Moments, Intermittency and Growth Indices for the Nonlinear Fractional Stochastic Heat Equation

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Abstract: We study the nonlinear fractional stochastic heat equation in the spatial domain $\mathbb{R}$ driven by space-time white noise. The initial condition is taken to be a measure on $\mathbb{R}$, such as the Dirac delta function, but this measure may also have non-compact support. Existence and uniqueness, as well as upper and lower bounds on all p-th moments ($p \geq 2$), are obtained. These bounds are uniform in the spatial variable, which answers an open problem mentioned in Conus and Khoshnevisan [10]. We improve the weak intermittency statement by Foondun and Khoshnevisan [15], and we show that the growth indices (of linear type) introduced in [10] are infinite. We introduce the notion of "growth indices of exponential type" in order to characterize the manner in which high peaks propagate away from the origin, and we show that the presence of a fractional differential operator leads to significantly different behavior compared with the standard stochastic heat equation.

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1 Introduction

In this paper, we consider the following nonlinear fractional stochastic heat equation:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - {}_xD_\delta^a\right) u(t,x) = \rho\left(u(t,x)\right)\dot{W}(t,x), & t \in \mathbb{R}_+^* := \left]0,+\infty\right[\,,\ x \in \mathbb{R}, \\ u(0,\circ) = \mu(\circ), \end{cases} \tag{1.1}$$

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where $a \in]0,2]$ is the order of the fractional differential operator ${}_xD_\delta^a$ and δ ($|\delta| \le a \wedge (2-a) := \min(a, 2-a)$) is its skewness, $\dot{W}$ is the space-time white noise, μ is the initial data (a measure), the function $\rho : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz continuous, and $\circ$ denotes the spatial dummy variable. The definition and properties of the fractional differential operator ${}_xD_\delta^a$ are recalled in Section 2.1.

This equation falls into a class of equations studied by Debbi and Dozzi [13]. According to [11, Theorem 11], even the linear form of (1.1) ($\rho \equiv 1$) does not have a solution if $a \le 1$, so they consider $a \in]1,2]$. If we focus on deterministic initial conditions, then in the setting of (1.1), they proved in [13, Theorem 1] that there is a unique random field solution if μ has a bounded density. Equation (1.1) is of particular interest since it is an extension of the classical *parabolic Anderson model* [5], in which $a = 2$ and $\delta = 0$, so ${}_xD_\delta^a$ is the operator $\partial^2/\partial x^2$, and $\rho(u) = \lambda u$ is a linear function. Foondun and Khoshnevisan [15] considered problem (1.1) with the operator ${}_xD_\delta^a$ replaced by the $L^2(\mathbb{R})$-generator $\mathcal{L}$ of a Lévy process. They proved the existence of a random field solution under the assumption that the initial data μ has a bounded and nonnegative density. In [9], the operator ${}_xD_\delta^a$ is replaced by the generator of a symmetric Lévy process and the authors prove that μ can be any finite Borel measure on $\mathbb{R}$. Recently, Balan and Conus [1, 2] studied the Anderson model with fractional Laplacian and bounded initial condition, and with Gaussian, spatially homogeneous noise that behaves in time like a fractional Brownian motion with Hurst index $H > 1/2$.

Following the approach of [6], in which the case $a = 2$ and $\delta = 0$ was considered, we begin by extending the above results (for the operator ${}_xD_\delta^a$) to allow a wider class of initial data: Let $\mathcal{M}(\mathbb{R})$ be the set of signed Borel measures on $\mathbb{R}$. From the Jordan decomposition, $\mu = \mu_+ - \mu_-$ where $\mu_\pm$ are two non-negative Borel measures with disjoint support, and denote $|\mu| = \mu_+ + \mu_-$. Then our admissible initial data is $\mu \in \mathcal{M}_a(\mathbb{R})$, where

$$\mathcal{M}_a(\mathbb{R}) := \left\{ \mu \in \mathcal{M}(\mathbb{R}) : \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |\mu|(\mathrm{d}x) \frac{1}{1+|x-y|^{1+a}} < +\infty \right\}, \quad \text{for } a \in]1,2].$$

We will also use the set $\mathcal{M}_{a,+}(\mathbb{R}) = \{\mu \in \mathcal{M}_a(\mathbb{R}) : \mu \text{ is non negative}\}$. For $\mu \in \mathcal{M}_a(\mathbb{R})$, we obtain estimates for the moments $\mathbb{E}(|u(t,x)|^p)$ for all $p \ge 2$. These estimates have the same structure as those that are given in [6, Theorem 2.4], but the kernel $\mathcal{K}$ that appears in this reference has quite different properties than those of the kernel $\mathcal{K}$ that appears in relation with equation (1.1): see Section 3.2.

Let us define the *upper and lower Lyapunov exponents of order* p by

$$\overline{m}_p(x) := \limsup_{t\to+\infty} \frac{1}{t} \log \mathbb{E}\left(|u(t,x)|^p\right), \quad \underline{m}_p(x) := \liminf_{t\to+\infty} \frac{1}{t} \log \mathbb{E}\left(|u(t,x)|^p\right), \tag{1.2}$$

for all $p \ge 2$ and $x \in \mathbb{R}$. If the initial data is constant, then $\underline{m}_p$ and $\overline{m}_p$ do not depend on x. In this case, the solution is called *fully intermittent* if $\underline{m}_2 > 0$ and $m_1 = 0$ by Carmona and Molchanov [5, Definition III.1.1, on p. 55]. For a detailed discussion of the meaning of this intermittency property, see [17]. Informally, it means that the sample paths of $u(t,x)$ exhibit "high peaks" separated by "large valleys".

Foondun and Khoshnevisan proved *weak intermittency* in [15], namely, for all $p \geq 2$,

$$\overline{m}_2(x) > 0 \,, \quad \text{and} \quad \overline{m}_p(x) < +\infty \quad \text{for all } x \in \mathbb{R},$$

under the conditions that $\mu(\mathrm{d}x) = f(x)\mathrm{d}x$ with $\inf_{x\in\mathbb{R}} f(x) > 0$ and $\inf_{x\neq 0} |\rho(x)/x| > 0$. We improve this result by showing in Theorem 3.4 that when $1 < a < 2$, $|\delta| < 2 - a$ (strict inequality) and $\mu \in \mathcal{M}_a(\mathbb{R})$ is nonnegative and nonvanishing, then for all $p \geq 2$,

$$\inf_{x\in\mathbb{R}} \underline{m}_p(x) > 0, \quad \text{and} \quad \sup_{x\in\mathbb{R}} \overline{m}_p(x) < +\infty.$$

For this, we need a growth condition on ρ, namely, that for some constants $l_\rho > 0$ and $\varsigma \geq 0$,

$$\rho(x)^2 \geq l_\rho^2 \left(\varsigma^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}. \tag{1.3}$$

In a forthcoming paper [8], this weak intermittency property will be extended to full intermittency by showing in addition that $m_1(x) \equiv 0$.

Our result answers an open problem stated by Conus and Khoshnevisan [10]. Indeed, for the case of the fractional Laplacian, which corresponds to our setting with $a \in\,]1, 2[$ and $\delta = 0$, they ask whether the function $t \mapsto \sup_{x\in\mathbb{R}} \mathbb{E}\left(|u(t,x)|^2\right)$ has exponential growth in t for initial data with exponential decay. Our answer is "yes" under the condition (1.3). In addition, under these conditions, if $\mu \in \mathcal{M}_{a,+}(\mathbb{R})$and $\mu \neq 0$, then for fixed $x \in \mathbb{R}$, the function $t \mapsto \mathbb{E}\left(|u(t,x)|^2\right)$ has at least exponential growth; see Remark 3.5.

We define the following *growth indices of exponential type*:

$$\underline{e}(p) := \sup\left\{\alpha > 0 : \lim_{t\to\infty} \frac{1}{t} \sup_{|x|\geq \exp(\alpha t)} \log \mathbb{E}\left(|u(t,x)|^p\right) > 0\right\}, \tag{1.4}$$

$$\overline{e}(p) := \inf\left\{\alpha > 0 : \lim_{t\to\infty} \frac{1}{t} \sup_{|x|\geq \exp(\alpha t)} \log \mathbb{E}\left(|u(t,x)|^p\right) < 0\right\}\,, \tag{1.5}$$

in order to give a proper characterization of the propagation speed of "high peaks". This concept is discussed in Conus and Khoshnevisan [10]. These authors define analogous indices $\underline{\lambda}(p)$ and $\overline{\lambda}(p)$, in which $|x| \geq \exp(\alpha t)$ is replaced by $|x| \geq \alpha t$, which we call *growth indices of linear type*.

Conus and Khoshnevisan [10] consider the case where ${}_xD_\delta^a$ is replaced by the generator $\mathcal{L}$ of a real-valued symmetric Lévy process $(X_t,\, t \geq 0)$. They showed in [10, Theorem 1.1 and Remark 1.2] that if the initial data μ is a nonnegative lower semicontinuous function with certain exponential decay at infinity, and if X_1 has exponential moments, then

$$0 < \underline{\lambda}(p) \leq \overline{\lambda}(p) < +\infty\,, \quad \text{for all } p \in [2, +\infty)\,.$$

For example, a Lévy process that satisfies this assumption is the "truncated symmetric stable process".

Among Lévy processes, stable processes constitute an important subclass with a self-similarity property. Infinitesimal generators of these processes are not covered by the results of [10], since even the second moment of X_1 does not exists. It turns out that the fractional differential operator ${}_xD_\delta^a$ is the infinitesimal generator of a (not necessarily symmetric) strictly stable process with $a \in]1, 2]$ (see Section 2.1), and we will see that when $1 < a < 2$, the presence of the fractional differential operator ${}_xD_\delta^a$ in (1.1) leads to significantly different behaviors of the speed of propagation of high peaks, compared to that obtained in [10].

Indeed, we show first that if the initial data has sufficiently rapid decay at $\pm\infty$, then $\overline{e}(p) < \infty$ (see (3.22) and (3.23)). Then we show that if $1 < a < 2$ and $|\delta| < 2 - a$ (meaning that the underlying stable process has both positive and negative jumps), then

$$\underline{e}(p) > 0 \ , \quad \text{for all } p \in [2, +\infty) \text{ and } \mu \in \mathcal{M}_{a,+}(\mathbb{R}),\ \mu \neq 0, \tag{1.6}$$

provided ρ satisfies condition (1.3). This conclusion applies, for instance, to the case where the initial data μ is the Dirac delta function. In particular, for well-localized initial data (for instance, $\mu \ge 0$ and $\int_{\mathbb{R}} |y|^\eta \mu(\mathrm{d}y) < \infty$ for some $\eta > 0$), $0 < \underline{e}(p) \le \overline{e}(p) < +\infty$, whereas for initial data that is bounded below ($\mu(dx) = f(x)dx$ with $f(x) > c > 0$, for all $x \in \mathbb{R}$), $\underline{e}(p) = \overline{e}(p) = +\infty$. See Theorem 3.6 for the precise statements. As a direct consequence, $\underline{\lambda}(p) = \overline{\lambda}(p) = +\infty$ for all $p \in [2, \infty[$.

The structure of this paper is as follows. After defining the operator ${}_xD_\delta^a$ and giving the meaning of (1.1) in Section 2, the main results are presented in Section 3: Existence and general bounds are given in Theorem 3.1. These bounds are expressed in terms of the kernel $\mathcal{K}$ mentioned above, for which explicit upper and lower bounds are given. These lead to our results on weak intermittency (Theorem 3.4) and growth indices (Theorem 3.6). Section 4 contains the proof of Theorem 3.1 and Section 5 presents the proofs of Theorems 3.4 and 3.6.

2 Preliminaries and notation

We begin by defining the differential operator ${}_xD_\delta^a$ that appears in the SPDE (1.1), then we shall give the rigorous meaning of the SPDE.

2.1 The Riesz-Feller fractional derivative

Let $\mathcal{F}f(\xi) = \int_{\mathbb{R}} \mathrm{d}x\, e^{-i\xi x} f(x)$ denote the Fourier transform. For $0 < a \le 2$ and $|\delta| \le \min(a, 2-a)$, the Riesz-Feller fractional derivative ${}_xD_\delta^a f$ of a smooth and integrable function f is defined (see [19, (2.2)]) by

$$\mathcal{F}({}_xD_\delta^a f)(\xi) = {}_\delta\psi_a(\xi)\, \mathcal{F}f(\xi), \qquad \text{where } {}_\delta\psi_a(\xi) = -|\xi|^a \exp(-i\pi\delta \operatorname{sgn}(\xi)/2). \tag{2.1}$$

When $a = 2$ (and therefore $\delta = 0$), this is simply the ordinary second derivative $\frac{d^2}{dx^2}$. For $1 < a < 2$ and $\delta \le 2 - a$, which is the case that we are most interested in, as stated in

[19, (2.8)] (in which integrals are understood in the sense of Cauchy principle values), this is equivalent to the more explicit formula

$$ {}_xD^a_\delta f(x) = c_a^+ \int_0^{+\infty} \frac{f(x+z)-f(x)-zf'(x)}{z^{1+a}}\,dz + c_a^- \int_{-\infty}^0 \frac{f(x+z)-f(x)-zf'(x)}{(-z)^{1+a}}\,dz, \tag{2.2} $$

where

$$ c_a^\pm = \frac{\Gamma(1+a)}{\pi}\sin\left((a\pm\delta)\frac{\pi}{2}\right) $$

and $\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\mathrm{d}t$ is Euler's Gamma function [21]. Indeed, taking the Fourier transform of the right-hand side of (2.2) leads to (2.1) by (2.3) below and elementary properties of Fourier transform.

From the point of view of a probabilist, the operator ${}_xD^a_\delta$ is the infinitesimal generator of a strictly a-stable Lévy process $X=(X_t,\, t\geq 0)$ (where "strictly" refers to the fact that the process is centered: see [23, Chapter 3]), with Lévy measure

$$ \nu_a(dz) = c_a^+\,\frac{dz}{z^{1+a}}\,1_{\{z>0\}} + c_a^-\,\frac{dz}{(-z)^{1+a}}\,1_{\{z<0\}}. $$

Indeed, the general form of the infinitesimal generator of X, given in terms of its Lévy measure in [23, Theorem 31.5, p. 208], is

$$ \mathcal{L}f(x) = c_a^+ \int_0^{+\infty} \frac{f(x+z)-f(x)-zf'(x)}{z^{1+a}}\,dz + c_a^- \int_{-\infty}^0 \frac{f(x+z)-f(x)-zf'(x)}{(-z)^{1+a}}\,dz, $$

where no "truncation function" nor additional drift term appears because of the centering, and the characteristic function of X_t, given by the Lévy-Khintchine formula [23, Theorem 8.1], is

$$ \exp\left[-t\int_{-\infty}^{+\infty}(e^{i\xi z}-1-i\xi z)\,\nu_a(dz)\right] = \exp(-t\ {}_\delta\psi_a(\xi)), \tag{2.3} $$

where the right-hand side can be obtained from the left-hand side via a direct calculation using the formula $\Gamma(a)\Gamma(1-a)=\pi/\sin(\pi a)$ [21, 5.5.3, p. 138] and the identity

$$ \int_0^{+\infty}\frac{e^{-qz}-1+qz}{z^{1+a}}\,dz = q^a\,\Gamma(-a), \qquad q\in\mathbb{C},\ 1<\Re(a)<2, $$

which follows from [21, 5.9.5, p. 140].

