Eccentric behaviors of the Brownian sheet along lines

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Abstract

Distinct excursion intervals of a Brownian motion (that correspond to a fixed level) have no common endpoints. What is the situation for distinct excursion sets of a Brownian sheet? These sets are termed Brownian bubbles in the literature, and this paper examines how bubbles from fixed or random levels come into contact with each other, by examining whether or not the Brownian sheet restricted to a specific type of curve can have a point of increase. At random levels, we show that points of increase can occur along horizontal lines, while at fixed levels, such a point of increase can occur at the corner of a broken line segment with a right-angle. In addition, the Hausdorff dimension of the set of points with this last property is shown to be 1/2 a.s.

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1 Introduction

The Brownian sheet is a centered, continuous Gaussian process $(W(t),\ t \in \mathbb{R}^2_+)$ indexed by the positive quadrant in the plane, with covariance given by

$$E(W(s_1, s_2)W(t_1, t_2)) = (s_1 \wedge t_1)(s_2 \wedge t_2).$$

It is one of the natural extensions of Brownian motion to higher dimensional time.

In this paper, we study behaviors of excursions of the Brownian sheet and of the sets in the non-negative quadrant on which these excursions occur. These behaviors have been the subject of several papers in recent years [4, 5, 10, 16]. Given a level $q \in \mathbb{R}$, the complement of the level set $\{t : W(t) = q\}$ is a random open set, the components of which we call *Brownian bubbles,* following [5], or q-bubbles if the level q is to be specified. A component of $\{t : W(t) > q\}$ will be termed an *upwards bubble,* and a component of $\{t : W(t) < q\}$ a *downwards bubble.* Upwards or downwards is the *direction* of the bubble.

These bubbles are natural analogues of excursion intervals from level q of Brownian motion. Unlike the beautiful excursion theory available for Brownian motion [8, 19, 20], there is no such theory for the Brownian sheet. One difficulty, that also makes the problem challenging, is that the boundary of a bubble is a very complicated random set (in contrast to an excursion interval), and distinct bubbles may share common boundary points [16, 4]. For instance, in [4], it was shown that it is possible to travel within a bubble along a monotone curve until reaching the boundary of the bubble, and then, continuing along this curve, to immediately enter another distinct bubble of opposite direction. More precisely, for any $q \in \mathbb{R}$, with positive probability, there exists a (random) monotone curve $\gamma : [-1, 1] \to [1, 2]^2$ such that $W(\gamma(-u)) < q < W(\gamma(u))$, for $0 < u \leq 1$.

In this paper, we consider the possibility of travelling horizontally or vertically within a bubble to a boundary point, and then, from this boundary point, travelling horizontally or vertically and immediately entering a distinct bubble. Concretely, we ask:

1. are there horizontal line segments $[t_1-h, t_1+h]\times\{t_2\}$ such that $[t_1-h, t_1[\,\times\{t_2\}$ and $]t_1, t_1+h]\times\{t_2\}$ are in distinct q-bubbles, for fixed or random q?

2. are there "broken line segments" $\{t_1\}\times\,]t_2, t_2+h] \cup\,]t_1, t_1+h]\times\{t_2\}$ such that $\{t_1\}\times\,]t_2, t_2+h]$ and $]t_1, t_1+h]\times\{t_2\}$ are in distinct q-bubbles, for fixed or random q?

These questions are of interest independently of considerations regarding the geometry of bubbles. Indeed, it is a celebrated classical result of Dvoretsky, Erdős and Kakutani [7] (see also [2]) that Brownian motion has no point of increase, where a *point of increase* of $(B(u),\ u \geq 0)$ is a time $u > 0$ such that for some $\varepsilon > 0$,

$$B(u-h) < B(u) < B(u+h), \qquad 0 < h < \varepsilon.$$

In the case of the Brownian sheet, for each $t_2 > 0$, $t_1 \mapsto W(t_1, t_2)$ is a (speed t_2) Brownian motion, so a.s., there is no t_1 such that $[t_1 - h, t_1[\times \{t_2\}$ and $]t_1, t_1 + h] \times \{t_2\}$ lie in distinct bubbles of opposite direction, even at random levels. This naturally raises the question of whether this might occur for some random t_2, possibly with the additional requirement that $W(t_1, t_2)$ take some fixed value q. Because the Ornstein-Uhlenbeck process $(U_v,\ v \geq 0)$ on Wiener space can be represented using the Brownian sheet as follows: $U_v(u) = e^{-v/2} W(u, v)$ [13, 15], this question is the same as asking whether the Ornstein-Uhlenbeck process on Wiener space hits the set of paths with points of increase. Our first result answers this question in the affirmative.

Theorem 1 *Fix $h > 0$. With positive probability, there exists $(t_1, t_2) \in [2, 3]^2$ such that*

$$W(t_1 - u, t_2) < W(t_1, t_2) < W(t_1 + u, t_2), \qquad 0 < u < h. \tag{1}$$

From this "positive probability" result, one can easily deduce an a.s. result: there exist a.s. random (t_1, t_2) and $h > 0$ such that (1) holds. These results imply that at random levels, one can pass from a downwards bubble to an upwards bubble along horizontal line segments.

Given this result, one could ask whether a similar result is possible with the additional request that $W(t_1, t_2) = q$. Our approach does not allow us to answer this question (see Remark 8). However, we can answer the question negatively if both bubbles are required to have the same direction.

Theorem 2 *Fix $q \in \mathbb{R}$. With probability one, there is no $(t_1, t_2) \in \mathbb{R}_+^2$ such that $W(t_1, t_2) = q$ and for some $h > 0$,*

$$[t_1 - h, t_1[\times \{t_2\} \qquad \textit{and} \qquad]t_1, t_1 + h] \times \{t_2\}$$

belong to distinct q-bubbles with the same direction.

Another natural question is whether at some random level, one can pass from some bubble to a distinct bubble of the same direction while moving along a horizontal line segment. For any fixed t_2, $t_1 \mapsto W(t_1, t_2)$ has countably many local maxima and minima, but it can be shown that a.s., none of these correspond to passing between distinct bubbles of the same direction. In a future paper, the authors plan to show that this is possible for random t_2.

With regard to question 2 concerning broken line segments, we shall prove the following result.

Theorem 3 *Fix $q \in \mathbb{R}$ and $h > 0$. With positive probability, there exist uncountably many points $(t_1, t_2) \in [2, 3]^2$ such that*

$$W(t_1, t_2 + u) < q < W(t_1 + u, t_2), \qquad 0 < u < h \tag{2}$$

(and, of course, $W(t_1, t_2) = q$ by continuity).

As for Theorem 1, it is possible to deduce an a.s. statement from Theorem 3. The estimates used in the proof of Theorem 3 enable us to show in addition that the Hausdorff dimension of the set of points with the property (2) is 1/2: see Theorem 9. On the other hand, there are a small number of such points with an additional property as described in the following theorem.

Theorem 4 *Fix $q \in I\!R$. With positive probability, there exists $(t_1, t_2) \in [2,3]^2$ such that for $0 < u < 1$,*

$$W(t_1, t_2 + u) < q < W(t_1 + u, t_2) \quad \text{and} \quad W(t_1 - u, t_2) > q.$$

For a point (t_1, t_2) as in the conclusion of this theorem, q is a local minimum of $s_1 \mapsto W(s_1, t_2)$, this local minimum occurs at t_1 and (t_1, t_2) is the lower extremity of a vertical segment that is contained in a downwards q-bubble. In other words, W has a point of increase along the broken line with corner at (t_1, t_2).

The result of Theorem 3 raises the question of whether it is possible to fix the value of t_1 and find some random t_2 so that (t_1, t_2) has property (2). We prove that this is not possible even if the level q is allowed to be random. For convenience, we consider broken lines going up and then to the right, rather than, as in Theorem 3, lines going down and to the right. By time inversion properties of the Brownian sheet (see [23, Chap.1]), it is sufficient to consider one case only.

Theorem 5 *Fix $t_1 > 0$. A.s., there do not exist $t_2 > 0$ and $h > 0$, such that*

$$W(t_1, (t_2 - u) \vee 0) \leq W(t_1, t_2) \leq W(t_1 + u, t_2), \qquad \text{for all } 0 \leq u \leq h.$$

In the next section, we state the main estimates (Lemma 6) and use these to prove Theorems 1 and 3. We also prove Theorem 5. Section 3 is concerned with Hausdorff dimensions. In Section 4 we give the proof of the estimates in Lemma 6. Finally, in Section 5, we prove Theorems 4 and 2.

2 Points of increase

Throughout this paper, $q \in I\!R$ is fixed. For $u \geq 0$, define

$$g(u) = u^{3/4}$$

(3/4 could be replaced by any number greater than 1/2), and for $t = (t_1, t_2) \in I\!R_+^2$ and $n \in I\!N$, define $W_R^{t,n} = (W_R^{t,n}(u),\ u \geq 0)$, $W_U^{t,n} = (W_U^{t,n}(v),\ v \geq 0)$ and $W_L^{t,n} =$

$(W_L^{t,n}(u),\ 0 \leq u \leq t_1 - 2^{-2n})$, where

$$\begin{aligned} W_R^{t,n}(u) &= W(t_1 + 2^{-2n} + u, t_2) - W(t_1 + 2^{-2n}, t_2), \\ W_U^{t,n}(v) &= W(t_1, t_2 + 2^{-2n} + v) - W(t_1, t_2 + 2^{-2n}), \\ W_L^{t,n}(u) &= W(t_1 - 2^{-2n} - u, t_2) - W(t_1 - 2^{-2n}, t_2) \end{aligned}$$

("L" for "left," "R" for "right" and "U" for "up").

For each $t \in [2,3]^2$, define events $F_0(t,n)$, $F_R(t,n)$, $F_U(t,n)$ and $F_L(t,n)$ by

$$\begin{aligned} F_0(t,n) = \{ & W(t_1 + 2^{-2n}, t_2) \in]q + 2^{-n}, q + 2^{-n+1}[, \\ & W(t_1, t_2 + 2^{-2n}) \in]q - 2^{-n+1}, q - 2^{-n}[, \\ & W(t_1 - 2^{-2n}, t_2) \in]q - 2^{-n+1}, q - 2^{-n}[\}, \end{aligned}$$

$$\begin{aligned} F_R(t,n) = \{ & W_R^{t,n}(\cdot) \text{ hits } 1 \text{ before hitting the graph of } g(\cdot) - 2^{-n} \\ & \text{and } W_R^{t,n}(u) \geq g(u) - 2^{-n},\ 0 \leq u \leq 1\}, \end{aligned}$$

$$\begin{aligned} F_U(t,n) = \{ & W_U^{t,n}(\cdot) \text{ hits } -1 \text{ before the graph of } -g(\cdot) + 2^{-n}, \\ & \text{and } W_U^{t,n}(v) \leq -g(v) + 2^{-n},\ 0 \leq v \leq 1\}, \end{aligned}$$

$$\begin{aligned} F_L(t,n) = \{ & W_L^{t,n}(\cdot) \text{ hits } -1 \text{ before the graph of } -g(\cdot) + 2^{-n} \\ & \text{and } W_L^{t,n}(u) \leq -g(u) + 2^{-n},\ 0 \leq u \leq 1\}, \end{aligned}$$

and finally,

$$F^B(t,n) = F_U(t,n) \cap F_0(t,n) \cap F_R(t,n), \qquad F^H(t,n) = F_L(t,n) \cap F_R(t,n)$$

("B" for "broken line" and "H" for "horizontal line").

