

A Feynman-Kac-type formula for the deterministic and stochastic wave equations and other p.d.e.'s

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Abstract

We establish a probabilistic representation for a wide class of linear deterministic p.d.e.s with potential term, including the wave equation in spatial dimensions 1 to 3. Our representation applies to the heat equation, where it is related to the classical Feynman-Kac formula, as well as to the telegraph and beam equations. If the potential is a (random) spatially homogeneous Gaussian noise, then this formula leads to an expression for the moments of the solution.

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1 Introduction

The purpose of this paper is to present a form of the Feynman-Kac formula which applies to a wide class of linear partial differential equations with a potential term, and, in particular, to the wave equation in dimensions $d \leq 3$. In the case of the heat equation, this gives an expression that differs from the classical Feynman-Kac formula. As an application, we consider a random potential term which is a spatially homogeneous Gaussian random field that is white in time. In this case, our approach provides a probabilistic representation for all product moments of the solution, which has already shown its usefulness for studying asymptotic behavior of these moments in the case of the stochastic wave equation (see [10]).

We begin by giving an informal derivation of the representation in the special case of the heat equation with potential, where we can contrast it with the classical Feynman-Kac formula. Consider the heat equation on \mathbb{R}^d with a deterministic potential $V(t, x)$:

$$\begin{aligned}\frac{\partial u(t, x)}{\partial t} &= \frac{1}{2}\Delta u(t, x) + V(t, x)u(t, x), \\ u(0, x) &= f(x).\end{aligned}\tag{1.1}$$

The classical Feynman-Kac formula for the solution ($u(t, x)$, $t \geq 0$, $x \in \mathbb{R}^d$) (see for instance [15]) states that, under appropriate conditions on V and f ,

$$u(t, x) = E_x^B \left[f(B_t) \exp \left(\int_0^t V(t-s, B_s) ds \right) \right]$$

where $(B_t, t \geq 0)$ is a Brownian motion in \mathbb{R}^d , and E_x^B is the expectation for Brownian motion started at $B_0 = x$.

We now heuristically derive an alternative probabilistic representation to (1.1), which will be rigorously established as a special case of the main result in section 3. We start by writing Duhamel's formula for the solution $u(t, x)$, using the Green's function, as follows:

$$u(t, x) = \int_{\mathbb{R}^d} p_t(x-y)f(y)dy + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)V(s, y)u(s, y)dyds,\tag{1.2}$$

where

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp \left(-\frac{|x|^2}{2t} \right).$$

We use (1.2) as the start of an iteration scheme. Substituting this expression for $u(s, y)$ back into the right hand side of (1.2) suggests the following series expansion for $u(t, x)$:

$$u(t, x) = \sum_{m=0}^{\infty} I_m(t, x),\tag{1.3}$$

where $I_0(t, x) = \int_{\mathbb{R}^d} p_t(x-y)f(y)dy$ and

$$I_{m+1}(t, x) = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)V(s, y)I_m(s, y)dyds.\tag{1.4}$$

We wish to write an explicit expression for $I_m(t, x)$. To begin with, let

$$w(t, x) = I_0(t, x).$$

For convenience, let $s_{n+1} = t$ and $y_{n+1} = x$. Then we have

$$I_m(t, x) = \int_{T_m(t)} \int_{\mathbb{R}^{md}} \left(\prod_{k=1}^m p_{s_{k+1}-s_k}(y_{k+1} - y_k) V(s_k, y_k) \right) w(s_1, y_1) d\bar{y} d\bar{s} \quad (1.5)$$

where

$$T_m(t) = \{(s_1, \dots, s_m) : 0 \leq s_1 \leq \dots \leq s_m \leq t\},$$

$d\bar{y} = dy_1 \cdots dy_m$ and $d\bar{s} = ds_1 \cdots ds_m$. An alternative derivation of the series expansion (1.3) for $u(t, x)$ starts by expanding the exponential in the classical Feynman-Kac formula as a Taylor series and confirming that the terms correspond to the expansion (1.3). However, we will make use of the iterative formula (1.4) later on.

A basic observation is that domain of integration $T_m(t)$ has volume $t^m/m!$, which, except for a missing exponential factor, is a Poisson probability. If $N(t)$ is a rate one Poisson process, then $P[N(t) = m] = t^m e^{-t}/m!$. Let $\tau_1 < \tau_2 < \dots$ be the times of the successive jumps of the Poisson process, and let $\tau_0 = 0$. It is well known that if we condition on $N_t = m$, then the vector (τ_1, \dots, τ_m) is uniformly distributed over the simplex $T_m(t)$. The time reversed sequence $t - \tau_m, \dots, t - \tau_1$ is also uniformly distributed on $T_m(t)$. Therefore, setting $s_k = t - \tau_{m+1-k}$ and replacing y_k by y_{m+1-k} , so that $y_0 = x$, we may rewrite the expression (1.5) for $I_m(t, x)$ as

$$e^t E^N \left[\int_{\mathbb{R}^{md}} \left(\prod_{k=1}^m p_{\tau_k - \tau_{k-1}}(y_k - y_{k-1}) V(t - \tau_k, y_k) \right) w(t - \tau_m, y_m) d\bar{y} 1_{\{N(t)=m\}} \right].$$

where E^N is the expectation with respect to the Poisson process. But we can also exploit the fact that $p_t(x)$ is the probability density for the increments of a d -dimensional Brownian motion. Thus

$$\begin{aligned} \int_{\mathbb{R}^{md}} \left(\prod_{k=1}^m p_{\tau_k - \tau_{k-1}}(y_k - y_{k-1}) V(t - \tau_k, y_k) \right) w(t - \tau_m, y_m) d\bar{y} \\ = E_x^B \left[\left(\prod_{k=1}^m V(t - \tau_k, B_{\tau_k}) \right) w(t - \tau_m, B_{\tau_m}) \right], \end{aligned}$$

where E_x^B denotes the expectation with respect to Brownian motion started at x , and therefore,

$$I_m(t, x) = e^t E_x^B E^N \left[\left(\prod_{k=1}^m V(t - \tau_k, B_{\tau_k}) \right) w(t - \tau_m, B_{\tau_m}) 1_{\{N(t)=m\}} \right].$$

Summing over m , we get

$$u(t, x) = e^t E_x^B E^N \left[\left(\prod_{k=1}^{N_t} V(t - \tau_k, B_{\tau_k}) \right) w(t - \tau_{N(t)}, B_{\tau_{N(t)}}) \right]. \quad (1.6)$$

The representation (1.6), unlike the classical Feynman-Kac formula, does not use the entire Brownian path but only the values at a finite (random) set of times. This allows us to extend this type of representation to equations where the differential operator is not the infinitesimal generator of a Markov process. All we will require is a Poisson process and an independent stochastic process whose one dimensional marginals give the Green's function for the differential operator. In particular, we will treat the case of the wave equation with potential in dimensions $d \leq 3$.

The outline of this paper is as follows. In Section 2, we describe the class of equations that we will consider and establish a series representation in the case of a bounded potential. In Section 3, we establish our Feynman-Kac-type formula analogous to (1.6), where the Brownian motion will be replaced by a suitable spatial motion that depends on the particular equation being considered. In Section 4, we give an application to the situation where the potential is a Gaussian random field whose covariance is formally given by

$$E \left[\dot{F}(t, x) \dot{F}(s, y) \right] = \delta_0(t - s) f(x - y).$$

In this equation, $\delta(\cdot)$ denotes the Dirac delta function, $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous on $\mathbb{R}^d \setminus \{0\}$ and the right-hand side is such that $f(x - y)$ is indeed a covariance. This type of covariance is widely used in the literature, including for instance [5, 6, 8, 17]. In this case, unlike the classical Feynman-Kac formula, the noise is too rough for the probabilistic representation of the solution to make sense. Instead, we establish in this section a formula for the second moment of the solution. Section 5 contains the extension to n -fold product moments. This formula makes use of a Poisson random measure combined with a spatial motion. The first two named authors have made use of this formula [10] to establish intermittency properties of the solution to the wave equation with potential.

We end by making a few comments on related literature. Probabilistic representations of the solution to deterministic p.d.e.'s abound. The closest related work seems to be results on random evolutions, surveyed for example in Hersch [12] and Pinsky [19]. These references give probabilistic representations for some hyperbolic equations, including the Poisson representation for the damped wave equation in one spatial dimension (also known as the telegraph equation) as developed by Marc Kac [13, 14]. Related also is the use of random flight models for the Boltzmann equation, as described, for example, in [19, 11]. We cannot quite find our approach represented in this literature. The use of Poisson probabilities is also implicit in other works: Albeverio and Hoegh-Krohn [1] and Albeverio, Blanchard, Coombe, Hoegh-Krohn, and Sirugue [2] have used the idea that the multiple integrals involved in the expansion of Feynman integrals are related to Poisson probabilities (see also [23]). All these works display the usefulness of probabilistic representations in studying problems of asymptotics, homogenization and perturbation theory for the deterministic p.d.e. For parabolic equations with random potentials, the classical Feynman-Kac formula has been a key tool, for example in the parabolic Anderson problem (see Carmona and Molchanov [3]) and in random waves (see Oksendal, Vage and Zhao [18]). We hope that our representation may play a similar role for other equations with a random potential.