In order to study the SPDE (1.1), we need the fundamental solution of the operator $\frac{\partial}{\partial t} - {}_xD^a_\delta$. According to the above discussion, this is

$$ {}_\delta G_a(t,x) := \mathcal{F}^{-1}\left[\exp\{\,{}_\delta\psi_a(\cdot)t\}\right](x) = \frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{d}\xi\ \exp\left\{i\xi x - t|\xi|^a e^{-i\pi\delta\,\mathrm{sgn}(\xi)/2}\right\}, \tag{2.4} $$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. This is (one way to represent) the density of the strictly a-stable random variable X_t with Lévy measure $t\nu_a$, as given in the representation

(C) in [27, p. 17]. For comparison, the expression in [23, Theorem 14.15] corresponds to the representation (A) in [27, p. 9]. For strictly stable laws, in particular the case $1 < a < 2$, these two representations are equivalent: see [27, (I.28), (I.26) and Theorem C.3]. Properties of ${}_\delta G_a$ will be given in Lemma 4.1 below. We refer to [19] for more details on these fractional differential operators.

In this paper, we denote the solution to the homogeneous equation

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - {}_x D_\delta^a\right) u(t,x) = 0, & t \in \mathbb{R}_+^*,\ x \in \mathbb{R}, \\ u(0,\circ) = \mu(\circ), \end{cases}$$

by

$$J_0(t,x) := \left({}_\delta G_a(t,\circ) * \mu\right)(x) = \int_{\mathbb{R}} \mu(\mathrm{d}y)\ {}_\delta G_a(t, x-y),$$

where "$*$" denotes the convolution in the space variable.

2.2 The stochastic PDE

Let $W = \{W_t(A),\ A \in \mathcal{B}_b(\mathbb{R}),\ t \geq 0\}$ be a space-time white noise defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{B}_b(\mathbb{R})$ is the collection of Borel sets with finite Lebesgue measure. Let $(\mathcal{F}_t,\ t \geq 0)$ be the filtration generated by W and augmented by the σ-field $\mathcal{N}$ generated by all P-null sets in $\mathcal{F}$:

$$\mathcal{F}_t = \sigma\left(W_s(A) : 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R})\right) \vee \mathcal{N}, \quad t \geq 0.$$

In the following, we fix this filtered probability space $\{\Omega, \mathcal{F}, (\mathcal{F}_t,\ t \geq 0), P\}$. We use $\|\cdot\|_p$ to denote the $L^p(\Omega)$-norm ($p \geq 1$). With this setup, W becomes a worthy martingale measure in the sense of Walsh [25], and $\iint_{[0,t]\times\mathbb{R}} X(s,y) W(\mathrm{d}s, \mathrm{d}y)$ is well-defined in this reference for a suitable class of random fields $\{X(s,y),\ (s,y) \in \mathbb{R}_+ \times \mathbb{R}\}$.

The rigorous meaning of the SPDE (1.1) uses the integral formulation

$$\begin{aligned} u(t,x) &= J_0(t,x) + I(t,x), \quad \text{where} \\ I(t,x) &= \iint_{[0,t]\times\mathbb{R}} {}_\delta G_a(t-s, x-y)\, \rho(u(s,y))\, W(\mathrm{d}s, \mathrm{d}y). \end{aligned} \tag{2.5}$$

Definition 2.1. A process $u = \left(u(t,x),\ (t,x) \in \mathbb{R}_+^* \times \mathbb{R}\right)$ is called a *random field solution* to (1.1) if:

(1) u is adapted, i.e., for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $u(t,x)$ is $\mathcal{F}_t$-measurable;

(2) u is jointly measurable with respect to $\mathcal{B}\left(\mathbb{R}_+^* \times \mathbb{R}\right) \times \mathcal{F}$;

(3) $\left({}_{\delta}G_a^2 \star ||\rho(u)||_2^2\right)(t,x) < +\infty$ for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, where "$\star$" denotes the simultaneous convolution in both space and time variables, that is,

$$\left({}_{\delta}G_a^2 \star ||\rho(u)||_2^2\right)(t,x) := \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \; {}_{\delta}G_a^2(t-s,x-y) \, ||\rho(u(s,y))||_2^2 \, ;$$

(4) For all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $u(t,x)$ satisfies (2.5) a.s.;

(5) The function $(t,x) \mapsto I(t,x)$ mapping $\mathbb{R}_+^* \times \mathbb{R}$ into $L^2(\Omega)$ is continuous.

Assume that the function $\rho : \mathbb{R} \mapsto \mathbb{R}$ is globally Lipschitz continuous with Lipschitz constant $\mathrm{Lip}_\rho > 0$. This implies a growth condition on ρ: for some constants $L_\rho > 0$ and $\overline{\varsigma} \geq 0$,

$$\rho(x)^2 \leq L_\rho^2 \left(\overline{\varsigma}^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}. \tag{2.6}$$

Note that one can take $\overline{\varsigma} = |\rho(0)|$ and $L_\rho \leq \sqrt{2}\,\mathrm{Lip}_\rho$ (and the inequality may even be strict). We shall also specially consider the linear case $\rho(u) = \lambda u$ with $\lambda \neq 0$, which is related to the *parabolic Anderson model* ($a = 2$). It is a special case of the following near-linear growth condition: for some constant $\varsigma \geq 0$,

$$\rho(x)^2 = \lambda^2 \left(\varsigma^2 + x^2\right), \qquad \text{for all } x \in \mathbb{R}\,. \tag{2.7}$$

For all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, define

$$\begin{aligned} \mathcal{L}_0\,(t,x;\lambda) &:= \lambda^2 \; {}_{\delta}G_a^2(t,x), \\ \mathcal{L}_n\,(t,x;\lambda) &:= \underbrace{\left(\mathcal{L}_0 \star \cdots \star \mathcal{L}_0\right)}_{n+1 \text{ factors } \mathcal{L}_0(\cdot,\circ;\lambda)} (t,x;\lambda), \quad \text{for } n \geq 1, \end{aligned} \tag{2.8}$$

and

$$\mathcal{K}\,(t,x;\lambda) := \sum_{n=0}^{\infty} \mathcal{L}_n\,(t,x;\lambda) \tag{2.9}$$

(the convergence of this series is established in Proposition 3.2). For $t \geq 0$, define

$$\mathcal{H}(t;\lambda) := \left(1 \star \mathcal{K}(\cdot,\circ;\lambda)\right)(t,x)$$

(notice that the right-hand side does not depend on x).

Let z_p be the universal constant in the Burkholder-Davis-Gundy inequality (in particular, $z_2 = 1$), and so $z_p \leq 2\sqrt{p}$ for all $p \geq 2$; see [4, Appendix]. Define

$$a_{p,\overline{\varsigma}} = \begin{cases} 2^{(p-1)/p} & \text{if } \overline{\varsigma} \neq 0 \text{ and } p > 2, \\ \sqrt{2} & \text{if } \overline{\varsigma} = 0 \text{ and } p > 2, \\ 1 & \text{if } p = 2. \end{cases}$$

We apply the following conventions to the kernel functions $\mathcal{K}(t,x;\lambda)$ (and similarly to $\mathcal{H}(t;\lambda)$):

$$\begin{aligned} \mathcal{K}(t,x) &:= \mathcal{K}(t,x;\lambda), & \overline{\mathcal{K}}(t,x) &:= \mathcal{K}\left(t,x;L_\rho\right), \\ \underline{\mathcal{K}}(t,x) &:= \mathcal{K}\left(t,x;l_\rho\right), & \widehat{\mathcal{K}}_p(t,x) &:= \mathcal{K}\left(t,x;a_{p,\bar\varsigma}\, z_p\, L_\rho\right), \quad \text{for } p\ge 2\,. \end{aligned}$$

3 Main results

3.1 Existence, uniqueness and moments

The following theorem extends the result of [6, Theorem 2.4] from $a=2$ to $a\in\,]1,2]$. In view of the related result [7, Theorem 2.3] and Remark 2.4 in this reference, the bounds in this theorem are not a surprise, though they do require a proof. The main effort will be to turn these abstract bounds into concrete estimates, via explicit upper and lower bounds on the functions $\mathcal{K}$ and $\mathcal{H}$ (see Section 3.2). For $\tau\ge t>0$ and $x,y\in\mathbb{R}$, define

$$\begin{aligned} \mathcal{I}(t,x,\tau,y;\varsigma,\lambda) :=& \lambda^2\int_0^t \mathrm{d}r\int_{\mathbb{R}}\mathrm{d}z\left[J_0^2(r,z)+\left(J_0^2(\cdot,\circ)\star\mathcal{K}(\cdot,\circ;\lambda)\right)(r,z)+\varsigma^2\left(\mathcal{H}(r;\lambda)+1\right)\right] \\ &\times\; {}_\delta G_a(t-r,x-z)\,{}_\delta G_a(\tau-r,y-z). \end{aligned}$$

Theorem 3.1 (Existence,uniqueness and moments). *Suppose that*

(i) $1<a\le 2$ *and* $|\delta|\le 2-a$*;*

(ii) the function ρ *is Lipschitz continuous and satisfies the growth condition* (2.6)*;*

(iii) the initial data are such that $\mu\in\mathcal{M}_a(\mathbb{R})$*.*

Then the stochastic PDE (1.1) *has a random field solution* $(u(t,x),\,(t,x)\in\mathbb{R}_+^*\times\mathbb{R})$*. Moreover:*
(1) $u(t,x)$ *is unique in the sense of versions;*
(2) $(t,x)\mapsto u(t,x)$ *is* $L^p(\Omega)$*-continuous for all integers* $p\ge 2$*;*
(3) For all even integers $p\ge 2$*, all* $\tau\ge t>0$ *and* $x,y\in\mathbb{R}$*,*

$$||u(t,x)||_p^2\le\begin{cases} J_0^2(t,x)+\left(\left[\bar\varsigma^2+J_0^2\right]\star\overline{\mathcal{K}}\right)(t,x), & \text{if } p=2\,,\\ 2J_0^2(t,x)+\left(\left[\bar\varsigma^2+2J_0^2\right]\star\widehat{\mathcal{K}}_p\right)(t,x), & \text{if } p>2\,,\end{cases} \tag{3.1}$$

and

$$\mathbb{E}\left[u(t,x)u\left(\tau,y\right)\right]\le J_0(t,x)J_0\left(\tau,y\right)+\mathcal{I}(t,x,\tau,y;\bar\varsigma,L_\rho)\,. \tag{3.2}$$

(4) If ρ *satisfies* (1.3)*, then for all* $\tau\ge t>0$ *and* $x,y\in\mathbb{R}$*,*

$$||u(t,x)||_2^2\ge J_0^2(t,x)+\left(\left(\underline{\varsigma}^2+J_0^2\right)\star\underline{\mathcal{K}}\right)(t,x), \tag{3.3}$$

and

$$\mathbb{E}\left[u(t,x)u\left(\tau,y\right)\right] \geq J_0(t,x)J_0\left(\tau,y\right) + \mathcal{I}(t,x,\tau,y;\underline{\varsigma},l_\rho)\ . \tag{3.4}$$

(5) If ρ satisfies (2.7), then for all $\tau \geq t > 0$ and $x,y\in\mathbb{R}$,

$$||u(t,x)||_2^2 = J_0^2(t,x) + \left(\left(\varsigma^2 + J_0^2\right)\star\mathcal{K}\right)(t,x), \tag{3.5}$$

and

$$\mathbb{E}\left[u(t,x)u\left(\tau,y\right)\right] = J_0(t,x)J_0\left(\tau,y\right) + \mathcal{I}(t,x,\tau,y;\varsigma,\lambda)\ . \tag{3.6}$$

The proof of this theorem is explained in Section 4.

3.2 Estimates on the kernel function $\mathcal{K}(t,x)$

Recall that if the partial differential operator is the heat operator $\frac{\partial}{\partial t} - \frac{\nu}{2}\Delta$, then

$$\mathcal{K}^{\text{heat}}(t,x;\lambda) = G_{\frac{\nu}{2}}(t,x)\left(\frac{\lambda^2}{\sqrt{4\pi\nu t}} + \frac{\lambda^4}{2\nu}\,e^{\frac{\lambda^4 t}{4\nu}}\,\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)\right), \tag{3.7}$$

where $\nu > 0$ and $\Phi(x)$ is the distribution function of a standard Normal random variable; see [6, Proposition 2.2]. When the partial differential operator is the wave operator $\frac{\partial^2}{\partial t^2} - \kappa^2\Delta$,

$$\mathcal{K}^{\text{wave}}(t,x;\lambda) = \frac{\lambda^2}{4} I_0\left(\sqrt{\frac{\lambda^2((\kappa t)^2 - x^2)}{2\kappa}}\right) 1_{\{|x|\leq\kappa t\}}, \tag{3.8}$$

where $\kappa > 0$ and $I_0(x)$ is the modified Bessel function of the first kind of order 0; see [7, Proposition 3.1].

Except in the above two cases, we do not have an explicit formula for the kernel function $\mathcal{K}(t,x)$ in (2.9). In order to make use of the moment formulas in (3.1) and (3.3), we derive upper and lower bounds on this kernel function in the following two propositions. We will need the two-parameter *Mittag-Leffler function* [22, Section 1.2]:

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad \alpha>0,\ \beta>0. \tag{3.9}$$

Let a^* be the dual of a: $1/a + 1/a^* = 1$. By Lemma 4.1 below (Property (ii)), the constant

$$\Lambda = {}_\delta\Lambda_a := \sup_{x\in\mathbb{R}}\ {}_\delta G_a(1,x) \tag{3.10}$$

is finite. In particular,

$${}_0\Lambda_a = {}_0G_a(1,0) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}\xi\ \exp\left(-|\xi|^a\right) = \frac{1}{a\pi}\int_0^\infty \mathrm{d}t\ e^{-t}\,t^{-1+1/a} = \frac{\Gamma\left(1+1/a\right)}{\pi}\ .$$

In the following, we often omit the dependence of ${}_\delta\Lambda_a$ on δ and a and simply write Λ instead of ${}_\delta\Lambda_a$. Define

$$\begin{aligned} &\gamma := \lambda^2 \Lambda\, \Gamma(1/a^*), \quad \overline{\gamma} := L_\rho^2\, \Lambda\, \Gamma(1/a^*), \\ &\underline{\gamma} := l_\rho^2\, \Lambda\, \Gamma(1/a^*), \quad \widehat{\gamma}_p := a_{p,\bar{\varsigma}}^2 z_p^2\, L_\rho^2\, \Lambda\, \Gamma(1/a^*), \quad \text{for } p \geq 2. \end{aligned} \tag{3.11}$$

Clearly, $\widehat{\gamma}_2 = \overline{\gamma}$.

