Criteria for hitting probabilities with applications to systems of stochastic wave equations

by

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Abstract: We develop several results on hitting probabilities of random fields which highlight the role of the dimension of the parameter space. This yields upper and lower bounds in terms of Hausdorff measure and Bessel-Riesz capacity, respectively. We apply these results to a system of stochastic wave equations in spatial dimension $k \geq 1$ driven by a d-dimensional spatially homogeneous additive Gaussian noise that is white in time and colored in space.

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1 Introduction

There have recently been several papers on hitting probabilities for systems of stochastic partial differential equations (SPDEs). The first seems to be [13], which studied mainly polarity of points for the Gaussian random field which is the solution of a system of linear heat equations in spatial dimension 1 driven by space-time white noise. Next, the paper [8] studied hitting probabilities for a nonlinear system of (reduced) wave equations in spatial dimension 1, and established upper and lower bounds on hitting probabilities in terms of Bessel-Riesz capacity.

The paper [4] considered a system of nonlinear heat equations in spatial dimension $k = 1$ with additive space-time white noise and established lower and upper bounds on the probability that the solution $(u(t, x),\ (t, x) \in \mathbb{R}_+ \times [0, 1])$ hits a set $A \subset \mathbb{R}^d$ in terms of capacity and Hausdorff measure, respectively. These results were extended to systems of the same heat equations but with multiplicative noise in [5]. The paper [6] extends these results to systems of non-linear heat equations in spatial dimensions $k \geq 1$, driven by spatially homogeneous noise that is white in time. Some other results on hitting probabilities for parabolic SPDEs with reflection are contained in the papers [21], [22], [7].

The objective of this paper is to begin a similar program for systems of stochastic wave equations, starting with the analogue of [4]. We note that properties of solutions of stochastic wave equations in spatial dimensions $k > 1$ are often much more difficult to obtain than their analogues for heat equations, due to the greater irregularity of the fundamental solution of the wave equation. One example of this is the study in [9] of Hölder continuity of sample paths for the 3-dimensional wave equation.

In [4], various conditions on the density of the random vector $(u(t, x),\ u(s, y))$ were identified that imply upper and lower bounds on hitting probabilities. The conditions were expressed using a "parabolic metric" and were designed to be applied to the stochastic heat equation driven by space-time white noise. They were applied there first to study the linear stochastic heat equation, and then the non-linear stochastic heat equation with additive noise was handled by appealing to Girsanov's theorem. Because of the absence of a suitable Girsanov's theorem for heat or wave equations in spatial dimensions $k > 1$ (a problem also noted in [6]), we will develop first some general results that will also be useful for nonlinear equations. In contrast with [4], these results are designed to be used for stochastic wave equations. We will apply them to linear wave equations in spatial dimension $k \geq 1$, driven by spatially homogeneous noise that is white in time. In work in progress, we intend to use these general results to study the non-linear stochastic wave equation with additive and/or multiplicative noise.

More precisely, we consider here the d–dimensional stochastic process $U = \{(u_i(t,x), i = 1,\dots,d), (t,x) \in [0,T] \times \mathbb{R}^k\}$ which solves the system of SPDEs

$$\frac{\partial^2 u_i}{\partial t^2}(t,x) - \Delta u_i(t,x) = \sum_{j=1}^{d} \sigma_{i,j} \dot{F}^j(t,x), \tag{1}$$

for $(t,x) \in]0,T] \times \mathbb{R}^k$, with initial conditions

$$u_i(0,x) = \frac{\partial u_i}{\partial t}(0,x) = 0. \tag{2}$$

Here, Δ denotes the Laplacian on $\mathbb{R}^k$, and $\sigma = (\sigma_{i,j})$ is a deterministic, invertible, $d \times d$ matrix. The noise process $\dot{F} := (\dot{F}^1, \dots, \dot{F}^d)$ is a centered (generalized) Gaussian process whose covariance is informally given by an expression such as

$$E(\dot{F}^i(t,x)\dot{F}^j(s,y) = \delta_{i,j}\, \delta(t-s)\, \|x-y\|^{-\beta}, \tag{3}$$

where $\delta_{i,j}$ denotes the Kronecker symbol, $\delta(\cdot)$ is the Dirac delta function at zero and $\beta > 0$. More precisely, let $C_0^\infty(\mathbb{R}^{k+1})$ denote the space of infinitely differentiable functions with compact support, and consider a family of centered Gaussian random vectors $F = (F(\varphi) = (F^1(\varphi), \dots, F^d(\varphi)), \varphi \in C_0^\infty(\mathbb{R}^{k+1})$, with covariance function

$$E(F(\varphi)F(\psi)) = \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^k} \Gamma(dx)\, (\varphi(t,\cdot) * \tilde{\psi}(t,\cdot))(x), \tag{4}$$

where $\tilde{\psi}(t,x) := \psi(t,-x)$ and Γ is a non-negative and non-negative definite tempered measure on $\mathbb{R}^k$. We note that (4) reduces to (3) if $\Gamma(dx) = \|x\|^{-\beta} dx$. By the Bochner-Schwartz theorem (see [17]), there exists a non-negative tempered measure μ on $\mathbb{R}^k$ (termed the *spectral measure* of F) such that $\Gamma = \mathcal{F}\mu$, where $\mathcal{F}$ denotes the Fourier transform. Elementary properties of the Fourier transform show that equation (4) can be written

$$E(F(\varphi)F(\psi)) = \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^k} \mu(d\xi)\, \mathcal{F}\varphi(t,\cdot)(\xi)\, \overline{\mathcal{F}\psi(t,\cdot)(\xi)}. \tag{5}$$

Let $G(t,x)$ be the fundamental solution of the wave equation. Generically, the solution u of (1) is given by

$$u_i(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-r, x-y) \sum_{j=1}^{d} \sigma_{i,j} M^j(dr,dy) \tag{6}$$

where $M = (M^1, \dots, M^d)$ is the martingale measure derived from $\dot{F}$ (see [3] for details). However, it is well-known that G is a function in dimensions $k \in \{1, 2\}$ only, so the stochastic integral in (6) should be interpreted in the sense of [2]. We note that according to (4) and (5),

$$E\left((u_i(t,x))^2\right) = \left(\sum_{j=1}^d \sigma_{i,j}^2\right) \int_0^t dr \int_{\mathbb{R}^k} \mu(d\xi)\ |\mathcal{F}G(t-r)(\xi)|^2,$$

and it is well-known (see [19]) that

$$\mathcal{F}G(t)(\xi) = \frac{\sin(t\|\xi\|)}{\|\xi\|}. \tag{7}$$

Following [2], [14], we note that when μ is not the null measure, the solution $u(t,x)$ of (1) is a random vector, and the right-hand side of (6) is well-defined, if and only if the following hypothesis is satisfied:

(H) $$0 < \int_{\mathbb{R}^k} \frac{\mu(d\xi)}{1+\|\xi\|^2} < \infty.$$

In this case, the process u given by (6) is a natural example of an anisotropic Gaussian process as considered in [20]. Notice that for the covariance density in (3), condition **(H)** is satisfied when $\beta \in \,]0, 2 \wedge k[$.

In Section 2 of this paper, we develop several results on hitting probabilities that are related to those of [4] but are appropriate for studying the wave equation in all spatial dimensions. Indeed, the results of this reference were tailored to the particularities of the heat equation in spatial dimension 1, while our results highlight the role of the spatial dimension and are applicable to the stochastic wave equation. Theorem 2.1 gives a lower bound on hitting probabilities, Proposition 2.3 and Theorem 2.4 give upper bounds. These three results apply to arbitrary stochastic processes, while Theorem 2.6 gives a refinement of the upper bound in the case of Gaussian processes. These results are used in Section 4, but will also be useful for studying nonlinear forms of (1), which is currently work in progress.

In Section 3, we give simple conditions on a Gaussian process $(X(t))$ that ensure an upper bound on the density function of $(X(t), X(s))$. This is related to a result in [8]. The upper bound is expressed in terms of the canonical metric of the Gaussian process.

In Section 4, the main effort is to obtain upper and lower bounds on the behavior of the canonical metric associated with the process u (Proposition 4.1). This is somewhat intricate for the lower bounds, mainly because the expression for $E((u(t,x) - u(s,y))^2)$ involves integrals of trigonometric functions, and these are not so easy to bound from below by positive quantities.

Section 4 ends with applying the results of Sections 2 and 3 on hitting probabilities to obtain Theorems 4.4 and 4.5. These yield the following type of bounds:

$$c\,\mathrm{Cap}_{d-\frac{2(k+1)}{2-\beta}}(A) \leq P\{u([t_0,T]\times[-M,M]^k)\cap A \neq \emptyset\} \leq C\,\mathcal{H}_{d-\frac{2(k+1)}{2-\beta}}(A), \quad (8)$$

where Cap_γ and $\mathcal{H}_\gamma$ denote capacity and Hausdorff measure, respectively (their definitions are recalled in Section 2). We note that the same dimensions appear on both the left- and right-hand sides of (8). This conclusion could also have been deduced from Theorem 7.6 in [20] or Theorem 2.1 in [1], which contain general results on hitting probabilities for anisotropic Gaussian processes. This is because our estimates on the canonical metric of u mentioned above, together with our Lemma 3.2, verify Conditions (C1) and (C2) in these two references. We note also that these estimates hint at the fact that condition (C3') of [20] should be satisfied by u.