2 Series representation for bounded potential

Our series and probabilistic representations will be for the integral equation

$$u(t, x) = w(t, x) + \int_0^t ds \int_{\mathbb{R}^d} S(s, dy) V(t - s, x - y) u(t - s, x - y), \quad (2.1)$$

where w is a bounded and measurable function and $S(s, \cdot)$ is a signed measure. In this section, the key assumption is the following:

Assumption A. For each $t \geq 0$, $S(t, dy)$ is a signed measure on \mathbb{R}^d satisfying

$$\sup_{t \in [0, T]} |S(t, \mathbb{R}^d)| < \infty \quad \text{for all } T > 0,$$

where $|S(t, \mathbb{R}^d)|$ denotes the total variation.

It is well known that a large class of linear partial differential equations of the form

$$Lu(t, x) = V(t, x)u(t, x)$$

can be recast, using their Green's functions, into this integral form. We briefly recall some illustrative examples that we consider later.

Example 2.1. (a) *The heat equation on \mathbb{R}^d .* Take $L = \frac{\partial}{\partial t} - \frac{1}{2}\Delta$ and $S(t, dy) = p_t(y)dy$. Then for a suitable initial condition $u(0, x) = f_0(x)$, the Green's function representation of the heat equation leads to the integral equation (2.1) with

$$w(t, x) = \int_{\mathbb{R}^d} f_0(x - y)p_t(y)dy.$$

(b) *The wave equation on \mathbb{R}^d for $d \leq 3$.* Take $L = \frac{\partial^2}{\partial t^2} - \Delta$ and

$$S(t, dy) = \begin{cases} \frac{1}{2}1_{\{|y|<t\}}dy & \text{if } d = 1, \\ \frac{1}{2\pi\sqrt{t^2-|y|^2}}1_{\{|y|<t\}}dy & \text{if } d = 2, \\ \frac{\sigma_t^{(2)}(dy)}{4\pi t} & \text{if } d = 3, \end{cases}$$

where $\sigma_t^{(2)}$ denotes the surface area on $\partial B(0, t)$ (the boundary of the ball centered at 0 with radius t). For all three values of d , $S(t, \mathbb{R}^d) = t$. The initial conditions are of the form $u(0, x) = f_0(x)$ and $\frac{\partial u}{\partial t}(0, x) = f_1(x)$ for given $f_0, f_1 : \mathbb{R}^d \rightarrow \mathbb{R}$. In this case, letting $*$ denote convolution,

$$w(t, x) = \frac{\partial}{\partial t}(S(t) * f_0)(x) + (S(t) * f_1)(x).$$

For $d \geq 4$, the fundamental solution of the wave equation is not a signed measure and so assumption A will not hold.

(c) *The wave equation with damping.* Take $L = \frac{\partial^2}{\partial t^2} + 2a\frac{\partial}{\partial t} - \Delta$ on \mathbb{R}^d . This also falls into the considered class when $d \leq 3$. Then

$$S(t, dy) = \begin{cases} \frac{e^{-at}}{2}\mathcal{I}_0(|a|\sqrt{t^2-y^2})1_{\{|y|<t\}}dy & \text{if } d = 1, \\ \frac{e^{-at}}{2\pi}\frac{\cosh(|a|\sqrt{t^2-|y|^2})}{\sqrt{t^2-|y|^2}}1_{\{|y|<t\}}dy & \text{if } d = 2, \\ \frac{e^{-at}}{4\pi}\left(\frac{\sigma_t^{(2)}(dy)}{t} + |a|\frac{\mathcal{I}_1(|a|\sqrt{t^2-|y|^2})}{\sqrt{t^2-|y|^2}}1_{\{|y|<t\}}dy\right) & \text{if } d = 3. \end{cases}$$

In these formulas, given for instance in [16] and [9], \mathcal{I}_0 and \mathcal{I}_1 are modified Bessel functions of the first kind and of orders 0 and 1, respectively. In these three dimensions, $S(t, dy)$ is a non-negative measure.

(d) *The beam equation.* In dimension $d = 1$, this is given by $L = \frac{\partial}{\partial t} + \frac{\partial^4}{\partial x^4}$ on \mathbb{R} . Then $S(t, dy) = q_t(y)dy$, where $q_t(y)$ has Fourier transform $\exp(-|\xi|^4 t)$ for $t > 0$. The smoothness and integrability of q_t , and hence assumption A, can be deduced from the Fourier transform (for example $|x^2 q_t(x)| \leq C\|\partial^2 \hat{q}/\partial \xi^2\|_1$).

We now give a series representation for the solution $u(t, x)$ of (2.1).

Proposition 2.2. *Let $S(t, dy)$ be a signed measure satisfying assumption A. Suppose that $V(t, y)$ and $w(t, x)$ are bounded measurable functions on $[0, T] \times \mathbb{R}^d$. Define $H_0(t, x) = w(t, x)$, and, for $m \geq 0$,*

$$H_{m+1}(t, x) = \int_0^t ds \int_{\mathbb{R}^d} S(s, dy) V(t-s, x-y) H_m(t-s, x-y). \quad (2.2)$$

Then the integral equation (2.1) has a unique solution satisfying $\sup_{t \leq T, x \in \mathbb{R}^d} E[|u(t, x)|^2] < \infty$ given by

$$u(t, x) = \sum_{m=0}^{\infty} H_m(t, x) \quad (2.3)$$

(the series converges uniformly on $[0, T] \times \mathbb{R}^d$).

Proof. We first check the convergence of the series (2.3). Set $M_m(s) = \sup_{z \in \mathbb{R}^d} |H_m(s, z)|$. Then

$$\begin{aligned} M_{m+1}(t) &\leq \sup_{z \in \mathbb{R}^d} \int_0^t ds \int_{\mathbb{R}^d} |S(s, dy)| \sup_{r, z} |V(r, z)| \sup_z |H_m(t-s, x-z)| \\ &\leq C(S, V) \int_0^t ds M_m(s). \end{aligned}$$

A simple induction argument shows that $M_m(s) < \infty$ for all m, s . Gronwall's lemma (see e.g. [6, Remark (6)]) now implies that $\sum_{m=0}^{\infty} H_m(t, x)$ converges uniformly on $[0, T] \times \mathbb{R}^d$.

Another Gronwall argument shows the uniqueness of solutions to (2.1). So it suffices to check that $\sum_{m=0}^{\infty} H_m(t, x)$ satisfies (2.1). This is the case, since

$$\begin{aligned} w(t, x) + \int_0^t ds \int_{\mathbb{R}^d} S(s, dy) V(t-s, x-y) \sum_{m=0}^{\infty} H_m(t-s, x-y) \\ = I_0(t, x) + \sum_{m=0}^{\infty} \int_0^t ds \int_{\mathbb{R}^d} S(s, dy) V(t-s, x-y) H_m(t-s, x-y) \\ = \sum_{m=0}^{\infty} H_m(t, x). \end{aligned}$$

Fubini's theorem, used for the first equality, applies by uniform convergence of the series and assumption A. \square

3 Probabilistic representation

For the probabilistic representation, we use the following additional assumption on the kernel $S(t, dy)$ used in the integral equation (2.1).

Assumption B. There exists a jointly measurable process $(\tilde{X}_t, t > 0)$ such that for each $t > 0$

$$P \left\{ -\tilde{X}_t \in dx \right\} = \frac{|S(t, dx)|}{|S(t, \mathbb{R}^d)|}.$$

(In the case where $S(t, A) = S(t, -A)$, the minus sign in front of \tilde{X}_t is not needed).

Example 3.1. (a) *The heat equation.* In this case, one can take $\tilde{X}_t = \sqrt{t} X_0$, where X_0 is a standard $N(0, I_d)$ random vector in \mathbb{R}^d . An alternative possibility is to let (\tilde{X}_t) be a standard Brownian motion in \mathbb{R}^d .

(b) *The wave equation.* In the three dimensional case, one can take $\tilde{X}_t = t \Theta_0$, where Θ_0 is chosen according to the uniform probability measure on $\partial B(0, 1)$. The one and two dimensional cases can be handled in a similar way.

(c) *The damped wave equation.* Kac [13, 14] pointed out a neat representation for solutions to the damped wave equation in dimension 1. Let $(N_a(t))$ be a rate a Poisson process and define $\tau_t = \int_0^t (-1)^{N_a(s)} ds$. If $w(t, x)$ solves the undamped wave equation $\frac{\partial^2 w}{\partial t^2} - \Delta w = 0$ for $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, then $u(t, x) = E[w(\tau_t, x)]$ solves the damped wave equation $\frac{\partial^2 u}{\partial t^2} + 2a \frac{\partial u}{\partial t} - \Delta u = 0$, and with the same initial conditions. Using this, one finds that the kernel $S_a(t, dy)$ for the damped equation can be written as $S_a(t, dy) = E[S(\tau_t, dy)1_{\{\tau_t > 0\}} + S(-\tau_t, dy)1_{\{\tau_t < 0\}}]$, where $S(t, dy)$ is the kernel for the wave equation. Now we satisfy assumption B by setting $\tilde{X}_t = |\tau_t| \Theta_0$, where Θ_0 is a uniform random variable on $[-1, 1]$, independent of $N_a(t)$.

(d) *The beam equation.* As in example (a), we can use scaling to set $\tilde{X}_t = t^{1/4} X_0$, where X_0 is chosen to have distribution $|S(1, dy)|$.

