proposition 3.1]. Because of the different natures of the heat and wave kernels, we use here Proposition 3.3 for this instead of [9, Proposition 3.4].

(3) In the first step of the Picard iteration scheme, the following property is useful: For all compact sets $K \subseteq \mathbb{R}_+ \times \mathbb{R}$,

$$\sup_{(t,x)\in K} \left(\left[1 + J_0^2\right] \star G_\kappa^2\right)(t, x) < +\infty.$$

For the heat equation, this property is discussed in [9, Lemma 3.9]. Here, Lemma 3.4 gives the desired result with minimal requirements on the initial data. This property, together with the calculation of the function $\mathcal{K}$ in Proposition 3.1, ensures that all the $L^p(\Omega)$-moments of $u(t,x)$ are finite. This property is also used to establish uniform convergence of the Picard iteration scheme, hence $L^p(\Omega)$–continuity of $(t,x)\mapsto I(t,x)$.

(4) As for the two-point correlation function, for $t'\ge t\ge 0$ and $x,\ x'\in\mathbb{R}$,

$$\begin{aligned}\mathbb{E}\left[u(t,x)u(t',x')\right] =& J_0(t,x)J_0(t',x')\\ &+\int_0^t \mathrm{d}s\int_{\mathbb{R}}\mathrm{d}z\ ||\rho(u(s,z))||_2^2\ G_\kappa(t-s,x-z)G_\kappa\left(t'-s,x'-z\right).\end{aligned}$$

Unless (2.4) holds, since we are going to bound ρ above or below via (2.2) or (2.3), we may as well replace $||\rho(u(s,z))||_2^2$ by $\lambda^2(\varsigma^2+||u(s,z)||_2^2)$ and then apply the moment formula (2.10). To calculate the double integral, replace the product of two G_κ functions by $\lambda^{-2}\mathcal{L}_0(T-s,X-z)$ using Lemma 3.5 and use the definitions of $\mathcal{H}(t)$ in (1.5) to see that it is equal to

$$\left[\left(\varsigma^2+J_0^2+(J_0^2\star\mathcal{K})+\varsigma^2(1\star\mathcal{K})\right)\star\mathcal{L}_0\right](T,X).$$

Apply property (3.4) to see that this is equal to $(J_0^2\star\mathcal{K})(T,X)+\varsigma^2\mathcal{H}(T)$. This gives (2.11), together with (2.7) and (2.9). This completes the proof of Theorem 2.3. □

The next two lemmas are needed already for formula (1.5).

Lemma 3.6. *For $a\ne 0$ and $t\ge 0$, $\int_0^t \mathrm{d}s\cosh(as)(t-s)=a^{-2}\left(\cosh(at)-1\right)$, $\int_0^t\mathrm{d}s\sinh(as)(t-s)=a^{-2}\left(\sinh(at)-at\right)$, and $\int_0^t \mathrm{d}s\ \sinh(as)(t-s)^2=a^{-3}\left(2\cosh(at)-a^2t^2-2\right)$.*

Lemma 3.7. *For $t\ge 0$ and $x\in\mathbb{R}$, we have that $\int_{\mathbb{R}}\mathrm{d}x\mathcal{K}(t,x)=|\lambda|(\kappa/2)^{1/2}\sinh\left(|\lambda|\,(\kappa/2)^{1/2}t\right)$ and $\left(1\star\mathcal{K}\right)(t,x)=\cosh\left(|\lambda|\,(\kappa/2)^{1/2}t\right)-1$.*

Proof. By a change of variable,

$$\int_{\mathbb{R}}\mathrm{d}x\ \mathcal{K}(t,x)=2\int_0^{|\lambda|\sqrt{\kappa/2}\,t}\mathrm{d}y\ \frac{\lambda^2}{4}\frac{\sqrt{2\kappa}}{|\lambda|}\frac{y}{\sqrt{\kappa t^2\lambda^2/2-y^2}}I_0(y).$$

Then the first statement follows from [23, (6) on p. 365] with $\nu=0$, $\sigma=1/2$ and $a=|\lambda|\,(\kappa/2)^{1/2}t$. The second statement is a simple application of the first. □

Proof of Corollary 1.1. In this case, $J_0(t,x)=w+\kappa\widetilde{w}t$. Formula for (1.8) follows from the moment formula (2.11) and the integrals in Lemmas 3.7 and 3.6. □

Proof of Corollary 1.2. In this case, $J_0(t,x)=G_\kappa(t,x)$ and so $\lambda^2J_0^2(t,x)=\mathcal{L}_0(t,x)$. By (3.8), we know that $J_0(t,x)J_0(t',x')=J_0^2(T,X)$. Hence, By (2.11) and Proposition 3.1,

$$\mathbb{E}\left[u(t,x)u(t',x')\right]=J_0^2(T,X)+\left(J_0^2\star\mathcal{K}\right)(T,X)+\varsigma^2\mathcal{H}\left(T\right)=\lambda^{-2}\mathcal{K}\left(T,X\right)+\varsigma^2\mathcal{H}\left(T\right),$$

which completes the proof of Corollary 1.2. □

Proof of Theorem 2.7. Clearly, $J_0(t,x) = w + \kappa\widetilde{w}t$.