Proposition 3.2 (Upper bound on $\mathcal{K}(t,x)$)**.** *Suppose that $a \in \,]1,2]$ and $|\delta| \leq 2-a$. The kernel function $\mathcal{K}(t,x)$ defined in (2.9) satisfies, for all $t \geq 0$ and $x \in \mathbb{R}$,*

$$\mathcal{K}(t,x) \leq {}_\delta G_a(t,x) \frac{\gamma}{t^{1/a}} E_{1/a^*,1/a^*}\left(\gamma t^{1/a^*}\right) \tag{3.12}$$

$$\leq \frac{C}{t^{1/a}}\, {}_\delta G_a(t,x) \left(1 + t^{1/a} \exp\left(\gamma^{a^*} t\right)\right), \tag{3.13}$$

where the constant $C = C(a,\delta,\lambda)$ can be chosen as

$$C(a,\delta,\lambda) := \gamma \sup_{t\geq 0} \frac{E_{1/a^*,1/a^*}\left(\gamma\, t^{1/a^*}\right)}{1 + t^{1/a} \exp\left(\gamma^{a^*} t\right)} < +\infty\,. \tag{3.14}$$

This proposition is proved in Section 4. For a lower bound on $\mathcal{K}(t,x)$, we need another family of kernel functions: for $a > 0$, $t > 0$ and $x \in \mathbb{R}$, define

$$g_a(t,x) := \frac{1}{\pi} \frac{t}{\left(t^{2/a} + x^2\right)^{\frac{a}{2}+\frac{1}{2}}}\,. \tag{3.15}$$

These functions have the same scaling property as ${}_\delta G_a(t,x)$ (see Lemma 4.1(iv) below):

$$g_a(t,x) = \frac{1}{t^{1/a}} g_a\left(1, \frac{x}{t^{1/a}}\right).$$

Note that $g_1(t,x)$ is the Poisson kernel (see, e.g., [26, p. 268]), which satisfies the semigroup property

$$\left(g_1(t-s,\cdot) * g_1(s,\cdot)\right)(x) = g_1(t,x), \qquad 0 \leq s \leq t,\ x \in \mathbb{R}.$$

For $a \in \,]1,2[$ and $|\delta| < 2-a$, define

$$\widetilde{C}_{a,\delta} := \inf_{(t,x)\in\mathbb{R}_+^*\times\mathbb{R}} \frac{{}_\delta G_a(t,x)}{\pi g_a(t,x)} > 0 \tag{3.16}$$

(for the strict positivity, see Lemma 5.1 below). Then let

$$\Upsilon(\lambda,a,\delta) := \frac{\lambda^2\, \widetilde{C}_{a,\delta}^2\, C_{a+1/2}^2\, \Gamma\left(1/a^*\right)}{\pi\, 2^{2(a+3+1/a)}}\,, \tag{3.17}$$

where

$$C_\nu := \frac{\Gamma(\nu)\Gamma(1/2)}{2\,\Gamma(\nu+1/2)}\,, \qquad \nu \geq 1/2\,. \tag{3.18}$$

Proposition 3.3 (Lower bound on $\mathcal{K}(t,x)$). *Fix $a \in]1,2[$ and $|\delta| < 2-a$ (note the strict inequality). Then for all $t>0$ and $x \in \mathbb{R}$,*

$$\mathcal{K}(t,x) \geq \pi^2\, \widetilde{C}_{a,\delta}^2\, \lambda^2\, \Gamma(1/a^*)\, g_a^2\,(t,x)\ E_{1/a^*,1/a^*}\left(\Upsilon(\lambda,a,\delta)\, t^{1/a^*}\right). \tag{3.19}$$

In particular, for all $t>0$ and $x \in \mathbb{R}$,

$$\left(1 \star \mathcal{K}\right)(t,x) \geq \frac{\pi^{1/2}\, \widetilde{C}_{a,\delta}^2\, \lambda^2\, \Gamma(1/a^*)\Gamma(a+1/2)}{\Gamma(1+a)}\, t^{1/a^*} E_{1/a^*,1+1/a^*}\left(\Upsilon(\lambda,a,\delta)\, t^{1/a^*}\right). \tag{3.20}$$

This proposition is proved in Section 5.1.

3.3 Growth indices and weak intermittency

Theorem 3.4 (Weak intermittency). *Suppose that $a \in]1,2]$ and $|\delta| \leq 2-a$.*
(1) If ρ satisfies (2.6) and $\mu \in \mathcal{M}_a(\mathbb{R})$, then for all even integers $p \geq 2$,

$$\sup_{x\in\mathbb{R}} \overline{m}_p(x) \leq \frac{1}{2}\left(16\, L_\rho^2\, \Lambda \Gamma(1/a^*)\right)^{a^*}\ p^{2+1/(a-1)}. \tag{3.21}$$

(2) Suppose ρ satisfies (1.3), $|\delta| < 2-a$ (strict inequality) and $\mu \in \mathcal{M}_{a,+}(\mathbb{R})$. If either $\mu \neq 0$ or $\underline{\varsigma} \neq 0$, then for all $p \geq 2$,

$$\inf_{x\in\mathbb{R}}\ \underline{m}_p(x) \geq \frac{p}{2}\Upsilon\left(l_\rho, a, \delta\right)^{a^*} > 0.$$

Note that if $a=2$, then (3.21) implies that for some constant C, we have $\overline{m}_p \leq Cp^3$, which recovers previous analyses (see [3], [6, Example 2.7], etc).

Remark 3.5. Fix $p \geq 2$. Clearly, Theorem 3.4 implies that for all $x \in \mathbb{R}$,

$$\liminf_{t\to\infty} \frac{1}{t} \sup_{y\in\mathbb{R}} \log \mathbb{E}\left(|u(t,y)|^p\right) \geq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left(|u(t,x)|^p\right) = \underline{m}_p(x) \geq \frac{p}{2}\Upsilon\left(l_\rho,a,\delta\right)^{a^*} > 0\,.$$

Hence, the function $t \mapsto \sup_{y\in\mathbb{R}} \mathbb{E}\left(|u(t,y)|^p\right)$ has at least exponential growth. This answers the second open problem stated by Conus and Khoshnevisan in [10]. Moreover, Theorem 3.4 implies that for all fixed $x \in \mathbb{R}$, the function $t \mapsto \mathbb{E}\left(|u(t,x)|^p\right)$ also has at least exponential growth.

Recall the definitions of the constants $\widehat{\gamma}_p$ and $\Upsilon(l_\rho,a,\delta)$ in (3.11) and (3.17), respectively.

Theorem 3.6 (Growth indices). *(1) Suppose that $a \in]1,2]$, $|\delta| \leq 2-a$ and ρ satisfies (2.6) with $\overline{\varsigma}=0$. If there are $C<\infty$, $\alpha>0$ and $\beta>0$ such that for all $(t,x) \in [1,\infty[\,\times\mathbb{R}$,*

$$|J_0(t,x)| \leq C(1+t^\alpha)(1+|x|)^{-\beta}. \tag{3.22}$$

Then

$$\overline{e}(p) \leq \frac{\widehat{\gamma}_p^{a^*}}{\beta} < +\infty. \tag{3.23}$$

In particular, if, for some $\eta > 0$, $\int_{\mathbb{R}} |\mu|(\mathrm{d}y)(1+|y|^\eta) < \infty$, then (3.22) and (3.23) are satisfied with $\beta = \min(\eta, 1+a)$.
(2) Suppose that $a \in \,]1,2[$ (note that $a \neq 2$), $|\delta| < 2-a$ (strict inequality) and ρ satisfies (1.3). For all $\mu \in \mathcal{M}_{a,+}(\mathbb{R})$, $\mu \neq 0$ and all $p \geq 2$, if $\underline{\varsigma} = 0$, then

$$\underline{e}(p) \geq \frac{\Upsilon\left(l_\rho, a, \delta\right)^{a^*}}{2(a+1)} > 0.$$

For these μ, if $\underline{\varsigma} = 0$ and there is $c > 0$ such that

$$J_0(t,x) \geq c, \qquad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{3.24}$$

or if $\underline{\varsigma} \neq 0$, then $\underline{e}(p) = \overline{e}(p) = +\infty$. In particular, $\underline{\lambda}(p) = \overline{\lambda}(p) = +\infty$ for all $p \geq 2$, and a sufficient condition for (3.24) is that $\mu(dx) = f(x)dx$ with $f(x) \geq c$, for all $x \in \mathbb{R}$.

The two theorems above are proved in Section 5.

Remark 3.7. In the case of the classical parabolic Anderson model, in which $a = 2$, $\delta = 0$ and $\rho(u) = \lambda u$, it was shown in [6] that $\underline{\lambda}(2) = \overline{\lambda}(2) = \lambda^2/2$ when the initial data has compact support (for instance). Here, it is natural to ask whether $\underline{e}(p) = \overline{e}(p)$ when $\rho(u) = \lambda u$, for instance for initial data with compact support. This remains an open question.

4 Proof of Theorem 3.1

We need some technical results. The proof of Theorem 3.1 will be presented at the end of this section.

The Green functions defined in (2.4) are densities of stable random variables. Some key properties are stated in the next lemma. Recall that a probability density function $f : \mathbb{R} \to \mathbb{R}_+$ is called *bell-shaped* if f is infinitely differentiable and its k-th derivative $f^{(k)}$ has exactly k zeros in its support for all k.

Lemma 4.1. *For $a \in \,]0,2]$, the following properties hold:*

(i) For fixed $t > 0$, the function ${}_\delta G_a(t,\cdot)$ is a bell-shaped density function. In particular, $\int_{\mathbb{R}} {}_\delta G_a(t,x)\mathrm{d}x = 1$.

(ii) The unique mode is located on the positive semi-axis $x > 0$ if $\delta > 0$, on the negative semi-axis $x < 0$ if $\delta < 0$, and at $x = 0$ if $\delta = 0$.

(iii) ${}_\delta G_a(t,x)$ *satisfies the semigroup property, i.e., for* $0<s<t$,

$${}_\delta G_a(t+s,x) = \left({}_\delta G_a(t,\circ) * {}_\delta G_a(s,\circ)\right)(x).$$

(iv) ${}_\delta G_a(t,x)$ *has the following scaling property: For all* $n\ge 0$,

$$\frac{\partial^n}{\partial x^n}\, {}_\delta G_a(t,x) = t^{-\frac{n+1}{a}}\, \frac{\partial^n}{\partial \xi^n}\, {}_\delta G_a(1,\xi)\bigg|_{\xi=t^{-1/a}x}. \tag{4.1}$$

(v) For $0<a<2$ *with* $a\ne 1$, *when* $x\to\pm\infty$,

$${}_\delta G_a(1,x) = \frac{1}{\pi}\sum_{j=1}^N |x|^{-aj-1}\frac{(-1)^{j+1}}{j!}\Gamma(aj+1)\sin\left(j(a\pm\delta)\pi/2\right) + O\left(|x|^{-a(N+1)-1}\right).$$

(vi) If $a\in\,]1,2]$, *then there exist finite constants* $K_{a,n}$ *such that*

$$\left|\frac{\partial^n}{\partial x^n}\, {}_\delta G_a(1,x)\right| \le \frac{K_{a,n}}{1+|x|^{1+n+a}}, \quad \text{for } n\ge 0. \tag{4.2}$$

Moreover, for all $T\ge t>0$, $n\ge 0$ *and* $x\in\mathbb{R}$,

$$\left|\frac{\partial^n}{\partial x^n}\, {}_\delta G_a(t,x)\right| \le t^{-\frac{n+1}{a}}\frac{K_{a,n}}{1+|t^{-1/a}x|^{1+n+a}} \le K_{a,n}\, t^{-\frac{n+1}{a}}\frac{(T\vee 1)^{1+\frac{n+1}{a}}}{1+|x|^{1+n+a}}. \tag{4.3}$$

(vii) $\lim_{t\downarrow 0}\, {}_\delta G_a(t,x) = \delta_0(x)$, *where* $\delta_0(x)$ *is the Dirac delta function with unit mass at zero.*

Proof. Most of these properties appear in several books [27, 24, 18]. We refer the interested readers to [13, Lemma 1] for Properties (i) (except the bell-shaped density), (iii) and (iv). Formula (v) can be found in [18], (5.9.3) in Sec. 5.9 and (5.8.6) in Sec. 5.8; for $x\to-\infty$, use (5.8.2c) in this reference (note that the formula in [13, Lemma 1(vii)] is not quite correct for $x\to-\infty$). The proof that the density is bell-shaped is due to Gawronski [16]. Property (ii) can be found in the summary part of [27, Section 2.7, p. 143–147].

Now we prove (vi). Property (4.2) follows from [13, Corollary 1]. By the scaling property (4.1) and (4.2),

$$\left|\frac{\partial^n}{\partial x^n}\, {}_\delta G_a(t,x)\right| \le t^{-\frac{n+1}{a}}\frac{K_{a,n}}{1+|t^{-1/a}x|^{1+n+a}} = t^{-\frac{n+1}{a}}\frac{K_{a,n}\, t^{1+\frac{n+1}{a}}}{t^{1+\frac{n+1}{a}}+|x|^{1+n+a}}.$$

Using the fact that the function $t\mapsto \frac{t}{t+z}$ is monotone increasing on $\mathbb{R}_+$, the above quantity is less than

$$t^{-\frac{n+1}{a}}\frac{K_{a,n}\,(T\vee 1)^{1+\frac{n+1}{a}}}{(T\vee 1)^{1+\frac{n+1}{a}}+|x|^{1+n+a}} \le t^{-\frac{n+1}{a}}\frac{K_{a,n}\,(T\vee 1)^{1+\frac{n+1}{a}}}{1+|x|^{1+n+a}}.$$

This proves (4.3).

Property (vii) follows easily by taking Fourier transforms $\mathcal{F}({}_\delta G_a(t,\cdot))(\xi) = \exp\left({}_\delta\psi_a(\xi)t\right) \to 1$ as $t\downarrow 0$. This completes the proof of Lemma 4.1. $\square$

Let $\mathcal{L}_n(t,x;\lambda)$, $\mathcal{K}(t,x;\lambda)$, and $\Lambda = {}_\delta\Lambda_a$ be defined in (2.8), (2.9), and (3.10), respectively. Recall that $1/a + 1/a^* = 1$.