Let $\tilde{X}^{(i)} = (\tilde{X}_t^{(i)}, t \geq 0)$, $i \geq 1$, be i.i.d. copies of $(\tilde{X}_t, t \geq 0)$, and let $(N(t), t \geq 0)$ be a rate one Poisson process independent of the $(\tilde{X}^{(i)})$. Let $0 < \tau_1 < \tau_2 < \dots$ be the jump times of $(N(t))$ and set $\tau_0 \equiv 0$. Define a process $X = (X_t, t \geq 0)$ as follows :

$$X_t = X_0 + \tilde{X}_t^{(1)} \text{ for } 0 \leq t \leq \tau_1,$$

and for $i \geq 1$,

$$X_t = X_{\tau_i} + \tilde{X}_{t-\tau_i}^{(i+1)}, \text{ for } \tau_i < t \leq \tau_{i+1}.$$

We use P_x to denote a probability under which, in addition, $X_0 = x$ with probability one. Informally, the process X follows $\tilde{X}^{(1)}$ during the interval $[0, \tau_1]$, then follows $\tilde{X}^{(2)}$ started at X_{τ_1} during $[\tau_1, \tau_2]$, then $\tilde{X}^{(3)}$ started at X_{τ_2} during $[\tau_2, \tau_3]$, etc. See Figure 1 for an illustration.

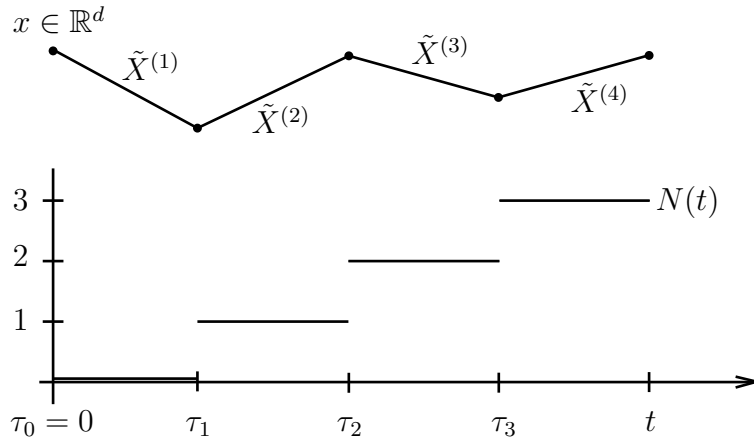


Figure 1: A sample path of the process X and of the Poisson process $(N(t))$.

Theorem 3.2. *Suppose that the kernel $S(t, dy)$ is a non-negative measure satisfying assumptions A and B. Suppose that $V(t, x)$ and $w(t, x)$ are bounded and measurable functions on $[0, T] \times \mathbb{R}^d$. Then $(u(t, x), t \leq T, x \in \mathbb{R}^d)$ defined by*

$$u(t, x) = e^t E_x \left[w \left(t - \tau_{N(t)}, X_{\tau_{N(t)}} \right) \prod_{i=1}^{N(t)} \left[S(\tau_i - \tau_{i-1}, \mathbb{R}^d) V(t - \tau_i, X_{\tau_i}) \right] \right] \quad (3.1)$$

(where, on $\{N(t) = 0\}$, the product is defined to take the value 1) is the solution of (2.1).

Proof. For $t \geq 0$ and $m \geq 0$, let

$$Y(m, t) = e^t 1_{\{N(t)=m\}} w(t - \tau_m, X_{\tau_m}) \prod_{i=1}^m \left[S(\tau_i - \tau_{i-1}, \mathbb{R}^d) V(t - \tau_i, X_{\tau_i}) \right].$$

Then $u(t, x) = \sum_{m=0}^{\infty} E_x[Y(m, t)]$. In order to show that $u(t, x)$ is the solution of (2.1), it suffices by Proposition 2.2 to show that $H_m(t, x) = E_x[Y(m, t)]$ for all m, t, x . We prove this by induction on m . For $m = 0$,

$$\begin{aligned} E_x[Y(m, t)] &= E_x \left[e^t 1_{\{N(t)=0\}} w(t, X_0) \right] \\ &= e^t w(t, x) P_x \{N(t) = 0\} = w(t, x) = H_0(t, x). \end{aligned}$$

Now fix $m \geq 1$ and suppose by induction that $H_{m-1}(t, x) = E_x[Y(m-1, t)]$, for all (t, x) . Set $\mathcal{F}_1 = \sigma\{X_{\tau_1}, \tau_1\}$. Then

$$\begin{aligned} E_x[Y(m, t)] &= E_x \left[S(\tau_1, \mathbb{R}^d) V(t - \tau_1, X_{\tau_1}) 1_{\{\tau_1 \leq t\}} e^{\tau_1} \right. \\ &\quad \times E_x \left[e^{t-\tau_1} 1_{\{N(t)-N(\tau_1)=m-1\}} w((t - \tau_1) - (\tau_m - \tau_1), X_{\tau_m}) \right. \\ &\quad \left. \left. \times \prod_{i=2}^m \left\{ S((\tau_i - \tau_1) - (\tau_{i-1} - \tau_1), \mathbb{R}^d) V((t - \tau_1) - (\tau_i - \tau_1), X_{\tau_i}) \right\} \middle| \mathcal{F}_1 \right] \right]. \end{aligned}$$

Note that, for $i \geq 1$,

$$X_{\tau_i} = X_{\tau_1} + \sum_{j=1}^{i-1} \tilde{X}_{\tau_{j+1}-\tau_j}^{(j+1)},$$

and the conditional expectation can be expressed using only the increments $\tau_i - \tau_1$ for $i \geq 1$. Using the strong Markov property of $(N(t))$ at time τ_1 and the independence of the families $X_t^{(i)}$, we deduce that

$$\begin{aligned} E_x[Y(m, t)] &= E_x \left[S(\tau_1, \mathbb{R}^d) V(t - \tau_1, X_{\tau_1}) e^{\tau_1} 1_{\{\tau_1 \leq t\}} Y_{m-1}(t - \tau_1, X_{\tau_1}) \right] \\ &= \int_0^t ds e^{-s} S(s, \mathbb{R}^d) e^s \int_{\mathbb{R}^d} \frac{S(s, dy)}{S(s, \mathbb{R}^d)} V(t - s, x - y) Y_{m-1}(t - s, x - y) \\ &= \int_0^t ds \int_{\mathbb{R}^d} S(s, dy) V(t - s, x - y) H_{m-1}(t - s, x - y) \\ &= H_m(t, x), \end{aligned}$$

by the induction hypothesis and (2.2). This completes the proof. \square

We have presented the simplest setting of the probabilistic representation, sufficient to treat our interest in the wave equation in dimensions $d \leq 3$ and the random potentials in the subsequent sections. However various extensions and variations of this representation are possible. We give a brief description here, leaving the details to the interested reader.

1. For a signed kernel $S(t, dy)$, we need to modify somewhat the representation. Write $S(t, dy) = S_+(t, dy) - S_-(t, dy)$ for the Hahn-Jordan decomposition into a difference of non-negative measures. Choose, if possible, subsets $A(t) \subseteq \mathbb{R}^d$ so that $S_+(t, A(t)) = S_-(t, A^c(t)) = 0$ and $(x, t) \rightarrow 1_{A(t)}(x)$ is measurable. (Note that this is certainly possible when $S(t, dy) = q_t(y)dy$ for continuous $(q_t(y), t > 0, y \in \mathbb{R}^d)$.) Let C_t be a counter defined by

$$C_t = \sum_{i=1}^{\infty} 1_{\{\tau_i \leq t\}} 1_{\{X_{\tau_{i-1}} - X_{\tau_i} \in A(\tau_i - \tau_{i-1})\}}.$$

Then the argument above leads to the representation

$$u(t, x) = e^t E_x \left[w \left(t - \tau_{N(t)}, X_{\tau_{N(t)}} \right) (-1)^{C_t} \prod_{i=1}^{N(t)} \left[|S(\tau_i - \tau_{i-1}, \cdot)| V(t - \tau_i, X_{\tau_i}) \right] \right],$$

where $|S(t, \cdot)|$ denotes the total variation of the measure $S(t, dy)$. This representation then covers the case of the beam equation in all dimensions $d \geq 1$.