We recall that a point $z \in \mathbb{R}^d$ is *polar* for u if for all $t_0 > 0$ and $M > 0$,

$$P\{z \in u([t_0,T]\times[-M,M]^k)\} = 0.$$

Notice, as a consequence of (8), that if $d < 2(k+1)/(2-\beta)$, then points *are not* polar for u, while if $d > 2(k+1)/(2-\beta)$, then points *are* polar for u. In the case where β is rational and $2(k+1)/(2-\beta) = d$ is an integer, then polarity of points in the critical dimension d is an open problem.

As mentioned above, in work in progress, we plan to extend these results to systems of nonlinear stochastic wave equations with additive noise but without using Girsanov's theorem. It is a separate endeavor to develop the estimates from Malliavin calculus needed for multiplicative noise, as was done in [5] for the heat equation, and which will also make use of the results in Section 2.

2 General results on hitting probabilities

Throughout this section, $V = \{v(x),\ x \in \mathbb{R}^m\}$, $m \in \mathbb{N}^*$, denotes an $\mathbb{R}^d$-valued stochastic process with continuous sample paths. We will fix a compact set $I \subset \mathbb{R}^m$ of positive Lebesgue measure and consider an arbitrary Borel set $A \subset \mathbb{R}^d$. Our aim is to give sufficient conditions on the stochastic process V which lead to lower and upper bounds on the hitting probabilities

$$P\{v(I) \cap A \neq \emptyset\}$$

in terms of the capacity and the Hausdorff measure of A, respectively, of a certain dimension. Here, $v(I)$ denotes the image of I under the (random) map $x \mapsto v(x)$.

We now introduce some notation and recall the definition of capacity and Hausdorff measure. For any $\gamma \in \mathbb{R}$, we define the *Bessel-Riesz kernels* by

$$K_\gamma(r) = \begin{cases} r^{-\gamma} & \text{if } \gamma > 0, \\ \log\left(\frac{c}{r}\right) & \text{if } \gamma = 0, \\ 1 & \text{if } \gamma < 0, \end{cases} \tag{9}$$

where c is a constant whose value will be specified later in the proof of Lemma 2.2. Then, for every Borel set $A \subset \mathbb{R}^d$, we define $\mathcal{P}(A)$ to be the set of probability measures on A. For $\mu \in \mathcal{P}(A)$, we set

$$\mathcal{E}_\gamma(\mu) = \int_A \int_A K_\gamma(\|x-y\|)\mu(dx)\mu(dy).$$

The *Bessel-Riesz capacity* of a Borel set $A \subset \mathbb{R}^d$ is defined as follows:

$$\mathrm{Cap}_\gamma(A) = \left[\inf_{\mu \in \mathcal{P}(A)} \mathcal{E}_\gamma(\mu)\right]^{-1}, \tag{10}$$

with the convention that $1/\infty = 0$.

The γ-dimensional *Hausdorff measure* of a Borel set $A \subset \mathbb{R}^d$ is defined by $\mathcal{H}_\gamma(A) = \infty$ if $\gamma < 0$, and for $\gamma \geq 0$,

$$\mathcal{H}_\gamma(A) = \liminf_{\varepsilon \to 0^+} \left\{ \sum_{i=1}^{\infty} (2r_i)^\gamma : A \subset \cup_{i=1}^{\infty} B_{r_i}(x_i), \sup_{i \geq 1} r_i \leq \varepsilon \right\}. \tag{11}$$

Here and throughout the paper, $B_r(x)$ denotes the open Euclidean ball centered at x and with radius r. Positive constants will be denoted most often by C or c, although their value may change from one line to the next. For a given subset $S \subset \mathbb{R}^n$ and $\nu > 0$, we denote by $S^{(\nu)}$ the ν-enlargement of S.

We begin by studying the lower bound for $P\{v(I) \cap A \neq \emptyset\}$.

Theorem 2.1 *Fix $N > 0$ and assume that the stochastic process V satisfies the following two hypotheses:*

(1) For any $x, y \in I$ with $x \neq y$, the vector $(v(x), v(y))$ has a density $p_{x,y}$, and there exist $\gamma, \alpha \in \,]0, \infty[$ such that

$$p_{x,y}(z_1, z_2) \leq C \frac{1}{\|x-y\|^\gamma} \exp\left(-\frac{c\|z_1 - z_2\|^2}{\|x-y\|^\alpha}\right),$$

for any $z_1, z_2 \in [-N, N]^d$, where C and c are positive constants independent of x and y.

(2) One of the next two conditions holds:

(P) The density p_x of $v(x)$ is continuous, bounded, and $p_x(w) > 0$ for any $x \in I$ and $w \in [-(N+1), N+1]^d$.

(P') For any compact set $K \subset \mathbb{R}^d$ and for any $x \in I$, $\inf_{w \in K} p_x(w) \geq c_0 > 0$.

Then there exists a positive and finite constant $c = c(N, \alpha, \gamma, I, m)$ such that, for all Borel sets $A \subset [-N, N]^d$,

$$P\{v(I) \cap A \neq \emptyset\} \geq c \ \mathrm{Cap}_{\frac{2}{\alpha}(\gamma - m)}(A). \tag{12}$$

Proof. Without loss of generality, we may assume that $\mathrm{Cap}_{\frac{2}{\alpha}(\gamma-m)}(A) > 0$, otherwise there is nothing to prove. Under this assumption, we necessarily have $\frac{2}{\alpha}(\gamma - m) < d$ and $A \neq \emptyset$ (see [12], Appendix C, Corollary 2.3.1, p. 525).

Assume first that A is a compact set. Following the scheme of the proof of Theorem 2.1 in [4], we consider three different cases.

Case 1: $\gamma - m < 0$. Let $z \in A$, $\varepsilon \in]0, 1[$, and set

$$J_\varepsilon(z) = \frac{1}{(2\varepsilon)^d} \int_I dx \ \mathbf{1}_{B_\varepsilon(z)}(v(x)).$$

We will prove that $E(J_\varepsilon(z)) \geq c_1$ and $E[(J_\varepsilon(z))^2] \leq c_2$ for some positive constants c_1, c_2. With this, by using the Paley-Zygmund inequality ([12], Chapter 3, Lemma 1.4.1), and noticing that $\mathrm{Cap}_\beta(A) = 1$ for $\beta < 0$, we obtain

$$\begin{aligned} P\{J_\varepsilon(z) > 0\} &\geq \frac{[E(J_\varepsilon(z))]^2}{E[(J_\varepsilon(z))^2]} \geq C \\ &= C \ \mathrm{Cap}_{\frac{2}{\alpha}(\gamma-m)}(A). \end{aligned}$$

But $P\{J_\varepsilon(z) > 0\}$ is bounded above by $P\{v(I) \cap A^{(\varepsilon)} \neq \emptyset\}$. Since A is compact and the trajectories of v are continuous, by letting ε tend to 0, we obtain (12).

The lower bound for $E(J_\varepsilon(z))$ is a direct consequence of assumption (2). To obtain the upper bound for $E[(J_\varepsilon(z))^2]$, we first use the hypothesis (1) to obtain

$$E[(J_\varepsilon(z))^2] \leq C \int_I dx \int_I dy \frac{1}{\|x - y\|^\gamma}.$$

Let $\rho_0 > 0$ be such that $I \subset B_{\frac{\rho_0}{2}}(0)$. Fix $x \in I$; after the change of variables $y \to x - y$ and by considering polar coordinates, we easily get

$$E[(J_\varepsilon(z))^2] \leq C \int_0^{\rho_0} \rho^{m-1-\gamma} d\rho.$$

The last integral is bounded by a finite positive constant $c(m,\gamma,I)$ Therefore we obtain $E[(J_\varepsilon(z))^2] \leq c_2$.