(1) If $|\bar{\varsigma}| + |w| + |\widetilde{w}| = 0$, then $J_0(t,x) \equiv 0$ and $\rho(0) = 0$, so $u(t,x) \equiv 0$ and the bound (2.12) is trivially true. If $|\bar{\varsigma}| + |w| + |\widetilde{w}| \neq 0$, then by (2.6), for all even integers $p \geq 2$,

$$||u(t,x)||_p^2 \leq 2\left(w + \kappa\widetilde{w}t\right)^2 + \left[2\left(w + \kappa\widetilde{w}t\right)^2 + \bar{\varsigma}^2\right]\widehat{\mathcal{H}}_p(t).$$

Hence, by (1.5), $\overline{m}_p \leq a_{p,\bar{\varsigma}} z_p L_\rho \sqrt{\kappa/2}\, p/2$. Then by (2.5) and the fact that $z_2 = 1$ and $z_p \leq 2\sqrt{p}$ for $p \geq 2$, we obtain (2.12).

(2) Note that the term $2\left(w + \kappa\widetilde{w}t\right)^2 + \bar{\varsigma}^2$ on the r.h.s. of the above inequality does not vanish since $|\bar{\varsigma}| + |w| + |\widetilde{w}| \neq 0$. By (2.8) and Corollary 1.1,

$$||u(t,x)||_2^2 \geq -\underline{\varsigma}^2 - \frac{4\kappa\widetilde{w}^2}{l_\rho^2} + \left(w^2 + \underline{\varsigma}^2 + \frac{4\kappa\widetilde{w}^2}{l_\rho^2}\right)\cosh\left(|\, l_\rho\,|\sqrt{\kappa/2}\, t\right).$$

Clearly, $|\underline{\varsigma}| + |w| + |\widetilde{w}| \neq 0$ implies that $\underline{m}_2 \geq |\, l_\rho\,|\sqrt{\kappa/2}$.

Part (3) is a consequence of (1) and (2). This completes the proof of Theorem 2.7. □

4 Proof of Theorem 2.9 (growth indices)

It will follow from Theorem 2.3 that we will be able to study separately the contributions of the initial position and the initial velocity. We consider the case $g \not\equiv 0$ and $\mu \equiv 0$ in Proposition 4.3, the case $g \equiv 0$ and $\mu \not\equiv 0$ in Proposition 4.6, then we combine the two to prove Theorem 2.9. We begin with some technical lemmas. Recall that $H(t)$ is the Heaviside function.

Lemma 4.1. *Let $f(t,x) = \frac{1}{2}\left(e^{-\beta|x-\kappa t|} + e^{-\beta|x+\kappa t|}\right)H(t)$. Then we have the following bounds:*

(1) Set $\sigma := \sqrt{\beta^2 + \frac{\lambda^2}{2\kappa}}$. For $\beta > 0$, $t \geq 0$ and $|x| \geq \kappa t$,

$$\left(f \star \mathcal{K}\right)(t,x) \leq \frac{\lambda^2\, t}{2(\sigma - \beta)} e^{-\beta|x| + \kappa\sigma t}.$$

(2) For $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $\beta > 0$ and $a, b \in \,]0,1[$,

$$\left(f \star \mathcal{K}\right)(t,x) \geq \begin{cases} \frac{1}{2} e^{-\beta\kappa t}\cosh(\beta|x|)\left(I_0\left(\sqrt{\frac{\lambda^2(\kappa^2t^2 - x^2)}{2\kappa}}\right) - 1\right) & \text{if } |x| \leq \kappa t, \\ \frac{\lambda^2 e^{-\beta|x|}}{2(1-a^2)\beta^2\kappa} I_0\left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\, b\,\kappa t\right) g(t\,; a, b, \beta, \kappa) & \text{if } |x| \geq \kappa t\,, \end{cases}$$

where the function $g\left(t\,; a, b, \beta, \kappa\right)$ is equal to

$$a\cosh\left(ab\beta\kappa t\right)\cosh\left((1-b)\beta\kappa t\right) - a\cosh\left(a\beta\kappa t\right) + \sinh\left((1-b)\beta\kappa t\right)\sinh\left(ab\beta\kappa t\right).$$