Lemma 4.2 (Theorem 1.3 p. 32 in [22]). *If $0 < \alpha < 2$, β is an arbitrary complex number and μ is an arbitrary real number such that*

$$\pi\alpha/2 < \mu < \pi \wedge (\pi\alpha) ,$$

then for an arbitrary integer $p \geq 1$ the following expression holds:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp\left(z^{1/\alpha}\right) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(|z|^{-1-p}\right), \quad |z| \to \infty, \quad |\arg(z)| \leq \mu .$$

Proposition 4.3. *For $1 < a \leq 2$, $|\delta| \leq 2 - a$ and $\lambda > 0$, we have the following properties:*

(i) $\mathcal{L}_n(t,x;\lambda)$ is non-negative and for all $n \geq 0$ and $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$,

$$\mathcal{L}_n(t,x;\lambda) \leq \; B_{n+1}(t;\lambda)\, {}_\delta G_a(t,x) \;, \tag{4.4}$$

where

$$B_n\left(t;\lambda\right) := \lambda^{2n} \Lambda^n \frac{\Gamma\left(1/a^*\right)^n}{\Gamma\left(n/a^*\right)}\, t^{-1+n/a^*} \qquad (n \geq 0,\; \lambda \in \mathbb{R}).$$

(ii) $B_n\left(t\,;\lambda\right) \geq 0$ and for all $m \in \mathbb{N}^*$, $\sum_{n=0}^\infty B_n\left(t\,;\lambda\right)^{1/m} < +\infty$.

(iii) For all $t > 0$ and $\lambda > 0$, the series $\sum_{n=1}^\infty \mathcal{L}_n(t,x;\lambda)$ converges uniformly over $x \in \mathbb{R}$ and hence $\mathcal{K}(t,x;\lambda)$ in (2.9) is well defined.

Proof. (i) Non-negativity is clear. The scaling property (4.1) and the definition of Λ in (3.10) imply that

$${}_\delta G_a(t,x) \leq t^{-1/a} \Lambda \;, \tag{4.5}$$

which establishes the case $n = 0$ in (4.4). Suppose by induction that the relation (4.4) holds up to $n - 1$. Then by (4.5), we have

$$\begin{aligned} \mathcal{L}_n(t,x;\lambda) = & \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, \mathcal{L}_{n-1}\left(t-s, x-y\right) \lambda^2 \, {}_\delta G_a^2\left(s,y\right) \\ \leq & \lambda^{2(n+1)} \Lambda^{n+1} \frac{\Gamma\left(1/a^*\right)^n}{\Gamma\left(n/a^*\right)} \int_0^t \mathrm{d}s \, \left(t-s\right)^{-1+n/a^*} s^{-1/a} \\ & \times \int_{\mathbb{R}} \mathrm{d}y \;\, {}_\delta G_a\left(t-s, x-y\right) \, {}_\delta G_a\left(s,y\right). \end{aligned}$$

The conclusion now follows from the semigroup property of ${}_\delta G_a(t,x)$ and Euler's Beta integral (see [21, 5.12.1, on p. 142]):

$$\int_0^t \mathrm{d}s\; s^{a-1} (t-s)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\, t^{a+b-1}, \qquad \text{with } \Re(a) > 0 \text{ and } \Re(b) > 0. \tag{4.6}$$

(ii) The non-negativity is clear. Denote $\beta := 1/a^*$. In order to show convergence of $\sum_n B_n(t\,;\lambda)^{1/m}$, we use the ratio test:

$$\left(\frac{B_n(t\,;\lambda)}{B_{n-1}(t\,;\lambda)}\right)^{1/m} = \left(\lambda^2\Lambda\Gamma(\beta)\,t^\beta\right)^{1/m}\left(\frac{\Gamma((n-1)/a^*)}{\Gamma(n/a^*)}\right)^{1/m}.$$

By the asymptotic expansion of the Gamma function ([21, 5.11.2, on p. 140]),

$$\frac{\Gamma((n-1)/a^*)}{\Gamma(n/a^*)} \approx \left(\frac{e}{\beta}\right)^\beta \left(1-\frac{1}{n}\right)^{(n-1)\beta}\frac{1}{n^\beta} \approx \frac{1}{(\beta n)^\beta},$$

for large n. Clearly, $\beta > 0$ since $1/a < 1$. Hence for all $t > 0$, for large n,

$$\left(\frac{B_n(t\,;\lambda)}{B_{n-1}(t\,;\lambda)}\right)^{1/m} \approx \left(\lambda^2\Lambda\Gamma(\beta)\,t^\beta\right)^{1/m}\frac{1}{(\beta n)^{\beta/m}},$$

and this goes to zero as $n \to +\infty$.

(iii) By (4.4) and (4.5),

$$\mathcal{L}_n(t,x;\lambda) \le B_{n+1}(t\,;\lambda)\,t^{-1/a}\Lambda,$$

so (ii) implies (iii). This completes the proof of Proposition 4.3. □

Proof of Proposition 3.2. The bound (3.12) follows from the fact that

$$\sum_{k=1}^{\infty}\frac{z^k}{\Gamma(\alpha k)} = zE_{\alpha,\alpha}(z)\,, \tag{4.7}$$

which can be easily seen from the definition, and the bound in Proposition 4.3 (i):

$$\begin{aligned}\mathcal{K}(t,x;\lambda) \le\ {}_\delta G_a(t,x)\sum_{n=1}^{\infty}B_n(t;\lambda) &= \frac{1}{t}\,{}_\delta G_a(t,x)\sum_{n=1}^{\infty}\frac{\left(\lambda^2\Lambda\Gamma(1/a^*)\,t^{1/a^*}\right)^n}{\Gamma(n/a^*)}\\ &= \lambda^2\Lambda\Gamma(1/a^*)t^{-1/a}\,{}_\delta G_a(t,x)E_{1/a^*,1/a^*}\left(\lambda^2\Lambda\Gamma(1/a^*)t^{1/a^*}\right).\end{aligned}$$

As for (3.13), we only need to show that the constant C defined in (3.14) is finite. Let

$$f(t) = \frac{E_{1/a^*,1/a^*}\left(\gamma\,t^{1/a^*}\right)}{1+t^{1/a}\exp\left(\gamma^{a^*}\,t\right)}.$$

By Lemma 4.2 with the real non-negative value $z = \gamma\,t^{1/a^*}$ and $p = 1$,

$$\gamma E_{1/a^*,1/a^*}\left(\gamma\,t^{1/a^*}\right) \le a^*\,\gamma^{a^*}\,t^{1/a}\exp\left(\gamma^{a^*}\,t\right) + O\left(\frac{1}{|t|^{2/a^*}}\right), \qquad t\to+\infty\,,$$

where we have used the convention that $1/\Gamma(0) = 0$ (see [21, 5.7.1, p.139]), therefore

$$\lim_{t\to+\infty} f(t) \le a^*\gamma^{a^*}.$$

Since $E_{\alpha,\alpha}(\cdot)$ is continuous (by uniform convergence of the series in (3.9)), we conclude that $\sup_{t\ge0} f(t) < +\infty$. This completes the proof of Proposition 3.2. □

The next proposition is in principle a consequence of certain calculations in [13]. It is however not stated explicitly there, so we include a proof for the convenience of the reader.

Proposition 4.4. *Fix $1 < a \le 2$, $|\delta| \le 2-a$ and $1/a + 1/a^* = 1$. There are three universal constants*

$$C_1 := \int_{\mathbb{R}} \frac{1-\cos(u)}{2\pi\cos(\pi\delta/2)|u|^a}\,\mathrm{d}u\ , \quad C_3 := \frac{a^*\Gamma(1+1/a)}{2^{1/a}\pi\cos^{1/a}(\pi\delta/2)},$$

and

$$C_2 := \left(2^{1/a}-1\right)C_3 + \frac{\Gamma(1+1/a)\sin^2(\pi\delta/2)}{2\pi a[\cos(\pi\delta/2)]^{2+1/a}}\ ,$$

such that

(i) for all $t > 0$ and $x, y \in \mathbb{R}$,

$$\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \left[\,{}_\delta G_a(t-r, x-z) - {}_\delta G_a(t-r, y-z)\right]^2 \le C_1|x-y|^{a-1}\ ; \tag{4.8}$$

(ii) for all $s, t \in \mathbb{R}_+^*$ with $s \le t$, and $x \in \mathbb{R}$,

$$\int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \left[\,{}_\delta G_a(t-r, x-z) - {}_\delta G_a(s-r, x-z)\right]^2 \le C_2(t-s)^{1-1/a} \tag{4.9}$$

and

$$\int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \left[\,{}_\delta G_a(t-r, x-z)\right]^2 \le C_3(t-s)^{1-1/a}\ . \tag{4.10}$$

Remark 4.5. This proposition is a generalization of [6, Proposition 3.5] for the heat equation. In fact, if we take $a = 2$ and $\delta = 0$, then ${}_\delta G_a(t,x) = G_2(t,x) = \frac{1}{\sqrt{4\pi t}}\exp\left(-x^2/(4t)\right)$. Let C_i', $i = 1, 2, 3$, be the optimal constants given in [6, Proposition 3.5] with $\nu = 2$. Then we have the following relations:

$$\frac{C_1'}{2} = C_1 = \frac{1}{2}\ , \qquad \frac{C_2'}{\sqrt{2}} = C_2 = \frac{\sqrt{2}-1}{\sqrt{2\pi}}, \qquad \frac{C_3'}{\sqrt{2}} = C_3 = \frac{1}{\sqrt{2\pi}}\ ,$$

where for C_1, we use the fact that $\int_{\mathbb{R}} \frac{1-\cos(u)}{u^2}\,\mathrm{d}u = 2\int_0^\infty \frac{\sin(u)}{u}\,\mathrm{d}u = \pi$; see [21, 4.26.12, on p. 122] for the last integral. This recovers the optimal inequalities of [6, Proposition 3.5], since ν appears in the right-hand side of the inequalities in this reference.

Proof of Proposition 4.4. (i) Note that

$$\mathcal{F}(\,{}_\delta G_a(t,\cdot))(\xi) := \int_{\mathbb{R}} \mathrm{d}x\, e^{-i\xi x}\,{}_\delta G_a(t,x) = \exp\left\{t\ {}_\delta\psi_a(\xi)\right\} = \exp\left\{-t|\xi|^a e^{-i\pi\delta\,\mathrm{sgn}(\xi)/2}\right\}.$$

By Plancherel's theorem, the left hand side of (4.8) equals

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \left| e^{-i\xi x-(t-r)|\xi|^a e^{i\pi\delta\,\mathrm{sgn}(\xi)/2}} - e^{-i\xi y-(t-r)|\xi|^a e^{i\pi\delta\,\mathrm{sgn}(\xi)/2}} \right|^2 \\
&\qquad = \frac{1}{2\pi}\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi\ \exp\{-2(t-r)|\xi|^a\cos(\pi\delta/2)\} \left| e^{-i\xi x} - e^{-i\xi y}\right|^2 \\
&\qquad = \frac{1}{\pi}\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi\ \exp\{-2(t-r)|\xi|^a\cos(\pi\delta/2)\} \left(1-\cos(\xi(x-y))\right).
\end{aligned}$$

After integrating over r, the above integral equals

$$\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{d}\xi\, \frac{1-\exp\{-2t|\xi|^a\cos(\pi\delta/2)\}}{2\cos(\pi\delta/2)|\xi|^a} \left(1-\cos(\xi(x-y))\right).$$

Use the change of variables $\xi = u/|x-y|$ to see that this is equal to

$$\frac{1}{\pi}|x-y|^{a-1}\int_{\mathbb{R}} \mathrm{d}u\, \frac{1-\exp(-2t|u|^a\cos(\pi\delta/2)/|x-y|^a)}{2\cos(\pi\delta/2)|u|^a}(1-\cos(u)) \le C_1'\,|x-y|^{a-1},$$

where

$$C_1' = \int_{\mathbb{R}} \frac{1-\cos(u)}{2\pi\cos(\pi\delta/2)|u|^a}\,\mathrm{d}u.$$

This proves (4.8).

(ii) Denote the left hand side of (4.9) by I. Apply Plancherel's theorem to I:

$$\begin{aligned}
I =& \frac{1}{2\pi}\int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi\, |\exp(-i\xi x-(t-r)|\xi|^a e^{i\pi\delta\,\mathrm{sgn}(\xi)/2}) \\
&\qquad\qquad - \exp(-i\xi x-(s-r)|\xi|^a e^{i\pi\delta\,\mathrm{sgn}(\xi)/2})|^2 \\
=& \frac{1}{2\pi}\int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi\, |\exp(-(t-r)|\xi|^a e^{i\pi\delta\,\mathrm{sgn}(\xi)/2}) - \exp(-(s-r)|\xi|^a e^{i\pi\delta\,\mathrm{sgn}(\xi)/2})|^2.
\end{aligned}$$

Denote $\beta := \pi\delta\,\mathrm{sgn}(\xi)/2 \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ and

$$A_{r,t} := (t-r)|\xi|^a\cos(\beta), \qquad B_{r,t} := (t-r)|\xi|^a\sin(\beta)\,.$$

Then

$$\begin{aligned}
&\left|\exp(-(t-r)|\xi|^a e^{i\pi\delta\,\mathrm{sgn}(\xi)/2}) - \exp(-(s-r)|\xi|^a e^{i\pi\delta\,\mathrm{sgn}(\xi)/2})\right|^2 \\
&\quad = \left|e^{-A_{r,t}}\cos(B_{r,t}) - ie^{-A_{r,t}}\sin(B_{r,t}) - e^{-A_{r,s}}\cos(B_{r,s}) + ie^{-A_{r,s}}\sin(B_{r,s})\right|^2 \\
&\quad = e^{-2A_{r,t}} + e^{-2A_{r,s}} - 2e^{-(A_{r,t}+A_{r,s})}\cos\left(B_{r,t}-B_{r,s}\right).
\end{aligned}$$

Now, according to [21, Equation 5.9.1, p.139], for $z\in\mathbb{C}$ with $\Re(z)>0$,

$$\int_{\mathbb{R}} \mathrm{d}\xi\, e^{-z|\xi|^a} = 2\int_0^\infty \mathrm{d}\xi\, e^{-z\xi^a} = 2z^{-1/a}\Gamma\left(1+1/a\right). \tag{4.11}$$

Hence, since $\cos(\beta) = \cos(\pi\delta/2)$, which does not depend on ξ,

$$\int_{\mathbb{R}} \mathrm{d}\xi \; e^{-2A_{r,t}} = \int_{\mathbb{R}} \mathrm{d}\xi \; e^{-2(t-r)\cos(\beta)|\xi|^a} = \frac{2^{1/a^*}\Gamma(1+1/a)}{\cos^{1/a}(\beta)} \frac{1}{(t-r)^{1/a}} \,. \tag{4.12}$$

Similarly,

$$\int_{\mathbb{R}} \mathrm{d}\xi \; e^{-2A_{r,s}} = \frac{2^{1/a^*}\Gamma(1+1/a)}{\cos^{1/a}(\beta)} \frac{1}{(s-r)^{1/a}} \,.$$