Let D_{2n} be the set of points in $[2,3]^2$ for which both coordinates are dyadic rationals of order $2n$. For $i,j \in \{0,\ldots,n\}$, let $E_{i,j}$ be the set of couples (s,t) of elements of D_{2n} such that

$$2^{-2(n-i+1)} \leq |s_1 - t_1| \leq 2^{-2(n-i)} \quad \text{and} \quad 2^{-2(n-j+1)} \leq |s_2 - t_2| \leq 2^{-2(n-j)}.$$

Finally, we consider the (partial) order $\leq$ on $\mathbb{R}_+^2$ defined by

$$s = (s_1, s_2) \leq t = (t_1, t_2) \qquad \Longleftrightarrow \qquad s_1 \leq t_1 \text{ and } s_2 \leq t_2.$$

We also write $s \ll t$ if $s_1 < t_1$ and $s_2 < t_2$.

Lemma 6 *Given $q \in \mathbb{R}$, there exist constants $K > 0$, $c > 0$ and $C > 0$ such that for all large $n \in \mathbb{N}$ and $t \in D_{2n}$,*

(a) $P(F_i(t,n)) \geq K\, 2^{-n}$, for $i \in \{0, R, U, L\}$;

(b) $P(F^B(t,n)) \geq K\,2^{-3n}$ *and* $P(F^H(t,n)) \geq K\,2^{-2n}$*;*
(c) for all $(s,t) \in E_{i,j}$ *with* $s \leq t$ *or* $t \geq s$*,*

$$P(F^B(s,n) \cap F^B(t,n)) \leq C\,2^{-(3n+i+j)}(2^{-i} \wedge 2^{-j});$$

(d) for all $(s,t) \in E_{i,j}$*,*

$$P(F^H(s,n) \cap F^H(t,n)) \leq C\,2^{-(2n+2i)}(\exp(-c(j-i)2^{-(n-j)}) \wedge 1).$$

Remark 7 (a) If $s,t \in [2,3]^2$ but neither $s \leq t$ nor $t \leq s$, then it follows from the definition that $F^B(s,n) \cap F^B(t,n) = \emptyset$. Indeed, if $s_1 < t_1$ and $s_2 > t_2$, for instance, then $W(t_1,s_2) > q$ on $F^B(s,n)$, while $W(t_1,s_2) < q$ on $F^B(t,n)$.

(b) Note the difference in the exponent of the first 2 on the right-hand sides of (c) and (d). A factor similar to the last exponential factor on the right-hand side of (d) could be included on the right-hand side of (c), to yield

$$P(F^B(s,n) \cap F(t,n)) \leq C\;2^{-(3n+i+j)}(2^{-i} \wedge 2^{-j})\exp(-c(n-i)2^{-(n-j)} - c(n-j)2^{-(n-i)}).$$

This extra exponential factor is crucial in the proof of Theorem 1 (horizontal lines) but is not needed in the proof of Theorem 3 (broken lines).

Assuming this lemma, we now prove Theorems 3 and 1. The method used in the proofs is known as the "second-moment argument."

PROOF OF THEOREM 3. We first show that with positive probability, there exists $t \in D_{2n}$ such that (2) holds. This will in particular allow the reader to compare this proof with the proof of Theorem 1 below. Let $X_n(\omega)$ be the number of elements $t \in D_{2n}$ such that $\omega \in F^B(t,n)$. We shall show that

$$E(X_n) \geq K\,2^n \qquad \text{and} \qquad E(X_n^2) \leq 8\,C\,2^{2n}. \tag{3}$$

Indeed, applying Lemma 6(b), we see that

$$E(X_n) = \sum_{t \in D_{2n}} P(F^B(t,n)) \geq (2^{2n})^2 K\,2^{-3n} = K\,2^n.$$

Moreover, noticing that the cardinality of $E_{i,j}$ is bounded by $(2^{2n})^2\,2^{2i}\,2^{2j}$, we can apply Lemma 6(c) to get

$$\begin{aligned} E(X_n^2) &= \sum_{s,t \in \mathbb{D}_{2n}} P(F^B(s,n) \cap F^B(t,n)) \\ &\leq \sum_{i=0}^{n}\sum_{j=0}^{n} \sum_{(s,t)\in E_{i,j}} C\,2^{-(3n+i+j)}(2^{-i} \wedge 2^{-j}) \end{aligned}$$

$$
\begin{aligned}
&\leq C\,2^n \sum_{i=0}^{n}\sum_{j=0}^{n} 2^{i+j}(2^{-i}\wedge 2^{-j}) \\
&\leq 2C\,2^n \sum_{i=0}^{n}\sum_{j=0}^{i} 2^{j} \\
&\leq 8C\,2^{2n}.
\end{aligned}
$$

This proves the inequalities in (3).

We now conclude from (3) that $E(X_n^2) \leq (8C/K^2)E(X_n)^2$, and an application of the Cauchy-Schwarz inequality to the right-hand side yields $P\{X_n > 0\} \geq \frac{K^2}{8C}$. By Fatou's lemma,

$$P\left(\limsup_{n\to\infty}\{X_n > 0\}\right) \geq \limsup_{n\to\infty} P\{X_n > 0\} \geq \frac{K^2}{8C} > 0.$$

Let $G = \limsup_{n\to\infty}\{X_n > 0\}$ and fix $w \in G$. There is a sequence $n_k \uparrow \infty$ such that $X_{n_k}(\omega) > 0$ for all k, that is, there exists a sequence $(t^{(k)}) \subset [2,3]^2$ such that $\omega \in F^B(t^{(k)}, n_k)$ for all k (note in passing that $\{t^{(k)}\}$ is totally ordered for $\leq$ by Remark 7(a)). By taking a subsequence, we can assume that $(t^{(k)})$ converges to $t \in [2,3]^2$. By construction, $W(t) = q$ and for $0 < u < 1$,

$$W(t_1+u, t_2) - q = \lim_{k\to\infty} W(t_1^{(k)}+u, t_2^{(k)}) - q \geq \lim_{k\to\infty} g(u - 2^{-n_k}) - 2^{n_k} = g(u) > 0,$$

and similarly, $W(t_1, t_2+u) - q < 0$. This proves that on G, there exists a point $(t_1, t_2) \in [2,3]^2$ such that property (2) holds. Building on the arguments just developped, we shall show in Theorem 9 that with positive probability, the Hausdorff dimension of the set of points with property (2) is 1/2, which implies that there are uncountably many such points. This proves Theorem 3. $\diamondsuit$

Proof of Theorem 1. Let $X_n(\omega)$ be this time the number of elements $t \in D_{2n}$ such that $\omega \in F^H(t,n)$. We shall show that

$$E(X_n) \geq K\,2^{2n} \qquad \text{and} \qquad E(X_n^2) \leq C(4+4c)2^{4n}. \tag{4}$$

Indeed, applying Lemma 6(b),

$$E(X_n) = \sum_{t\in D_{2n}} P(F^H(t,n)) \geq (2^{2n})^2 K\,2^{-2n} = K\,2^{2n}.$$

Moreover, by Lemma 6(d) and the bound on the cardinality of $E_{i,j}$,

$$
\begin{aligned}
E(X_n^2) &= \sum_{s,t\in \mathbb{D}_{2n}} P(F^H(s,n)\cap F^H(t,n)) \\
&\leq \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{(s,t)\in E_{i,j}} C\,2^{-(2n+2i)}(\exp(-c(j-i)2^{-(n-j)})\wedge 1) \\
&\leq C\,2^{2n}\sum_{i=0}^{n}\sum_{j=0}^{n} 2^{2j}(\exp(-c(j-i)2^{-(n-j)})\wedge 1).
\end{aligned}
$$

We split the sum into two parts, according as $i \geq j$ or $i < j$. When $i \geq j$, the last factor equals 1 and the first part of the sum becomes

$$\sum_{i=0}^{n} \sum_{j=0}^{i} 2^{2j} \leq 2 \sum_{i=0}^{n} 2^{2i} \leq 4\, 2^{2n}.$$

When $i < j$, the exponential plays a crucial role and the second part of the sum becomes

$$\begin{aligned} \sum_{i=0}^{n} \sum_{j=i+1}^{n} 2^{2j} \exp(-c(j-i)2^{-(n-j)}) &= \sum_{j=1}^{n} \sum_{i=0}^{j-1} 2^{2j} \exp(-c(j-i)2^{-(n-j)}) \\ &= \sum_{j=1}^{n} 2^{2j} \sum_{i=1}^{j} \exp(-c\, i\, 2^{-(n-j)}). \end{aligned}$$

The sum over i is geometric, equal to

$$\frac{\exp(-c2^{-(n-j)}) - \exp(-c(j+1)2^{-(n-j)})}{1 - \exp(-c\, 2^{-(n-j)})}.$$

Because the numerator is ≤ 1 and the denominator is $\geq c\; 2^{-(n-j)-1}$, we conclude that

$$E(X_n^2) \leq C\; 2^{2n} \left(4 2^{2n} + c \sum_{j=0}^{n} 2^{2j} \cdot 2^{n-j+1} \right) \leq C(4+4c)2^{4n}.$$

This proves the inequalities in (4). The remainder of the proof is similar to the end of the proof of Theorem 3 and is therefore omitted. ◊

Remark 8 One could attempt to use the same method to prove existence of points of increase along horizontal lines at a fixed level q, say. In this case, one would define

$$F_0^H(t,n) = F_L(t,n) \cap F_0(t,n) \cap F_R(t,n).$$

As in Lemma 6(b), we would check that

$$P(F_0^H(t,n)) \geq K\, 2^{-3n},$$

and as in Lemma 6(c), for all $(s,t) \in E_{i,j}$,

$$P(F_0^H(s,n) \cap F_0^H(t,n)) \leq C\; 2^{-(3n+2i)}(2^{-i} \wedge 2^{-j}) \exp(-c(n-i)2^{-(n-j)} - c(n-j)2^{-(n-i)}).$$

Letting $X_n(\omega)$ be the number of elements $t \in D_{2n}$ such that $\omega \in F_0^H(t,n)$, the reader may check that one gets, as in the proof of Theorem 1, that

$$E(X_n) \geq K\, 2^n \qquad \text{and} \qquad E(X_n^2) \leq C' 2^{2n} \log n.$$

One cannot conclude from these bounds that $E(X_n)^2/E(X_n^2)$ is bounded below, so the "second-moment argument" with these estimates fails.