Proof. (1) Because $f(t,\circ)$ and $\mathcal{K}(t,\circ)$ are even functions, it suffices to consider the case $x \le -\kappa t$. In this case, $y \le -\kappa s$ implies that $f(s,y) = \frac{1}{2}\left(e^{\beta(y-\kappa s)} + e^{\beta(y+\kappa s)}\right) H(s)$. Because

$$I_0(z) \le \cosh(z) \le e^{|z|}, \quad \text{for all } z \in \mathbb{R}, \tag{4.1}$$

which can be seen from the formula $I_0(z) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta \, \cosh(z\cos(\theta))$ (see [29, (10.32.1)]),

$$\begin{aligned}
\left(f \star \mathcal{K}\right)(t,x) &\le \frac{\lambda^2}{4}\int_0^t \mathrm{d}s \int_{x-\kappa(t-s)}^{x+\kappa(t-s)} \mathrm{d}y \, \frac{1}{2}\left(e^{\beta(y-\kappa s)} + e^{\beta(y+\kappa s)}\right) \exp\left(\sqrt{\frac{\lambda^2[\kappa^2(t-s)^2 - (x-y)^2]}{2\kappa}}\right) \\
&= \frac{\lambda^2}{8}\int_0^t \mathrm{d}s \, \left(e^{\beta(x-\kappa(t-s))} + e^{\beta(x+\kappa(t-s))}\right) \int_{-\kappa s}^{\kappa s} \mathrm{d}y \, \exp\left(-\beta y + \sqrt{\frac{\lambda^2[\kappa^2 s^2 - y^2]}{2\kappa}}\right).
\end{aligned}$$

The function $\psi(y) := -\beta y + \left[\lambda^2(\kappa^2 s^2 - y^2)/(2\kappa)\right]^{1/2}$ achieves its maximum at $y = -\sigma^{-1}\beta\kappa s \in [-\kappa s, \kappa s]$, and $\max_{|y|\le\kappa s}\psi(y) = \sigma\kappa s$, so

$$\begin{aligned}
\left(f \star \mathcal{K}\right)(t,x) &\le \frac{\lambda^2 \kappa\, t}{4}\int_0^t \mathrm{d}s \, \left(e^{\beta(x-\kappa t)+\kappa(\sigma+\beta)s} + e^{\beta(x+\kappa t)+\kappa(\sigma-\beta)s}\right) \\
&\le \frac{\lambda^2 t}{4(\sigma-\beta)}\left(e^{\beta(x-\kappa t)+\kappa(\sigma+\beta)t} + e^{\beta(x+\kappa t)+\kappa(\sigma-\beta)t}\right) = \frac{\lambda^2 t}{2(\sigma-\beta)} e^{\beta x + \kappa\sigma t}.
\end{aligned}$$

(2) We consider two cases. *Case I:* $|x| \le \kappa t$. As shown in Figure 2, we decompose the space-time convolution into three parts S_i corresponding to the three integration regions D_i, $i = 1,2,3$:

$$\left(f \star G_\kappa\right)(t,x) = \sum_{i=1}^3 S_i = \sum_{i=1}^3 \frac{1}{2}\iint_{D_i} \mathrm{d}s\mathrm{d}y \, f(s,y).$$

Clearly, $\left(f \star \mathcal{K}\right)(t,x) \ge S_3$. Because

$$f\left(s,y\right) \ge \frac{1}{2}\left(e^{-\beta(\kappa t - x)} + e^{-\beta(\kappa t + x)}\right), \quad \text{for all } (s,y) \in D_3,$$

we see that

$$S_3 \ge \frac{2}{\lambda^2} e^{-\beta\kappa t}\cosh\left(\beta x\right)\left(\mathcal{L}_0 \star \mathcal{K}\right)(t,x).$$

Then apply (3.4).