For the third term, notice that

$$\begin{aligned} e^{-(A_{r,t}+A_{r,s})}\cos(B_{r,t}-B_{r,s}) &= \exp\left(-\left(\frac{t+s}{2}-r\right)2\,|\xi|^a\cos(\beta)\right)\cos\left((t-s)\,|\xi|^a\sin(\beta)\right) \\ &= \Re\left[\exp\left\{-\left[\left(\frac{t+s}{2}-r\right)2\cos(\beta)+i(t-s)\sin(\beta)\right]|\xi|^a\right\}\right]. \end{aligned}$$

Apply (4.11) with $z = \left(\frac{t+s}{2}-r\right)2\cos(\beta)+i(t-s)\sin(\beta)$:

$$\begin{aligned} &\int_{\mathbb{R}} \mathrm{d}\xi \;\exp\left\{-\left[\left(\frac{t+s}{2}-r\right)2\cos(\beta)+i(t-s)\sin(\beta)\right]|\xi|^a\right\} \\ &\qquad = 2\Gamma(1+1/a)\left[\left(\frac{t+s}{2}-r\right)2\cos(\beta)+i(t-s)\sin(\beta)\right]^{-1/a}. \end{aligned}$$

Apply Lemma 4.6 below with $c=1/a$, $b=\left(\frac{t+s}{2}-r\right)2\cos(\beta)$ and $x=(t-s)^2\sin^2(\beta)$:

$$\begin{aligned} \left[\left(\frac{t+s}{2}-r\right)2\cos(\beta)+i(t-s)\sin(\beta)\right]^{-1/a} \geq\; & \frac{1}{2^{1/a}\cos^{1/a}(\beta)}\frac{1}{((t+s)/2-r)^{1/a}} \\ & - \frac{(a+1)\sin^2(\beta)}{2a^2\left[2\cos(\beta)\right]^{2+1/a}}\frac{(t-s)^2}{((t+s)/2-r)^{2+1/a}}. \end{aligned}$$

Hence,

$$\begin{aligned} 2\int_{\mathbb{R}} \mathrm{d}\xi\; e^{-(A_{r,t}+A_{r,s})}\cos(B_{r,t}-B_{r,s}) \geq & 2^{1+1/a^*}\frac{\Gamma(1+1/a)}{\cos^{1/a}(\beta)}\frac{1}{((t+s)/2-r)^{1/a}} \\ & - \frac{2\Gamma(1+1/a)(a+1)\sin^2(\beta)}{a^2\left[2\cos(\beta)\right]^{2+1/a}}\frac{(t-s)^2}{((t+s)/2-r)^{2+1/a}}. \end{aligned}$$

Integrating over r and then applying Lemma 4.7 below, we get

$$\begin{aligned} I \leq & \frac{\Gamma(1+1/a)}{2^{1/a}\pi\cos^{1/a}(\beta)}\int_0^s \mathrm{d}r\left(\frac{1}{(t-r)^{1/a}}+\frac{1}{(s-r)^{1/a}}-\frac{2}{\left[(t+s)/2-r\right]^{1/a}}\right) \\ & + \frac{\Gamma(1+1/a)(a+1)\sin^2(\beta)}{\pi a^2\left[2\cos(\beta)\right]^{2+1/a}}\int_0^s \mathrm{d}r\frac{(t-s)^2}{((t+s)/2-r)^{2+1/a}} \end{aligned}$$

$$\leq C_2(t-s)^{1/a^*},$$

where $1/a^* + 1/a = 1$.

As for (4.10), from (4.12), we have

$$\int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \, [\,{}_\delta G_a(t-r, x-z)]^2 = \frac{1}{2\pi} \int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}\xi \, e^{-2(t-r)|\xi|^a \cos(\beta)}$$
$$= \frac{\Gamma(1+1/a)}{2^{1/a}\pi \cos^{1/a}(\beta)} \int_s^t \mathrm{d}r \, \frac{1}{(t-r)^{1/a}} = \frac{a^*\Gamma(1+1/a)}{2^{1/a}\pi \cos^{1/a}(\beta)} (t-s)^{1/a^*} \, . \quad (4.13)$$

This completes the proof of Proposition 4.4. □

Lemma 4.6. *Suppose $b > 0$ and $c \in \,]0,1]$. Then for all $x \geq 0$,*

$$\Re\left((b \pm i\sqrt{x})^{-c}\right) \geq \frac{1}{b^c} - \frac{c(1+c)}{2} \frac{x}{b^{2+c}} \, .$$

Proof. Let $\theta = \arctan(\sqrt{x}/b) \in \; [0, \pi/2[$ and denote $f(x) := \Re\left((b \pm i\sqrt{x})^{-c}\right)$. Then

$$f(x) = \left(b^2 + x\right)^{-c/2} \cos\left(c\,\theta\right).$$

Because $\cos(\theta) \geq 1 - \theta^2/2$ and $\arctan(y) \leq y$ for $y \geq 0$, we have that

$$\cos\left(c\,\theta\right) \geq 1 - \frac{c^2\,\theta^2}{2} \geq 1 - \frac{c^2 x}{2b^2}.$$

By Taylor's theorem, for some $\zeta \in [0, x]$,

$$\left(b^2 + x\right)^{-c/2} = b^{-c} - \frac{1}{2} c x (b^2 + \zeta)^{-1-c/2} \geq b^{-c} - \frac{1}{2} c x b^{-2-c}.$$

Combining the above two lower bounds proves the lemma. □

Lemma 4.7. *For all $t \geq s \geq 0$ and $a \in \,]1,2]$, we have*

$$\int_0^s \mathrm{d}r \left(\frac{1}{(t-r)^{1/a}} + \frac{1}{(s-r)^{1/a}} - \frac{2}{((t+s)/2 - r)^{1/a}} \right) \leq a^*(2^{1/a} - 1) \; (t-s)^{1/a^*} \; ,$$

and

$$\int_0^s \mathrm{d}r \frac{(t-s)^2}{\left((t+s)/2 - r\right)^{2+1/a}} \leq \frac{a}{a+1} \, 2^{1+1/a} \, (t-s)^{1/a^*},$$

where a^* is the dual of a: $1/a + 1/a^* = 1$.

Proof. Clearly,

$$\frac{1}{a^*}\int_0^s \mathrm{d}r\left(\frac{1}{(t-r)^{1/a}}+\frac{1}{(s-r)^{1/a}}-\frac{2}{((t+s)/2-r)^{1/a}}\right)$$
$$= s^{1/a^*}+t^{1/a^*}-(t-s)^{1/a^*}+2^{1/a}(t-s)^{1/a^*}-2^{1/a}(t+s)^{1/a^*}\,.$$

We shall to prove that

$$(t-s)^{-1/a^*}\left[s^{1/a^*}+t^{1/a^*}-(t-s)^{1/a^*}+2^{1/a}(t-s)^{1/a^*}-2^{1/a}(t+s)^{1/a^*}\right]$$

is bounded from above for all $0\le s\le t$, or, equivalently, that

$$g(r):=\frac{r^{1/a^*}+1-(1-r)^{1/a^*}+2^{1/a}(1-r)^{1/a^*}-2^{1/a}(1+r)^{1/a^*}}{(1-r)^{1/a^*}}$$

is bounded for all $r\in[0,1]$. Clearly, $g(0)=0$ and $\lim_{r\uparrow 1} g(r)=2^{1/a}-1$ (by applying L'Hospital's rule once). Hence $\sup_{r\in[0,1]} g(r)<\infty$. In addition,

$$g'(r)=\frac{\left((1+r)^{1/a}+(1+1/r)^{1/a}\right)-2^{1+1/a}}{a^*(1-r)^{2-1/a}(1+r)^{1/a}},$$

and notice that for all $r\in\,]0,1]$,

$$(1+r)^{1/a}+(1+1/r)^{1/a}\ge 2\left[(1+r)(1+1/r)\right]^{1/(2a)}=2\left(\sqrt{r}+\frac{1}{\sqrt{r}}\right)^{1/a}\ge 2^{1+1/a}\,.$$

Hence $g'(r)\ge 0$ for $r\in\,]0,1]$ and $\sup_{r\in[0,1]} g(r)=\lim_{r\uparrow 1} g(r)=2^{1/a}-1$. Therefore, the first inequality is proved with the constant $a^*(2^{1/a}-1)$.

As for the second inequality, we have that

$$\int_0^s \mathrm{d}r\frac{(t-s)^2}{((t+s)/2-r)^{2+1/a}}=\frac{a}{a+1}\,2^{1+1/a}\,\frac{\left((t+s)^{1+1/a}-(t-s)^{1+1/a}\right)}{(t+s)^{1+1/a}}(t-s)^{1/a^*}$$
$$\le\frac{a}{a+1}\,2^{1+1/a}\,(t-s)^{1/a^*},$$

which completes the proof of the Lemma 4.7. ☐

The following proposition is useful to prove the $L^p(\Omega)$–continuity of $I(t,x)$.

Proposition 4.8. *Suppose that $a\in\,]1,2]$ and $|\delta|\le 2-a$. Fix $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$. Denote*

$$B:=B_{t,x}=\left\{(t',x')\in\mathbb{R}_+^*\times\mathbb{R}:\ 0\le t'\le t+1/2,\ |x-x'|\le 1\right\}.$$

Then there exists a constant $A>0$ such that for all $(t',x')\in B$, $s\in[0,t'[$ and $|y|\ge A$,

$${}_{\delta}G_a\left(t'-s,x'-y\right)\le\ {}_{-\delta}G_a\left(t+1-s,x-y\right)+\ {}_{\delta}G_a\left(t+1-s,x-y\right).$$

Proof. The case where $a = 2$ is proved in [6, Proposition A.3], so we only need to prove the case where $1 < a < 2$. Denote $F(t, x) := {}_{\delta}G_a(t, x) + {}_{-\delta}G_a(t, x)$. Suppose the mode of the density ${}_{\delta}G_a(1, x)$ is located at $m \in \mathbb{R}$. By the scaling property, the mode of the density ${}_{\delta}G_a(t, x)$ is located at $t^{1/a}m$. Hence, when $x \geq t^{1/a}|m|$ (resp. $x \leq -t^{1/a}|m|$), the function $x \mapsto F(t, x)$ is decreasing (resp. increasing).

Fix $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$. Assume that $|y-x| > 1+(t+1/2)^{1/a}|m|$. From the above observation, we deduce that for all $(t', x') \in B$,

$$ {}_{\delta}G_a(t' - s, x' - y) \leq F(t' - s, |y - x| - |x - x'|) \leq F(t' - s, |y - x| - 1). \tag{4.14} $$

Apply Lemma 4.1(v) with $N = 1$ and use the scaling property of ${}_{\delta}G_a(t, x)$ to get

$$ F(t, x) = 2\frac{\Gamma(a + 1)}{\pi} \sin\left(\frac{\pi}{2}a\right) \cos\left(\frac{\pi}{2}\delta\right) t\,|x|^{-1-a} + O\left(t^2\,|x|^{-1-2a}\right). $$

Because $|\delta| \leq 2 - a$ and $a \in]1, 2[$, we see that $\sin(\pi a/2)\cos(\pi\delta/2) \neq 0$. Hence,

$$ \begin{aligned} \frac{F(t + 1 - s, x - y)}{F(t' - s, |y - x| - 1)} &= \frac{(t + 1 - s)|x - y|^{-1-a} + O\left((t + 1 - s)^2\,|x - y|^{-1-2a}\right)}{(t' - s)\,||y - x| - 1|^{-1-a} + O\left((t' - s)^2||y - x| - 1|^{-1-2a}\right)} \\ &= \frac{t + 1 - s}{t' - s}\,\frac{|x - y|^{-1-a} + O\left((t + 1 - s)\,|x - y|^{-1-2a}\right)}{||y - x| - 1|^{-1-a} + O\left((t' - s)\,||y - x| - 1|^{-1-2a}\right)}. \end{aligned} $$

Now it is clear that

$$ \lim_{|y|\to+\infty}\ \inf_{(t',x')\in B,\, s\in[0,t'[}\ \frac{|x - y|^{-1-a} + O\left((t + 1 - s)|x - y|^{-1-2a}\right)}{||y - x| - 1|^{-1-a} + O\left((t' - s)\,||y - x| - 1|^{-1-2a}\right)} = 1, $$

which implies that

$$ \begin{aligned} \lim_{|y|\to+\infty}\ \inf_{(t',x')\in B,\, s\in[0,t'[}\ \frac{F(t + 1 - s, x - y)}{F(t' - s, |y - x| - 1)} &\geq \inf_{(t',x')\in B,\, s\in[0,t'[} \frac{t + 1 - s}{t + 1/2 - s} \\ &= \frac{t + 1}{t + 1/2} = 1 + \frac{1}{2t + 1} > 1, \end{aligned} $$

where we have used the fact that $s \mapsto (t + 1 - s)/(t + 1/2 - s)$ is increasing. Hence, by (4.14), we can choose a large constant A such that for all $|y| \geq A$, $(t', x') \in B$ and $s \in [0, t'[$, the inequality

$$ \frac{F(t + 1 - s, x - y)}{{}_{\delta}G_a(t' - s, x' - y)} \geq 1 + \frac{1}{2(t + 1)} > 1 $$

holds. This completes the proof of Proposition 4.8. □

Lemma 4.9. *For all $m, n \in \mathbb{N}$, there exist polynomials $\{P_i^{(n,m)}(x),\ i = 0, \dots, n\}$, such that:*

(1) the $P_i^{(n,m)}(x)$ are of degree $\leq i$ and they satisfy

$$ \frac{\partial^{n+m}}{\partial t^n \partial x^m}\,{}_{\delta}G_a(t, x) = \frac{1}{(at)^n} \sum_{i=0}^{n} P_i^{(n,m)}(x) \frac{\partial^{i+m}}{\partial x^{i+m}}\,{}_{\delta}G_a(t, x)\,; $$

(2) For fixed $t > 0$, the partial derivative $\frac{\partial^{n+m}}{\partial t^n \partial x^m}\, {}_\delta G_a(t,\cdot)$ as a function of x is smooth and integrable.