PROOF OF THEOREM 5. The scaling properties of the Brownian sheet imply that $(s_1, s_2) \mapsto W(s_1 t_1, s_2)/\sqrt{t_1}$ is again a Brownian sheet, and therefore it suffices to prove the theorem in the case where $t_1 = 1$. Similarly, $(s_1, s_2) \mapsto \sqrt{v}\, W(1, s_2/v)$ is a Brownian sheet, and therefore it suffices to show that for all $h > 0$, $P(F_h) = 0$, where

$$F_h = \{\exists t_2 \geq 1 : W(1, (t_2 - u) \vee 0) \leq W(t_1, t_2) \leq W(t_1 + u, t_2),\ \forall u \in [0, h]\}.$$

For $k \in I\!N$, let $F_{h,k}$ be defined in the same way as F_h, but with the additional restriction that $t_2 \in [1+kh, 1+(k+1)h]$. It will even suffice to prove that $P(F_{h,k}) = 0$, for all $h > 0$ and $k \in I\!N$.

Fix $h > 0$ and $k \in I\!N$. Then $F_{h,k} \subset G_{h,k}$, where

$$\begin{aligned} G_{h,k} &= \{\exists v \in [0, h] : W(1, 1 + kh + v) = \sup_{0 \leq u \leq v} W(1, 1 + kh + u) \qquad (5)\\ &\qquad \text{and } W(1 + u, 1 + kh + v) - W(1, 1 + kh + v) \geq 0,\ \forall u \in [0, h]\}. \qquad (6) \end{aligned}$$

Note that the set D of all $v \in [0, h]$ with property (5) has the same law as the zero set of a Brownian motion [19, Chap.VI, Theorem 2.3]. As such, its Hausdorff measure with respect to the function $t \mapsto (2t \log\log(1/t))^{\frac{1}{2}}$ is finite a.s. (see [18, 22]), and in particular, its Hausdorff measure with respect to the function $\sqrt{t}$ is zero a.s. Therefore, for any $\varepsilon > 0$, we can cover D by a sequence (I_j) of (random) intervals such that $\sum_j |I_j|^{\frac{1}{2}} < \varepsilon$ a.s. Note that the I_j are measurable relative to the sigma-field generated by the process $W(1, \cdot)$. In addition,

$$G_{h,k} \subset \cup_j H_{h,k,j}, \qquad (7)$$

where

$$H_{h,k,j} = \{\exists v \in I_j : W(1 + u, 1 + kh + v) - W(1, 1 + kh + v) \geq 0,\ \forall u \in [0, h]\}.$$

Fix j. We shall show that there is a constant K such that

$$P(H_{h,k,j} \mid W(1, \cdot)) \leq K |I_j|^{\frac{1}{2}} \quad \text{a.s.} \qquad (8)$$

Indeed, if $I_j = [u_2, v_2]$ and $B_{h,k,j}(u) = W(1+u, 1+kh+u_2) - W(1, 1+kh+u_2)$, then

$$W(1 + u, 1 + kh + v) - W(1, 1 + kh + v) = B_{h,k,j}(u) + \Delta_{]1,1+u]\times]1+kh+u_2,1+kh+v]} W,$$

where for $s_1 \leq t_1$ and $s_2 \leq t_2$,

$$\Delta_{]s_1,t_2]\times]s_2,t_2]} W = W(t_1, t_2) - W(s_1, t_2) - W(t_1, s_2) + W(s_1, s_2),$$

and so

$$H_{h,k,j} \subset \left\{ \inf_{0 \leq u \leq h} B_{h,k,j}(u) + \sup_{0 \leq u \leq h,\ u_2 \leq v \leq v_2} \Delta_{]1,1+u]\times]1+kh+u_2,1+kh+v]} W \geq 0 \right\}.$$

By the independent increments and scaling properties of the Brownian sheet, the event on the right-hand side has the same conditional probability given $W(1,\cdot)$ as

$$\left\{\sqrt{h}\,\sqrt{1+kh+u_2}\inf_{0\le u\le 1} B(u)+\sqrt{h}\,\sqrt{v_2-u_2}\sup_{0\le u\le 1,\ 0\le v\le 1}\tilde{W}(u,v)\ge 0\right\},$$

where $B(\cdot)$ is a standard Brownian motion and $\tilde{W}$ is an independent Brownian sheet. The probability of this event is bounded by

$$\int_{I\!R} P\left\{\inf_{0\le u\le 1} B(u)\ge -x\sqrt{v_2-u_2}\right\} P\left\{\sup_{0\le u\le 1,\ 0\le v\le 1}\tilde{W}(u,v)\in[x,x+dx]\right\}. \tag{9}$$

From the reflection principle, $P\{\inf_{0\le u\le 1} B(u)\ge -\alpha\}=P\{|B(1)|\le\alpha\}\le 2\varphi(0)\alpha$, where $\varphi(\cdot)$ is the standard $N(0,1)$ density function, and therefore, the expression in (9) is bounded by

$$C|I_j|^{\frac{1}{2}}E\left(\sup_{0\le u\le 1,\ 0\le v\le 1}\tilde{W}(u,v)\right).$$

The expectation is finite by [17, Lemma 1.2], and so (8) is proved.

From (7) and (8), we conclude that

$$P(G_{h,k})\le E\left(\sum_j P(H_{h,k,j}\mid W(1,\cdot))\right)\le KE\left(\sum_j |I_j|^{\frac{1}{2}}\right)\le K\varepsilon,$$

and as $\varepsilon>0$ is arbitrary, this implies that $P(G_{h,k})=0$. This completes the proof of Theorem 5. ◊

3 Hausdorff dimensions

Theorem 9 *Fix $q\in I\!R$ and $h>0$. With probability one, the Hausdorff dimension of*

$$\mathcal{I}=\{(t_1,t_2)\in I\!R_+^2: W(t_1,t_2+u)<q<W(t_1+u,t_2),\ 0<u<h\}$$

is equal to $\frac{1}{2}$.

Proof. Given the estimates of Lemma 6, this proof goes along the same lines as that of [16, Proposition 2.1]. We first prove that the Hausdorff dimension is $\le 1/2$. By scaling properties of the Brownian sheet, it suffices to consider the case $h=1$. We can also replace $\mathcal{I}$ by $\mathcal{I}\cap[2,3]^2$. For $s\in D_{2n}$, define squares

$$I_n(s)=[s_1,s_1+2^{-2n}]\times[s_2,s_2+2^{-2n}] \tag{10}$$

and events

$$F_n(s)=\{\exists t\in I_n(s): W(t_1,t_2+u)<q<W(t_1+u,t_2),\ 0<u\le 1\}.$$

We shall show below that there is $K > 0$ such that

$$P(F_n(s)) \le K n^3\, 2^{-3n}, \text{ for all large } n \in I\!N \text{ and } s \in D_{2n}. \tag{11}$$

Assuming this, we now prove the theorem. Notice that for each $n \in I\!N$ and $\omega \in \Omega$, $\{I_n(s) : s \in D_{2n}$ and $\omega \in F_n(s)\}$ is a (random) covering of $\mathcal{I} \cap [2,3]^2$, and for $\alpha > 0$,

$$\begin{aligned} E\left(\sum_{s \in D_{2n} :\, \omega \in F_n(s)} (\text{diam } I_n(s))^\alpha \right) &\le \sum_{s \in D_{2n}} 2^{-\alpha(2n+\frac{1}{2})} P(F_n(s)) \\ &\le K 2^{4n}\, 2^{-2\alpha n} n^3\, 2^{-3n} \\ &= K n^3\, 2^{(1-2\alpha)n}. \end{aligned}$$

For $\alpha > 1/2$, we conclude from Fatou's lemma that

$$\liminf_{n \to \infty} \sum_{s \in D_{2n} :\, \omega \in F_n(s)} (\text{diam } I_n(s))^\alpha = 0 \quad a.s.,$$

and therefore the Hausdorff dimension of $\mathcal{I} \cap [2,3]^2$ is $\le \alpha$ a.s., for all $\alpha > 1/2$, and thus is $\le 1/2$ a.s.

We now prove (11). Let

$$W_R^s(u) = W(s_1 + u, s_2) - W(s_1, s_2), \qquad W_U^s(v) = W(s_1, s_2 + v) - W(s_1, s_2),$$

and define

$$G_n(s) = \{|W(s) - q| < n 2^{-n}\} \cap \tilde{G}_n(s),$$

where

$$\tilde{G}_n(s) = \{W_R^s(u) > -n 2^{-n+1} \text{ and } W_U^s(u) < n 2^{-n+1},\ 0 < u < \frac{1}{2}\}.$$

Then $F_n(s) \setminus G_n(s)$ is contained in

$$(F_n(s) \cap \{|W(s) - q| \ge n 2^{-n}\}) \cup (F_n(s) \cap \{|W(s) - q| < n 2^{-n}\} \cap \tilde{G}_n(s)^c). \tag{12}$$

On $F_n(s)$, there is $t \in I_n(s)$ such that $W(t) = q$, and therefore the first term in the union is contained in

$$\{ \sup_{t \in I_n(s)} |W(t) - W(s)| \ge n 2^{-n}\}, \tag{13}$$

and the probability of this event is bounded by $K e^{-cn^2}$ by [17, Lemma 1.3].

On $\{|W(s) - q| < n 2^{-n}\} \cap \tilde{G}_n(s)^c$, there is $u \in]0, 1/2]$ such that either

$$W(s_1 + u, s_2) < q + n 2^{-n} - n 2^{-n+1} = q - n 2^{-n},$$

or $W(s_1, s_2+u) > q + n2^{-n}$. Therefore, the second term on the right-hand side of (12) is contained in

$$\left\{ \sup_{0<u<\frac{1}{2}} \sup_{0<v<2^{-2n}} |W(s_1+u, s_2+v) - W(s_1+u, s_2)| > n2^{-n} \right\} \tag{14}$$

$$\cup \left\{ \sup_{0<u<2^{-2n}} \sup_{0<v<\frac{1}{2}} |W(s_1+u, s_2+v) - W(s_1, s_2+v)| > n2^{-n} \right\}. \tag{15}$$

Scaling properties of the Brownian sheet imply that the probability of this event is not greater than

$$2P\left\{ \sup_{3<u<\frac{7}{2}} \sup_{0<v<1} |W(u,v)| > n \right\} \leq 2Ke^{-cn^2}$$

by [17, Lemma 1.3]. We have therefore shown that there is $K > 0$ such that for all n,

$$P(F_n(s) \setminus G_n(s)) \leq Ke^{-cn^2}.$$

We now estimate $P(F_n(s) \cap G_n(s))$. This is of course not greater than $P(G_n(s))$, and by the independent increments property of the Brownian sheet, this is bounded by

$$P\{|W(s) - q| < n2^{-n}\} \cdot (Kn2^{-n+1})^2 \leq K'n^3\,2^{-3n}.$$

In conclusion, for large n,

$$\begin{aligned} P(F_n(s)) &\leq P(F_n(s) \setminus G_n(s)) + P(F_n(s) \cap G_n(s)) \\ &\leq Kn^3\,2^{-3n}, \end{aligned}$$

and (11) is proved.