Case 2: $0 < \frac{2}{\alpha}(\gamma - m) < d$. Let $\mu \in \mathcal{P}(A)$. Let $g_\varepsilon = \frac{1}{(2\varepsilon)^d}\mathbf{1}_{B_\varepsilon(0)}$ and

$$J_\varepsilon(\mu) = \frac{1}{(2\varepsilon)^d}\int_I dx \int_A \mu(dz)\ \mathbf{1}_{B_\varepsilon(0)}\left(v(x) - z\right) = \int_I dx\left(g_\varepsilon * \mu\right)\left(v(x)\right).$$

Clearly, assumption (2) implies that $E\left(J_\varepsilon(\mu)\right) \geq c_1$, for a constant c_1 which does not depend on μ or ε. Moreover,

$$\begin{aligned} E[(J_\varepsilon(\mu))^2] = &\int_I dx \int_I dy \int_{\mathbb{R}^d} dz_1 \int_{\mathbb{R}^d} dz_2 \\ &\times \left(g_\varepsilon * \mu\right)(z_1)\left(g_\varepsilon * \mu\right)(z_2) p_{x,y}(z_1, z_2). \end{aligned}$$

By hypothesis (1), Lemma 2.2 below and Theorem B.1 in [4], this is bounded by

$$\begin{aligned} &C \int_{\mathbb{R}^d} dz_1 \int_{\mathbb{R}^d} dz_2 \left(g_\varepsilon * \mu\right)(z_1)\left(g_\varepsilon * \mu\right)(z_2) K_{\frac{2}{\alpha}(\gamma-m)}(\|z_1 - z_2\|) \\ &= C\ \mathcal{E}_{\frac{2}{\alpha}(\gamma-m)}\left(g_\varepsilon * \mu\right) \\ &\leq C\ \mathcal{E}_{\frac{2}{\alpha}(\gamma-m)}(\mu). \end{aligned}$$

By choosing μ such that $\mathcal{E}_{\frac{2}{\alpha}(\gamma-m)}(\mu) \leq 2/\mathrm{Cap}_{\frac{2}{\alpha}(\gamma-m)}(A)$, we obtain

$$E[(J_\varepsilon(\mu))^2] \leq \frac{C}{\mathrm{Cap}_{\frac{2}{\alpha}(\gamma-m)}(A)},$$

and this yields (12) by a similar argument as in Case 1.

Case 3: $\gamma - m = 0$. The proof is done exactly in the same way as for Case 2, by applying Theorem B.2 in [4] instead of Theorem B.1.

Now let A be a Borel set included in $[-N, N]^d$. It is well-known that

$$\mathrm{Cap}_\beta(A) = \sup_{F \subset A,\ F \text{ compact}} \mathrm{Cap}_\beta(F), \tag{13}$$

(see for instance [10], Chapter 3). Therefore, for any compact set $F \subset A$, we have

$$P\{v(I) \cap A \neq \emptyset\} \geq P\{v(I) \cap F \neq \emptyset\} \geq c\ \mathrm{Cap}_{\frac{2}{\alpha}(\gamma-m)}(F).$$

This yields (12) by taking the supremum over such F and using (13).

The proof of the theorem is complete. □

In order to end the study of the lower bounds, we prove a technical lemma which was used in the proof of Theorem 2.1 to relate joint densities with Bessel-Riesz kernels.

Lemma 2.2 *Fix $\alpha, \gamma \in]0, \infty[$. There exists a constant $C := C(N, \alpha, \gamma, I, m)$ such that for any $a \in]-N, N[$,*

$$\int_I dx \int_I dy \frac{1}{\|x-y\|^\gamma} \exp\left(-\frac{a^2}{\|x-y\|^\alpha}\right) \le C K_{\frac{2}{\alpha}(\gamma-m)}(a). \tag{14}$$

Proof. Fix $\rho_0 > 0$ such that $I \subset B_{\frac{\rho_0}{2}}(0)$. Fix $x \in I$ and consider the change of variables $z = a^{-\frac{2}{\alpha}}(x-y)$. Denoting by $\mathcal{I}$ the left-hand side of (14), we have

$$\mathcal{I} \le C(I)\ a^{-\frac{2}{\alpha}(\gamma-m)} \int_{B_{\frac{\rho_0}{a^{2/\alpha}}}(0)} dz\ \frac{1}{\|z\|^\gamma} \exp\left(-\frac{1}{\|z\|^\alpha}\right).$$

Let

$$\mathcal{J} = \int_{B_{\frac{\rho_0}{a^{2/\alpha}}}(0)} dz\ \frac{1}{\|z\|^\gamma} \exp\left(-\frac{1}{\|z\|^\alpha}\right).$$

Using polar coordinates, we have $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$, with

$$\mathcal{J}_1 = \int_0^{\frac{\rho_0}{N^{2/\alpha}}} d\rho\ \rho^{m-1-\gamma} \exp\left(-\frac{1}{\rho^\alpha}\right),$$
$$\mathcal{J}_2 = \int_{\frac{\rho_0}{N^{2/\alpha}}}^{\frac{\rho_0}{a^{2/\alpha}}} d\rho\ \rho^{m-1-\gamma} \exp\left(-\frac{1}{\rho^\alpha}\right).$$

Clearly $\mathcal{J}_1 \le C(\rho_0, N)$. In order to study $\mathcal{J}_2$, we bound the exponential by 1 and we consider three different cases.

Case 1: If $m - \gamma < 0$, then

$$\mathcal{J}_2 \le (\gamma - m)^{-1} \left(\frac{\rho_0}{N^{\frac{2}{\alpha}}}\right)^{m-\gamma} \le C(N, \alpha, \gamma, \rho_0, m).$$

Case 2: If $m - \gamma > 0$, then

$$\mathcal{J}_2 \le (m - \gamma)^{-1} \left(\frac{\rho_0}{a^{\frac{2}{\alpha}}}\right)^{m-\gamma} \le C(\gamma, \rho_0, m) a^{\frac{2}{\alpha}(\gamma-m)}.$$

Case 3: If $m - \gamma = 0$, then

$$\mathcal{J}_2 \le \frac{2}{\alpha} \log\left(\frac{N}{a}\right).$$

Since $\mathcal{I} \le C(I) a^{-\frac{2}{\alpha}(\gamma-m)} \mathcal{J}$, we reach the conclusion using the definition of $K_\beta(a)$ for $\beta, a \in \mathbb{R}$ (see (9); in the case where $m - \gamma = 0$, the constant c in (9) must be chosen sufficiently large). $\square$

We now study upper bounds for the hitting probabilities. For this, we fix $\delta > 0$, $\varepsilon \in\,]0,1[$, $j_1, \dots, j_m \in \mathbb{Z}$, and we set $j = (j_1, \dots, j_m)$ and

$$R_j^\varepsilon = \prod_{l=1}^{m} \left[j_l \varepsilon^{\frac{1}{\delta}}, (j_l+1)\varepsilon^{\frac{1}{\delta}} \right]. \tag{15}$$

The next statement is an extension to higher dimensions of Theorem 3.1 in [4].

Proposition 2.3 *Let $D \subset \mathbb{R}^d$ and $\gamma > 0$. We assume that there exists a positive constant c such that, for all small $\varepsilon \in\,]0,1[$, $z \in D^{(1)}$, and any set R_j^ε such that $R_j^\varepsilon \cap I \neq \emptyset$,*

$$P\left\{ v(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset \right\} \le c\, \varepsilon^\gamma. \tag{16}$$

Then, there exists a positive constant C such that, for any Borel set $A \subset D$,

$$P\left\{ v(I) \cap A \neq \emptyset \right\} \le C \mathcal{H}_{\gamma - \frac{m}{\delta}}(A). \tag{17}$$

Proof. We suppose $\gamma - \frac{m}{\delta} \ge 0$, otherwise $\mathcal{H}_{\gamma-\frac{m}{\delta}}(A) = \infty$ and therefore (17) obviously holds. Clearly, by the additive property of probability,

$$P\left\{ v(I) \cap B_\varepsilon(z) \neq \emptyset \right\} \le \sum_{j : R_j^\varepsilon \cap I \neq \emptyset} P\left\{ v(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset \right\},$$

for any $\varepsilon > 0$. Since I is bounded, the number of terms in the sum on the right-hand side of this inequality is bounded by a multiple of $\varepsilon^{-\frac{m}{\delta}}$. Hence

$$P\left\{ v(I) \cap B_\varepsilon(z) \neq \emptyset \right\} \le C \varepsilon^{-\frac{m}{\delta}} P\left\{ v(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset \right\}.$$

Then, using (16), we obtain

$$P\left\{ v(I) \cap B_\varepsilon(z) \neq \emptyset \right\} \le C \varepsilon^{\gamma - \frac{m}{\delta}}. \tag{18}$$

This yields (17) by a *covering argument*, as it is shown in the proof of Theorem 3.1 in [4]. For the sake of completeness we sketch this argument.

Fix $\varepsilon \in\,]0,1[$ sufficiently small and consider a sequence of open balls $(B_n,\ n \ge 1)$ with respective radii $r_n \in\,]0, \varepsilon]$, such that $B_n \cap A \neq \emptyset$, $A \subset \cup_{n\ge1} B_n$ and

$$\sum_{n\ge1} (2r_n)^{\gamma - \frac{m}{\delta}} \le \mathcal{H}_{\gamma-\frac{m}{\delta}}(A) + \varepsilon.$$

Then, by (18),

$$\begin{aligned} P\left\{v(I) \cap A \neq \emptyset\right\} &\leq \sum_{n \geq 1} P\left\{v(I) \cap B_n \neq \emptyset\right\} \\ &\leq C \sum_{n \geq 1} (2r_n)^{\gamma - \frac{m}{\delta}} \\ &\leq C \left(\mathcal{H}_{\gamma - \frac{m}{\delta}}(A) + \varepsilon\right). \end{aligned}$$

Finally, we let $\varepsilon \downarrow 0$ to conclude. □

In the next theorem, we give sufficient conditions on the process V for the assumptions of Proposition 2.3 to be satisfied and therefore to ensure (17).