Proof. Part (2) is a direct consequence of (1) and (i) and (vi) in Lemma 4.1. We now prove (1). It is clearly true for $n = m = 0$: in this case, $P_0^{(0,0)}(x) \equiv 1$. Moreover, if $n = 0$, then it is trivially true, with $P_0^{(0,m)}(x) = 1$. Consider the case $n = 1$ and $m = 0$. Using the scaling properties twice, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\, {}_\delta G_a(t,x) &= \left[-\frac{1/a}{t^{1+1/a}}\, {}_\delta G_a\left(1,\xi\right) - \frac{1}{t^{1/a}} \frac{\partial\, {}_\delta G_a(1,\xi)}{\partial \xi} \frac{x/a}{t^{1+1/a}} \right]\Bigg|_{\xi = t^{-1/a}x} \\
&= -\frac{1}{at}\left(\frac{1}{t^{1/a}}\, {}_\delta G_a\left(1, \frac{x}{t^{1/a}}\right) + \frac{x}{t^{2/a}} \frac{\partial\, {}_\delta G_a(1,\xi)}{\partial \xi}\bigg|_{\xi = t^{-1/a}x} \right) \\
&= -\frac{1}{at}\left({}_\delta G_a(t,x) + x \frac{\partial}{\partial x}\, {}_\delta G_a(t,x) \right).
\end{aligned}$$

So in this case, $P_0^{(1,0)}(x) = -1$ and $P_1^{(1,0)}(x) = -x$. Now suppose that (1) is true for $n, m \in \mathbb{N}$. It is easy to see that (1) is true also for $n, m+1$ with

$$P_i^{(n,m+1)}(x) = P_i^{(n,m)}(x) + \frac{\mathrm{d}}{\mathrm{d}x} P_{i+1}^{(n,m)}(x), \quad \text{for } i = 0,\dots,n-1, \qquad P_n^{(n,m+1)}(x) = P_n^{(n,m)}(x),$$

so $P_i^{(n,m+1)}(x)$ is a polynomial of degree $\le i$.

Now assume by induction that $n \ge 1$ and the property is true for $\tilde{n} \le n$ and all $m \ge 0$. We shall establish the property for $n+1$ and m. By the induction assumption, we have

$$\frac{\partial^{n+1+m}}{\partial t^{n+1} \partial x^m} {}_\delta G_a(t,x) = \frac{-na}{(at)^{n+1}} \sum_{i=0}^{n} P_i^{(n,m)}(x) \frac{\partial^{i+m}}{\partial x^{i+m}} {}_\delta G_a(t,x) + \frac{1}{(at)^n} \sum_{i=0}^{n} P_i^{(n,m)}(x) \frac{\partial^{1+i+m}}{\partial t \partial x^{i+m}} {}_\delta G_a(t,x).$$

Then replace $\frac{\partial^{1+i+m}}{\partial t \partial x^{i+m}}\, {}_\delta G_a(t,x)$ by the following sum using the induction assumption:

$$\frac{\partial^{1+i+m}}{\partial t \partial x^{i+m}}\, {}_\delta G_a(t,x) = \frac{1}{at}\left(P_0^{(1,i+m)}(x) \frac{\partial^{i+m}}{\partial x^{i+m}}\, {}_\delta G_a(t,x) + P_1^{(1,i+m)}(x) \frac{\partial^{i+m+1}}{\partial x^{i+m+1}}\, {}_\delta G_a(t,x) \right).$$

Finally, after grouping terms one can choose the following polynomials:

$$P_0^{(n+1,m)}(x) = -\, na P_0^{(n,m)}(x) + P_0^{(n,m)}(x) P_0^{(1,m)}(x),$$

which is a polynomial of order 0,

$$P_i^{(n+1,m)}(x) = -\, na P_i^{(n,m)}(x) + P_i^{(n,m)}(x) P_0^{(1,i+m)}(x) + P_{i-1}^{(n,m)}(x) P_1^{(1,i+m-1)}(x),$$

which are polynomials of degree $\le i$, for $i = 1,\dots,n$, and

$$P_{n+1}^{(n+1,m)}(x) = P_n^{(n,m)}(x) P_1^{(1,n+m)}(x),$$

which are polynomials of degree $\le n+1$. This completes the proof of Lemma 4.9. □

Lemma 4.10. *Suppose that $a \in]1,2]$ and $\mu \in \mathcal{M}_a(\mathbb{R})$.*

(1) The function $J_0(t,x) = ({}_\delta G_a(t,\cdot) * \mu)(x)$ belongs to $C^\infty\left(\mathbb{R}_+^* \times \mathbb{R}\right)$.

(2) For all compact sets $K \subset \mathbb{R}_+^* \times \mathbb{R}$ and $v \in \mathbb{R}$,

$$\sup_{(t,x)\in K} \left(\left[v^2 + J_0^2\right] \star \mathcal{K}\right)(t,x) < \infty. \tag{4.15}$$

In fact, for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$,

$$\left(J_0^2 \star \mathcal{K}\right)(t,x) \le C'(t \vee 1)^{2(1+1/a)} t^{1-2/a} \left[t^{-1/a} + \exp\left(\gamma^{a^*} t\right)\right], \tag{4.16}$$

where

$$C' := C A_a^2 K_{a,0}^2 \max\left(a^*, \frac{\Gamma(1/a^*)^2}{\Gamma(2/a^*)}\right), \tag{4.17}$$

$C = C(a,\delta,\lambda)$ is defined in (3.14), $K_{a,0}$ is defined in (4.2), and

$$A_a := \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} \frac{|\mu|(\mathrm{d}z)}{1+|y-z|^{1+a}}. \tag{4.18}$$

Proof. (1) Fix $0 < t \le T$ and $n, m \in \mathbb{N}$. By Lemma 4.9 and (4.3),

$$\left|\frac{\partial^{n+m}}{\partial t^n \partial x^m}\, {}_\delta G_a(t,x)\right| \le \frac{1}{(at)^n} \sum_{i=0}^n \left|P_i^{(n,m)}(x)\right| K_{a,i+m}\, t^{-(i+m+1)/a} \frac{(T\vee 1)^{1+(i+m+1)/a}}{1+|x|^{1+i+m+a}}.$$

Since the polynomials $P_i^{(n,m)}(x)$ are of degree $\le i$, for some finite constant $C > 0$ depending on a, m, n and T, the above bound reduces to

$$\left|\frac{\partial^{n+m}}{\partial t^n \partial x^m}\, {}_\delta G_a(t,x)\right| \le C \frac{g(t)}{1+|x|^{m+1+a}}, \quad \text{with } g(t) := \sum_{i=0}^n t^{-n-(i+m+1)/a}.$$

Hence, for $0 < t_1 < t_2 \le T$,

$$\int_{t_1}^{t_2} \mathrm{d}s \int_{\mathbb{R}} \mu(\mathrm{d}z) \left|\frac{\partial^{n+m}}{\partial s^n \partial x^m}\, {}_\delta G_a(s,z)\right| < +\infty. \tag{4.19}$$

By Fubini's theorem and induction, it is now possible to conclude that $J_0(\cdot,\circ) \in C^\infty(\mathbb{R}_+^* \times \mathbb{R})$. Indeed, the first step of this induction argument is:

$$\begin{aligned} J_0(t_2,x) - J_0(t_1,x) &= \int_{\mathbb{R}} \mu(\mathrm{d}y)\left({}_\delta G_a(t_2,x-y) - {}_\delta G_a(t_1,x-y)\right) \\ &= \int_{\mathbb{R}} \mu(\mathrm{d}y) \int_{t_1}^{t_2} \mathrm{d}t \frac{\partial}{\partial t}\, {}_\delta G_a(t,x-y) = \int_{t_1}^{t_2} \mathrm{d}t \int_{\mathbb{R}} \mu(\mathrm{d}y) \frac{\partial}{\partial t}\, {}_\delta G_a(t,x-y), \end{aligned}$$

where we have used Fubini's theorem, which applies by (4.19). This shows that

$$\frac{\partial}{\partial t}J_0(t,x) = \int_{\mathbb{R}} \mu(\mathrm{d}y)\frac{\partial}{\partial t}\,{}_\delta G_a(t,x-y),$$

and higher derivatives are obtained by induction. This proves (1).

(2) By (4.3), for $0 < s \le t$,

$$|J_0\,(s,y)\,| \le A_a\,K_{a,0}\,(t\vee 1)^{1+1/a}s^{-1/a}, \tag{4.20}$$

where A_a is defined in (4.18). Let $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$. By (3.13), and by replacing one factor $|J_0(s,y)|$ of $J_0^2(s,y)$ by the above bound, we have that

$$\begin{aligned}\left(J_0^2\star\mathcal{K}\right)(t,x) \le C\int_0^t \mathrm{d}s\left(\frac{1}{(t-s)^{1/a}} + \exp\left(\gamma^{a^*}(t-s)\right)\right)\int_{\mathbb{R}}\mathrm{d}y\ {}_\delta G_a\left(t-s,x-y\right)\\ \times A_aK_{a,0}(t\vee 1)^{1+1/a}s^{-1/a}\left|\int_{\mathbb{R}}\mu(\mathrm{d}z)\,{}_\delta G_a(s,y-z)\right|,\end{aligned}$$

where the constant $C := C(a,\delta,\lambda)$ is defined in (3.14). Integrate over $\mathrm{d}y$ using the semigroup property, and then integrate over $\mu(\mathrm{d}z)$:

$$\left(J_0^2\star\mathcal{K}\right)(t,x) \le CA_aK_{a,0}(t\vee 1)^{1+1/a}|J_0(t,x)|\int_0^t\mathrm{d}s\,\frac{1}{s^{1/a}}\left[\frac{1}{(t-s)^{1/a}}+\exp\left(\gamma^{a^*}(t-s)\right)\right]. \tag{4.21}$$

Apply (4.20) to $J_0(t,x)$. The integral over s gives

$$\int_0^t\mathrm{d}s\left[\frac{1}{s^{1/a}(t-s)^{1/a}}+\frac{1}{s^{1/a}}\exp\left(\gamma^{a^*}(t-s)\right)\right] \le \int_0^t\mathrm{d}s\left[\frac{1}{s^{1/a}(t-s)^{1/a}}+\frac{1}{s^{1/a}}\exp\left(\gamma^{a^*}t\right)\right].$$

Use (4.6) to see that this is equal to

$$t^{1-2/a}\frac{\Gamma\left(1-1/a\right)^2}{\Gamma\left(2-2/a\right)} + a^*t^{1/a^*}\exp\left(\gamma^{a^*}t\right) = t^{1/a^*}\left[\frac{1}{t^{1/a}}\frac{\Gamma\left(1/a^*\right)^2}{\Gamma\left(2/a^*\right)} + a^*\exp\left(\gamma^{a^*}t\right)\right]. \tag{4.22}$$

Hence, combining the above facts proves (4.16).

For (4.15), we consider the case $\mu(dx) = v\,dx$, for which $\mu\in\mathcal{M}_a\left(\mathbb{R}\right)$ and $J_0(t,x)\equiv v$. Together with (4.16) for all $\mu\in\mathcal{M}_a\left(\mathbb{R}\right)$, we obtain (4.15). This completes the proof of Lemma 4.10. □

Proof of Theorem 3.1. The proof follows the same six steps as those in the proof of [6, Theorem 2.4] with some minor changes:

(1) Both proofs rely on estimates on the kernel function $\mathcal{K}(t,x)$. Instead of an explicit formula as for the heat equation case (see [6, Proposition 2.2]), Proposition 3.2 ensures the finiteness and provides a bound on the kernel function $\mathcal{K}(t,x)$.

(2) In the Picard iteration scheme (i.e., Steps 1–4 in the proof of [6, Theorem 2.4]), we need to check the $L^p(\Omega)$-continuity of the stochastic integral, which then guarantees that at the next step, the integrand is again in $\mathcal{P}_2$, via [6, Proposition 3.4]. Here, the statement of [6, Proposition 3.4] is still true by replacing in its proof [6, Propositions 3.5 and A.3] by Propositions 4.4 and 4.8, respectively. Note that when applying Proposition 4.8, we need to replace the G_ν^2 in [6, (3.8)] by $({}_{-\delta}G_a + {}_{\delta}G_a)^2 \le 2\,{}_{-\delta}G_a^2 + 2\,{}_{\delta}G_a^2$.

(3) In the first step of the Picard iteration scheme, the following property is useful: For all compact sets $K \subseteq \mathbb{R}_+ \times \mathbb{R}$,

$$\sup_{(t,x)\in K} \left(\left[1+J_0^2\right] \star {}_{\delta}G_a^2\right)(t,x) < +\infty.$$

For the heat equation, this property is discussed in [6, Lemma 3.9]. Here, Lemma 4.10 gives the desired result with minimal requirements on the initial data. This property, together with the calculation of the upper bound on the function $\mathcal{K}$ in Proposition 3.2, guarantees (as in [6, Lemmas 3.3 and 3.7]) that all the $L^p(\Omega)$-moments of $u(t,x)$ are finite. This property is also used to establish uniform convergence of the Picard iteration scheme, hence $L^p(\Omega)$–continuity of $(t,x) \mapsto I(t,x)$.

The proofs of (3.1)–(3.4) are identical to those of the corresponding properties in [6, Theorem 2.4], and (3.5) and (3.6) are direct consequences of the preceding statements.

This completes the proof of Theorem 3.1. □

5 Proofs of Theorems 3.4 and 3.6

We begin with the upper bound in Theorem 3.4.

Proof of Theorem 3.4(1). Recall from (3.11) that $\widehat{\gamma}_p = a_{p,\bar\varsigma}^2 z_p^2\, L_\rho^2\, \Lambda\Gamma(1/a^*)$, and $a^* = a/(a-1)$. By (3.1), (4.16) and (4.20), for all $x \in \mathbb{R}$,

$$\overline{m}_p(x) = \limsup_{t\to\infty} \frac{\log ||u(t,x)||_p^p}{t} \le \frac{\widehat{\gamma}_p^{a^*} p}{2} = \frac{p}{2}\left[a_{p,\bar\varsigma}^2 z_p^2\, L_\rho^2\, \Lambda\, \Gamma\left(1/a^*\right)\right]^{a^*}.$$

Since $a_{p,\bar\varsigma} \le 2$ and $z_p \le 2\sqrt{p}$, (3.21) follows. □

5.1 Lower bound on $\mathcal{K}(t,x)$ (Proposition 3.3)

We need some properties of $g_a(t,x)$ defined in (3.15).

Lemma 5.1. *Suppose that $a \in \,]1,2[$ and $|\delta| < 2-a$. Then the constant $\widetilde{C}_{a,\delta}$ defined in (3.16) is strictly positive, and so for all $t>0$ and $x\in\mathbb{R}$,*

$$ {}_{\delta}G_a(t,x) \ge \widetilde{C}_{a,\delta}\, \pi\, g_a(t,x) = \frac{\widetilde{C}_{a,\delta}\, t}{\left(t^{2/a}+x^2\right)^{\frac{a}{2}+\frac{1}{2}}}. \tag{5.1}$$

Proof. By the scaling property of both ${}_\delta G_a$ and $g_a(t,x)$,

$$\inf_{(t,x)\in\mathbb{R}_+^*\times\mathbb{R}} \frac{{}_\delta G_a(t,x)}{\pi g_a(t,x)} = \frac{1}{\pi}\inf_{y\in\mathbb{R}} \frac{{}_\delta G_a(1,y)}{g_a(1,y)} .$$

Let $f(y) = {}_\delta G_a(1,y)/g_a(1,y)$. In the case where $1<a<2$ and $|\delta|<2-a$, both ${}_\delta G_a(1,y)$ and $g_a(1,y)$ have tails at $\pm\infty$ with polynomial decay of the same rate as $|y|^{-1-a}$: see [27, p. 143] (we use here the fact that $|\delta|\neq 2-a$). Since $y\mapsto {}_\delta G_a(t,y)$ is unimodal (see [18, Lemma 5.10.1]), we conclude that $f(y)>0$ for all $y\in\mathbb{R}$, and that $\lim_{y\to\pm\infty} f(y)>0$. Therefore, $\inf_{y\in\mathbb{R}} f(y)>0$, and this completes the proof of Lemma 5.1. □

Lemma 5.2. *Let $f_{b,\nu}(x) = f(x) := (b^2+x^2)^{-\nu-1/2}$ with $b>0$ and $\nu\geq 1/2$. Then*

$$\mathcal{F}[f](z) = \int_{\mathbb{R}} \mathrm{d}x\, e^{-izx} f(x) \geq C_\nu\, b^{-2\nu} \exp\left(-b|z|\right) , \tag{5.2}$$

for all $b>0$ and $z\in\mathbb{R}$, where the constant $C_\nu>0$ is given in (3.18).