We now prove that the Hausdorff dimension of $\mathcal{I}$ is $\geq 1/2$. Let $B_n(\omega) = \{t \in D_{2n} : \omega \in F^B(t,n)\}$. We begin by showing that for $\alpha < 1/2$, there is $p > 0$ and $k < \infty$ such that for all $n \in I\!\!N$,

$$P\left\{ \frac{1}{\text{card}(B_n)} Y_n < k \right\} \geq p,$$

where

$$Y_n = \sum_{\substack{s,t \in B_n \\ s \neq t}} \frac{1}{|s-t|^\alpha}.$$

First, observe that card $B_n = X_n$, where X_n is defined in the proof of Theorem 3, and using (3) and the definition of the $E_{i,j}$, we see that

$$\frac{1}{E(X_n)^2} E(Y_n) \leq \frac{2^{-2n}}{K^2} \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{(s,t) \in E_{i,j}} \frac{P\{s \in B_n,\ t \in B_n\}}{(2^{-2(n-i+1)} \wedge 2^{-2(n-j+1)})^\alpha}.$$

Now $s \in B_n$ and $t \in B_n$ if and only if $F^B(s,n) \cap F^B(t,n)$ occurs, so by Lemma 6(c), this is bounded by

$$\begin{aligned} &\frac{4C}{K^2} 2^{2n} \sum_{i=0}^{n} \sum_{j=0}^{n} 2^{2i}\, 2^{2j}\, 2^{-(3n+i+j)} (2^{-i} \wedge 2^{-j})(2^{2(n-i)} \vee 2^{2(n-j)})^{\alpha} \\ &\leq \frac{8C}{K^2} 2^{(-1+2\alpha)n} \sum_{i=0}^{n} \sum_{j=0}^{i} 2^{(1-2\alpha)j} \\ &\leq \tilde{C}\ 2^{(-1+2\alpha)n}\, 2^{(1-2\alpha)n} \\ &= \tilde{C} \end{aligned}$$

(the last inequality uses the fact that $\alpha < 1/2$). Using the elementary inequality $P\{X > \lambda E(X)\} \geq (1-\lambda)^2 E(X)^2 / E(X^2)$ for a non-negative random variable Y (see e.g. [9, p.8]), we conclude from (3) that there is $c > 0$ (not depending on n) such that $P(X_n > cE(X_n)) > c$, and from the above that there is $L < \infty$ such that

$$P\left\{ \frac{1}{E(X_n)^2} Y_n \leq L \right\} > 1 - \frac{c}{2}.$$

Therefore, $P\{Y_n \leq Lc^{-2} X_n^2\} \geq c/2$. Set $F = \limsup_{n\to\infty} \{Y_n \leq Lc^{-2} X_n^2\}$. By Fatou's lemma, $P(F) \geq c/2$, and on F,

$$\frac{1}{(\text{card } B_n)^2} Y_n \leq Lc^{-2}$$

occurs for infinitely many n. Because all limits of sequences (t_n), with $t_n \in B_n$ for all n, belong to $\mathcal{I} \cap [2,3]^2$, we conclude following [12, Chap. II §3, p. 160-162] that on F, the capacitary dimension of $\mathcal{I} \cap [2,3]^2$, therefore its Hausdorff dimension by Frostman's Lemma [Kahane, Chap. 3 §3], is at least α.

In fact, c actually depends on q, through the constants in Lemma 6(a) and (c). However, inspection of the proof of these two statements in Lemma 6 (given in the next section) shows that the constant in (c) of this Lemma does not depend on q, and for q in a bounded interval, the constant in (a) can be chosen independently of q. Writing $\mathcal{I}_q$ instead of $\mathcal{I}$ to emphasize the dependence on q, notice that scaling properties of the Brownian sheet imply that $P\{\dim(\mathcal{I}_q \cap [2k,3k]^2) \geq \alpha\} \geq P\{\dim(\mathcal{I}_{q/k} \cap [2,3]^2) \geq \alpha\} \geq c/2$ for all $k \in \mathbb{N}^*$. We conclude from the zero-one law of Orey and Pruitt [17] that $P\{\dim \mathcal{I}_q \geq \alpha\} = 1$. $\diamondsuit$

Remark 10 Fix $q \in \mathbb{R}$ and $h > 0$, and let $\mathcal{I}$ be as in Theorem 9. Observe that any pair $\{s,t\}$ of points in $\mathcal{I}$ such that $\max(|s_1 - t_1|, |s_2 - t_2|) < h$ must satisfy either $s \ll t$ or $t \ll s$. Indeed, if for instance, $s_1 > t_1$ and $s_2 \leq t_2$, then (s_1, t_2) would satisfy $W(s_1,t_2) \leq q$ (because $s \in \mathcal{I}$) and $W(s_1,t_2) > q$ (because $t \in \mathcal{I}$), a contradiction.

As mentioned at the end of the proof of Theorem 3, an obvious consequence of Theorem 9 is that $\mathcal{I}$ is uncountable and therefore cannot consist only of isolated points. If R is a square with sides of length h that contains an infinite subset $\mathcal{I}_R$ of $\mathcal{I}$, then by the above, $\mathcal{I}_R$ must be totally ordered for $\leq$ and is therefore contained in a monotone curve. It might be tempting to conjecture that $\mathcal{I}$ *contains* a monotone curve, but this is not the case. Indeed, any such curve would be contained in the level set $\{t : W(t) = q\}$, but this is not possible by [3, Theorem 1]: any Jordan arc in the level set must be nowhere differentiable, whereas a monotone curve has a tangent at many points. In fact, the discussion above, along with [3, Theorem 3], implies that $\mathcal{I}$ is an uncountable but totally disconnected set.

Remark 11 The same arguments as in the proof of the upper-bound in Theorem 9 show that for $h > 0$, the Hausdorff dimension of the set

$$B = \{(t_1, t_2) \in \mathbb{R}_+^2 : W(t_1 - u, t_2) < W(t_1, t_2) < W(t_1 + u, t_2),\ 0 < u < h\}$$

is ≤ 1. However, the bound in Lemma 6(d) is not sufficient to yield a lower bound on the Hausdorff dimension of this set.

4 Proof of Lemma 6

(a) Fix $t \in D_{2n}$. We first show that there is $K > 0$ such that $P(F_0(t,n)) \geq K 2^{-n}$. Let

$$\begin{array}{ll} Z_1 = W(t_1 - 2^{-2n}, t_2), & Z_2 = W(t_1, t_2) - W(t_1 - 2^{-2n}, t_2) \\ Z_3 = W(t_1 + 2^{-2n}, t_2) - W(t_1, t_2), & Z_4 = W(t_1, t_2 + 2^{-2n}) - W(t_1, t_2). \end{array}$$

Because of the independence of increments of the Brownian sheet over disjoint rectangles, Z_1, Z_2, Z_3 and Z_4 are independent Gaussian random variables, and

$$\begin{aligned} F_0(t,n) &= \{Z_1 + Z_2 + Z_3 \in]q + 2^{-n}, q + 2^{-n+1}[, \\ &\qquad Z_1 + Z_2 + Z_4 \in]q - 2^{-n+1}, q - 2^{-n}[, \quad Z_1 \in]q - 2^{-n+1}, q - 2^{-n}[\,\}. \end{aligned}$$

Let X_1, X_2, X_3 and X_4 be independent standard $N(0,1)$ random variables, and set $\alpha_{t,n} = ((t_1 - 2^{-2n}) \cdot t_2)^{1/2}$. Then (Z_1, Z_2, Z_3, Z_4) has the same law as

$$(\alpha_{t,n} X_1,\ 2^{-n}\sqrt{t_2} X_2,\ 2^{-n}\sqrt{t_2} X_3,\ 2^{-n}\sqrt{t_1} X_4),$$

and therefore

$$\begin{aligned} P(F_0(t,n)) &\geq P\{\alpha_{t,n} X_1 + 2^{-n}\sqrt{t_2}(X_2 + X_3) \in]q + 2^{-n}, q + 2^{-n+1}[, \\ &\qquad \alpha_{t,n} X_1 + 2^{-n}(\sqrt{t_2}\, X_2 + \sqrt{t_1}\, X_4) \in]q - 2^{-n+1}, q - 2^{-n}[, \\ &\qquad \alpha_{t,n} X_1 \in]q - 2^{-n+1} q - 2^{-n}[,\ X_2 \in [-1, 1]\}. \end{aligned}$$

Because $1 \leq \alpha_{t,n} \leq 9$, we use independence and the fact that the standard Gaussian density is bounded below over the interval $[-5,5]$ to see that this is bounded below by

$$\int_{(q-2^{-n+1})/\alpha_{t,n}}^{(q-2^{-n})/\alpha_{t,n}} dx_1\ f_{X_1}(x_1) \int_{-1}^{1} dx_2\ f_{X_2}(x_2) \cdot k_0 \geq K\ 2^{-n},$$

where f_{X_1} (resp. f_{X_2}) denotes the density function of X_1 (resp. X_2). For $i \in \{R, U, L\}$, $P(F_i(t,n))$ is bounded below by

$$\begin{aligned} P\{B \text{ hits } \frac{1}{9} \text{ before the graph of } \frac{1}{9}(g(\cdot) - 2^{-n}) \\ \text{and } B_u \geq \frac{1}{9}(g(u) - 2^{-n}),\ 0 \leq u \leq 1\}, \end{aligned}$$

where $B = (B_u)$ is a standard Brownian motion. This probability is $\geq K\,2^{-n}$ by Lemma 12 below.

(b) By the independent increments property of W, $F_U(t,n)$, $F_0(t,n)$ and $F_R(t,n)$ are independent and similarly, $F_L(t,n)$ and $F_R(t,n)$ are independent. Therefore, $P(F^B(t,n)) \geq K\,2^{-3n}$ and $P(F^H(t,n)) \geq K\ 2^{-2n}$ by (a).

(c) Observe that for $(s,t) \in E_{i,j}$ with $s \leq t$,

$$\begin{aligned} & F^B(s,n) \cap F^B(t,n)) \\ & \quad \subset F_0(s,n) \cap \tilde{F}_U(s,t,n) \cap \tilde{F}_R(s,t,n) \cap F_0(t,n) \cap F_U(t,n) \cap F_R(t,n), \end{aligned} \tag{16}$$

where

$$\begin{aligned} \tilde{F}_U(s,t,n) &= \left\{ W_U^{s,n}(u) \leq -g(u) + 2^{-n},\ 0 \leq u \leq \frac{t_2 - s_2}{2} \right\} \\ \tilde{F}_R(s,t,n) &= \left\{ W_R^{s,n}(u) \geq g(u) - 2^{-n},\ 0 \leq u \leq \frac{t_1 - s_1}{2} \right\}. \end{aligned}$$

Notice that $F_U(t,n)$ and $F_R(t,n)$ are independent of each other and of the other events in (16), and the probability of each is bounded by $P\{B(u) \geq -2^{-n},\ 0 \leq u \leq 1\}$, where $(B(u))$ is a standard Brownian motion, therefore, from the reflection principle, by $C\ 2^{-n}$ [19, Chap.III.§3].