Theorem 2.4 *Let $D \subset \mathbb{R}^d$. Assume that the stochastic process V satisfies the following two conditions:*

(1) For any $x \in \mathbb{R}^m$, the random vector $v(x)$ has a density p_x, and

$$\sup_{z \in D^{(2)}} \sup_{x \in I^{(1)}} p_x(z) \leq C.$$

(2) There exists $\delta \in]0, 1]$ and a constant C such that, for any $q \in [1, \infty[$, $x, y \in I^{(1)}$,

$$E\left(\|v(x) - v(y)\|^q\right) \leq C\|x - y\|^{q\delta}.$$

Then for any $\gamma \in]0, d[$, the inequality (16) holds, and consequently, for every Borel set $A \subset D$,

$$P\left\{v(I) \cap A \neq \emptyset\right\} \leq C\mathcal{H}_{\gamma - \frac{m}{\delta}}(A). \tag{19}$$

Proof. We keep the notations of Proposition 2.3 and write $x_j^\varepsilon = (j_l \varepsilon^{\frac{1}{\delta}}, l = 1, \ldots, m)$. For any $z \in D^{(1)}$ and R_j^ε such that $R_j^\varepsilon \cap I \neq \emptyset$, set

$$Y_j^\varepsilon = \|v(x_j^\varepsilon) - z\|, \ Z_j^\varepsilon = \sup_{x \in R_j^\varepsilon} \|v(x) - v(x_j^\varepsilon)\|.$$

By applying the version of Kolmogorov's criterion as it is stated in [15], Theorem 2.1, page 26, using assumption (2), we obtain

$$E\left(\left(Z_j^\varepsilon\right)^q\right) \leq C\|x - x_j^\varepsilon\|^{\alpha q},$$

for any $q \in [1, \infty[$ and $\alpha \in]0, \delta - \frac{m}{q}[$. Hence

$$E\left(\left(Z_j^\varepsilon\right)^q\right) \leq C\varepsilon^{\gamma_0 q}, \tag{20}$$

with $\gamma_0 < 1 - \frac{m}{q\delta}$.

Let $\gamma \in]0, d[$. We first prove that

$$P\{Z_j^\varepsilon \geq \frac{1}{2} Y_j^\varepsilon\} \leq C\varepsilon^\gamma. \tag{21}$$

For this, we consider the decomposition

$$P\{Z_j^\varepsilon \geq \frac{1}{2} Y_j^\varepsilon\} \leq P\{Y_j^\varepsilon \leq \varepsilon^{\frac{\gamma}{d}}\} + P\{Z_j^\varepsilon \geq \frac{1}{2}\varepsilon^{\frac{\gamma}{d}}\},$$

and then give upper bounds for each term on the right-hand side.

Clearly, from the boundedness of the density stated in assumption (1),

$$P\{Y_j^\varepsilon \leq \varepsilon^{\frac{\gamma}{d}}\} \leq C\varepsilon^\gamma,$$

and by Markov's inequality along with (20),

$$P\{Z_j^\varepsilon \geq \frac{1}{2}\varepsilon^{\frac{\gamma}{d}}\} \leq C\varepsilon^{q(\gamma_0 - \frac{\gamma}{d})}.$$

Therefore,

$$P\{Z_j^\varepsilon \geq \frac{1}{2} Y_j^\varepsilon\} \leq C\left(\varepsilon^\gamma + \varepsilon^{q(\gamma_0 - \frac{\gamma}{d})}\right),$$

for any $\gamma_0 < 1 - \frac{m}{q\delta}$. Since $\gamma \in]0, d[$, we can choose $\gamma_0 < 1$ and q arbitrarily large such that $\frac{\gamma}{d} < \gamma_0 < 1 - \frac{m}{q\delta}$. Hence, we obtain (21).

If $v(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset$, then $Y_j^\varepsilon \leq \varepsilon + Z_j^\varepsilon$. Therefore,

$$\begin{aligned} P\{v(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset\} &\leq P\{Y_j^\varepsilon \leq \varepsilon + Z_j^\varepsilon\} \\ &\leq P\{Z_j^\varepsilon \geq \frac{1}{2} Y_j^\varepsilon\} + P\{Y_j^\varepsilon \leq 2\varepsilon\} \\ &\leq C\left(\varepsilon^\gamma + \varepsilon^d\right) \\ &\leq C\varepsilon^\gamma, \end{aligned}$$

since $\gamma \in]0, d[$. This proves (16) for any $\gamma \in]0, d[$. According to Proposition 2.3, we obtain (19). $\square$

The remainder of this section is devoted to extending the validity of (19) to $\gamma = d$ in the case where V belongs to a particular class of Gaussian processes. For this class, we will prove that, instead of (20), the following stronger property holds:

For any $\varepsilon \in]0,1[$, for each $j \in \mathbb{Z}^m$ with $R_j^\varepsilon \cap I \neq \emptyset$, and every $q \in [1,\infty[$, there is $C > 0$ such that

$$E\left(\sup_{x\in R_j^\varepsilon} \|v(x) - v(x_j^\varepsilon)\|^q\right) \leq C\varepsilon^q. \tag{22}$$

Then, we will show that $P\{v(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset\} \leq C\varepsilon^d$ (see Theorem 2.6 below). Together with Proposition 2.3, this will yield the desired improvement.

We first give a sufficient condition which applies to arbitrary continuous stochastic processes V.

Lemma 2.5 *Let $\nu \in]0,1]$. Suppose that for any $\varepsilon \in]0,1[$ sufficiently small,*

$$E\left(\int_{B_\varepsilon(x)} dy \int_{B_\varepsilon(x)} d\bar{y} \left[\exp\left\{\frac{\|v(y) - v(\bar{y})\|}{\|y-\bar{y}\|^\nu}\right\}\right]\right) \leq C\varepsilon^{2m}, \tag{23}$$

where C is a positive constant. Let $S_\varepsilon^\nu(x) = \{y \in \mathbb{R}^m : \|x-y\| \leq \varepsilon^{\frac{1}{\nu}}\}$.

Then, for any $q \in [1,\infty[$, there exists $\bar{C} > 0$ such that for all small $\varepsilon > 0$,

$$E\left(\sup_{y\in S_\varepsilon^\nu(x)} \|v(x) - v(y)\|^q\right) \leq \bar{C}\varepsilon^q. \tag{24}$$

Proof. By (23), $B(\omega) < \infty$ a.s., where

$$B(\omega) = \int_{S_\varepsilon^\nu(x)} dy \int_{S_\varepsilon^\nu(x)} d\bar{y} \left[\exp\left\{\frac{\|v(y)(\omega) - v(\bar{y})(\omega)\|}{\|y-\bar{y}\|^\nu}\right\}\right].$$

We apply the Garsia-Rodemich-Rumsey lemma (cf. [18], exercise 2.4.1, pg. 60) to the functions $\psi(x) = e^x - 1$, $p(x) = x^\nu$ and functions $f : S_\varepsilon^\nu(x) \subset \mathbb{R}^m \to \mathbb{R}^d$ given by the sample paths of the process V restricted to the parameter set $S_\varepsilon^\nu(x)$, to obtain

$$\|v(x) - v(y)\| \leq 8 \int_0^{2\|x-y\|} \psi^{-1}\left(\frac{C_1 B(\omega)}{u^{2m}}\right) \nu u^{\nu-1} du,$$

where C_1 is a positive constant which depends only on m. Consequently, for any $q \in [1,\infty[$,

$$E\left(\sup_{y\in S_\varepsilon^\nu(x)} \|v(x) - v(y)\|^q\right) \leq 8E\left(\int_0^{2\varepsilon^{\frac{1}{\nu}}} \psi^{-1}\left(\frac{C_1 B(\omega)}{u^{2m}}\right) \nu u^{\nu-1} du.\right)^q.$$

We notice that since $\psi^{-1}(x) = \ln(1+x)$ is an increasing function on $[0,\infty)$, the constant C_1 above can be taken arbitrarily large. In the sequel, we will fix $q \in [1, \infty[$ and we will assume that $C_1 \geq (e^{q-1} - 1)\, C_2^{-1} 2^{2m}$, where C_2 is the square of the volume of the unit ball in $\mathbb{R}^m$. Then

$$B(\omega) \geq C_2 \varepsilon^{\frac{2m}{\nu}} \geq \frac{e^{q-1} - 1}{C_1} u^{2m},$$

for any $u \in [0, 2\varepsilon^{\frac{1}{\nu}}]$.