2. If, instead of being real-valued, $u(t, x) = (u_1(t, x), \dots, u_n(t, x)) \in \mathbb{R}^n$, and $V(t, x)$ is an $n \times n$ matrix, so that (2.1) is in fact a system of p.d.e.'s, then the formula in Theorem 3.2 still holds, provided the matrix product in (3.1) is ordered according to increasing values of i .
3. We have treated for simplicity the case of spatially homogeneous equations on \mathbb{R}^d . However, in principle, suitable changes should allow representations for inhomogeneous equations, or equations in domains with suitable boundary conditions.
4. For any $\lambda > 0$, one can replace the potential V by $\lambda^{-1}V$ and use a Poisson process of rate λ to obtain an alternative representation. For example, rewriting the heat equation (1.1) as

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \lambda \left[\frac{V}{\lambda} \right] u,$$

we would get the representation

$$u(t, x) = e^{\lambda t} E_x \left[w \left(t - \tau_{N(t)}, X_{\tau_{N(t)}} \right) (-1)^{C_t} \prod_{i=1}^{N(t)} \left[|S(\tau_i - \tau_{i-1}, \cdot)| \lambda^{-1} V(t - \tau_i, X_{\tau_i}) \right] \right] \quad (3.2)$$

where X_t starts over at the times of a rate λ Poisson process. For large λ , these representations, when using a Markovian X , become close to the classical Feynman-Kac formula. For example, we can further rewrite (1.1) as

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \lambda \left[1 + \frac{V}{\lambda} \right] u - \lambda u. \quad (3.3)$$

Due to the term $-\lambda u$ in (3.3), the Green's function $e^{-\lambda t} p_t(y)$ of $Lu = \frac{\partial}{\partial t} - \frac{1}{2}\Delta u + \lambda u$ gives rise to a factor $e^{-\lambda t}$ inside the expectation in (3.2), which cancels the factor $e^{\lambda t}$ which is outside of the expectation. Regarding $[1 + (V/\lambda)]$ as our potential term, we find

$$u(t, x) = E_x^B \left[w(t - \tau_{N_\lambda(t)}, B_{\tau_{N_\lambda(t)}}) \exp \left(\sum_{m=1}^{N_\lambda(t)} \log(1 + \lambda^{-1} V(t - \tau_m, B_{\tau_m})) \right) \right].$$

Now letting $\lambda \rightarrow \infty$ and using $\log(1 + x) \approx x$, the integrand involves a Riemann sum approximation to the integral in the classical Feynman-Kac formula.

4 Second moments for random potentials

4.1 The random potentials

We are now going to consider a class of linear equations driven by spatially homogeneous Gaussian noise $\dot{F}(t, x)$, whose covariance is formally given by

$$E \left[\dot{F}(t, x) \dot{F}(s, y) \right] = \delta_0(t - s) f(x - y).$$

In this equation, $\delta(\cdot)$ denotes the Dirac delta function, and $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous on $\mathbb{R}^d \setminus \{0\}$. More precisely, let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of Schwartz test functions (see [20]). On a given probability space, we define a Gaussian process $F = (F(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^{d+1}))$ with mean zero and covariance functional

$$E[F(\varphi)F(\psi)] = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \varphi(t, x) f(x - y) \psi(t, y).$$

Since this is a covariance, it is well-known [20, Schwartz, Chap. VII, Théorème XVII] that f must be symmetric and be the Fourier transform of a non-negative tempered measure μ on \mathbb{R}^d , termed the spectral measure : $f = \mathcal{F}\mu$. In this case, F extends to a worthy martingale measure $M = (M_t(B), t \geq 0, B \in \mathcal{B}_b(\mathbb{R}^d))$ in the sense of [21], with covariation measure Q defined by

$$Q([0, t] \times A \times B) = \langle M(A), M(B) \rangle_t = t \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy 1_A(x) f(x - y) 1_B(y),$$

and dominating measure $K = Q$ (see [8, 6]). By construction, $t \mapsto M_t(B)$ is a continuous martingale and

$$F(\varphi) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi(t, x) M(dt, dx),$$

where the stochastic integral is as defined in [21].

Assumption C. For each $t > 0$, $S(t, dy)$ is a non-negative measure and takes values in the space of distributions with rapid decrease [20, Chap.VIII, §5]. Moreover, it satisfies

$$\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}S(s, \cdot)(\xi)|^2 < \infty \quad (4.1)$$

and

$$\lim_{h \downarrow 0} \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) \sup_{t < r < t+h} |\mathcal{F}S(r)(\xi) - \mathcal{F}S(t)(\xi)|^2 = 0,$$

where μ is the spectral measure of the Gaussian process F .

Consider the stochastic integral equation

$$u(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}^d} S(t-s, x-y) u(s, y) F(ds, dy), \quad (4.2)$$

where $w(t, x)$ is a random field satisfying appropriate conditions (see below).

Our motivation is the case where $S(t, dy)$ is the Green's function for a partial differential operator L , and the study of the stochastic p.d.e. $Lu = u\dot{F}$, with stationary initial conditions independent of \dot{F} . This s.p.d.e. can be recast into this integral form with $w(t, x)$ being the solution of $Lw = 0$ with the same initial conditions as $u(t, x)$. In this context, $(w(t, \cdot), M_t(\cdot))$ is stationary in x , or, more precisely, has property (S) of Dalang [6, Definition 5.1].

The stochastic integral in (4.2) needs defining. If $S(s, y)$ is a smooth function, as in the heat equation, then we can use the stochastic integral with respect to a worthy martingale measure introduced in [21]. In this case, (4.2) has a unique solution provided $w(t, x)$ is a predictable process such that $\sup_{t \leq T, x \in \mathbb{R}^d} E[w^2(t, x)] < \infty$.

If $S(s, \cdot)$ is a singular measure, as in the case of the 3-dimensional wave equation that we are particularly interested in, then we use the integral introduced in Dalang [6]. We briefly describe his construction, that uses an approximation to the identity. Choose $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\psi \geq 0$, the support of ψ is contained in the unit ball of \mathbb{R}^d and $\int_{\mathbb{R}^d} \psi(x) dx = 1$. For $\ell \geq 1$, set $\psi_\ell(x) = \ell^d \psi(\ell x)$, so that $\psi_\ell \rightarrow \delta_0$ as $\ell \rightarrow \infty$. The stochastic integral in (4.2) is the L^2 -limit of the usual stochastic integrals

$$\int_0^t \int_{\mathbb{R}^d} S_\ell(t-s, x-y) u(s, y) F(ds, dy),$$

where $S_\ell(t, x)$ is the convolution $\int S(t, dy) \psi_\ell(x-y)$. While studying the s.p.d.e. $Lu = u\dot{F}$ as above, this convergence was established in [6]. Assumption C (in particular, the fact that $S(t, dy)$ is non-negative) is also used in the definition of the stochastic integral.

The following proposition gives a straightforward extension of [6, Theorem 13] that we need to handle the integral equation (4.2).

Proposition 4.1. *Fix $T > 0$. Suppose $(w(t, x), (t, x) \in [0, T] \times \mathbb{R}^d)$ is a predictable process such that $\sup_{t \leq T, x \in \mathbb{R}^d} E[w^2(t, x)] < \infty$ and that $(w(t, x))$ has property (S) of [6, Definition 5.1]. If Assumption C holds, then (4.2) has a unique square-integrable solution $(u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d)$. Moreover, this solution is L^2 -continuous and if $\sup_{t \leq T, x \in \mathbb{R}^d} E[|w(t, x)|^p] < \infty$ for some $p \geq 2$, then*

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} E[|u(t, x)|^p] < \infty.$$

Proof. The proof is identical to that of [6, Theorem 13], except that $u_0(t, x)$ there is set to $w(t, x)$ rather than to 0, and u_{n+1} in the Picard iteration is defined by

$$u_{n+1}(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}^d} S(t-s, x-y) u_n(s, y) F(ds, dy)$$

rather than by equation (52) of [6]. All the arguments in that proof apply, essentially without change, to this situation. \square

In the terminology of [9], the solution $u(t, x)$ given by Proposition 4.1 is a *random field solution* of (4.2), that is, a solution that is defined for every t and x (as opposed to a *function-valued solution*, defined only for all t and almost all x , that would not be adequate for our purposes).

In fact, it is shown in [6] that (4.1) is even a necessary condition for (4.2) to have a solution satisfying $\sup_{t \leq T, x \in \mathbb{R}^d} E[u^2(t, x)] < \infty$.

In the cases of the heat and wave equations, [6] gives equivalent conditions to (4.1) involving only μ or the function f in the covariance structure.

4.2 The series representation

In this subsection, we work under assumptions A and C. We assume that $w(t, x)$ satisfies the properties indicated in Proposition 4.1.

We shall show that there is a series representation for the solution $u(t, x)$ of (4.2), analogous to (2.3), but with the deterministic integral replaced by the stochastic integral as in (4.2). Define $I_0(t, x) = w(t, x)$, and, for $m \geq 0$,

$$I_{m+1}(t, x) = \int_0^t \int_{\mathbb{R}^d} S(t-s, x-y) I_m(s, y) F(ds, dy). \quad (4.3)$$

Proposition 4.2. *Suppose that $w(t, x)$ satisfies the properties indicated in Proposition 4.1. Then the series*

$$u(t, x) = \sum_{m=0}^{\infty} I_m(t, x) \quad (4.4)$$

converges in L^2 uniformly over $(t, x) \in [0, T] \times \mathbb{R}^d$ and is the unique solution to (4.2).