Case II: $|x| \ge \kappa t$. Similar to the proof of part (1), one can assume that $x \le -\kappa t$. Then

$$\left(f \star \mathcal{K}\right)(t,x) = \frac{\lambda^2}{8}\int_0^t \mathrm{d}s \int_{-\kappa s}^{\kappa s} \mathrm{d}y \, I_0\left(\sqrt{\frac{\lambda^2(\kappa^2 s^2 - y^2)}{2\kappa}}\right)\left(e^{\beta(x-y-\kappa(t-s))} + e^{\beta(x-y+\kappa(t-s))}\right).$$

Fix $a, b \in \left]0,1\right[$. Then

$$\left(f \star \mathcal{K}\right)(t,x) \ge \frac{\lambda^2}{4}\int_{bt}^t \mathrm{d}s \int_{-a\kappa s}^{a\kappa s} \mathrm{d}y \, I_0\left(\sqrt{\frac{\lambda^2(\kappa^2 s^2 - y^2)}{2\kappa}}\right) e^{\beta(x-y)}\cosh(\beta\kappa(t-s))$$

$$\geq \frac{\lambda^2 e^{\beta x}}{4} I_0\left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\, b\,\kappa t\right) \int_{bt}^{t} \mathrm{d}s \int_{-a\kappa s}^{a\kappa s} \mathrm{d}y\ \cosh(\beta\kappa(t-s)) e^{-\beta y}\,.$$

Since

$$\int_{bt}^{t} \mathrm{d}s \int_{-a\kappa s}^{a\kappa s} \mathrm{d}y\ \cosh(\beta\kappa(t-s)) e^{-\beta y} = \frac{2}{\beta} \int_{bt}^{t} \mathrm{d}s\ \cosh(\beta\kappa(t-s)) \sinh(a\beta\kappa s),$$

part (2) is proved by an application of the following integral: For $a \neq c$, $t > 0$ and $b \in [0,1]$,

$$\begin{aligned}&\int_{bt}^{t} \mathrm{d}s\ \cosh\left(a(t-s)\right) \sinh\left(cs\right)\\ &\quad = \left(a^2-c^2\right)^{-1} \Big(c\cosh(bct)\cosh\left(a(1-b)t\right) - c\cosh(ct) + a\sinh(bct)\sinh\left(a(1-b)t\right)\Big),\end{aligned}$$

which can be proved by using the formula $\cosh(x)\sinh(y) = \frac{1}{2}\left(\sinh(x+y)+\sinh(-x+y)\right)$. This completes the proof of Lemma 4.1. □

Lemma 4.2. *The kernel function $\mathcal{K}(t,x)$ defined in (1.4) is strictly increasing in t for $x \in \mathbb{R}$ fixed and decreasing in $|x|$ for $t > 0$ fixed. Moreover, for all $(s,y) \in [0,t] \times \mathbb{R}$, we have that*

$$\frac{\lambda^2}{2} G_\kappa\left(s,y\right) \leq \mathcal{K}\left(s,y\right) \leq \frac{\lambda^2}{2} I_0\left(|\lambda|\sqrt{\kappa/2}\, t\right) G_\kappa\left(s,y\right).$$

Proof. The first statement is true by (1.3). As for the inequalities, the upper bound follows from the first part. The lower bound is clear since $I_0(0) = 1$ by (1.3). □

Proposition 4.3. *Suppose that $\mu \equiv 0$. Fix $\beta > 0$. Then:*

(1) Suppose $|\rho(u)| \leq L_\rho\,|u|$ with $L_\rho \neq 0$ and let $g(x)$ be a measurable function such that for some constant $C > 0$, $|g(x)| \leq Ce^{-\beta|x|}$ for almost all $x \in \mathbb{R}$. Then (2.13) holds with $\beta_1 \wedge \beta_2$ there replaced by β.

(2) Suppose $|\rho(u)| \geq l_\rho\,|u|$ with $l_\rho \neq 0$ and let $g(x)$ be a measurable function such that for some constant $c > 0$, $|g(x)| \geq c\, e^{-\beta|x|}$ for almost all $x \in \mathbb{R}$. Then (2.14) holds with $\beta_1' \wedge \beta_2'$ there replaced by β.

In particular, if $g(x)$ satisfies both Conditions (1) and (2), and $\rho(u) = \lambda u$ with $\lambda \neq 0$, then (2.15) holds.

Proof. (1) Let $J_0(t,x) = \frac{1}{2}\left(g(x-\kappa t) + g(x+\kappa t)\right) H(t)$. By the assumptions on $g(x)$,

$$|J_0(t,x)|^2 \leq \frac{C^2}{2}\left(e^{-2\beta|x-\kappa t|} + e^{-2\beta|x+\kappa t|}\right) H(t), \quad \text{for almost all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

We first consider the case $p > 2$. By the moment bound (2.6) and Lemma 4.1 (1), for $|x| \geq \kappa t$,

$$||u(t,x)||_p^2 \leq 2J_0^2(t,x) + C' t \exp\left(-2\beta|x| + \kappa\sigma t\right),$$

for some constant $C' > 0$, where $\sigma := \left[4\beta^2 + (2\kappa)^{-1} a_{p,\bar{\varsigma}}^2\, z_p^2\, L_\rho^2\right]^{1/2}$. Consider $\alpha > \kappa$. Because the supremum over $|x| \geq \alpha t$ of the right-hand side is attained at $|x| = \alpha t$,

$$\lim_{t\to\infty} \frac{1}{t} \sup_{|x|\geq \alpha t} \log ||u(t,x)||_p^p \leq -2\alpha\beta + \kappa\sigma, \quad \text{for } \alpha > \kappa.$$