Jensen's inequality applied first to the convex function $\varphi_1(x) = x^q$, $x \in \mathbb{R}$ and the integral with respect to the measure $\mu(du) = u^{\nu-1} du$, and then to the concave function $\varphi_2(x) = \ln^q(1+x)$, $x \in [e^{q-1} - 1, \infty[$ and to the expectation operator, yield

$$\begin{aligned} E\left(\sup_{y \in S_\varepsilon^\nu(x)} \|v(x) - v(y)\|^q \right) &\leq 8\varepsilon^{q-1} \\ &\quad \times \int_0^{2\varepsilon^{\frac{1}{\nu}}} E\left[\ln^q \left(1 + \frac{C_1 B(\omega)}{u^{2m}} \right) \right] u^{\nu-1} du \\ &\leq C\varepsilon^{q-1} \int_0^{2\varepsilon^{\frac{1}{\nu}}} \ln^q \left(1 + \frac{C_3 \varepsilon^{\frac{2m}{\nu}}}{u^{2m}} \right) \nu u^{\nu-1} du, \end{aligned}$$

with $C_3 = C_1 C$. With the change of variable $u \to \frac{u^\nu}{\varepsilon}$, we have

$$\begin{aligned} \int_0^{2\varepsilon^{\frac{1}{\nu}}} \ln^q \left(1 + \frac{C_3 \varepsilon^{\frac{2\mu}{\nu}}}{u^{2m}} \right) \nu u^{\nu-1} du &= \varepsilon \int_0^{2^\nu} \ln^q \left(1 + \frac{C_3}{w^{\frac{2m}{\nu}}} \right) dw \\ &= \bar{C}\varepsilon. \end{aligned}$$

This proves (24). $\square$

We can now sharpen the result of Theorem 2.4 in the case of Gaussian processes.

Theorem 2.6 *Assume that the stochastic process $V = \{v(x),\ x \in \mathbb{R}^m\}$ is continuous, Gaussian, centered, with independent, identically distributed components $\{v_i(x),\ x \in \mathbb{R}^m\}$, $i = 1, \ldots, d$, and $\inf_{x \in I^{(1)}} \mathrm{Var}(v_1(x)) > 0$. Fix $\delta \in]0, 1]$ and suppose that for any $\varepsilon > 0$ small enough and any R_j^ε (defined in (15)) such that $R_j^\varepsilon \cap I \neq \emptyset$,*

$$E\left(\int_{R_j^\varepsilon} dy \int_{R_j^\varepsilon} d\bar{y} \left[\exp\left\{ \frac{\|v(y) - v(\bar{y})\|}{\|y - \bar{y}\|^\delta} \right\} \right] \right) \leq C\varepsilon^{\frac{2m}{\delta}}. \tag{25}$$

Then for every $z \in \mathbb{R}^d$ and R_j^ε as before,

$$P\{v(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset\} \leq C\varepsilon^d. \tag{26}$$

Consequently, for any Borel set $A \subset \mathbb{R}^d$,

$$P\{v(I) \cap A \neq \emptyset\} \leq C\mathcal{H}_{d-\frac{m}{\delta}}(A). \tag{27}$$

Proof. By Lemma 2.5, assumption (25) implies (22). We use this property and adapt the proof of Proposition 4.4 of [4]. First, for any $z \in \mathbb{R}^d$, we write

$$P\{v(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset\} = P\left\{\inf_{x\in R_j^\varepsilon} \|v(x) - z\| \leq \varepsilon\right\}.$$

Next, we write the condition $\|v(x) - z\| \leq \varepsilon$ in terms of two independent random variables, as follows. Set

$$c_j^\varepsilon(x) = \frac{E\left(v_1(x)v_1(x_j^\varepsilon)\right)}{\mathrm{Var}(v_1(x_j^\varepsilon))}.$$

Because V is a Gaussian process,

$$E\left(v(x)|v(x_j^\varepsilon)\right) = c_j^\varepsilon(x)v(x_j^\varepsilon).$$

Set

$$Y_j^\varepsilon = \inf_{x\in R_j^\varepsilon} \|c_j^\varepsilon(x)v(x_j^\varepsilon) - z\|, \qquad Z_j^\varepsilon = \sup_{x\in R_j^\varepsilon} \|v(x) - c_j^\varepsilon(x)v(x_j^\varepsilon)\|.$$

Again because V is a Gaussian process, these two random variables are independent, and

$$P\{\inf_{x\in R_j^\varepsilon} \|v(x) - z\| \leq \varepsilon\} \leq P\{Y_j^\varepsilon \leq \varepsilon + Z_j^\varepsilon\}. \tag{28}$$

Our next aim is to prove that for any $r \geq 0$,

$$P(Y_j^\varepsilon \leq r) \leq Cr^d, \tag{29}$$

For this, we first notice that by independence of the components of V,

$$P(Y_j^\varepsilon \leq r) \leq \prod_{i=1}^d P(G_{j,i}^\varepsilon),$$

where

$$G_{j,i}^\varepsilon = \{\inf_{x\in R_j^\varepsilon} |c_j^\varepsilon(x)v_i(x_j^\varepsilon) - z_i| \leq r\}.$$

By setting $e_j^\varepsilon = \inf_{x\in R_j^\varepsilon} c_j^\varepsilon(x)$, we have

$$P(G_{j,i}^\varepsilon) \leq P\left(v_i(x_j^\varepsilon) \in B_{\frac{r}{e_j^\varepsilon}}(z)\right).$$

Since V is centered and $\inf_{x\in I^{(1)}} \mathrm{Var}(v_1(x)) > 0$ by hypothesis, Schwarz's inequality and (22) yield

$$
\begin{aligned}
|c_j^\varepsilon(x) - 1| &= \frac{\left|E\left[v_1(x_j^\varepsilon)\left(v_1(x) - v_1(x_j^\varepsilon)\right)\right]\right|}{\mathrm{Var}(v_1(x_j^\varepsilon))} \\
&\leq C\left(\frac{E\left(\left[v_1(x) - v_1(x_j^\varepsilon)\right]^2\right)}{\mathrm{Var}(v_1(x_j^\varepsilon))}\right)^{\frac{1}{2}} \\
&\leq C\varepsilon, \qquad (30)
\end{aligned}
$$

for any $x \in R_j^\varepsilon$. This implies $\frac{r}{e_j^\varepsilon} \leq Cr$, and since the density of $v_i(x_j^\varepsilon)$ is bounded, we get

$$
P\left(v_i(x_j^\varepsilon) \in B_{\frac{r}{e_j^\varepsilon}}(z)\right) \leq Cr,
$$

and therefore (29) holds.

By (29) and the independence of Y_j^ε and Z_j^ε,

$$
P\{Y_j^\varepsilon \leq \varepsilon + Z_j^\varepsilon\} \leq CE\left[\left(\varepsilon + Z_j^\varepsilon\right)^d\right].
$$

Consider the decomposition $Z_j^\varepsilon = Z_j^{\varepsilon,1} + Z_j^{\varepsilon,2}$, where

$$
Z_j^{\varepsilon,1} = \sup_{x\in R_j^\varepsilon} \|v(x) - v(x_j^\varepsilon)\|, \qquad Z_j^{\varepsilon,2} = \|v(x_j^\varepsilon)\| \sup_{x\in R_j^\varepsilon} |1 - c_j^\varepsilon(x)|.
$$

By (22), we have $E\left(\left|Z_j^{\varepsilon,1}\right|^d\right) \leq C\varepsilon^d$. Moreover, by (30) and (25),

$$
E\left(\left\|Z_j^{\varepsilon,2}\right\|^d\right) \leq C\varepsilon^d E\left(\|v(x_j^\varepsilon)\|^d\right) \leq C\varepsilon^d.
$$

This completes the proof of (26). Finally, (27) follows from Proposition 2.3.

□

3 Joint densities of Gaussian processes

Consider a Gaussian family of centered, $\mathbb{R}^d$–valued random vectors, indexed by a compact metric space $(\mathbb{T}, d)$, that we denote by $X = (X_t,\ t \in \mathbb{T})$. We suppose that the component processes $(X_t^i,\ t \in \mathbb{T})$, $i = 1, \ldots, d$, are independent. We also assume mean-square continuity, that is, by letting

$$
\delta(s,t) = \left(E\left(\|X_t - X_s\|^2\right)\right)^{\frac{1}{2}}
$$

denote the canonical (pseudo)-metric associated with X, we have $\delta(s,t) \to 0$ as $d(s,t) \to 0$.

Let $p_{s,t}(z_1, z_2)$ denote the joint density of (X_s, X_t) at $(z_1, z_2) \in \mathbb{R}^{2d}$. The purpose of this section is to establish upper bounds of exponential type for $p_{s,t}(z_1, z_2)$. We notice that these conditions, and in particular, condition (c) below, are easily verified in many examples.