Proof. We first check the L^2 -convergence of the series in (4.4). Set

$$M_m(t) = \sup_{x \in \mathbb{R}^d} E[I_m(t, x)^2].$$

By [6, Theorem 2],

$$M_m(t) \leq \int_0^t ds M_{m-1}(s) \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}S(t-s, \cdot)(\xi)|^2,$$

By (4.1) and [6, 7, Lemma 15], we conclude that

$$\sum_{m=0}^{\infty} M_m(s)^{1/2} < \infty,$$

which establishes the L^2 -convergence of the series. Set $u_n(t, x) = \sum_{m=0}^n I_m(t, x)$. Then $u_n(t, x) \rightarrow u(t, x)$ in L^2 , and by [6, Theorem 2], as $n \rightarrow \infty$,

$$\int_0^t \int_{\mathbb{R}^d} S(t-s, x-y) u_n(s, y) F(ds, dy) \xrightarrow{L^2} \int_0^t \int_{\mathbb{R}^d} S(t-s, x-y) u(s, y) F(ds, dy).$$

Therefore,

$$\begin{aligned}
& w(t, x) + \int_0^t \int_{\mathbb{R}^d} S(t-s, x-y) u(s, y) F(ds, dy) \\
&= \lim_{n \rightarrow \infty} \left(w(t, x) + \int_0^t \int_{\mathbb{R}^d} S(t-s, x-y) u_n(s, y) F(ds, dy) \right) \\
&= \lim_{n \rightarrow \infty} \left(I_0(t, x) + \sum_{m=0}^n \int_0^t \int_{\mathbb{R}^d} S(t-s, x-y) I_m(s, y) F(ds, dy) \right) \\
&= \lim_{n \rightarrow \infty} \sum_{m=0}^{n+1} I_m(t, x) \\
&= u(t, x),
\end{aligned}$$

showing that $u(t, x)$ solves (4.2). □

The successive terms in (4.4) are orthogonal in L^2 , that is $E[I_m(t, x)I_{m'}(s, y)] = 0$ whenever $m \neq m'$. The series is therefore a chaos expansion for the noise F . The orthogonality can be checked by induction on m and m' , using the fact that the covariance between I_m and $I_{m'}$ reduces, as in (4.5) below, to an expression involving the covariance between I_{m-1} and $I_{m'-1}$.

4.3 The probabilistic representation of second moments

In this subsection, we work under assumptions A, B, and C. We make the same assumptions on $w(t, x)$ as in subsection 4.2.

Let $(N(t), t \geq 0)$ be a rate one Poisson process. Using two independent i.i.d. families $(\tilde{X}^{(i,1)}, i \geq 1)$ and $(\tilde{X}^{(i,2)}, i \geq 1)$, construct, as in Section 3, two processes $X^1 = (X_t^1, t \geq 0)$ and $X^2 = (X_t^2, t \geq 0)$ which renew themselves at the same set of jump times τ_i of the process N , and which start, under P_{x_1, x_2} , at x_1 and x_2 respectively. See Figure 2 for an illustration.

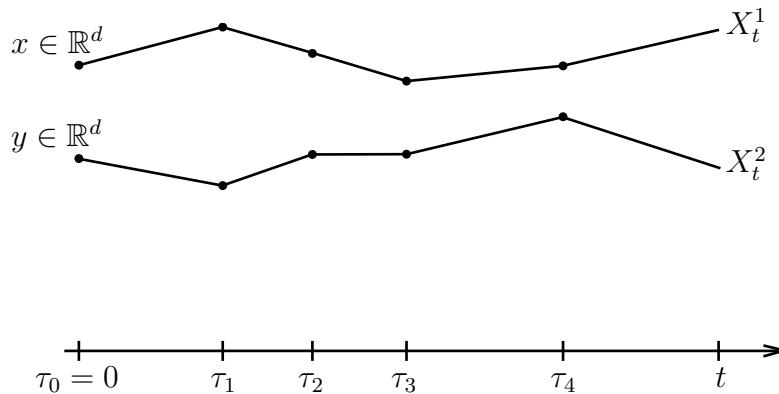


Figure 2: A sample path of the processes X^1 and X^2 .

Theorem 4.3. *Let $u(t, x)$ be the solution of (4.2) given in Proposition 4.2. Then*

$$E[u(t, x)u(t, y)] = e^t E_{x,y} \left[w \left(t - \tau_{N(t)}, X_{\tau_{N(t)}}^1 \right) w \left(t - \tau_{N(t)}, X_{\tau_{N(t)}}^2 \right) \right. \\ \left. \times \prod_{i=1}^{N(t)} (S(\tau_i - \tau_{i-1}, \mathbb{R}^d))^2 f(X_{\tau_i}^1 - X_{\tau_i}^2) \right].$$

Proof. Observe that by Proposition 4.2,

$$E[u(t, x)u(t, y)] = \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} E[I_m(t, x)I_{m'}(t, y)] = \sum_{m=0}^{\infty} E[I_m(t, x)I_m(t, y)],$$

using the orthogonality of the terms in the series. For $m \geq 1$, using the smoothed kernels $S_\ell(t, x)$ defined earlier, we have

$$\begin{aligned} & E[I_m(t, x)I_m(t, y)] \\ &= E \left[\int_0^t \int_{\mathbb{R}^d} S(t-s, x-z) I_{m-1}(s, z) F(ds, dz) \int_0^t \int_{\mathbb{R}^d} S(t-s, y-z) I_{m-1}(s, z) F(ds, dz) \right] \\ &= \lim_{\ell \rightarrow \infty} E \left[\int_0^t \int_{\mathbb{R}^d} S_\ell(t-s, x-z) I_{m-1}(s, z) F(ds, dz) \int_0^t \int_{\mathbb{R}^d} S_\ell(t-s, y-z) I_{m-1}(s, z) F(ds, dz) \right] \\ &= \lim_{\ell \rightarrow \infty} \int_0^t ds \int_{\mathbb{R}^d} dz_1 \int_{\mathbb{R}^d} dz_2 S_\ell(t-s, x-z_1) S_\ell(t-s, y-z_2) f(z_1 - z_2) E[I_{m-1}(s, z_1) I_{m-1}(s, z_2)] \\ &= \int_0^t ds \int_{\mathbb{R}^d} S(t-s, x-dz_1) \int_{\mathbb{R}^d} S(t-s, y-dz_2) f(z_1 - z_2) E[I_{m-1}(s, z_1) I_{m-1}(s, z_2)], \end{aligned} \quad (4.5)$$

where we have used the Lebesgue Differentiation Theorem [22, Chapter 7, Exercise 2] in the final step. We shall now show by induction that

$$E[I_m(t, x)I_m(t, y)] = J(m, t, x, y), \quad m \geq 0, \quad (4.6)$$

where

$$J(m, t, x, y) = e^t E_{x,y} \left[\mathbf{1}_{\{N(t)=m\}} w \left(t - \tau_m, X_{\tau_m}^1 \right) w \left(t - \tau_m, X_{\tau_m}^2 \right) \right. \\ \left. \times \prod_{i=1}^m \{ S(\tau_i - \tau_{i-1}, \mathbb{R}^d)^2 f(X_{\tau_i}^1 - X_{\tau_i}^2) \} \right].$$

For $m = 0$,

$$J(0, t, x, y) = e^t w(t, x) w(t, y) P_{x,y} \{ N(t) = 0 \} = E[I_0(t, x)I_0(t, y)].$$

We suppose now that (4.6) holds for $m - 1$. By the Markov property at τ_1 , arguing as in Theorem 3.2, we have, choosing $\mathcal{F}_1 = \sigma\{\tau_1, X_{\tau_1}^1, X_{\tau_1}^2\}$,

$$\begin{aligned} J(m, t, x, y) &= E_{x,y} \left[\mathbf{1}_{\{\tau_1 \leq t\}} e^{\tau_1} f(X_{\tau_1}^1 - X_{\tau_1}^2) S(\tau_1, \mathbb{R}^d)^2 \right. \\ &\quad \times e^{(t-\tau_1)} E_{x,y} [\mathbf{1}_{\{N(t)-N(\tau_1)=m-1\}} w(t - \tau_m, X_{\tau_m}^1) w(t - \tau_m, X_{\tau_m}^2) \\ &\quad \times \prod_{i=2}^m [S(\tau_i - \tau_{i-1}, \mathbb{R}^d)^2 f(X_{\tau_i}^1 - X_{\tau_i}^2) | \mathcal{F}_1]] \\ &= E_{x,y} \left[\mathbf{1}_{\{\tau_1 \leq t\}} e^{\tau_1} f(X_{\tau_1}^1 - X_{\tau_1}^2) S(\tau_1, \mathbb{R}^d)^2 J(m-1, t - \tau_1, X_{\tau_1}^1, X_{\tau_1}^2) \right] \\ &= \int_0^t ds \int_{\mathbb{R}^d} S(s, x - dz_1) \int_{\mathbb{R}^d} S(s, y - dz_2) f(z_1 - z_2) J(m-1, t - s, z_1, z_2). \end{aligned}$$

The conclusion now follows from (4.5) and the induction hypothesis. \square

Remark 4.4. By multiplying the integral formulas (4.2) for $u(t, x)$ and $u(t, y)$ and taking expectations, one expects formally the integral equation

$$\begin{aligned} & E[u(t, x)u(t, y)] \\ &= w(t, x)w(t, y) + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S(t-s, x-dz_1)S(t-s, y-dz_2)f(z_1-z_2)E[u(s, z_1)u(s, z_2)]. \end{aligned}$$

This new integral equation on \mathbb{R}^{2d} is of the same form as (2.1). This leads to an alternative derivation, by applying Theorem 3.2, of the representation for second moments given above. However, we have used the argument above as it will generalize to higher moments.

5 Moments of order n

In this section, we work again under assumptions A, B, and C. In addition to the assumptions on $w(t, x)$ made in subsection 4.2, we assume that $\sup_{t \leq T} E[|w(t, x)|^p] < \infty$, for all $T, p > 0$, which ensures that the solutions have finite p -th moments.