Set

$$\begin{aligned} Y_1 &= W\left(s_1, \frac{s_2 + t_2}{2}\right), \\ Y_2 &= W\left(\frac{s_1 + t_1}{2}, s_2\right) - W(s_1, s_2), \\ Y_3 &= W(t_1 + 2^{-n}, t_2) - Y_1 - Y_2. \end{aligned}$$

The event $F_0(t,n)$ is contained in the event

$$\{W(t_1 + 2^{-2n}, t_2) \in]q + 2^{-n}, q + 2^{-n+1}[\}.$$

This event can be written

$$\{Y_3 \in]q + 2^{-n} - Y_1 - Y_2, q + 2^{-n+1} - Y_1 - Y_2[\},$$

and therefore its conditional probability given $\mathcal{F}_{(s_1,(s_2+t_2)/2)} \vee \mathcal{F}_{((s_1+t_1)/2,t_2)}$ is bounded above by the (unconditional) probability that Y_3 lie in some given interval of length 2^{-n}. Because $(s,t) \in E_{i,j}$, Var $Y_3 \geq (2^{-2(n-i)} \vee 2^{-2(n-j)})/8$, and so this probability is $\leq C(2^{-i} \wedge 2^{-j})$. The remaining three events $F_0(s,n)$, $\tilde{F}_U(s,t,n)$ and $\tilde{F}_R(s,t,n)$ are independent,

$$P(F_0(s,n)) \leq P\{W(s_1 + 2^{-2n}, s_2) \in]q + 2^{-n}, q + 2^{-n+1}[\} \leq C\ 2^{-n},$$

and

$$\begin{aligned} P(\tilde{F}_U(s,t,n)) &\leq P\{B(u) \leq 2^{-n},\ 0 \leq u \leq 2^{-2(n-j+2)}\} \leq C\ 2^{-j}, \\ P(\tilde{F}_R(s,t,n)) &\leq P\{B(u) \geq -2^{-n},\ 0 \leq u \leq 2^{-2(n-i+2)}\} \leq C\ 2^{-i} \end{aligned}$$

(we have again used the reflection principle, along with Brownian scaling). Putting together the bounds above, we conclude that

$$P(F^B(s,n) \cap F^B(t,n)) \leq C\ 2^{-3n}\, 2^{-i}\, 2^{-j}\, (2^{-i} \wedge 2^{-j}),$$

which is the estimate in (c).

(d) Fix $(s,t) \in E_{i,j}$ such that $s \leq t$ (the other relative positions of s and t are bounded similarly). Define

$$\begin{aligned} \hat{F}_R(s,t,n) &= \{W_R^{s,n}(u) \geq g(u) - 2^{-n},\ 0 \leq u \leq \frac{t_1 - s_1}{2}\}, \\ \hat{F}_L(t,s,n) &= \{W_L^{t,n}(u) \leq -g(u) + 2^{-n},\ 0 \leq u \leq \frac{t_1 - s_1}{2}\}, \\ G_R(t,s,n) &= \{W_R^{t,n}(\cdot) \text{ hits 1 before } -2^{-n}, \\ &\qquad W_R^{s,n}(t_1 - s_1 + \cdot) \text{ hits 1 before } -2^{-n}\}. \end{aligned}$$

Observe that

$$F^H(s,n) \cap F^H(t,n) \subset F_L(s,n) \cap \hat{F}_R(s,t,n) \cap \hat{F}_L(t,s,n) \cap G_R(t,s,n),$$

and that the first three events on the right-hand side are mutually independent. Considerations in the proof of (c) show that

$$P(F_L(s,n)) \leq C\, 2^{-n}, \quad P(\hat{F}_R(s,t,n)) \leq C\, 2^{-i}, \quad P(\hat{F}_L(t,s,n)) \leq C\, 2^{-i}.$$

Let

$$H(s,t,n) = F_L(s,n) \cap \hat{F}_R(s,t,n) \cap \hat{F}_L(t,s,n) \cap \{W_R^{s,n}(t_1 - s_1) > -2^{-n}\}.$$

Then $P(H(s,t,n)) \le C\,2^{-n-2i}$, and therefore the conclusion will follow if we prove that

$$P(G_R(t,s,n)|H(s,t,n)) \le K\,2^{-n}(\exp(-c(j-i)2^{-(n-j)}) \wedge 1). \tag{17}$$

When $j \le i$, the right-hand side is equal to 2^{-n}. Because

$$G_R(t,s,n) \subset \{W_R^{t,n}(\cdot) \text{ hits } 1 \text{ before } -2^{-n}\}$$

and the event on the right-hand side is independent of $H(s,t,n)$ and its probability is $\le 2^{-n}$, the inequality (17) is satisfied in this case.

Assume now that $i < j$. Set

$$\begin{aligned}
Y &= W(t_1,s_2) - W\left(\frac{s_1+t_1}{2} + 2^{-2n}, s_2\right),\\
\hat{Y} &= E\left(Y|W_L^{t,n}(u),\ 0 \le u \le \frac{t_1+s_1}{2}\right) = \frac{s_2}{t_2}\,W_L^{t,n}\left(\frac{s_1+t_1}{2} - 2^{-2n}\right),\\
Z &= W_R^{s,n}(t_1-s_1)\\
&= W_R^{s,n}\left(\frac{s_1+t_1}{2}\right) + \hat{Y} + (Y-\hat{Y}) + (W(t_1+2^{-2n},s_2) - W(t_1,s_2)).
\end{aligned}$$

Then

$$G_R(t,s,n) = \{W_R^{t,n} \text{ hits } 1 \text{ before } -2^{-n},\ W_R^{(t_1,s_2),n}(\cdot) + Z \text{ hits } 1 \text{ before } -2^{-n}\}.$$

Notice that $(W_R^{t,n}(\cdot), W_R^{(t_1,s_2),n}(\cdot))$ is independent of $\sigma(H(s,t,n)) \vee \sigma(Z)$, and therefore, for each z, the event

$$G(t,s,n;z) = \{W_R^{t,n}(\cdot) \text{ hits } 1 \text{ before } -2^{-n}, W_R^{(t_1,s_2),n}(\cdot) \text{ hits } 1-z \text{ before } -z-2^{-n}\}$$

is independent of $\sigma(H(s,t,n)) \vee \sigma(Z)$. It follows that

$$P(G_R(t,s,n)|H(s,t,n)) = \int_{-2^{-n}}^{+\infty} P(G(t,s,n;z))\, f_{Z|H(s,t,n)}(z)\, dz, \tag{18}$$

where $f_{Z|H(s,t,n)}$ denotes the conditional density of Z given $H(s,t,n)$. Let $G'(t,s,n;z)$ be defined as $G(t,s,n;z)$ but with $1-z$ replaced by $1/2$. For $z \le 1/2$, $G(t,s,n;z) \subset G'(t,s,n;z)$, and $P(G'(t,s,n;z))$ is a non-decreasing function of z, which is therefore bounded by $P(G'(t,s,n;z \vee 2^{-n}))$.

Given $H(s,t,n)$, the law of $W_R^{s,n}((s_1+t_1)/2)$ is that of a standard Brownian motion at time $s_2^{1/2}(t_1-s_1)/2$ conditioned not to have hit -2^{-n}, and the law of Y is also, but at time $t_2^{1/2}(t_1-s_1)/2$. Because $Y-\hat{Y}$ is independent of $H(s,t,n)$, its law given $H(s,t,n)$ is still normal, with mean 0 and variance $s_2(1-s_2/t_2)(t_1-s_1)/2$. It follows therefore from Lemmas 14 and 15 below that the conditional density of $2^{n-i}\,Z$ given $H(s,t,n)$ is bounded by

$$\psi(x) = K(|x|^3 \vee 1)e^{-x^2/2},$$

where K is a constant that does not depend on n, i or j, and therefore the conditional probability in (18) is no greater than

$$\int_{-2^{-i}}^{+\infty} (P(G'(t,s,n;(2^{-(n-i)}x) \vee 2^{-n})) \vee 1)\, \psi(x)\, dx. \tag{19}$$

Let $k_0 = \sup_{x \geq 0} x^{1/2}\, 2^{-x}$. Then the integral in (19) can be split into two integrals, the first over $x \leq (n-i)^{1/2}/(16k_0)$, the second over $x > (n-i)^{1/2}/(16k_0)$. By Lemma 13 below, the first integral is bounded by

$$\begin{aligned}
&\int_{-2^{-i}}^{+\infty} 2^{-n} \left(\frac{(2^{-(n-i)}x) \vee 2^{-n}}{2^{-(n-j+1)}} \right)^{c2^{-(n-j)}/\sqrt{s_2}} \psi(x)\, dx \\
&\quad = 2^{-n} (2^{i-j})^{c2^{-(n-j)}/\sqrt{s_2}} \int_{-2^{-i}}^{+\infty} (x \vee 2^{-i})^{c2^{-(n-j)}/\sqrt{s_2}} \psi(x)\, dx \\
&\quad \leq K\; 2^{-n}\, 2^{-c(j-i)2^{-(n-j)}/\sqrt{s_2}},
\end{aligned} \tag{20}$$

while the second integral is bounded by

$$\int_{(n-i)^{1/2}/(16k_0)}^{+\infty} P\{W_R^{t,n}(\cdot) \text{ hits } 1 \text{ before } -2^{-n}\}\, \psi(x)\, dx \leq K 2^{-n} e^{-(n-i)/C}.$$

For $c < 1/C$, the right-hand side is

$$\leq K\; 2^{-n} \exp(-c(n-i)2^{-(n-j)}). \tag{21}$$

We observe that

$$2^{-c(n-i)2^{-(n-j)}} \leq 2^{-c(j-i)2^{-(n-j)}},$$

and, from (19), (20) and (21), we conclude that (17) holds with $i < j$. This completes the proof of Lemma 6. $\diamondsuit$

Lemma 12 *Fix $\alpha > \frac{1}{2}$ and $\kappa > 0$. Set $g_\alpha(u) = u^\alpha$ and let $B = (B_u,\ u \geq 0)$ be a standard Brownian motion started at zero. There is a constant $c > 0$ ($c = c(\alpha, \kappa)$) such that for all $0 < \varepsilon < 1$,*

$$P\{B \text{ hits } \kappa \text{ before the graph of } \kappa(g_\alpha(\cdot) - \varepsilon) \text{ and } B_u > \kappa(g_\alpha(u) - \varepsilon),\ 0 \leq u \leq 1\} \geq c\,\varepsilon. \tag{22}$$

Proof. Because $\kappa > 0$ is fixed, we assume to simplify the notations that $\kappa = 1$ and we write $g(u)$ instead of $g_\alpha(u)$. Define

$$\tau_\varepsilon = \inf\{u > 0 : B_u = 1 \text{ or } B_u \leq g(u) - \varepsilon\}.$$

The probability in (22) is bounded below by

$$P(\{\tau_\varepsilon \leq 1,\ B_{\tau_\varepsilon} = 1\} \cap \{B_u > g(u) - \varepsilon,\ \tau_\varepsilon < u \leq 1\})$$
$$\geq E\left(1_{\{\tau_\varepsilon \leq \frac{1}{2},\ B_{\tau_\varepsilon}=1\}} P\{B_u \geq g(u) - \varepsilon,\ \tau_\varepsilon < u < \tau_\varepsilon + 1 | \mathcal{F}_{\tau_\varepsilon}\}\right).$$

On $\{\tau_\varepsilon \leq \frac{1}{2},\ B_{\tau_\varepsilon} = 1\}$, the conditional probability is bounded below by

$$P\{1 + B_u > g(\frac{1}{2} + u),\ 0 < u < 1\},$$

which is a positive number that does not depend on ε. It is therefore sufficient to show that

$$P\{\tau_\varepsilon \leq \frac{1}{2},\ B_{\tau_\varepsilon} = 1\} \geq c\,\varepsilon.$$