Proposition 3.1 *Suppose that*

(a) $\sigma_{t,i}^2 := Var(X_t^i) > 0$, *for any* $i = 1, \ldots, d$ *and for all* $t \in \mathbb{T}$,

(b) $\mathrm{Corr}(X_s^i, X_t^i) < 1$, *for any* $i = 1, \ldots, d$, $s, t \in \mathbb{T}$ *with* $s \neq t$,

(c) there exists $\eta > 0$ *and a positive constant* $C > 0$ *such that, for all* $s, t \in \mathbb{T}$,
$$\sup_{i \in \{1,\ldots,d\}} |\sigma_{t,i}^2 - \sigma_{s,i}^2| \leq C \left(\delta(s,t)\right)^{1+\eta}.$$

Fix $M > 0$. *Then, there exists* $C > 0$ *such that for all* $s, t \in \mathbb{T}$ *with* $s \neq t$ *and* $z_1, z_2 \in [-M, M]^d$,
$$p_{s,t}(z_1, z_2) \leq \frac{C}{\left(\delta(s,t)\right)^d} \exp\left(-\frac{c\|z_1 - z_2\|^2}{(\delta(s,t))^2}\right),$$
for some positive and finite constants C *and* c.

Proof. Notice that (a), (b) and the independence of the components yield the existence of $p_{s,t}$.

Fix $i = 1, \ldots, d$, and denote by $p_{s,t}^i(z_1, z_2)$, $p_{t|s}^i(\cdot|z_2)$ and $p_s^i(\cdot)$ the joint density of (X_s^i, X_t^i) at (z_1, z_2), the conditional density of X_t^i given $X_s^i = z_2$ and the marginal density of X_s^i, respectively. It is well-known (linear regression) that
$$p_{t|s}^i(z_1|z_2) = \frac{1}{\tau_{s,t}\sqrt{2\pi}} \exp\left(-\frac{|z_1 - m_{s,t} z_2|^2}{2\tau_{s,t}^2}\right),$$
where
$$\tau_{s,t}^2 = \sigma_t^2(1 - \rho_{s,t}^2), \quad \rho_{s,t} = \frac{\sigma_{s,t}}{\sigma_s \sigma_t}, \quad m_{s,t} = \frac{\sigma_{s,t}}{\sigma_s^2}, \quad \sigma_{s,t} = E(X_s^i X_t^i),$$
and, for the sake of simplicity, we have omitted the index i. Since
$$p_{s,t}^i(z_1, z_2) = p_{t|s}^i(z_1|z_2) p_s^i(z_2),$$

the triangle inequality along with the elementary bound $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ yields

$$\begin{aligned} p^i_{s,t}(z_1, z_2) &\leq \frac{1}{2\pi\sigma_s\tau_{s,t}} \exp\left(-\frac{|z_1 - z_2|^2}{4\tau^2_{s,t}}\right) \\ &\quad \times \exp\left(\frac{|z_2|^2|1 - m_{s,t}|^2}{2\tau^2_{s,t}}\right) \exp\left(-\frac{|z_2|^2}{2\sigma^2_s}\right). \end{aligned}$$

By hypotheses (a) and (c), $s \mapsto \sigma^2_s$ is bounded above and bounded below by a positive constant, therefore for $z_2 \in [-M, M]$,

$$p^i_{s,t}(z_1, z_2) \leq \frac{C}{\tau_{s,t}} \exp\left(-\frac{|z_1 - z_2|^2}{4\tau^2_{s,t}}\right) \exp\left(\frac{M^2|1 - m_{s,t}|^2}{2\tau^2_{s,t}}\right).$$

The conclusion now follows from Lemma 3.2 below and the independence of the components of X. □

Lemma 3.2 *With the same assumptions and notations as in Proposition 3.1, there exist constants $0 < c_1 < c_2 < \infty$ such that for all $s, t \in \mathbb{T}$,*

(1) $c_1\ \delta(s,t) \leq \tau_{s,t} \leq c_2\ \delta(s,t)$,

(2) $|1 - m_{s,t}| \leq c_2\ \delta(s,t)$.

Proof. A simple calculation gives

$$\sigma^2_t\sigma^2_s - \sigma^2_{s,t} = \frac{1}{4}\left[\delta(s,t)^2 - (\sigma_t - \sigma_s)^2\right]\left[(\sigma_t + \sigma_s)^2 - \delta(s,t)^2\right], \tag{31}$$

(see [13], equation (3.1)). Therefore, by hypothesis (c) of Proposition 3.1,

$$1 - \rho^2_{s,t} \leq \frac{C}{\sigma^2_s\sigma^2_t}\ \delta(s,t)^2.$$

From assumptions (a) and (c), it follows that there is a positive constant $c_2 < \infty$ such that for all $s, t \in \mathbb{T}$,

$$\tau^2_{s,t} \leq c^2_2\ \delta(s,t)^2,$$

which proves the upper bound in assertion (1).

For the lower bound in (1), we note that for s near t, the second factor on the right-hand side of (31) is bounded below since $\delta(s,t) \to 0$ as $d(s,t) \to 0$. Further, by hypotheses (a) and (c),

$$\begin{aligned} \delta(s,t)^2 - (\sigma_t - \sigma_s)^2 &= \delta(s,t)^2 - \frac{(\sigma^2_t - \sigma^2_s)^2}{(\sigma_t + \sigma_s)^2} \\ &\geq \delta(s,t)^2 - \tilde{c}_1\delta(s,t)^{2+2\eta} \\ &\geq c_1\delta(s,t)^2, \end{aligned}$$

for s near t. This proves the lower bound in (1) when $\delta(s,t)$ is sufficiently small.

In order to extend this inequality to all $s,t \in \mathbb{T}$, it suffices to observe that by hypothesis (b),

$$\sigma_t^2 \sigma_s^2 - \sigma_{s,t}^2 > 0,$$

if $s \neq t$, and by hypothesis (c), for $\varepsilon > 0$, there is $c' > 0$ such that $\sigma_t^2 \sigma_s^2 - \sigma_{s,t}^2 > c'$ for $\delta(s,t) \geq \varepsilon$. This proves the lower bound in assertion (1).

In order to prove assertion (2), observe that

$$|1 - m_{s,t}| = \frac{|\sigma_s^2 - \sigma_{t,s}|}{\sigma_s^2},$$

and

$$\begin{aligned} |\sigma_s^2 - \sigma_{t,s}| &= \left|\delta(s,t)^2 + E\left((X_s - X_t)X_t\right)\right| \\ &\leq \delta(s,t)^2 + \delta(s,t)\sigma_t \leq c\ \delta(s,t). \end{aligned}$$

This completes the proof. □

4 Hitting probabilities for the stochastic wave equation: the Gaussian case

In this section, we consider the solution to equation (1), which is the d–dimensional Gaussian random field defined by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-r, x-y)\sigma M(ds,dy), \quad (t,x) \in [0,T] \times \mathbb{R}^k. \tag{32}$$

Since σ is invertible, we may assume as in [4] that σ is the identity matrix. Notice that, in this case, $u(t,x) = (u_1(t,x), \ldots, u_d(t,x))$, with

$$u_i(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-r, x-y) M^i(ds,dy), \quad (t,x) \in [0,T] \times \mathbb{R}^k,$$

$i = 1, \ldots, d$, and therefore, the component processes $(u_i(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^k)$, $i = 1, \ldots, d$, are i.i.d.

Most of the results of this section require the following hypothesis:

$(\mathbf{H}_\beta)$ The spectral covariance measure μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^k$ and its density is given by

$$f(\xi) = \|\xi\|^{-k+\beta},\ \beta \in\ ]0, 2 \wedge k[.$$

Equivalently, $\Gamma(dx) = C(k,\beta)\|x\|^{-\beta}dx$ (see [11]). Notice that $(\mathbf{H}_\beta)$ implies **(H)**.

In the sequel we fix a strictly positive real number t_0. We first aim for lower bounds on hitting probabilities. For this, we intend to apply Theorem 2.1. The required upper bound on the joint densities will be obtained by combining Proposition 3.1 and the next two results.