Proof. Note that the function $f(x)$ is an even function, so its Fourier transform is real-valued, instead of complex-valued, which allows us to bound this transform from below. Indeed, by [14, (7), p. 11], we have that

$$\mathcal{F}[f](z) = \left(\frac{|z|}{b}\right)^{\nu} \frac{\sqrt{\pi}}{2^\nu \Gamma\left(\nu+1/2\right)} K_\nu\left(b|z|\right), \quad \text{for } \Re(b)>0 \text{ and } \nu>-1/2,$$

where $K_\nu(x)$ is the modified Bessel function of the second kind. Equivalently, we need to prove that the function

$$\mathbb{R}_+\times\mathbb{R}\ni(b,z)\mapsto \left(\frac{|z|}{b}\right)^{\nu} \frac{\sqrt{\pi}}{2^\nu \Gamma\left(\nu+1/2\right)} K_\nu\left(b|z|\right) b^{2\nu}\exp\left(b|z|\right)$$

is uniformly bounded away from zero. By choosing $u=b|z|$, we reduce this problem to bounding the following function

$$\mathbb{R}_+\ni u\mapsto \frac{\sqrt{\pi}}{2^\nu \Gamma\left(\nu+1/2\right)} h(u) \tag{5.3}$$

away from zero, where $h(u) := u^\nu e^u K_\nu(u)$. By the differential formula for $x^{\pm\nu}K_\nu(x)$ (see, for instance, [20, 51:10:4, p. 532]),

$$h'(u) = e^u u^\nu \left(K_\nu(u) - K_{\nu-1}(u)\right).$$

By the integral representation of $K_\nu(z)$ in [21, 10.32.9, p. 252],

$$K_\nu(u) - K_{\nu-1}(u) = \frac{1}{2}\int_0^\infty e^{-u\cosh(t)}\left(e^{\nu t} - e^{-(\nu-1)t}\right)\left(1-e^{-t}\right)\mathrm{d}t \geq 0$$

since $\nu \geq 1/2$. Hence, $h'(u) > 0$ and

$$\inf_{u\in\mathbb{R}_+} h(u) = \lim_{u\to 0} h(u) = 2^{\nu-1}\Gamma(\nu)\,,$$

where we have used the property $K_\nu(u) \sim \frac{1}{2}\Gamma(\nu)(\frac{1}{2}u)^{-\nu}$ as $u \downarrow 0$ (see [21, 10.30.2, p. 252]). Therefore,

$$C_\nu = \inf_{u\in\mathbb{R}_+} \frac{\sqrt{\pi}}{2^\nu\Gamma\left(\nu+1/2\right)} h(u) = \frac{\Gamma(\nu)\Gamma(1/2)}{2\Gamma(\nu+1/2)}\,,$$

This completes the proof of Lemma 5.2. □

In the next lemma, we gather some properties of the function $g_a(t,x)$.

Lemma 5.3. *(1) For $a>0$ and $t>0$, $g_a(t,x-y) \geq \pi 2^{-(a+1)} t^{1/a} g_a(t,x) g_a(t,y)$.*

(2) For $t>0$ and $z\in\mathbb{R}$, $\mathcal{F}[g_a^2(t,\cdot)](z) \geq C_{a+1/2}\,\pi^{-2} t^{-1/a}\exp(-t^{1/a}|z|)$.

(3) For all $a>0$, $t\geq s>0$ and $x\in\mathbb{R}$, we have

$$\left(g_a^2\left(t-s,\cdot\right) * g_a^2\left(s,\cdot\right)\right)(x) \geq \frac{C_{a+1/2}^2}{\pi^3\, 2^{2a+3}} (ts)^{-1/a} (t-s)^{1/a} g_a^2(t-s,x),$$

where the constant $C_{a+1/2}$ is defined in (3.18).

(4) For $t\geq r\geq t/2>0$, $g_a(r,x) \geq (t/r)^{1/a}\, 2^{-1-1/a} g_a(t,x)$.

Proof. (1) Because $1+(u-v)^2 \leq 1+2u^2+2v^2 \leq (1+2u^2)(1+2v^2)$, $g_a(t,x-y)$ is bounded from below by $\pi t^{1/a} g_a\left(t,\sqrt{2}\,x\right) g_a\left(t,\sqrt{2}\,y\right)$. Then use the inequality $g_a\left(t,\sqrt{2}\,x\right) \geq 2^{-(a+1)/2} g_a(t,x)$.

(2) We apply Lemma 5.2 with $\nu = a+1/2$ and $b = t^{1/a}$.

(3) By (1),

$$\left(g_a^2\left(t-s,\cdot\right) * g_a^2\left(s,\cdot\right)\right)(x) \geq \pi^2 2^{-2(a+1)} (t-s)^{2/a} g_a^2\left(t-s,x\right) \int_{\mathbb{R}} \mathrm{d}y\, g_a^2\left(t-s,y\right) g_a^2\left(s,y\right).$$

The integral can be bounded using Plancherel's identity and (2):

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{d}y\, g_a^2\left(t-s,y\right) g_a^2\left(s,y\right) &\geq \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}z \frac{C_{a+1/2}^2}{\pi^4} ((t-s)s)^{-1/a} \exp\{-|z|((t-s)^{1/a}+s^{1/a})\} \\
&= \frac{C_{a+1/2}^2}{2\pi^5} ((t-s)s)^{-1/a} \frac{2}{(t-s)^{1/a}+s^{1/a}} \\
&\geq \frac{C_{a+1/2}^2}{2\pi^5} ((t-s)s)^{-1/a}\, t^{-1/a}.
\end{aligned}$$

This proves (3).

(4) Notice that for $t \ge r \ge t/2 > 0$,

$$
\begin{aligned}
g_a(r,x) &= \frac{r^{-1/a}}{\pi}\left(1+\frac{x^2}{r^{2/a}}\right)^{-(a+1)/2} \ge \frac{r^{-1/a}}{\pi}\left(1+\frac{x^2}{(t/2)^{2/a}}\right)^{-(a+1)/2} \\
&\ge \frac{r^{-1/a}}{\pi}(t/2)^{1+1/a}((t/2)^{2/a}+x^2)^{-(a+1)/2} \\
&= \frac{r^{-1/a}}{\pi 2^{1+1/a}} t^{1+1/a}((t/2)^{2/a}+x^2)^{-(a+1)/2} \ge \frac{r^{-1/a}}{2^{1+1/a}} t^{1/a} g_a(t,x).
\end{aligned}
$$

□

Proof of Proposition 3.3. Denote

$$
\left(g_a^2\right)^{\star n}(t,x) := \underbrace{\left(g_a^2 \star \cdots \star g_a^2\right)}_{n \text{ factors } g_a^2}(t,x)\ .
$$

Notice that by (2.9) and (5.1),

$$
\mathcal{K}(t,x;\lambda) = \sum_{n=0}^{\infty} \left(\lambda^2\, {}_\delta G_a^2\right)^{\star(n+1)}(t,x) \ge \sum_{n=0}^{\infty} \left(\lambda^2\, \widetilde{C}_{a,\delta}^2\, \pi^2\, g_a^2\right)^{\star(n+1)}(t,x)\ . \tag{5.4}
$$

We now bound space-time convolutions of g_a^2 with itself. We claim that

$$
\left(\lambda^2 g_a^2\right)^{\star(n+1)}(t,x) \ge \frac{\lambda^{2(n+1)}\, \Theta_a^n\, \Gamma(1/a^*)^{n+1}}{\Gamma((n+1)/a^*)} t^{n/a^*} g_a^2(t,x) \qquad \text{for all } n \ge 0, \tag{5.5}
$$

where

$$
\Theta_a := C_{a+1/2}^2\, \pi^{-3}\, 2^{-2(a+3+1/a)}.
$$

The case where $n = 0$ is clear. Consider $n \ge 1$ and assume by induction that (5.5) holds for $n-1$. By the induction hypothesis and Lemma 5.3(3),

$$
\begin{aligned}
\left(\lambda^2 g_a^2\right)^{\star(n+1)}(t,x) &\ge \frac{\lambda^{2(n+1)}\Theta_a^{n-1}\Gamma(1/a^*)^n}{\Gamma(n/a^*)} \int_0^t \mathrm{d}s\; (t-s)^{(n-1)/a^*} (g_a^2(t-s,\cdot) * g_a^2(s,\cdot))(x) \\
&\ge K_n\, t^{-1/a} \int_0^t \mathrm{d}s\; g_a^2(t-s,x)\, [(t-s)s]^{-1/a}\, (t-s)^{\frac{n-1}{a^*}+\frac{2}{a}},
\end{aligned} \tag{5.6}
$$

where

$$
K_n = \frac{\lambda^{2(n+1)}\Theta_a^{n-1}\, \Gamma(1/a^*)^n\, C_{a+1/2}^2}{\Gamma(n/a^*)\, \pi^3\, 2^{2a+3}}.
$$

Notice that $t - s \ge t/2$ for $0 \le s \le t/2$, so we apply Lemma 5.3(4) to see that

$$
\left(\lambda^2 g_a^2\right)^{\star(n+1)}(t,x) \ge \frac{K_n}{2^{2+2/a}}\, t^{1/a} g_a^2(t,x) \int_0^{t/2} \mathrm{d}s\, [s(t-s)]^{-1/a} (t-s)^{\frac{n-1}{a^*}}. \tag{5.7}
$$

For $0 \le s \le t/2$, we have $t-s \ge s$, so we replace the last factor $t-s$ by s to see that

$$\left(\lambda^2 g_a^2\right)^{\star(n+1)}(t,x) \ge \frac{K_n}{2^{2+2/a}}\, t^{1/a} g_a^2(t,x) \int_0^{t/2} \mathrm{d}s\, [s(t-s)]^{-1/a} s^{\frac{n-1}{a^*}}.$$

Use the change of variables $s \mapsto t-s$ in (5.7) and add to this last integral to see that

$$\left(\lambda^2 g_a^2\right)^{\star(n+1)}(t,x) \ge \frac{K_n}{2^{2+2/a}}\, t^{1/a} g_a^2(t,x) \frac{1}{2} \int_0^{t} \mathrm{d}s\, [s(t-s)]^{-1/a} s^{\frac{n-1}{a^*}}.$$

Then apply Euler's Beta integral (4.6). This proves (5.5).

Therefore, by (5.4), (5.5) and (3.9),

$$\begin{aligned}
\mathcal{K}(t,x;\lambda) &\ge \sum_{n=0}^{+\infty} \left(\lambda^2\, \widetilde{C}_{a,\delta}^2\, \pi^2 g_a^2\right)^{\star(n+1)}(t,x) \\
&\ge \pi^2\, \widetilde{C}_{a,\delta}^2\, \lambda^2\, \Gamma(1/a^*) g_a^2(t,x) \sum_{n=0}^{+\infty} \frac{(\lambda^2\, \widetilde{C}_{a,\delta}^2\, \pi^2\, \Theta_a\, \Gamma(1/a^*) t^{1/a^*})^n}{\Gamma((n+1)/a^*)} \\
&= \pi^2\, \widetilde{C}_{a,\delta}^2\, \lambda^2\, \Gamma(1/a^*) g_a^2(t,x) E_{1/a^*,1/a^*}(\lambda^2\, \widetilde{C}_{a,\delta}^2\, \pi^2\, \Theta_a\, \Gamma(1/a^*) t^{1/a^*}).
\end{aligned}$$

This proves the statement (3.19) in Proposition 3.3.

As for (3.20), notice that by (3.19),

$$(1 \star \mathcal{K})(t,x) = \int_0^t ds \int_{\mathbb{R}} dy\, \mathcal{K}(s,y) \ge C \int_0^t ds\, E_{1/a^*,1/a^*}\left(\Upsilon(\lambda,a,\delta)\, s^{1/a^*}\right) \int_{\mathbb{R}} dy\, g_a^2(s,y),$$

where $C = \pi^2\, \widetilde{C}_{a,\delta}^2\, \lambda^2\, \Gamma(1/a^*)$. By [21, 5.12.3, p.142] and Euler's beta integral,

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{d}y\, g_a^2(s,y) &= \frac{s^{-1/a}}{\pi^2} 2 \int_0^\infty \mathrm{d}z \frac{1}{(1+z^2)^{a+1}} = \frac{s^{-1/a}}{\pi^2} \int_0^\infty \mathrm{d}y \frac{1}{y^{1/2}(1+y)^{a+1}} \\
&= \frac{\Gamma(a+1/2)}{\Gamma(1+a)\pi^{3/2}}\, s^{-1/a}.
\end{aligned}$$

By [22, (1.99), p. 24],

$$\int_0^t \mathrm{d}s\, s^{-1/a}\, E_{1/a^*,1/a^*}\left(\Upsilon(\lambda,a,\delta)\, s^{1/a^*}\right) = t^{1/a^*} E_{1/a^*,1+1/a^*}\left(\Upsilon(\lambda,a,\delta)\, t^{1/a^*}\right).$$

This establishes (3.20) and completes the proof of Proposition 3.3. □

5.2 Proofs of Theorems 3.6 and 3.4(2)

Lemma 5.4. *Suppose that* $a \in \,]1,2[$, $|\delta| < 2-a$ *and* $\mu \in \mathcal{M}_{a,+}(\mathbb{R})$, $\mu \ne 0$. *Then for all* $\epsilon > 0$, *there exists a constant* C *such that for all* $t \ge 0$ *and* $x \in \mathbb{R}$,

$$\left({}_{\delta}G_a(t,\cdot) * \mu\right)(x) \ge C\, 1_{[\epsilon,\infty[}(t)\, g_a(t,x).$$