Notice that $-2\alpha\beta + \kappa\sigma < 0 \Leftrightarrow \alpha > \kappa\frac{\sigma}{2\beta}$. Since $\kappa\frac{\sigma}{2\beta} > \kappa$, we conclude that $\overline{\lambda}(p) \leq \kappa\frac{\sigma}{2\beta}$, which is the formula in (2.13) (with $\beta_1 \wedge \beta_2$ there replaced by β) for $p > 2$. For the case $p = 2$, we simply replace z_p and $a_{p,\bar{\varsigma}}$ by 1 (see (2.5)).

(2) Note that $\underline{\lambda}(p) \geq \underline{\lambda}(2)$ and $||u||_p \geq ||u||_2$ for $p \geq 2$, so we only need to consider $p = 2$. Assume first that $\rho(u) = \lambda u$. Since $|g(x)| \geq c\, e^{-\beta|x|}$ a.e.,

$$J_0^2(t,x) \geq \frac{c^2}{4}\left(e^{-2\beta|x-\kappa t|} + e^{-2\beta|x+\kappa t|}\right).$$

If $|x| \leq \kappa t$, by (2.8), Lemma 4.2 and Lemma 4.1,

$$||u(t,x)||_2^2 \geq \left(J_0^2 \star \mathcal{K}\right)(t,x) \geq \frac{c^2}{4} e^{-2\beta\kappa t} \cosh(2\beta|x|) \left(I_0\left(\sqrt{\frac{\lambda^2(\kappa^2t^2 - x^2)}{2\kappa}}\right) - 1\right).$$

Then use the following asymptotic formula for $I_0(x)$ (see, [29, (10.30.4)]):

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \quad \text{as } x \to \infty, \tag{4.2}$$

to see that for $0 \leq \alpha < \kappa$,

$$\lim_{t\to+\infty} \frac{1}{t} \sup_{|x|\geq\alpha t} \log ||u(t,x)||_2^2 \geq -2\beta\kappa + 2\beta\alpha + |\lambda|\sqrt{\frac{\kappa^2-\alpha^2}{2\kappa}}.$$

Then

$$h(\alpha) := -2\beta\kappa + 2\beta\alpha + \frac{|\lambda|}{\sqrt{2\kappa}}\sqrt{\kappa^2-\alpha^2} \geq 0 \quad \Leftrightarrow \quad \kappa\,\frac{8\kappa\beta^2 - \lambda^2}{8\kappa\beta^2+\lambda^2} \leq \alpha \leq \kappa.$$

As α tends to κ from the left side, $h(\alpha)$ remains positive. Therefore, $\underline{\lambda}(2) \geq \kappa$.

If $x \leq -\kappa t$, again, by Lemma 4.1,

$$||u(t,x)||_2^2 \geq \frac{c^2\lambda^2 e^{-2\beta|x|}}{4(1-a^2)(2\beta)^2\kappa} I_0\left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}b\kappa t\right) g(t\,; a, b, 2\beta, \kappa), \quad \text{for all } a, b \in \,]0,1[.$$

For large t, replace both $\cosh(Ct)$ and $\sinh(Ct)$ by $\exp(Ct)/2$, with $C \geq 0$, to see that

$$g(t\,; a, b, 2\beta, \kappa) \geq C' \exp\left(2(1+(a-1)b)t\beta\kappa\right),$$

for some constant $C' > 0$. Hence, for $\alpha > \kappa$, by (4.2),

$$\lim_{t\to\infty} \frac{1}{t} \sup_{|x|\geq \alpha t} \log ||u(t,x)||_2^2 \geq \sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}} b\kappa - 2\beta\alpha + 2(1-(1-a)b)\beta\kappa \ .$$

Solve the inequality

$$h(\alpha) := \sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}} b\kappa - 2\beta\alpha + 2(1-(1-a)b)\beta\kappa > 0$$

to get

$$\alpha < \left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\frac{b}{2\beta} + 1 - (1-a)b\right)\kappa.$$

Since $a \in \,]0,1[$ is arbitrary, we can choose

$$a := \underset{a\in\,]0,1[}{\arg\max} \left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}\frac{b}{2\beta} + 1 - (1-a)b\right) = \left(1 + \frac{\lambda^2}{8\kappa\beta^2}\right)^{-1/2} .$$