Proposition 4.1 *Assume* $(\mathbf{H}_\beta)$. *Fix* $M > 0$. *Then, there exist positive constants* C_1, C_2 *such that, for any* $(t,x),(s,y) \in [t_0,T]\times[-M,M]^k$,

$$C_1\left(|t-s| + \|x-y\|\right)^{2-\beta} \le E\left(\|u_{t,x} - u_{s,y}\|^2\right) \le C_2\left(|t-s| + \|x-y\|\right)^{2-\beta}. \tag{33}$$

Proof. The structure of this proof is similar to that of Lemma 4.2 in [4], but the methods for obtaining the estimates differ substantially. Without loss of generality, we will assume that $d=1$. Let $R(x) = E(u(t,x)u(t,0))$ with $t \ge t_0$. Then

$$E\left((u_{t,x} - u_{t,y})^2\right) = 2\left(R(0) - R(x-y)\right).$$

Following the steps of the proof of Remark 5.2 in [9], with the dimension $k=3$ replaced by an arbitrary value of k, and therefore the Riesz kernel $\|\xi\|^{-(3-\beta)}$ replaced by $\|\xi\|^{-(k-\beta)}$, we obtain

$$R(0) - R(x) \le C\|x\|^{2-\beta}. \tag{34}$$

We next fix $y \in \mathbb{R}^k$ and consider increments in time. Let $t_0 \le s < t \le T$. Using (5) and (7),

$$E\left((u(t,y) - u(s,y))^2\right) = S_1(s,t) + S_2(s,t),$$

with

$$S_1(s,t) = \int_0^s dr \int_{\mathbb{R}^k} \frac{d\xi}{\|\xi\|^{k-\beta}} \frac{|\sin((t-r)\|\xi\|) - \sin((s-r)\|\xi\|)|^2}{\|\xi\|^2},$$

$$S_2(s,t) = \int_s^t dr \int_{\mathbb{R}^k} \frac{d\xi}{\|\xi\|^{k-\beta}} \frac{\sin^2((t-r)\|\xi\|)}{\|\xi\|^2}.$$

With the changes of variables $r \to s-r$ and $\xi \to (t-s)\xi$, along with the trigonometric formula $\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$, we obtain

$$\begin{aligned} S_1(s,t) &\le 4\int_0^s dr \int_{\mathbb{R}^k} \frac{d\xi}{\|\xi\|^{k-\beta+2}} \sin^2\left(\frac{(t-s)\|\xi\|}{2}\right) \\ &= 4\int_0^s dr (t-s)^{2-\beta} \int_{\mathbb{R}^k} \frac{dv}{\|v\|^{k-\beta+2}} \sin^2\left(\frac{\|v\|}{2}\right) \\ &\le C|t-s|^{2-\beta}. \end{aligned}$$

For the term $S_2(s,t)$, we consider the changes of variables $r \to t-r$ and then $\xi \to r\xi$, which easily yield

$$\begin{aligned} S_2(s,t) &\le \int_0^{t-s} dr\ r^{2-\beta} \int_{\mathbb{R}^k} \frac{dv}{\|v\|^{k-\beta+2}} \sin^2 \|v\| \\ &\le C|t-s|^{3-\beta}. \end{aligned}$$

Hence, we have proved

$$E\left((u(t,y)-u(s,y))^2\right) \le C|t-s|^{2-\beta}, \tag{35}$$

with a positive constant C depending only on T. With (34) and (35), we have established the upper bound in (33).

We now prove the lower bound in (33) using several steps.

Step 1. Assume $s = t \ge t_0$ and $x \ne y$. The arguments in the proof of Theorem 5.1(a) in [9] can be trivially extended to any spatial dimension k. Therefore, there is a positive constant c_1 such that for any $x, y \in [-M, M]^k$,

$$E\left((u(t,x)-u(t,y))^2\right) \ge c_1|x-y|^{2-\beta}. \tag{36}$$

Step 2. We show that, for arbitrary $x, y \in [-M, M]^k$ and $t_0 \le s \le t \le T$,

$$E\left((u(t,x)-u(s,y))^2\right) \ge c|t-s|^{2-\beta}. \tag{37}$$

Indeed the left-hand side of this inequality is equal to

$$R_1(s,t;x,y) + R_2(s,t;x,y),$$

with

$$\begin{aligned} R_1(s,t;x,y) &= \int_0^s dr \int_{\mathbb{R}^k} \frac{d\xi}{\|\xi\|^{k-\beta}} \\ &\qquad \times \left|\mathcal{F}G(t-r,x-\cdot)(\xi) - \mathcal{F}G(s-r,y-\cdot)(\xi)\right|^2, \\ R_2(s,t;x,y) &= \int_s^t dr \int_{\mathbb{R}^k} \frac{d\xi}{\|\xi\|^{k-\beta}} \left|\mathcal{F}G(t-r,x-\cdot)(\xi)\right|^2. \end{aligned}$$

Since $R_2(s,t;x,y)$ is positive, we can neglect its contribution. (We notice that

$$R_2(s,t;x,y) \ge C|t-s|^{3-\beta},$$

for some positive constant C. For $k = 3$, this is shown in the proof of Theorem 5.1 in [9], and it is easy to check that the arguments go through any dimension.)

By developing the integrand in $R_1(s,t;x,y)$, we find

$$\begin{aligned}
&\|\xi\|^2 \left|\mathcal{F}G(t-r,x-\cdot)(\xi) - \mathcal{F}G(s-r,y-\cdot)(\xi)\right|^2 \\
&\quad = \left|\sin((t-r)\|\xi\|) - e^{i\xi\cdot(y-x)}\sin((s-r)\|\xi\|)\right|^2 \\
&\quad = \frac{1-\cos(2(t-r)\|\xi\|)}{2} + \frac{1-\cos(2(s-r)\|\xi\|)}{2} \\
&\qquad - \cos(\xi\cdot(y-x))\left[\cos((t-s)\|\xi\|) - \cos((t+s-2r)\|\xi\|)\right].
\end{aligned}$$

After integrating this last expression with respect to the variable r, we obtain a positive quantity which is the sum of the following three terms:

$$\begin{aligned}
A_1 &= s\left[1-\cos((t-s)\|\xi\|)\cos(\xi\cdot(y-x))\right], \\
A_2 &= \frac{\sin((s+t)\|\xi\|)}{2\|\xi\|}\left(\cos(\xi\cdot(y-x)) - \cos((t-s)\|\xi\|)\right), \\
A_3 &= \frac{\sin(2(t-s)\|\xi\|)}{4\|\xi\|} - \frac{\sin((t-s)\|\xi\|)}{2\|\xi\|}\cos(\xi\cdot(y-x)).
\end{aligned}$$

For the integration with respect to the variable ξ, we restrict the domain to the set

$$D_0 = \{\xi\in\mathbb{R}^k : \|\xi\|(t-s)\geq 1,\ \cos((t-s)\|\xi\|)\geq 0\}.$$

Note that on D_0, we have $A_1\geq 0$. In fact,

$$\begin{aligned}
A_1 &= s\left[1-\cos((t-s)\|\xi\|) + \cos((t-s)\|\xi\|)(1-\cos(\xi\cdot(y-x)))\right] \\
&\geq s\left[1-\cos((t-s)\|\xi\|)\right].
\end{aligned}$$

Moreover,

$$|A_2+A_3| \leq \frac{2}{\|\xi\|}.$$

Thus, with the change of variables $\xi\to(t-s)\xi$, we easily obtain

$$\begin{aligned}
\int_{D_0}\frac{d\xi}{\|\xi\|^{k-\beta+2}}A_1 &\geq s\int_{D_0}\frac{d\xi}{\|\xi\|^{k-\beta+2}}\left[1-\cos((t-s)\|\xi\|)\right] \\
&= s|t-s|^{2-\beta}\int_{\{\|w\|\geq 1;\ \cos(\|w\|)\geq 0\}}\frac{dw}{\|w\|^{k-\beta+2}}(1-\cos(\|w\|)) \\
&\geq c_2|t-s|^{2-\beta}.
\end{aligned}$$

Similarly,

$$\int_{\{\|\xi\|(t-s)\geq 1\}}\frac{d\xi}{\|\xi\|^{k-\beta+2}}|A_2+A_3| \leq c_3|t-s|^{3-\beta}.$$

Therefore, by the triangular inequality, we obtain

$$R_1(s,t;x,y) \geq c_2|t-s|^{2-\beta} - c_3|t-s|^{3-\beta} \geq \frac{c_2}{2}|t-s|^{2-\beta},$$

if $|t-s| \leq \frac{c_2}{2c_3}$. This proves (37) for small values of $|t-s|$.

To extend the validity of (37) to arbitrary values of $|t-s|$, we notice that $R_1(s,t;x,y)$ is a continuous and positive function of its arguments and therefore, it is bounded below on $\{(s,t;x,y) \in [t_0,T]^2 \times [-M,M]^{2k} : |t-s| \geq \varepsilon\}$ by some constant c_ε, for any $\varepsilon > 0$. Hence, if $2T > |t-s| > \frac{c_2}{2c_3}$, we also have

$$R_1(s,t;x,y) \geq c|t-s|^{2-\beta},$$

for some c sufficiently small.