In the case where $u(t, x)$ solves a first order equation driven by the Gaussian noise \dot{F} , written in the form $\frac{\partial u}{\partial t} = \tilde{L}u + u\dot{F}$, where \tilde{L} is a differential operator in the space variables, then a formal calculation suggests that the n -th moment $m(t, x_1, \dots, x_n) = E[u(t, x_1) \dots u(t, x_n)]$ should satisfy

$$\frac{\partial m}{\partial t} = \tilde{L}_{x_1, \dots, x_n} m + \frac{1}{2} m \sum_{i \neq j}^n f(x_i - x_j)$$

(this formula is proved for discrete space in [3, Section II.3]). Here, $\tilde{L}_{x_1, \dots, x_n}$ stands for the sum of the operator \tilde{L} applied to each variable x_i . The equation for m is again of the same potential type considered in section 2, and can be recast as an integral equation using a multiple product kernel constructed out of the kernel $S(t, dy)$ for $\frac{\partial}{\partial t} - \tilde{L}$. Theorem 3.2 then leads to a probabilistic representation for m . This argument does not seem to apply for second order equations (in time) or directly for integral equations. However, as we shall now explain, it is possible to find a representation, analogous to the one for second moments, that holds for the higher moments of the general integral equation (4.2).

We start with an informal discussion of the representation for higher moments. The second moments were given in terms of a pair of processes, both of which were renewed at the times τ_i of a single Poisson process N . The situation for the n -th moment is somewhat analogous. Instead of two processes, we will use n processes X^1, \dots, X^n . For each pair of indices $\rho = \{\rho_1, \rho_2\}$, we create a Poisson process $N_t(\rho)$. The renewal times of the the process X^i will be the union of the Poisson times arising from the processes $N_t(\rho)$, such that the index i is contained in the pair of indices ρ .

More precisely we let \mathcal{P}_n denote the set of unordered pairs from $\mathcal{L}_n = \{1, \dots, n\}$ and for $\rho \in \mathcal{P}_n$, we write $\rho = \{\rho_1, \rho_2\}$, with $\rho_1 < \rho_2$. Note that $\text{card}(\mathcal{P}_n) = n(n-1)/2$. Let $(N_t(\rho), \rho \in \mathcal{P}_n)$ be independent rate one Poisson processes. For $A \subseteq \mathcal{P}_n$ let $N_t(A) = \sum_{\rho \in A} N_t(\rho)$. This defines a Poisson random measure such that for fixed A , $(N_t(A), t \geq 0)$ is a Poisson process with intensity $\text{card}(A)$. Let $\sigma_1 < \sigma_2 < \dots$ be the jump times of $(N_t(\mathcal{P}_n), t \geq 0)$, and $R^i = \{R_1^i, R_2^i\}$ be the pair corresponding to time σ_i . Two possible representations of this Poisson random measure are shown in Figure 3.

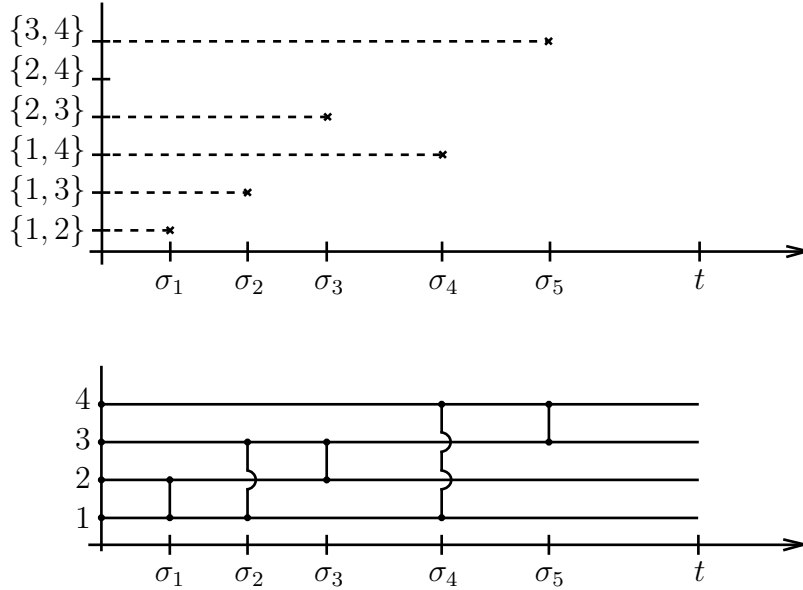


Figure 3: Two equivalent representations of the Poisson random measure $(N_t(\cdot))$: the top representation is simply the superposition of the Poisson processes $(N_t(\rho))$, $\rho \in \mathcal{P}_n$; in the bottom representation, two elements of \mathcal{L}_n are joined at time σ_i if they constitute the pair R^i .

For $\ell \in \mathcal{L}_n$, let $\mathcal{P}^{(\ell)} \subseteq \mathcal{P}_n$ be the set of pairs that contain ℓ , so that $\text{card}(\mathcal{P}^{(\ell)}) = n - 1$. Let $\tau_1^\ell < \tau_2^\ell < \dots$ be the jump times of $(N_t(\mathcal{P}^{(\ell)}), t \geq 0)$. We write $N_t(\ell)$ instead of $N_t(\mathcal{P}^{(\ell)})$. Note that

$$\sum_{\rho \in \mathcal{P}_n} N_t(\rho) = N_t(\mathcal{P}_n) = \frac{1}{2} \sum_{\ell \in \mathcal{L}_n} N_t(\ell).$$

We now define the motion process needed. For $\ell \in \mathcal{L}_n$ and $i \geq 0$, let $(\tilde{X}_t^{\ell, (i)}, t \geq 0)$ be i.i.d. copies of the process (\tilde{X}_t) defined before Example 3.1. Set

$$X_t^\ell = \begin{cases} X_0^\ell + \tilde{X}_t^{\ell, (1)}, & 0 \leq t \leq \tau_1^\ell, \\ X_{\tau_i^\ell}^\ell + \tilde{X}_{t-\tau_i^\ell}^{\ell, (i+1)}, & \tau_i^\ell < t < \tau_{i+1}^\ell. \end{cases}$$

This motion process is illustrated in Figure 4.

It will be useful to define X_t^ℓ for certain $t < 0$. For given $(t_1, x_1), \dots, (t_n, x_n)$, under the measure $P_{(t_1, x_1), \dots, (t_n, x_n)}$ we set

$$X_t^\ell = \tilde{X}_{t+t_\ell}^{\ell, (0)} \quad \text{for } -t_\ell \leq t \leq 0.$$

Finally we set $\tau_0^\ell = -t_\ell$. The following theorem gives a formula for the n -th moments, and it is the main result of this section.

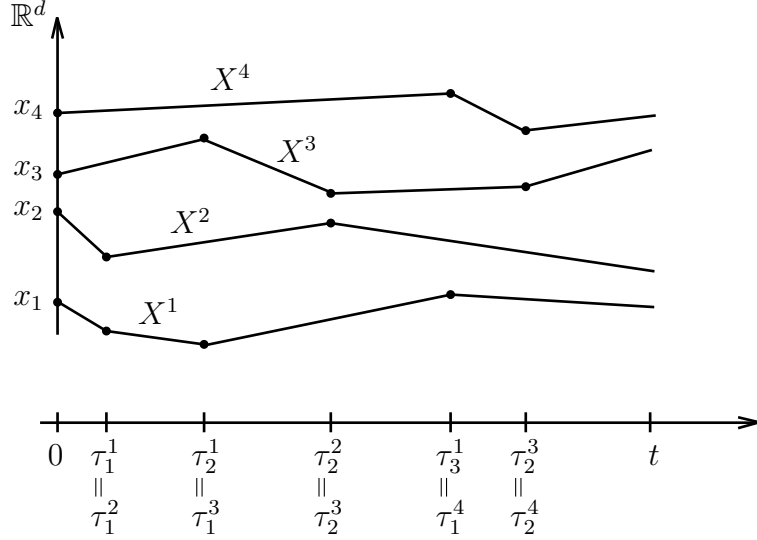


Figure 4: Illustration of the motion processes X^ℓ in the case where $n = 4$ and $X_0^\ell = x^\ell$, $\ell = 1, \dots, 4$.

Theorem 5.1. *The n -th moments are given by*

$$\begin{aligned}
& E[u(t, x_1) \cdots u(t, x_n)] \tag{5.1} \\
&= e^{tn(n-1)/2} E_{(0, x_1), \dots, (0, x_n)} \left[\prod_{i=1}^{N_t(\mathcal{P}_n)} f(X_{\sigma_i}^{R_i^1} - X_{\sigma_i}^{R_i^2}) \right. \\
&\quad \left. \times \prod_{\ell \in \mathcal{L}_n} \prod_{i=1}^{N_i(\ell)} S(\tau_i^\ell - \tau_{i-1}^\ell, \mathbb{R}^d) \cdot \prod_{\ell \in \mathcal{L}_n} w(t - \tau_{N_t(\ell)}^\ell, X_{\tau_{N_t(\ell)}^\ell}^\ell) \right].
\end{aligned}$$

The proof of this theorem requires some preliminaries. Let $I_m(t, x)$, $m \geq 0$, be as defined in (4.3). For $0 \leq s \leq t$, set

$$I_{m+1}(s, t, x) = \int_0^s \int_{\mathbb{R}^d} S(t-r, x-y) I_m(r, y) F(dr, dy),$$

so that $I_m(t, t, x) = I_m(t, x)$ for $m \geq 1$. For $m = 0$ and $0 \leq s < t$, we set $I_0(s, t, x) = I_0(t, t, x) = w(t, x)$. Let

$$I(s; (m_i, t_i, x_i), i = 1, \dots, n) = E \left[\prod_{i=1}^n I_{m_i}(s, t_i, x_i) \right],$$

for $m_i \geq 0$, $s \leq \min(t_1, \dots, t_n)$, $x_i \in \mathbb{R}^d$, $i = 1, \dots, n$. We begin by giving an inductive expression for this expectation.