Proof. Denote $J_0(t,x) = \left({}_\delta G_a(t,\cdot) * \mu \right)(x)$. By the lower bound on ${}_\delta G_a(t,x)$ in (5.1), Lemma 5.3(1) and the scaling property of $g_a(t,x)$, we have

$$\begin{aligned} J_0(t,x) &\geq \widetilde{C}_{a,\delta}\,\pi \int_{\mathbb{R}} \mu(\mathrm{d}y)\, g_a(t,x-y) \\ &\geq \frac{\widetilde{C}_{a,\delta}\,\pi^2}{2^{a+1}} t^{1/a} g_a(t,x) \int_{\mathbb{R}} \mu(\mathrm{d}y)\, g_a(t,y) \\ &= \frac{\widetilde{C}_{a,\delta}\,\pi}{2^{a+1}}\, g_a(t,x) \int_{\mathbb{R}} \mu(\mathrm{d}y) \left(1+\frac{y^2}{t^{2/a}}\right)^{-\frac{a+1}{2}} . \end{aligned}$$

The above integrand is non-decreasing with respect to t. Hence

$$\begin{aligned} J_0(t,x) &\geq \frac{\widetilde{C}_{a,\delta}\,\pi}{2^{a+1}}\, 1_{\{t\geq\epsilon\}}\, g_a(t,x) \int_{\mathbb{R}} \mu(\mathrm{d}y) \left(1+\frac{y^2}{\epsilon^{2/a}}\right)^{-\frac{a+1}{2}} \\ &= \frac{\widetilde{C}_{a,\delta}\,\pi^2\,\epsilon^{1/a}}{2^{a+1}} 1_{\{t\geq\epsilon\}}\, g_a(t,x) \int_{\mathbb{R}} \mu(\mathrm{d}y)\, g_a(\epsilon,y) . \end{aligned}$$

Since the function $y \mapsto g_a(\epsilon,y)$ is strictly positive and μ is nonnegative and non-vanishing, the integral is positive. Finally, we can take $C := \widetilde{C}_{a,\delta}\,\pi^2\,\epsilon^{1/a}\,2^{-(a+1)} \int_{\mathbb{R}} \mu(\mathrm{d}y)\, g_a(\epsilon,y)$. ☐

Lemma 5.5. *Suppose $\beta > 1$. For all $x \in \mathbb{R}$,*

$$\min_{y\in\mathbb{R}} \left(|x-y|^\beta + |y|\right) \geq \begin{cases} \beta^{\frac{\beta}{1-\beta}} + \left| |x| - \beta^{\frac{1}{1-\beta}} \right| & \text{if } |x| \geq \beta^{\frac{1}{1-\beta}}, \\ |x|^\beta & \text{otherwise.} \end{cases}$$

Proof. Fix $x \in \mathbb{R}$ and set $f(y) = |x-y|^\beta + |y|$. Assume first that $x \geq 0$. By studying the sign of the derivative of $f'(y)$, we find that if $x \geq \beta^{\frac{1}{1-\beta}}$, then f achieves its minimum at $y = x - \beta^{\frac{1}{1-\beta}}$. If $0 \leq x \leq \beta^{\frac{1}{1-\beta}}$, then f achieves it minimum at 0. The case $x < 0$ is treated similarly. ☐

Proof of Theorem 3.6. (1) In the following, we use C to denote some nonnegative constant, which may depend on a, δ and L_ρ, and can change from line to line. Fix $p \geq 2$. By (4.21) and (4.22), when $t \geq 1$,

$$\left(J_0^2 \star \widehat{\mathcal{K}}_p\right)(t,x) \leq Ct^2 \left(1+\exp(\widehat{\gamma}_p^{a^*} t)\right) |J_0(t,x)| ,$$

where the constant $\widehat{\gamma}_p$ is defined in (3.11). By (3.1) with $\varsigma = 0$ and (3.22), for $\alpha \geq 0$,

$$\lim_{t\to+\infty} \frac{1}{t} \sup_{|x|\geq\exp(\alpha t)} \log ||u(t,x)||_p^2 = \lim_{t\to+\infty} \frac{1}{t} \sup_{|x|\geq\exp(\alpha t)} \log \left(J_0^2 \star \widehat{\mathcal{K}}_p\right)(t,x) \leq \widehat{\gamma}_p^{a^*} - \alpha\beta.$$

Now, $\widehat{\gamma}_p^{a^*} - \alpha\beta < 0$ if and only if $\alpha > \beta^{-1}\,\widehat{\gamma}_p^{a^*}$. Therefore,

$$\overline{e}(p) := \inf\left\{\alpha > 0 : \lim_{t\to\infty}\frac{1}{t}\sup_{|x|\geq \exp(\alpha t)}\log \mathbb{E}\left(|u(t,x)|^p\right) < 0\right\} \leq \frac{\widehat{\gamma}_p^{a^*}}{\beta} < +\infty.$$

Concerning the sufficient condition for (3.22), suppose that for some $\eta > 0$, $\int_{\mathbb{R}} |\mu|(dy)(1 + |y|^\eta) < \infty$. We consider first the case where $\eta \in\,]0, 1+a[$. Then by (4.3),

$$|J_0(t,x)| \leq \int_{\mathbb{R}} |\mu|(dy)\, \frac{K_{a,0}(1+t)}{1+|x-y|^{1+a}} \leq C K_{a,0}(1+t) \sup_{y\in\mathbb{R}} \left[(1+|y|)(1+|x-y|)^{(1+a)/\eta}\right]^{-\eta}.$$

Let $\tilde{\beta} = (1+a)/\eta > 1$. Notice that

$$(1+|x-y|^{\tilde{\beta}})(1+|y|) \geq 1 + |x-y|^{\tilde{\beta}} + |y|.$$

By Lemma 5.5, we see that

$$|J_0(t,x)| \leq \tilde{C}(1+t)\frac{1}{1+|x|^\eta},$$

which is condition (3.22) with $\beta = \eta$.

Now consider the case where $\eta \geq 1+a$. Notice that if $\eta > 1+a$, then we generally do not expect (3.22) to hold with $\beta = \eta$, since for instance, $J_0(t,x) \sim 1/|x|^{1+a}$ as $|x| \to \infty$ when $\mu = \delta_0$. Observe that

$$|J_0(t,x)| \leq \int_{\mathbb{R}} \frac{|\mu|(dy)}{1+|x-y|^{1+a}}\ \ {}_{\delta}G_a(t,x-y)\left(1+|x-y|^{1+a}\right).$$

From (4.3), we deduce that for $t \geq 1$,

$${}_{\delta}G_a(t,x-y)\left(1+|x-y|^{1+a}\right) \leq K_{a,n}\, t.$$

Let $\varphi = \eta/(1+a)$, so that $\varphi \geq 1$. Since for some $\tilde{c} > 0$,

$$\begin{aligned}(1+|x-y|^2)(1+|y|^{2\varphi}) &\geq \frac{1}{2} + |x-y|^2 + \frac{1}{2} + |y|^{2\varphi} \geq \tilde{c}\left[1+|x-y|^2+|y|^2\right]\\ &\geq \tilde{c}\left(1+\frac{x^2}{2}\right),\end{aligned}$$

we see that for all $t \geq 1$ and $x \in \mathbb{R}$, there is $c > 0$ such that

$$\begin{aligned}|J_0(t,x)| &\leq C K_{a,n}\, t \int_{\mathbb{R}} \frac{|\mu|(dy)}{[(1+|x-y|^2)(1+|y|^{2\varphi})]^{(1+a)/2}}\ (1+|y|^\eta)\\ &\leq \tilde{C}\frac{t}{(1+x^2)^{(1+a)/2}}\int_{\mathbb{R}} |\mu|(dy)\,(1+|y|^\eta),\end{aligned}$$

which implies (3.22) with $\beta = 1 + a$.

(2) We only need to consider the case $p = 2$ because $\underline{e}(p) \geq \underline{e}(2)$ for $p \geq 2$. Assume first that $\underline{\varsigma} = 0$. Fix any $\epsilon \in]0, t/2[$, choose a constant $\widehat{C}$ according to Lemma 5.4 such that

$$J_0(t,x) = \left({}_{\delta}G_a(t,\cdot) * \mu \right) \geq \widehat{C}\, 1_{[\epsilon,\infty[}(t)\, g_a\,(t,x) =: I_\epsilon(t,x).$$

By (3.3),

$$||u(t,x)||_2^2 \geq J_0^2(t,x) + \left(J_0^2 \star \underline{\mathcal{K}}\right)(t,x) \geq \left(I_\epsilon^2 \star \underline{\mathcal{K}}\right)(t,x).$$

Set $\Upsilon = \Upsilon\,(l_\rho, a, \delta)$ (see (3.17)). By Proposition 3.3 and Lemma 5.3(3),

$$\begin{aligned}
\left(I_\epsilon^2 \star \underline{\mathcal{K}}\right)(t,x) &\geq \widehat{C}^2 C \int_0^{t-\epsilon} \mathrm{d}s\, E_{1/a^*,1/a^*}\left(\Upsilon s^{1/a^*}\right) \int_{\mathbb{R}} \mathrm{d}y\, g_a^2\,(t-s, x-y)\, g_a^2\,(s,y) \\
&\geq \frac{\widehat{C}^2\, C\, C_{a+1/2}^2}{\pi^3\, 2^{2a+3}} t^{-1/a} \int_0^{t-\epsilon} \mathrm{d}s\, E_{1/a^*,1/a^*}\left(\Upsilon s^{1/a^*}\right) s^{-1/a}(t-s)^{1/a} g_a^2(t-s,x).
\end{aligned}$$

Notice that

$$g_a^2(t-s,x) \geq (t-s)^2 t^{-2} g_a^2(t,x).$$

Hence,

$$\left(I_\epsilon^2 \star \underline{\mathcal{K}}\right)(t,x) \geq \frac{\widehat{C}^2\, C\, C_{a+1/2}^2}{\pi^3\, 2^{2a+3}} t^{-2-1/a} g_a^2(t,x) \int_0^{t-\epsilon} \mathrm{d}s\, E_{1/a^*,1/a^*}\left(\Upsilon s^{1/a^*}\right) s^{-1/a}(t-s)^{2+1/a}.$$

The above integral is bounded from below by

$$E_{1/a^*,1/a^*}\left(\Upsilon\,(t-2\epsilon)^{1/a^*}\right) \int_{t-2\epsilon}^{t-\epsilon} \mathrm{d}s\, s^{-1/a}(t-s)^{2+1/a} \geq E_{1/a^*,1/a^*}\left(\Upsilon\,(t-2\epsilon)^{1/a^*}\right) \frac{\epsilon^{2+1/a}}{(t-\epsilon)^{1/a}}\, \epsilon.$$

Therefore, we have

$$\left(I_\epsilon^2 \star \underline{\mathcal{K}}\right)(t,x) \geq \overline{C}\, g_a^2\,(t,x)\; t^{-2-1/a}\,(t-\epsilon)^{-1/a} E_{1/a^*,1/a^*}\left(\Upsilon(t-2\epsilon)^{1/a^*}\right), \tag{5.8}$$

where

$$\overline{C} = \frac{\epsilon^{3+1/a}\, \widehat{C}^2\, C\, C_{a+1/2}^2}{\pi^3\, 2^{2a+3}}\,.$$

Because $x \mapsto g_a(t,x)$ is an even function, decreasing for $x \geq 0$, we deduce that for all $\beta > 0$,

$$\sup_{|x|>\exp(\beta t)} ||u(t,x)||_2^2 \geq \overline{C}\, g_a^2\,(t, \exp(\beta t))\; t^{-2-1/a}(t-\epsilon)^{-1/a} E_{1/a^*,1/a^*}\left(\Upsilon(t-2\epsilon)^{1/a^*}\right).$$

Because $a > 0$, there exists $t_0 \geq 0$ such that for all $t \geq t_0$, $t^{2/a} \leq e^{2\beta t}$, so

$$g_a^2\,(t, \exp(\beta t)) = \frac{1}{\pi^2} \frac{t^2}{\left(t^{2/a} + e^{2\beta t}\right)^{a+1}} \geq \frac{1}{\pi^2} \frac{t^2}{2^{a+1} e^{2\beta(a+1)t}}.$$

Finally, by the asymptotic expansion of the Mittag-Leffler function in Lemma 4.2,

$$\lim_{t\to\infty}\frac{1}{t}\sup_{|x|>\exp(\beta t)}\log||u(t,x)||_2^2 \geq \Upsilon^{a^*} - 2\beta(a+1)\;. \tag{5.9}$$

Therefore,

$$\begin{aligned}\underline{e}(2) &= \sup\left\{\beta>0:\lim_{t\to+\infty}\frac{1}{t}\sup_{|x|>\exp(\beta t)}\log||u(t,x)||_2^2>0\right\}\\ &\geq \sup\left\{\beta>0:\Upsilon^{a^*}-2\beta(a+1)>0\right\} = \frac{\Upsilon^{a^*}}{2(a+1)}\;.\end{aligned}$$

Now let us consider the case where there is $c>0$ with $J_0 \geq c$, or $\underline{\varsigma} \neq 0$. In this case, by (3.3) and Proposition 3.3,

$$||u(t,x)||_2^2 \geq (c^2+\underline{\varsigma}^2)\left(1\star\underline{\mathcal{K}}\right)(t,x) \geq Ct^{1/a^*}E_{1/a^*,1+1/a^*}\left(\Upsilon t^{1/a^*}\right).$$

This lower bound does not depend on x and hence, by Lemma 4.2, we get (5.9) with the right-hand side replaced by Υ^{a^*}. This completes the proof of Theorem 3.6. □

Proof of Theorem 3.4 (2). If $\underline{\varsigma}\neq 0$, then from (3.3) and Proposition 3.3, for some constant $C>0$,

$$||u(t,x)||_2^2 \geq \underline{\varsigma}^2\left(1\star\underline{\mathcal{K}}\right)(t,x) \geq C\,\underline{\varsigma}^2\,t^{1/a^*}E_{1/a^*,1+1/a^*}\left(\Upsilon(l_\rho,a,\delta)t^{1/a^*}\right),$$

where the constant $\Upsilon(l_\rho,a,\delta)$ is defined in (3.17). Then use the asymptotic expansion of $E_{\alpha,\beta}(z)$ in Lemma 4.2 to obtain

$$\underline{m}_2(x) \geq \Upsilon\left(l_\rho,a,\delta\right)^{a^*}. \tag{5.10}$$

If $\underline{\varsigma}=0$, then from (3.3), (5.8) and the asymptotics of $E_{\alpha,\beta}(z)$ in Lemma 4.2, we obtain, via the calculation that led to (5.9), but without replacing x by $\exp(\beta t)$, the same lower bound as (5.10). Note that this lower bound does not depend on x. This proves the statement (2) with $p=2$. For $p>2$, we use Hölder's inequality

$$\mathbb{E}\left[|u(t,x)|^2\right] \leq \mathbb{E}\left[|u(t,x)|^p\right]^{2/p}.$$

Hence, $\underline{m}_p(x) \geq \frac{p}{2}\,\underline{m}_2(x)$. This completes the proof of Theorem 3.4. □

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