In this case, the critical growth rate is $\alpha = b\kappa\left[1+\lambda^2/(8\kappa\beta^2)\right]^{1/2} + (1-b)\kappa$. Finally, since b can be arbitrarily close to 1, we have that $\underline{\lambda}(2) \geq \kappa\left[1+\lambda^2/(8\kappa\beta^2)\right]^{1/2}$, and for the general case $|\rho(u)| \geq l_\rho |u|$, we have that $\underline{\lambda}(p) \geq \underline{\lambda}(2) \geq \kappa\left[1+l_\rho^2/(8\kappa\beta^2)\right]^{1/2}$. This completes the proof of Proposition 4.3. ☐

Now, let us consider the case where $g(x) \equiv 0$. We shall first study the case where $\mu(\mathrm{d}x) = e^{-\beta|x|}\mathrm{d}x$ with $\beta > 0$. In this case, $J_0(t,x)$ is given by the following lemma.

Lemma 4.4. *Suppose that $\mu(\mathrm{d}x) = e^{-\beta|x|}\mathrm{d}x$ with $\beta > 0$. For all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and $z > 0$,*

$$\left(\mu * 1_{\{|\cdot|\leq z\}}\right)(x) = \begin{cases} 2\beta^{-1}e^{-\beta|x|}\sinh(\beta z) & |x| \geq z, \\ 2\beta^{-1}\left(1 - e^{-\beta z}\cosh(\beta x)\right) & |x| \leq z. \end{cases}$$

In particular, we have that

$$J_0(t,x) = \begin{cases} \beta^{-1}e^{-\beta|x|}\sinh(\beta\kappa t) & |x| \geq \kappa t, \\ \beta^{-1}\left(1 - e^{-\beta\kappa t}\cosh(\beta x)\right) & |x| \leq \kappa t. \end{cases}$$

The proof is straightforward, and is left to the reader (see also [8, Lemma 4.4.5]).

Lemma 4.5. *Suppose that $\mu \in \mathcal{M}_G^\beta(\mathbb{R})$ with $\beta > 0$. Set $h(t,x) = (\mu * G_\kappa(t,\cdot))(x)$ and $\sigma = [\beta^2 + (2\kappa)^{-1}\lambda^2]^{1/2}$. Then for all $t \geq 0$ and $x \in \mathbb{R}$,*

$$|h(t,x)| \leq C\exp\left(\beta\kappa t - \beta|x|\right), \quad \textit{with } C = 1/2\int_{\mathbb{R}} |\mu|(\mathrm{d}x)\, e^{\beta|x|} ,$$

and

$$\left(|h| \star \mathcal{K}\right)(t,x) \leq \frac{\lambda^2 t}{2(\sigma-\beta)} e^{-\beta|x|+\sigma\kappa t}.$$

Proof. Considering the first inequality, observe that

$$e^{\beta|x|}\left|\left(\mu * G_\kappa(t,\cdot)\right)(x)\right| \le \frac{1}{2}\int_{x-\kappa t}^{x+\kappa t}|\mu|(\mathrm{d}y)\, e^{\beta|x|} \le \frac{1}{2}\int_{x-\kappa t}^{x+\kappa t}|\mu|(\mathrm{d}y)\, e^{\beta|x-y|}e^{\beta|y|}$$
$$\le \frac{1}{2}e^{\beta\kappa t}\int_{x-\kappa t}^{x+\kappa t}|\mu|(\mathrm{d}y)\, e^{\beta|y|} \le \frac{1}{2}e^{\beta\kappa t}\int_{\mathbb{R}}|\mu|(\mathrm{d}y)\, e^{\beta|y|}.$$

For the second inequality, set $f(t,x) = e^{\beta\kappa t-\beta|x|}$. Then by (4.1),

$$\begin{aligned}
(f\star\mathcal{K})(t,x) &= \frac{\lambda^2}{4}\int_0^t \mathrm{d}s\, e^{\beta\kappa(t-s)}\int_{-\kappa s}^{\kappa s}\mathrm{d}y\, \exp\left(-\beta|x-y| + \sqrt{\frac{\lambda^2\left(\kappa^2 s^2-y^2\right)}{2\kappa}}\right)\\
&\le \frac{\lambda^2}{4}\int_0^t \mathrm{d}s\, e^{\beta\kappa(t-s)}\int_{-\kappa s}^{\kappa s}\mathrm{d}y\, \exp\left(-\beta|x| + \beta|y| + \sqrt{\frac{\lambda^2\left(\kappa^2 s^2-y^2\right)}{2\kappa}}\right)\\
&\le \frac{\lambda^2}{2}e^{-\beta|x|}\int_0^t \mathrm{d}s\, e^{\beta\kappa(t-s)}\int_0^{\kappa s}\mathrm{d}y\, \exp\left(\beta y + \sqrt{\frac{\lambda^2\left(\kappa^2 s^2-y^2\right)}{2\kappa}}\right).
\end{aligned}$$