Lemma 5.2. *Suppose $m_1 + \dots + m_n = m$.*

(a) *If $m = 0$, then*

$$I(s; (0, t_i, x_i), i = 1, \dots, n) = \prod_{i=1}^n w(t_i, x_i).$$

(b) If $m \geq 1$, then

$$\begin{aligned}
& I(s; (m_i, t_i, x_i), i = 1, \dots, n) \\
&= \sum_{\rho \in \mathcal{P}_n: m_{\rho_1} \cdot m_{\rho_2} > 0} \int_0^s dr \int_{\mathbb{R}^d} S(t_{\rho_1} - r, dy_1) \int_{\mathbb{R}^d} S(t_{\rho_2} - r, dy_2) f(x_{\rho_1} - y_1 - x_{\rho_2} + y_2) \\
&\quad \times E \left[\prod_{i=1}^2 I_{m_{\rho_i}-1}(r, r, x_{\rho_i} - y_i) \cdot \prod_{k \in \mathcal{L}_n \setminus \rho} I_{m_k}(r, t_k, x_k) \right]. \tag{5.2}
\end{aligned}$$

Proof. Part (a) follows immediately from the definitions. For part (b), if $m = 1$, then $n - 1$ of the m_i are equal to 0 and so $n - 1$ of the $I_{m_i}(s, t_i, x_i)$ are deterministic. The one $I_{m_i}(s, t_i, x_i)$ with $m_i = 1$ is a martingale with mean zero, implying that $I(s; (m_i, t_i, x_i), i = 1, \dots, n) = 0$. The expression in formula (5.2) is also equal to 0 since there is no $\rho \in \mathcal{P}_n$ such that $m_{\rho_1} \cdot m_{\rho_2} > 0$.

If $m \geq 2$, we distinguish two cases. The first case is where all but one of the m_i are zero. In this case, $I(s; (m_i, t_i, x_i), i = 1, \dots, n)$ and expression (5.2) vanish, for the same reasons as in the case $m = 1$. We now consider the second case, in which there is at least one $\rho \in \mathcal{P}_n$ with $m_{\rho_1} \cdot m_{\rho_2} > 0$.

Using the smoothed kernels $S_\ell = \psi_\ell * S$, as in section 4.1, we define

$$I_{m+1}^\ell(s, t, x) = \int_0^s \int_{\mathbb{R}^d} S_\ell(t - r, x - y) I_m(r, y) F(dr, dy).$$

For fixed (t_i, x_i) , $s \mapsto I_{m_i}^\ell(s, t_i, x_i)$ is a martingale, and according to [21, Thm. 2.5], if $m_i > 0$ and $m_j > 0$, then the mutual variation process of $I_{m_i}^\ell(\cdot, t_i, x_i)$ and $I_{m_j}^\ell(\cdot, t_j, x_j)$ is

$$\begin{aligned}
s \mapsto & \int_0^s dr \int_{\mathbb{R}^d} dy_1 S_\ell(t_i - r, x_i - y_1) \int_{\mathbb{R}^d} dy_2 S_\ell(t_j - r, x_j - y_2) \\
& \times f(y_1 - y_2) I_{m_i-1}(r, r, y_1) I_{m_j-1}(r, r, y_2).
\end{aligned}$$

We now apply Itô's formula [4, Theorem 5.10] to the function $f(a_1, \dots, a_n) = a_1 \cdots a_n$ and the n martingales $I_{m_i}^\ell(\cdot, t_i, x_i)$, $i = 1, \dots, n$. Note that

$$\frac{\partial^2 f}{\partial a_i^2} = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial a_i \partial a_j} = \prod_{k \in \{1, \dots, n\} \setminus \{i, j\}} a_k \quad \text{if } i \neq j.$$

The stochastic integrals terms given by Itô's formula have mean zero, because the $I_{m_i}(\cdot, \cdot, \cdot)$ have bounded moments of all orders, so taking expectations we reach

$$\begin{aligned}
& E \left[\prod_{i=1}^n I_{m_i}^\ell(s, t_i, x_i) \right] \\
&= \sum_{\rho \in \mathcal{P}_n: m_{\rho_1} \cdot m_{\rho_2} > 0} \int_0^s dr \int_{\mathbb{R}^d} dy_1 S_\ell(t_{\rho_1} - r, x_{\rho_1} - y_1) \int_{\mathbb{R}^d} dy_2 S_\ell(t_{\rho_2} - r, x_{\rho_2} - y_2) \\
&\quad \times f(y_1 - y_2) E \left[I_{m_{\rho_1}-1}(r, r, y_1) I_{m_{\rho_2}-1}(r, r, y_2) \prod_{k \in \mathcal{L}_n \setminus \rho} I_{m_k}^\ell(r, t_k, x_k) \right]. \tag{5.3}
\end{aligned}$$

The variables $I_{m_k}(r, t_k, x_k)$ and $I_{m_k}^\ell(r, t_k, x_k)$ are both bounded in L^p for all p and continuous in L^2 in (r, x_k) , so that they are continuous in L^p in the variables (r, x_k) . This implies that the expectation in (5.3) is continuous in (r, x_1, \dots, x_n) . Using the change of variables $z_1 = x_{\rho_1} - y_1$ and $z_2 = x_{\rho_2} - y_2$, we let $\ell \rightarrow \infty$ in (5.3). The left-hand side converges to $I(s; (m_i, t_i, x_i), i = 1, \dots, n)$ and the right-hand side converges to formula (5.2), completing the proof. \square

Define

$$\begin{aligned} & J(t; (m_i, t_i, x_i), i = 1, \dots, n) \\ &= e^{tn(n-1)/2} E_{(t_1, x_1), \dots, (t_n, x_n)} \left[\mathbb{1}_{\{N_t(\ell) = m_\ell, \ell \in \mathcal{L}_n\}} \prod_{i=1}^{\frac{1}{2}(m_1 + \dots + m_n)} f(X_{\sigma_i}^{R_i^1} - X_{\sigma_i}^{R_i^2}) \right. \\ & \quad \left. \times \prod_{\ell \in \mathcal{L}_n} \prod_{i=1}^{m_i} S(\tau_i^\ell - \tau_{i-1}^\ell, \mathbb{R}^d) \cdot \prod_{\ell \in \mathcal{L}_n} w(t - \tau_{m_\ell}^\ell, X_{\tau_{m_\ell}^\ell}^\ell) \right]. \end{aligned}$$

The next aim is to show that these expectations satisfy a similar inductive formula.

Lemma 5.3. *Suppose $m_1 + \dots + m_n = m$.*

(a) *If $m = 0$, then*

$$J(t; (0, t_\ell, x_\ell), \ell = 1, \dots, n) = \prod_{\ell=1}^n w(t + t_\ell, x_\ell).$$

(b) *If $m \geq 1$, then $J(t; (m_\ell, t_\ell, x_\ell), \ell \in \mathcal{L}_n)$ is equal to*

$$\begin{aligned} & \sum_{\rho \in \mathcal{P}_n: m_{\rho_1} \cdot m_{\rho_2} > 0} \int_0^t ds \int_{\mathbb{R}^d} S(t_{\rho_1} + s, dy_1) \int_{\mathbb{R}^d} S(t_{\rho_2} + s, dy_2) f(x_{\rho_1} - y_1 - x_{\rho_2} + y_2) \\ & \quad \times J(t - s; (m_{\rho_i} - 1, 0, x_{\rho_i} - y_i), i = 1, 2; (m_\ell, s + t_\ell, x_\ell), \ell \in \mathcal{L}_n \setminus \rho). \end{aligned} \quad (5.4)$$

Proof. Part (a) follows immediately from the definitions. For part (b), in the case that only one of the m_i are non-zero then $J(t; (m_\ell, t_\ell, x_\ell), \ell \in \mathcal{L}_n) = 0$ since $P_{(t_1, x_1), \dots, (t_n, x_n)} \{N_t(\ell) = m_\ell, \ell \in \mathcal{L}_n\} = 0$, and formula (5.4) is also equal to 0 since there is no $\rho \in \mathcal{P}_n$ with $m_{\rho_1} \cdot m_{\rho_2} > 0$.