Set $\sigma_\varepsilon = \inf\{u > 0 : B_u \in \{-\varepsilon, 1\}\}$. Then $\tau_\varepsilon \leq \sigma_\varepsilon$, and $B_{\sigma_\varepsilon} = 1$ on $\{B_{\tau_\varepsilon} = 1\}$. Therefore,

$$P\{\tau_\varepsilon \leq \frac{1}{2},\ B_{\tau_\varepsilon} = 1\} = P\{\tau_\varepsilon \leq \frac{1}{2},\ B_{\tau_\varepsilon} = 1 | B_{\sigma_\varepsilon} = 1\}\, P\{B_{\sigma_\varepsilon} = 1\}.$$

The second factor is equal to ε [19, Chap.II, Prop.(3.8)], while according to Williams' path decomposition theorem [19, Chap.VI, Prop.(3.13)(iv)], the first is equal to

$$P_\varepsilon\{\rho_{1+\varepsilon} \leq \frac{1}{2},\ \chi_u > g(u),\ 0 < u < \rho_{1+\varepsilon}\},$$

where $(\chi_u,\ u \geq 0)$ is (under P_ε) a Bessel(3) process started at ε and $\rho_{1+\varepsilon} = \inf\{u > 0 : \chi_u = 1 + \varepsilon\}$. This probability is bounded below by

$$P_\varepsilon\{\rho_2 \leq \frac{1}{2},\ \chi_u > g(u),\ 0 < u < \rho_2\} \geq P_0\{\rho_2 \leq \frac{1}{2},\ \chi_u > g(u),\ 0 < u < \rho_2\}.$$

Because $\alpha > \frac{1}{2}$, according to a result of M. Motoo [14], $g(\cdot)$ is a lower escape function for the Bessel(3) process started at 0 (see also [11, Example 5.4.7]), and therefore this last probability is a positive number that does not depend on ε. This completes the proof. $\diamondsuit$

Lemma 13 *There are $K > 0$ and $c > 0$ such that: for all $s, t \in [2, 3]^2$ with $s_1 \leq t_1$, $s_2 < t_2$ and $t_1 - s_1 \leq \frac{1}{2}$, for all large n and $x \in [2^{-n}, \frac{1}{16}]$,*

$$P\{W_R^{t,n}(\cdot) \textit{ hits } 1 \textit{ before } -2^{-n},\ W_R^{(t_1,s_2),n}(\cdot) \textit{ hits } 1 \textit{ before } -x - 2^{-n}\}$$
$$\leq K\, 2^{-n} \left(\frac{x}{\sqrt{t_2 - s_2}}\right)^{c\sqrt{t_2 - s_2}/\sqrt{s_2}}.$$

Figure 1: The set $D(n,x)$.

PROOF. Fix $s,t \in [2,3]^2$. The law of $(W_R^{t,n}(\cdot),\ W_R^{(t_1,s_2),n}(\cdot))$ is the same as that of

$$(\sqrt{s_2}\,B_1,\ \sqrt{s_2}\,B_1 + \sqrt{t_2 - s_2}\,B_2),$$

where (B_1, B_2) is a standard planar Brownian motion started at the origin. Let

$$\begin{aligned} D(n,x) &= \{(b_1,b_2) \in I\!\!R^2 : -x - 2^{-n} < \sqrt{s_2}\,b_1 < 1, \\ &\qquad -2^{-n} < \sqrt{s_2}\,b_1 + \sqrt{t_2 - s_2}\,b_2 < 1\}. \end{aligned}$$

Then $D(n,x)$ is a parallelogram (see Figure 1), with one vertex at the point

$$I = \left(\frac{-x - 2^{-n}}{\sqrt{s_2}}, \frac{x}{\sqrt{t_2 - s_2}} \right).$$

Let L_1 be the union of the two boundary segments of $D(n,x)$ that meet at I, and L_2 the union of the other two boundary segments. The probability in the statement of the lemma is bounded above by

$$P_{(0,0)}\{(B_1,B_2) \text{ exits } D(n,x) \text{ via } L_2\}. \tag{23}$$

The Euclidean distance from I to the origin is bounded by

$$\frac{x + 2^{-n}}{\sqrt{s_2}} + \frac{x}{\sqrt{t_2 - s_2}} = \left(\left(1 + \frac{2^{-n}}{x} \right) \frac{\sqrt{t_2 - s_2}}{\sqrt{s_2}} + 1 \right) \frac{x}{\sqrt{t_2 - s_2}} \leq \frac{3x}{\sqrt{t_2 - s_2}}$$

(we have used the fact that $2^{-n} \leq x$), while the distance from I to

$$J = \left(\frac{1}{\sqrt{s_2}}, \frac{-2^{-n} - 1}{\sqrt{t_2 - s_2}} \right)$$

is bounded below by

$$\frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{s_2}} + \frac{2^{-n} + 1}{\sqrt{t_2 - s_2}} \right) \geq \frac{1}{\sqrt{2}} \frac{1}{\sqrt{t_2 - s_2}}.$$

We now restate the problem in a new coordinate system with origin at $0' = I$ and coordinate axes in the directions

$$b_1' = (1, \frac{-\sqrt{s_2}}{\sqrt{t_2 - s_2}}),\ b_2' = (\frac{\sqrt{s_2}}{\sqrt{t_2 - s_2}}, 1).$$

The probability in (23) is bounded above by

$$P_{(3x/\sqrt{t_2-s_2},2^{-n})}\{(B_1,B_2) \text{ exits } D' \text{ via } L'_2\}, \tag{24}$$

where

$$D' = \{(b'_1,b'_2) \in \mathbb{R}^2 : \ 0 < \sqrt{t_2-s_2}\, b'_1 + \sqrt{s_2}\, b'_2 < \frac{1}{\sqrt{3}}, \ 0 < b'_2 < \frac{1}{\sqrt{3}}\}$$

and L'_2 is the union of the two boundary segments of D' that do not contain the origin (of the new coordinate system).

If the first constraint in the definition of D' were not present, the probability in (24) would reduce to a hitting probability for real Brownian motion. In order to evaluate the influence of the constraint $0 < \sqrt{t_2-s_2}\, b'_1 + \sqrt{s_2}\, b'_2$, we use a method first developed by Spitzer [21] (whose results were later significantly extended in [6] and [1]) to study moments of escape times from cones, rather than the probability of exiting a truncated cone through a specific subset of the boundary, as is required here. We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and consider the image of D' under the transformation $\psi_\alpha : \mathbb{C} \to \mathbb{C}$ defined by

$$\psi_\alpha(z) = z^{\pi/(\pi-\alpha)},$$

where $\alpha \in]0,\pi/2[$ is defined by $\tan\alpha = \sqrt{t_2-s_2}/\sqrt{s_2}$. Under this transformation, which is conformal in D', (B_1,B_2) becomes a time-changed Brownian motion $(B_{\tau_t},\ t \geq 0)$ [19, Chap.V,§2, Thm. (2.5)].

The image of D' under ψ_α contains the rectangle

$$R = \left] -\frac{1}{2}, \frac{1}{2}\left(\frac{1+3x}{\sqrt{t_2-s_2}}\right)^{\pi/\pi-\alpha} \right[\times \left] 0, \frac{1}{2\sqrt{3}} \right[.$$

Let $S = (3x/\sqrt{t_2-s_2}, 2^{-n})$ and $S' = \psi_\alpha(S)$. If $S' = (b'_1,b'_2)$, we estimate b'_1 and b'_2 as follows. Let (r,θ) be the polar coordinates of S, and note that $r\sin\theta = 2^{-n}$ and

$$r \leq \frac{3x}{\sqrt{t_2-s_2}} + 2^{-n} \leq \frac{4x}{\sqrt{t_2-s_2}}.$$

The polar coordinates of S' are

$$(r^{\pi/(\pi-\alpha)}, \frac{\pi}{\pi-\alpha}\,\theta).$$

Use the elementary inequality $1/(1-x) \leq 1+2x$ to see that the b'_2-coordinate of S' is

$$r^{\pi/(\pi-\alpha)} \sin\left(\frac{\pi}{\pi-\alpha}\,\theta\right) \leq (r\ \sin\theta)\, r^{2\alpha/\pi}\, \frac{\sin\left(\frac{\pi}{\pi-\alpha}\theta\right)}{\sin\theta} \leq c\, 2^{-n} r^{2\alpha/\pi}. \tag{25}$$

The b_1'-coordinate of S' satisfies

$$0 \leq b_1' \leq r^{\pi/(\pi-\alpha)} \cos\left(\frac{\pi}{\pi-\alpha}\,\theta\right) \leq \left(\frac{4x}{\sqrt{t_2-s_2}}\right)^{\pi/(\pi-\alpha)}.$$

The distance of S' to the left boundary segment of R is therefore at least $1/2$. The distance to the right boundary segment is at least

$$\frac{1}{2}\left(\frac{1+3x}{\sqrt{t_2-s_2}}\right)^{\frac{\pi}{\pi-\alpha}} - \left(\frac{4x}{\sqrt{t_2-s_2}}\right)^{\frac{\pi}{\pi-\alpha}} = \frac{1}{2}\left(\frac{1+3x}{\sqrt{t_2-s_2}}\right)^{\frac{\pi}{\pi-\alpha}}\left(1-2\left(\frac{4x}{1+3x}\right)^{\frac{\pi}{\pi-\alpha}}\right).$$

The last factor is $\geq 1/2$ because $x \leq 1/16$, while the preceding factor is ≥ 1. Therefore, this distance is $\geq 1/4$. Finally, the distance of S' to the top boundary segment of R is at least $1/4$. Therefore, the probability, starting from S_1', that the motion (B_{τ_t}) exits R though its upper, left or right boundary segments, which is the same as for planar Brownian motion started at S', is bounded by a constant times the distance of S' to the lower boundary segment of R. By (25), this means that the probability in (24) is

$$\leq K \cdot 2^{-n}\left(\frac{x}{\sqrt{t_2-s_2}}\right)^{2\alpha/\pi}.$$

Because $\alpha = \arctan(\sqrt{t_2-s_2}/\sqrt{s_2})$ and $\arctan x \leq x$ for $x \geq 0$, we get the inequality stated in the lemma with $c = 2/\pi$. $\diamondsuit$

Lemma 14 *Fix $K_i > 0$, $n_i \in \mathbb{N}^*$, $c > 0$ and let X_i, $i = 1, 2$, be independent random variables with density bounded by $K_i(|x|^{n_i} \vee 1)\exp(-x^2/c)$. Then there is a constant K such that the density of $X_1 + X_2$ is bounded by $K(|x|^{n_1+n_2} \vee 1)\exp(-x^2/(2c))$.*

PROOF. Notice that $P\{X_1 + X_2 \in dz\}$ is equal to

$$\int_{\mathbb{R}} dx\ f_{X_1}(x) f_{X_2}(z-x) \leq K_1 K_2 \int_{\mathbb{R}} dx\,(|x|^{n_1} \vee 1)e^{-x^2/c}(|z-x|^{n_2} \vee 1)e^{-(z-x)^2/c}.$$