The function $\psi(y) := \beta y + \left[\lambda^2\left(\kappa^2 s^2 - y^2\right)/(2\kappa)\right]^{1/2}$ achieves its maximum at $y = \sigma^{-1}\beta\kappa s \in [0,\kappa s]$, and $\max_{y\in[0,\kappa s]}\psi(y) = \sigma\kappa s$, so

$$(f\star\mathcal{K}) \le \frac{\lambda^2\kappa t}{2}e^{-\beta|x|}\int_0^t \mathrm{d}s\, e^{\beta\kappa(t-s)+\sigma\kappa s} \le \frac{\lambda^2 t}{2(\sigma-\beta)}e^{-\beta|x|+\sigma\kappa t}.$$

This completes the proof. ∎

Proposition 4.6. *Suppose that $g\equiv 0$. Fix $\beta>0$.*

(1) If $|\rho(u)|\le L_\rho|u|$ with $L_\rho\ne 0$ and $\mu\in\mathcal{M}_G^\beta(\mathbb{R})$, then $\overline{\lambda}(p)$ satisfies (2.13) with $\beta_1\wedge\beta_2$ there replaced by β.

(2) Suppose that $|\rho(u)|\ge l_\rho|u|$ with $l_\rho\ne 0$ and $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$. If for some constant $c>0$, $f(x)\ge ce^{-\beta|x|}$ for all almost all $x\in\mathbb{R}$, then (2.14) holds with $\beta_1'\wedge\beta_2'$ there replaced by β.

In particular, if μ satisfies both Conditions (1) and (2), and $\rho(u)=\lambda u$ with $\lambda\ne 0$, then (2.15) holds.

Proof. (1) Let $p>2$ be an even integer. Let $h(t,x)$ be the function defined in Lemma 4.5. Notice that the first bound in Lemma 4.5 is satisfied by $h^2(t,x)$ provided β is replaced by 2β. By (2.6) and Lemma 4.5, we see that for some constant $C'>0$,

$$||u(t,x)||_p^2 \le 2h^2(t,x) + C't\exp\left(-2\beta|x|+\kappa\sigma t\right),$$

where $\sigma = \left[4\beta^2 + a_{p,\overline{\varsigma}}^2\, z_p^2\ \ L_\rho^2\,/(2\kappa)\right]^{1/2}$. Then it is clear that

$$\lim_{t\to\infty} \frac{1}{t} \sup_{|x|\ge \alpha t} \log ||u(t,x)||_p^p \le -2\beta\alpha + \kappa\sigma.$$

Solve the inequality $-2\beta\alpha+\kappa\sigma>0$ to get $\overline{\lambda}(p) \le \kappa\frac{\sigma}{2\beta}$. For the case $p=2$, simply replace z_p and $a_{p,\overline{\varsigma}}$ by 1.

(2) Suppose that $f(x) \ge e^{-\beta|x|}$ for almost all $x\in\mathbb{R}$ (i.e., set $c=1$). By (2.8) and (2.10), we may only consider the case where $\rho(u)=\lambda u$. Denote $J_0(t,x) = (e^{-\beta|\cdot|} * G_\kappa(t,\cdot))(x)$. We first consider the case where $|x|\le \kappa t$. As shown in Figure 2, split the integral that defines $\left(J_0^2 \star \mathcal{K}\right)(t,x)$ over the three regions I, II, and III, so that

$$||u(t,x)||_2^2 \ge \left(J_0^2\star\mathcal{K}\right)(t,x) = S_1+S_2+S_3 \ge S_3.$$

For arbitrary $a,b\in\,]0,1[$, we see that

$$\begin{aligned}
S_3 &\ge \frac{\lambda^2}{4}\int_{bt}^t \mathrm{d}s \int_{-a\kappa s}^{a\kappa s} \mathrm{d}y\ J_0^2\left(t-s,x-y\right) I_0\left(\sqrt{\frac{\lambda^2\left((\kappa s)^2-y^2\right)}{2\kappa}}\right)\\
&\ge \frac{\lambda^2}{4}\int_{bt}^t \mathrm{d}s\ I_0\left(\sqrt{\frac{\lambda^2\left(1-a^2\right)}{2\kappa}}\ \kappa s\right)\int_{-a\kappa s}^{a\kappa s}\mathrm{d}y\ J_0^2\left(t-s,x-y\right)\\
&\ge \frac{\lambda^2}{4} I_0\left(\sqrt{\frac{\lambda^2\left(1-a^2\right)}{2\kappa}}\ \kappa bt\right)\int_{bt}^t \mathrm{d}s\int_{-ab\kappa t}^{ab\kappa t}\mathrm{d}y\ J_0^2\left(t-s,x-y\right).
\end{aligned}$$