We now suppose that $m \geq 2$ and that there is at least one $\rho \in \mathcal{P}_n$ with $m_{\rho_1} \cdot m_{\rho_2} > 0$. In this case $\{N_t(\ell) = m_\ell, \ell \in \mathcal{L}_n\} \subset \{\sigma_1 \leq t\}$, and we are going to use the Markov property of $N_t(\mathcal{P}_n)$ at time σ_1 . Indeed, choosing $\mathcal{F}_1 = \sigma\{\sigma_1, R^1, X_{\sigma_1}^{R_1^1}, X_{\sigma_1}^{R_1^2}\}$, we may rewrite $J(t; (m_\rho, t_\ell, x_\ell), \ell \in \mathcal{L}_n)$ as

$$\begin{aligned} & \sum_{\rho \in \mathcal{P}_n: m_{\rho_1} \cdot m_{\rho_2} > 0} E_{(t_1, x_1), \dots, (t_n, x_n)} \left[\mathbb{1}_{\{\sigma_1 \leq t, R^1 = \rho\}} f(X_{\sigma_1}^{\rho_1} - X_{\sigma_1}^{\rho_2}) e^{tn(n-1)/2} \prod_{\ell \in \rho} S(\tau_1^\ell - \tau_0^\ell, \mathbb{R}^d) \right. \\ & \quad \times E_{(t_1, x_1), \dots, (t_n, x_n)} [\mathbb{1}_{\{N_t(\ell) - N_{\sigma_1}(\ell) = m_\ell, \ell \in \mathcal{L}_n \setminus \rho\} \cap \{N_t(\ell) - N_{\sigma_1}(\ell) = m_\ell - 1, \ell \in \rho\}} \prod_{\ell=2}^m f(X_{\sigma_i}^{R_i^1} - X_{\sigma_i}^{R_i^2}) \\ & \quad \left. \times \prod_{\ell \in \mathcal{L}_n \setminus \rho} \prod_{i=1}^{m_i} S(\tau_i^\ell - \tau_{i-1}^\ell, \mathbb{R}^d) \cdot \prod_{\ell \in \rho} \prod_{i=2}^{m_i} S(\tau_i^\ell - \tau_{i-1}^\ell, \mathbb{R}^d) \cdot \prod_{\ell \in \mathcal{L}_n} w(t - \tau_{m_\ell}^\ell, X_{\tau_{m_\ell}^\ell}^\ell) | \mathcal{F}_1 \right]. \end{aligned}$$

Note that at time σ_1 , on $\{R^1 = \rho\}$, the processes X^{ρ_i} start afresh from $X_{\sigma_1}^{\rho_i}$, $i = 1, 2$, while for $\ell \in \mathcal{L}_n \setminus \rho$, X^ℓ has seen no jump from $-t_\ell$ to σ_1 , that is for $\sigma_1 + t_\ell$ units of time. Using the strong Markov property at σ_1 , the conditional expectation above multiplied by $e^{(t-\sigma_1)n(n-1)/2}$ is equal to

$$J(t - \sigma_1; (m_{\rho_i} - 1, 0, X_{\sigma_1}^{\rho_i}), i = 1, 2; (m_\ell, \sigma_1 + t_\ell, x_\ell), \ell \in \mathcal{L}_n \setminus \rho).$$

Therefore, $J(t; (m_\ell, t_\ell, x_\ell), \ell \in \mathcal{L}_n)$ is equal to

$$\begin{aligned} \sum_{\rho \in \mathcal{P}_n: m_{\rho_1} \cdot m_{\rho_2} > 0} E_{(t_1, x_1), \dots, (t_n, x_n)} [1_{\{\sigma_1 \leq t, R^1 = \rho\}} e^{\sigma_1 n(n-1)/2} f(X_{\sigma_1}^{\rho_1} - X_{\sigma_1}^{\rho_2}) \prod_{\ell \in \rho} S(\tau_1^\ell - \tau_0^1, \mathbb{R}^d) \\ \times J(t - \sigma_1; (m_{\rho_i} - 1, 0, X_{\sigma_1}^{\rho_i}), i = 1, 2; (m_\ell, \sigma_1 + t_\ell, x_\ell), \ell \in \mathcal{L}_n \setminus \rho)]. \end{aligned}$$

The variable σ_1 is exponential with mean $2/(n(n-1))$ and the variable R is independent and uniformly distributed over \mathcal{P}_n . Taking the expectation over $\sigma_1, R^1, X_{\sigma_1}^{\rho_i}$ we reach (5.4). \square

Proof of Theorem 5.1. We note that it suffices to prove, when $t_i \geq t$, that

$$I(t; (m_1, t_1, x_1), \dots, (m_n, t_n, x_n)) = J(t; (m_1, t_1 - t, x_1), \dots, (m_n, t_n - t, x_n)). \quad (5.5)$$

Indeed, in this case, by Proposition 4.2,

$$\begin{aligned} E[u(t, x_1) \cdots u(t, x_n)] &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} E(I_{m_1}(t, t, x_1) \cdots I_{m_n}(t, t, x_n)) \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} I(t; (m_1, t, x_1), \dots, (m_n, t, x_n)) \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} J(t; (m_1, 0, x_1), \dots, (m_n, 0, x_n)), \end{aligned}$$

and this is equal to the expression in (5.1).

Let $m = m_1 + \cdots + m_n$. We are going to prove (5.5) by induction on m . If $m = 0$, then (5.5) follows from Lemma 5.2(a) and Lemma 5.3(a), since both sides of (5.5) are equal to $w(t_1, x_1) \cdots w(t_n, x_n)$. Now assume inductively that (5.5) holds for $m - 1 \geq 0$. By Lemma 5.3(b),

$$\begin{aligned} J(t; (m_\ell, t_\ell - t, x_\ell), \ell \in \mathcal{L}_n) \\ = \sum_{\rho \in \mathcal{P}_n: m_{\rho_1} \cdot m_{\rho_2} > 0} \int_0^t ds \int_{\mathbb{R}^d} S(t_{\rho_1} - t + s, dy_1) \int_{\mathbb{R}^d} S(t_{\rho_2} - t + s, dy_2) \\ \times f(x_{\rho_1} - y_1 - x_{\rho_2} + y_2) \\ \times J(t - s; (m_{\rho_i} - 1, 0, x_{\rho_i} - y_i), i = 1, 2; (m_\ell, s + t_\ell - t, x_\ell), \ell \in \mathcal{L}_n - \rho). \end{aligned} \quad (5.6)$$

By the induction hypothesis, the last factor $J(t - s; \dots)$ is equal to

$$I(t - s; (m_{\rho_i} - 1, t - s, x_{\rho_i} - y_i), i = 1, 2; (m_\ell, t_\ell, x_\ell), \ell \in \mathcal{L}_n \setminus \rho).$$

Now use the change of variables $r = t - s$ and Lemma 5.2(b) to see that the right-hand side of (5.6) is equal to $I(t; (m_\ell, t_\ell, x_\ell), \ell \in \mathcal{L}_n)$. This completes the proof. \square

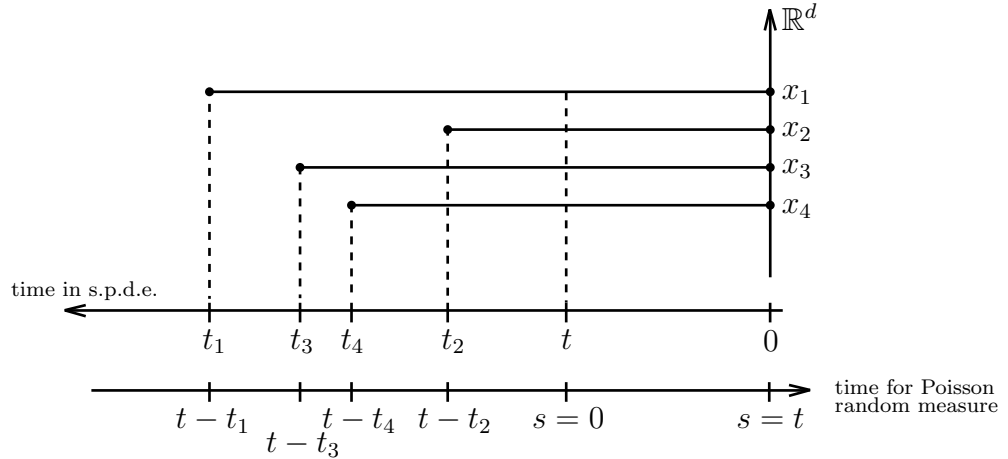


Figure 5: Illustration for equality (5.5), with the direction of time for the s.p.d.e. and for the Poisson random measure.

Remark 5.4. The intuition behind equality (5.5) is the following. Suppose $n = 4$ and consider space-time positions $(t_1, x_1), \dots, (t_4, x_4)$, as in Figure 5. The quantity $I(t; (m_\ell, t_\ell, x_\ell), \ell = 1, \dots, 4)$ is the expected product of iterated integrals, where the left-most integral is up to time $t \leq \min(t_1, \dots, t_4)$ and the order of the iterated integrals are m_1, \dots, m_4 .

On the other hand, time s for the Poisson random measure runs in the opposite direction as in the s.p.d.e. (see Figure 5). In the quantity $J(t, (m_\ell, t_\ell - t, x_\ell), \ell = 1, \dots, 4)$, the process X^ℓ starts at negative time $t - t_\ell$, and there are no Poisson pairs during negative time. During the time interval $s = 0$ to $s = t$, the number of Poisson pairs containing x_ℓ is set to m_ℓ . With these constraints, $I(t; (m_\ell, t_\ell, x_\ell), \ell = 1, \dots, 4) = J(t, (m_\ell, t_\ell - t, x_\ell), \ell = 1, \dots, 4)$ as stated in (5.5).

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