Elementary algebra shows that this equals

$$K_1 K_2 e^{-z^2/c} \int_{\mathbb{R}} dx\,(|x|^{n_1} \vee 1)(|z-x|^{n_2} \vee 1)\exp\left(-\frac{2(x-z/2)^2}{c} + \frac{z^2}{2c}\right).$$

For fixed z, use the change of variables $y = x - z/2$ to see that this equals

$$K_1 K_2 e^{-z^2/(2c)} \int_{\mathbb{R}} dy\,(|y + \frac{z}{2}|^{n_1} \vee 1)(|y - \frac{z}{2}|^{n_2} \vee 1)e^{-2y^2/c}$$

$$\leq K_1 K_2 e^{-z^2/(2c)} \left(\frac{c}{2}\right)^{1/2} E((|\sqrt{\frac{c}{2}}Y + \frac{z}{2}|^{n_1} \vee 1)(|\sqrt{\frac{c}{2}}Y - \frac{z}{2}|^{n_2} \vee 1)),$$

where Y is a $N(0,1)$ random variable. For $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$, one can develop the powers using Newton's rule, to see that the expectation is bounded by

$$K(|z|^{n_1+n_2} \vee 1).$$

$\diamondsuit$

Lemma 15 *Let B be a standard Brownian motion. There is $K > 0$ such that, for all $0 < \varepsilon < 1$, the conditional density of $B(1)$ given that B has not hit $-\varepsilon$ during the time interval $[0,1]$ is bounded by $K(|x| \vee 1)\exp(-x^2/2)$.*

PROOF. Let φ be the density of a $N(0,1)$ random variable. By the reflection principle, for $x > -\varepsilon$,

$$\begin{aligned} P\{B(1) \in [x, x+dx] \mid B \text{ has not hit } -\varepsilon\} &= \frac{(\varphi(x) - \varphi(x+2\varepsilon))dx}{2P\{-\varepsilon < B(1) < 0\}} \\ &= \varphi(x)\frac{1 - \exp(-2x\varepsilon - 2\varepsilon^2)}{2\int_{-\varepsilon}^{0} \varphi(y)dy}\, dx. \end{aligned}$$

¿From the elementary inequality $1 - e^{-x} \leq x$, we see that the numerator is bounded above by

$$2x\varepsilon + 2\varepsilon^2 = 2\varepsilon(x+\varepsilon) \leq 2\varepsilon(x+1) \leq 4\varepsilon(|x| \vee 1),$$

while the denominator is bounded below by $2\varphi(-1)\varepsilon$, therefore the fraction is bounded above by $K(|x| \vee 1)$, with $K = 2/\varphi(-1)$. $\diamondsuit$

5 Other properties

We shall now prove Theorem 4. The proof relies on the following lemma.

Lemma 16 *Fix $q \in \mathbb{R}$, and set*

$$\begin{aligned} \tilde{F}_0(t,n) = \{W(t_1 + 2^{-2n}, t_2) &\in\ ]q + 2^{-n}, q + 2^{-n+1}[, \\ W(t_1, t_2 + 2^{-2n}) &\in\ ]q - 2^{-n+1}, q - 2^{-n}[, \\ W(t_1 - 2^{-2n}, t_2) &\in\ ]q + 2^{-n}, q + 2^{-n+1}[, \\ W(t_1, t_2) &\in\ ]q - 2^{-n+1}, q - 2^{-n}[, \\ W(t_1, t_2 - 2^{-2n}) &\in\ ]q + 2^{-n}, q + 2^{-n+1}[\}. \end{aligned}$$

and

$$\tilde{F}_L(t,n) = \{W_L^{t,n}(u) \geq g(u) - 2^{-n},\ 0 \leq u \leq 1\}.$$

There exist a constant $K_1 > 0$ such that for all large n and $t = (t_1, t_2) \in [2,3]^2$,

$$P(\tilde{F}_L(t,n) \cap F_U(t,n) \cap \tilde{F}_0(t,n) \cap F_R(t,n)) \geq K_1\, 2^{-4n}.$$

PROOF. By the independent increments property of the Brownian sheet, $F_U(t,n)$, $F_R(t,n)$ and $\tilde{F}_L(t,n) \cap \tilde{F}_0(t,n)$ are independent. Moreover, $P(F_U(t,n)) \geq K\, 2^{-n}$ and $P(F_R(t,n)) \geq K\, 2^{-n}$ by Lemma 6(a), so we only need to show that $P(\tilde{F}_L(t,n) \cap \tilde{F}_0(t,n)) \geq C\, 2^{-2n}$.

Let $\mathcal{F}(t_1,t_2) = \sigma(W(s_1,s_2),\ s_1 \leq t_1,\ s_2 \leq t_2)$ and $\mathcal{G}(t_1,t_2) = \sigma(W(s_1,t_2),\ s_1 \leq t_1$. Recall that $W_R^t(\cdot)$ and $W_U^t(\cdot)$ are defined in the proof of Theorem 9, and define the events

$$\begin{aligned} F_1 &= \{W(t_1 - 2^{-2n}, t_2) \in]q + 2^{-n}, q + 2^{-n+1}[\}, \\ F_2 &= \{W(t_1, t_2) \in]q - 2^{-n+1}, q - 2^{-n}[\}, \\ F_3 &= \{W(t_1, t_2 - 2^{-2n}) \in]q + 2^{-n}, q + 2^{-n+1}[\}, \\ F_4 &= \{W(t_1, t_2) + W_R^t(2^{-2n}) \in]q + 2^{-n}, q + 2^{-n+1}[\}, \\ F_5 &= \{W(t_1, t_2) + W_U^t(2^{-2n}) \in]q - 2^{-n+1}, q - 2^{-n}[\}. \end{aligned}$$

Then

$$\tilde{F}_0(t,n) = F_1 \cap \cdots \cap F_5.$$

Notice that

$$P(F_5 \mid \mathcal{F}_t) = P\{2^n(q - W(t)) + Z \in] -2, -1[\mid W(t)\},$$

where Z is $N(0,1)$ and independent of $\mathcal{F}(t)$. On F_2, $1 \leq 2^n(q - W(t)) \leq 2$, so on F_2, this conditional probability is bounded below by $P\{Z \in I\}$, where $I \subset [-1,1]$ is some interval an length 1. In other words, on F_2,

$$P(F_5 \mid \mathcal{F}_t) \geq \varphi(1),$$

where $\varphi(\cdot)$ is the $N(0,1)$ density function.

Notice also that F_4 and F_5 are conditionally independent given $\mathcal{F}(t)$, and reasoning as above, we find that on F_2,

$$P(F_4 \mid \mathcal{F}_t) \geq \varphi(4).$$

Observe now that

$$W(t_1, t_2 - 2^{-2n}) = \frac{t_2 - 2^{-2n}}{t_2} W(t) + \left(t_1 \frac{t_2 - 2^{-2n}}{t_2}\right)^{\frac{1}{2}} 2^{-n} Y,$$

where Y is $N(0,1)$ and independent of $\mathcal{G}(t)$. Because W restricted to $([0,t_1] \times \{t_2\}) \cup (\{t_1\} \times [0,t_2])$ is a Markov process (it is a Brownian motion as one moves to the right along the horizontal segment, and then a Brownian bridge as one continues down the vertical segment),

$$P(F_3 \mid \mathcal{G}(t)) = P\{2^n(q - W(t)) + \frac{2^{-n}}{t_2} W(t) + \left(t_1 \frac{t_2 - 2^{-2n}}{t_2}\right)^{\frac{1}{2}} Y \in]1,2[\mid W(t)\},$$

so on F_2, this is bounded below by $P\{Y \in I\}$, where $I \subset [1,5]$ is a fixed interval of length 1. In other words, on F_2,

$$P(F_3 \mid \mathcal{G}(t)) \geq \varphi(5).$$

Also, reasoning as above, we conclude that on F_1,

$$P(F_2 \mid \mathcal{G}(t_1 - 2^{-2n}, t_2)) \geq c_1 > 0.$$

Putting together the estimates above yields a positive constant $\tilde{c}$ such that

$$P(\tilde{F}_L(t,n) \cap \tilde{F}_0(t,n)) \geq \tilde{c} P(\tilde{F}_L(t,n) \cap F_1),$$

and so we must show that there is $c > 0$ such that $P(\tilde{F}_L(t,n) \cap F_1) \geq c\, 2^{-2n}$.

♢

Proof of Theorem 4. Set

$$F(t,n) = \tilde{F}_L(t,n) \cap F_U(t,n) \cap \tilde{F}_0(t,n) \cap F_R(t,n)$$

and let $X_n(\omega)$ be the number of elements $t \in D_{2n}$ such that $\omega \in F(t,n)$. From Lemma 16, we conclude that

$$E(X_n) = \sum_{t \in D_{2n}} P(F(t,n)) \geq (2^{2n})^2 K_1\, 2^{-4n} = K_1.$$

Notice that for $s,t \in D_{2n}$ with $s \neq t$, $F(s,n) \cap F(t,n) = \emptyset$. Indeed, if $s_1 < t_1$ and $s_2 < t_2$, then $\omega \in F(s,n)$ implies $W(s_1,t_2) < q$ while $\omega \in F(t,n)$ implies $W(s_1,t_2) > q$. If $s_1 < t_1$ and $s_2 > t_2$, then $\omega \in F(s,n)$ implies $W(t_1,s_2) > q$ while $\omega \in F(t,n)$ implies $W(t_1,s_2) < q$. If $s_1 = t_1$ and $s_2 < t_2$, then $\omega \in F(s,n)$ implies $W(s_1, t_2 + 2^{-2n}) < q$ while $\omega \in F(t,n)$ implies $W(s_1, t_2 - 2^{-2n}) > q$. If $s_1 < t_1$ and $s_2 = t_2$, then $\omega \in F(s,n)$ implies $W(t_1,s_2) > q$ while $\omega \in F(t,n)$ implies $W(t_1,s_2) < q$. Since s and t can be interchanged, this shows that $F(s,n) \cap F(t,n) = \emptyset$, when $s \neq t$. Therefore

$$P\{X_n > 0\} = P(\cup_{t \in D_{2n}} F(t,n)) = \sum_{t \in D_{2n}} P(F(t,n)) = E(X_n) \geq K > 0.$$

The remainder of the proof is similar to the end of the proof of Theorem 3 and is therefore omitted. ♢

We now prove Theorem 2. The proof of this theorem uses the following lemma.