Clearly, for (s,y) in Region III of Figure 2, $|x-y|\le\kappa(t-s)$ and so by Lemma 4.4,

$$J_0\left(t-s,x-y\right) = \left(1-e^{-\beta\kappa(t-s)}\cosh\left(\beta(x-y)\right)\right)/\beta.$$

Using the inequalities $(a+b)^2 \ge \frac{a^2}{2}-b^2$ and $\cosh^2(x) = \frac{1}{2}\left(\cosh(2x)+1\right) \ge \frac{1}{2}\cosh(2x)$,

$$J_0^2\left(t-s,x-y\right) \ge \frac{1}{4\beta^2}\, e^{-2\beta\kappa(t-s)}\cosh(2\beta(x-y)) - \frac{1}{\beta^2}\,.$$

Hence,

$$\int_{bt}^t \mathrm{d}s\int_{-ab\kappa t}^{ab\kappa t}\mathrm{d}y\ J_0^2\left(t-s,x-y\right) \ge \frac{\left(1-e^{-2(1-b)\beta\kappa t}\right)\cosh(2\beta x)\sinh(2ab\beta\kappa t)}{8\beta^4\kappa} - \frac{2a(1-b)b\kappa t^2}{\beta^2}\,.$$

Therefore, by (4.2),

$$\lim_{t\to+\infty}\frac{1}{t}\sup_{|x|\ge\alpha t}\log||u(t,x)||_2^2 \ge 2\beta\alpha + 2ab\beta\kappa + b|\lambda|\sqrt{\kappa/2}\,\sqrt{1-a^2}\ > 0, \tag{4.3}$$

for $\alpha \le \kappa$ and all $a, b \in \,]0,1[\,$, which implies that $\underline{\lambda}(2) \ge \kappa$. As for the case where $|x| \ge \kappa t$, for all $a, b \in \,]0,1[$, by Lemma 4.4,

$$
\begin{aligned}
||u(t,x)||_2^2 &\ge \left(J_0^2 \star \mathcal{K}\right)(t,x) \\
&= \frac{\lambda^2}{16\beta^2}\int_0^t \mathrm{d}s \sinh^2(\beta\kappa(t-s)) \int_{-\kappa s}^{\kappa s} \mathrm{d}y\, e^{-2\beta|x-y|} I_0\left(\sqrt{\frac{\lambda^2(\kappa^2 s^2 - y^2)}{2\kappa}}\right) \\
&\ge \frac{\lambda^2 e^{-2\beta|x|+2a\kappa bt\beta}}{32\beta^3}\left(\frac{\sinh(2(1-b)\beta\kappa t)}{4\beta\kappa} - \frac{1}{2}(1-b)t\right) I_0\left(\sqrt{\frac{\lambda^2(1-a^2)}{2\kappa}}b\kappa t\right).
\end{aligned}
$$

Therefore, for $\alpha > \kappa$, we obtain the same inequality as (4.3). The rest argument is exactly the same as the proof of part (2) of Proposition 4.3. This completes the proof of Proposition 4.6. □

Proof of Theorem 2.9. Let $J_{0,1}(t,x)$ (resp. $J_{0,2}(t,x)$) be the homogeneous solutions obtained with the initial data g and 0 (resp. 0 and μ). Clearly, $J_0(t,x) = J_{0,1}(t,x) + J_{0,2}(t,x)$. For (1), we use the fact that $J_0^2(t,x) \le 2J_{0,1}^2(t,x) + 2J_{0,2}^2(t,x)$. By (2.6), we simply choose the larger of the upper bounds between Proposition 4.3 (1) and Proposition 4.6 (1). As for (2), because both g and μ are assumed nonnegative, $J_0^2(t,x) \ge J_{0,1}^2(t,x) + J_{0,2}^2(t,x)$. Hence, by (2.8), we only need to take the larger of the lower bounds between Proposition 4.3 (2) and Proposition 4.6 (2). Part (3) is a direct consequence of (1) and (2). When the initial data have compact support, both (1) and (2) hold for all $\beta_i > 0$ with $i = 1, 2$. Then letting these β_i's tend to $+\infty$ proves (4). This completes the proof of Theorem 2.9. □

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