Lemma 17 *Let $B = (B_u)$ be a standard Brownian motion, and for each $x > 0$, set $\tau_x = \inf\{u > 0 : B_u = x\}$. Given $\varepsilon > 0$, there are positive constants K and C such that for all large n, the event "there is $x \in [n^2\, 2^{-n}, n^{-6}]$ such that $\tau_x \leq n^2 x^2$ and $\inf_{u \leq \tau_x} B(u) \geq -Kx$" has probability at least $1 - C\, 2^{-n(1-\varepsilon)}$.*

PROOF. For $k = 0, 1, \dots, [n(1-\varepsilon/2)]$, set

$$\tau^{(k)} = \inf\{u > 0 : B_u \in \{-Kn^2\, 2^{-n+k}, n^2\, 2^{-n+k}\}\}.$$

Observe that

$$P_0\{B \text{ hits } -Kn^2 2^{-n} \text{ before } n^2 2^{-n}\} = \frac{1}{K+1}, \tag{26}$$

$$P_{-Kn^2 2^{-n+k}}\{B \text{ hits } -Kn^2\, 2^{-n+k+1} \text{ before } n^2\, 2^{-n+k+1}\} = \frac{K+2}{2K+2}, \tag{27}$$

and

$$\begin{aligned} P\{\tau^{(k)} \geq n^6\, 2^{-2n+2k}\} &\leq P\left\{\sup_{0<u<n^6\, 2^{-2n+2k}} |B(u)| \leq Kn^2\, 2^{-n+k}\right\} \\ &= P\left\{\sup_{0<u<1} |B(u)| \leq \frac{K}{n}\right\} \\ &\leq Ce^{-cn^2}. \end{aligned} \tag{28}$$

Notice that the complement of the event described in the lemma is contained in

$$\begin{aligned} &\bigcap_{k=0}^{[n(1-\varepsilon/2)]} (\{\tau^{(k)} \geq n^6\, 2^{-2n+2k}\} \cup \{B_{\tau^{(k)}} = -Kn^2\, 2^{-n+k}\}) \\ &\quad \subset \left(\bigcap_{k=0}^{[n(1-\varepsilon/2)]} \{B_{\tau^{(k)}} = -Kn^6\, 2^{-n+k}\}\right) \cup \left(\bigcup_{k=0}^{[n(1-\varepsilon/2)]} \{\tau_k \geq n^6\, 2^{-2n+2k}\}\right). \end{aligned}$$

By (28), the second term has probability bounded by Cne^{-cn^2}, while by (26), (27) and the strong Markov property, the probability of the first term is bounded by

$$\frac{1}{K}\left(\frac{K+2}{2K+2}\right)^{n/(1-\varepsilon/2)} \leq 2^{-n(1-\varepsilon)}$$

provided K is large enough. This proves the lemma. $\diamondsuit$

PROOF OF THEOREM 2. In view of scaling properties of the Brownian sheet, it is sufficient to show that there does not exist $(t_1, t_2) \in [2, 3]^2$ such that

(i) $W(t_1, t_2) = q$;

(ii) $W(t_1 + u, t_2) > q$, $0 < u \leq 1$;

(iii) $W(t_1 - u, t_2) > q$, $0 < u \leq 1$;

(iv) $(t_1 - 1, t_2)$ and $(t_1 + 1, t_2)$ belong to distinct components of $L_+(q)$.

For $s = (s_1, s_2) \in D_{2n}$, let $F_n(s)$ be the event just described but with the additional requirement $(t_1, t_2) \in I_n(s)$, where $I_n(s)$ is defined in (10). It suffices to show that $\sup_{s\in D_{2n}} P(F(s,n)) = o(2^{-4n})$. For $i \in \{L, R\}$, let

$$\tau_i(n) = \inf\{u \geq 0 : W_i^s(u) = n^{-3}\},$$

and set

$$\begin{aligned} G_n(s) \quad = \quad & \{|W(s) - q| \leq n2^{-n}\} \cap \{W_R^s(u) \geq -n2^{-n},\ 0 \leq u \leq 1\} \\ & \cap \{W_L^s(u) \geq -n2^{-n},\ 0 \leq u \leq 1\} \cap \{\tau_R(n) \vee \tau_L(n) \leq n^{-4}\}. \end{aligned}$$

Then

$$P(F_n(s)) \leq P(F_n(s) \cap G_n(s)) + P(F_n(s) \setminus G_n(s)).$$

Observe that the second term on the right-hand side is bounded by the sum of four terms:

$$P(F_n(s) \cap \{|W(s)| > n2^{-n}\}), \tag{29}$$

$$P(F_n(s) \cap \{W_R^s(\cdot) \text{ hits } -n2^{-n} \text{ before time } 1\}), \tag{30}$$

$$P(F_n(s) \cap \{W_L^s(\cdot) \text{ hits } -n2^{-n} \text{ before time } 1\}), \tag{31}$$

$$P\{\tau_R(n) \vee \tau_L(n) > n^{-4}\}. \tag{32}$$

By (i), the probability in (29) is bounded by the probability in (13), therefore by Ke^{-cn^2}. Similarly, the probability in (30) is bounded by the probability of the event in (14), therefore by Ke^{-cn^2}, and a similar bound holds for (31). Finally, (32) is bounded by

$$2P\{\tau_R(n) > n^{-4}\} = P\{\sup_{0\leq u\leq 1} |B(u)| < \frac{1}{n}\} \leq Ke^{-cn^2}.$$

It therefore only remains to show that $P(F_n(s) \cap G_n(s)) = o(2^{-4n})$.

Fix $\varepsilon > 0$ and choose $K > 0$ and $C > 0$ so that the event $H(n,k)$ described in Lemma 17 has probability at least $1 - C\,2^{-n(1-\varepsilon)}$. For $i \in \{U, D\}$, set

$$H_i(n) = \{\exists x \in [n^2 2^{-n}, n^{-4}] : \tau_x^i \leq n^2 x^2,\ \inf_{u\leq\tau_x^i} W_i^s(u) \geq -Kx\}.$$

Here, τ_x^i is defined in the same way as τ_x in Lemma 17, but with B replaced by $W_i^{(s_1-\tau_L(n),s_2)}$.

Note that $H_U(n)$ is independent of $H_D(n)$ and $G_n(s)$, and $H_D(n)$ is "essentially" independent of $G_n(s)$. Therefore, by Lemma 17,

$$P(G_n(s) \cap H_U(n)^c \cap H_D(n)^c) \leq (n2^{-n})^3\,(C2^{-n(1-\varepsilon)})^2 = o(2^{-4n}).$$

It remains now to show that

$$P(F_n(s) \cap G_n(s) \cap H_U(n)) = o(2^{-4n}) \quad \text{and} \quad P(F_n(s) \cap G_n(s) \cap H_D(n)) = o(2^{-4n}).$$

We only examine the first probability, as the other is "similar."

Let $x_U(n)$ be the smallest $x \in [n^2 2^{-n}, n^{-4}]$ that garantees the occurence of $H_U(n)$, and set $\tau_U(n) = \tau^U_{x_U(n)}$. Note that $\tau_U(n)$ is a stopping time relative to $W_U(\cdot)$. Let Γ be the union of the three segments

$$\begin{aligned}
\Gamma_1 &= \{s_1 - \tau_L(n)\} \times [s_2, s_2 + \tau_U(n)], \\
\Gamma_2 &= [s_1 - \tau_L(n), s_1 + \tau_R(n)] \times \{s_2 + \tau_U(n)\}, \\
\Gamma_3 &= \{s_1 + \tau_R(n)\} \times [s_2, s_2 + \tau_U(n)],
\end{aligned}$$

and notice that on $F_n(s) \cap G_n(s) \cap H_U(n)$, $W|_\Gamma$ hits 0. Define

$$\eta(u, v) = \Delta_{]s_1 - \tau_L(n), s_1 - \tau_L(n) + u] \times]s_2, s_2 + v]} W,$$

and observe that on $F_n(s) \cap G_n(s) \cap H_U(n)$, for $(t_1, t_2) \in \Gamma_1$,

$$\begin{aligned}
W(t_1, t_2) &= W(s) + W^s_L(\tau_L(s)) + W_U^{(s_1 - \tau_L(n), s_2)}(t_2 - s_2) \\
&\geq (q - n2^{-n}) + n^{-3} - K x_U(n) \\
&\geq q - n2^{-n} + n^{-3} - K n^{-4} \\
&\geq q + \frac{1}{2} n^3
\end{aligned}$$

for large n, and therefore $W|_{\Gamma_1} > 0$. On the same event, for $(t_1, t_2) \in \Gamma_2$,

$$\begin{aligned}
W(t_1, t_2) &= W(s) + W^s_{L\,\mathrm{or}\,R}(|t_1 - s_1|) + W_U^{(s_1 - \tau_L(n), s_2)}(\tau_U(n)) \\
&\qquad + \eta(t_1 - s_1 + \tau_L(n), \tau_U(n)) \\
&\geq (q - n2^{-n}) - n2^{-n} + x_U(n) + \eta(t_1 - s_1 + \tau_L(n), \tau_U(n)) \\
&\geq q - \frac{1}{2} x_U(n) + \eta(t_1 - s_1 + \tau_L(n), \tau_U(n)),
\end{aligned}$$

for n large, while for $(t_1, t_2) \in \Gamma_3$,

$$\begin{aligned}
W(t_1, t_2) &= W(s) + W^s_R(\tau_R(n)) + W_U^{(s_1 - \tau_L(n), s_2)}(t_2 - s_2) \\
&\qquad + \eta(\tau_R(n) + \tau_L(n), t_2 - s_2) \\
&\geq q - n2^{-n} + n^{-3} - K x_U(n) + \eta(\tau_R(n) + \tau_L(n), t_2 - s_2) \\
&\geq q - n2^{-n} + n^{-3} - K n^{-4} + \eta(\tau_R(n) + \tau_L(n), t_2 - s_2) \\
&\geq q - \frac{1}{2} n^{-3} + \eta(\tau_R(n) + \tau_L(n), t_2 - s_2).
\end{aligned}$$

It follows that $P(F_n(s) \cap G_n(s) \cap H_U(n))$ is bounded by the sum of two terms:

$$\begin{aligned}
P\{ \inf_{s_1 - \tau_L(n) < t_1 < s_1 + \tau_R(n)} \eta(t_1 - s_1 + \tau_L(n), \tau_U(n)) < -\frac{1}{2} x_U(n), \\
\tau_R(n) \vee \tau_L(n) \leq n^{-4},\ \tau_U(n) \leq n^{-6}\}
\end{aligned}$$

$$\leq P\left\{(2n^{-4})^{\frac{1}{2}}(n^2X_U^2(n))^{\frac{1}{2}}\sup_{t\in[0,1]^2} W(t) > \frac{1}{2}x_U(n)\right\}$$
$$= P\left\{\sup_{t\in[0,1]^2} W(t) > \frac{1}{2\sqrt{2}}n\right\}$$
$$\leq e^{-cn^2},$$

and

$$P\{\inf_{s_2<t_2<s_2+\tau_U(n)} \eta(\tau_R(n)+\tau_L(n), t_2-s_2) < -\frac{1}{2}n^{-3},$$
$$\tau_R(n)\vee\tau_L(n)\leq n^{-4},\ \tau_U(n)\leq n^{-6}\}$$

$$\leq P\left\{(2n^{-4})^{\frac{1}{2}}(n^{-6})^{\frac{1}{2}}\sup_{t\in[0,1]^2} W(t) > \frac{1}{2}n^{-3}\right\}$$
$$= P\left\{\sup_{t\in[0,1]^2} W(t) > \frac{1}{2\sqrt{2}}n^2\right\}$$
$$\leq e^{-cn^4}.$$

This completes the proof of the theorem. $\diamondsuit$

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