Step 3. Suppose $|t-s| \geq \left[\frac{c_1}{4C_2}\right]^{\frac{1}{2-\beta}} |x-y|$, where c_1 appears in (36) and C_2 in the right hand-side of (33). By Step 2, we clearly have

$$\begin{aligned}
E\left((u(t,x)-u(s,y))^2\right) &\geq c|t-s|^{2-\beta} \\
&\geq c\left(\frac{|t-s|}{2} + \frac{1}{2}\left(\frac{c_1}{4C_2}\right)^{\frac{1}{2-\beta}} |x-y|\right)^{2-\beta} \\
&\geq C_3\left(|t-s|+|x-y|\right)^{2-\beta}.
\end{aligned}$$

Step 4. Suppose $|t-s| \leq \left[\frac{c_1}{4C_2}\right]^{\frac{1}{2-\beta}} |x-y|$. Then,

$$\begin{aligned}
E\left((u(t,x)-u(s,y))^2\right) &\geq \frac{1}{2}E\left((u(t,x)-u(t,y))^2\right) - E\left((u(t,y)-u(s,y))^2\right) \\
&\geq \frac{1}{2}c_1|x-y|^{2-\beta} - C_2|t-s|^{2-\beta} \\
&\geq \frac{c_1}{4}|x-y|^{2-\beta} \\
&\geq \frac{c_1}{4}\left(\frac{|x-y|}{2} + \frac{1}{2}\left[\frac{4C_2}{c_1}\right]^{\frac{1}{2-\beta}} |t-s|\right)^{2-\beta} \\
&\geq C_4\left(|t-s|+|x-y|\right)^{2-\beta}.
\end{aligned}$$

With this, the lower bound in (33) is proved. □

Remark 4.1 (a) As mentioned in the Introduction, Proposition 4.1, together with Lemma 3.2, establishes conditions (C1) and (C2) of [20] for the process U.

(b) A consequence of the preceding proposition is that the sample paths of (32) are Hölder continuous, jointly in (t,x), of exponent $\gamma \in]0, \frac{2-\beta}{2}[$, but they are not Hölder continuous of exponent $\gamma > \frac{2-\beta}{2}$. We refer the reader to [9] for a similar result on the solution to a nonlinear stochastic wave equation in spatial dimension $k = 3$.

The next proposition is a further step towards proving that the process U satisfies the assumptions of Proposition 3.1. We denote by $\sigma^2_{t,x}$ the common variance of $u_i(t,x)$, $i = 1, \dots, d$.

Proposition 4.2 *Assume that condition* $(\mathbf{H})$ *is satisfied. Fix* $(t,x),(s,y) \in [t_0,T] \times \mathbb{R}^k$. *Then,*

(i) $\sigma^2_{t,x} \geq C(t_0 \wedge t_0^3) > 0$,

(ii) $|\sigma^2_{t,x} - \sigma^2_{s,y}| \leq C|t-s|$.

If, in addition, we assume that for $k' < k$, *all* k'*–dimensional submanifolds of* $\mathbb{R}^k$ *are sets with null* μ*-measure, then*

(iii) for any $(t,x) \neq (s,y)$ *and* $i = 1, \dots, d$,

$$\mathrm{Corr}\left(u^i(t,x), u^i(s,y)\right) < 1.$$

Proof. The variance of $u(t,x)$ is

$$\sigma^2_{t,x} = \int_0^t dr \int_{\mathbb{R}^k} \mu(d\xi)\, \frac{\sin^2((t-r)\|\xi\|)}{\|\xi\|^2}, \tag{38}$$

and satisfies

$$C(t \wedge t^3) \leq \sigma^2_{t,x} \leq \bar{C}\left(t + t^3\right) \tag{39}$$

(see for instance [16], Lemma 8.6). This proves (i).

Assumption **(H)** implies that

$$\sup_{r \in [0,T]} \int_{\mathbb{R}^k} \mu(d\xi)\, \frac{\sin^2(r\|\xi\|)}{\|\xi\|^2} \leq C.$$

Hence, assuming $t_0 \leq s < t \leq T$, we obtain

$$|\sigma^2_{t,x} - \sigma^2_{s,y}| = \int_s^t dr \int_{\mathbb{R}^k} \mu(d\xi)\, \frac{\sin^2(r\|\xi\|)}{\|\xi\|^2} \leq C(t-s),$$

which yields the conclusion (ii) of the proposition.

We now prove (iii) by checking that for any $(t,x) \neq (s,y)$ in $[t_0,T] \times \mathbb{R}^k$,

$$\sigma^2_{t,x}\sigma^2_{s,y} - \sigma^2_{t,x;s,y} > 0,$$

where $\sigma_{t,x;s,y}$ denotes the covariance of $u_i(t,x)$ and $u_i(s,y)$ for any $i = 1,\dots,d$.

Case 1: $s < t$. If $\sigma^2_{t,x}\sigma^2_{s,y} - \sigma^2_{t,x;s,y}$ were equal to zero, then the random variables $u^i(t,x)$ and $u^i(s,y)$ would have correlation equal to 1; therefore, there would be $\lambda \in \mathbb{R}$ such that $u^i(t,x) = \lambda u^i(s,y)$ a.s., and, in particular, we would have

$$E\left(\left(u^i(t,x) - \lambda u^i(s,y)\right)^2\right) = 0.$$

The left-hand side of this equality is

$$\int_s^t dr \int_{\mathbb{R}^k} \mu(d\xi)\, |\mathcal{F}G(t-r, x-\cdot)(\xi)|^2 \\ + \int_0^s dr \int_{\mathbb{R}^k} \mu(d\xi)\, |\mathcal{F}G(s-r, x-\cdot)(\xi) - \lambda \mathcal{F}G(s-r, y-\cdot)(\xi)|^2 ,$$

which is bounded below, as in (39), by $C((t-s)\wedge(t-s)^3)$. This leads to a contradiction.

Case 2: $s = t$, $x \neq y$. We start as in the preceding case by assuming that $\sigma^2_{t,x}\sigma^2_{t,y} - \sigma^2_{t,x;t,y} = 0$ and hence

$$E\left(\left(u^i(t,x) - \lambda u^i(t,y)\right)^2\right) = 0$$

for some $\lambda \in \mathbb{R}$. The left-hand side is equal to

$$\int_0^t dr \int_{\mathbb{R}^k} \mu(d\xi) \left|e^{i\xi\cdot x} - \lambda e^{i\xi\cdot y}\right|^2 |\mathcal{F}G(r,\cdot)(\xi)|^2 .$$

If $\lambda = 1$, then the integrand vanishes when $\cos[\xi\cdot(x-y)] = 1$ or $\sin(r\|\xi\|) = 0$, which occurs on a $(k-1)$–dimensional manifold of $\mathbb{R}^k$. Hence, by the assumption on μ, we reach a contradiction.

If $\lambda \neq 1$, then

$$\int_0^t dr \int_{\mathbb{R}^k} \mu(d\xi) \left|e^{i\xi\cdot x} - \lambda e^{i\xi\cdot y}\right|^2 |\mathcal{F}G(r,\cdot)(\xi)|^2 \\ \geq \int_0^t dr \int_{\mathbb{R}^k} \mu(d\xi)\ (1-\lambda)^2\ \frac{\sin^2(r\|\xi\|)}{\|\xi\|^2}.$$

This last integrand vanishes only when $\sin(r\|\xi\|) = 0$. Thus we also get a contradiction in this case. The proof of the proposition is now complete. □

We can now obtain the required properties on densities, as follows.

Proposition 4.3 *Assume* $(\mathbf{H}_\beta)$. *Fix* $M, N > 0$ *and* $(t,x), (s,y) \in [t_0, T] \times [-M, M]^k$ *with* $(t,x) \neq (s,y)$.

(a) *Let $p_{t,x;s,y}(\cdot,\cdot)$ denote the joint density of the random vector $(u(t,x),u(s,y))$. Then*

$$p_{t,x;s,y}(z_1,z_2) \le \frac{C}{(|t-s|+|x-y|)^{\frac{d(2-\beta)}{2}}} \exp\left(-\frac{c\|z_1-z_2\|^2}{(|t-s|+|x-y|)^{2-\beta}}\right), \tag{40}$$

for any $z_1, z_2 \in [-N,N]^d$, with C and c positive constants not depending on (t,x), (s,y).

(b) *Let $p_{t,x}$ denote the density of the random vector $u(t,x)$. Then, for each $(t,x) \in [t_0,T] \times \mathbb{R}^k$ and $z \in [-N,N]^d$,*

$$p_{t,x}(z) \ge C, \tag{41}$$

and

$$\sup_{z\in[-N,N]^d} \sup_{(t,x)\in[t_0,T]\times\mathbb{R}^k} p_{t,x}(z) \le C. \tag{42}$$

Proof. By Propositions 4.2 and 4.1, we see that the process U satisfies the hypotheses of Proposition 3.1 with $\eta = \frac{\beta}{2-\beta}$. Thus, we have statement (a).

The density $p_{t,x}$ is given by

$$p_{t,x}(z) = \frac{1}{(2\pi\sigma_{t,x}^2)^{\frac{d}{2}}} \exp\left(-\frac{\|z\|^2}{2\sigma_{t,x}^2}\right),$$

with $\sigma_{t,x}^2$ as in (38). By (39), we obtain both (41) and (42). □

The next theorem gives lower bounds on hitting probabilities.

Theorem 4.4 *Assume $(\mathbf{H}_\beta)$. Let I, J be compact subsets of $[t_0,T]$ and $\mathbb{R}^k$, respectively, each with positive Lebesgue measure. Fix $N>0$. Then:*

(1) *There exists a positive constant $c = c(I,J,N,\beta,k,d)$ such that, for any Borel set $A \subset [-N,N]^d$,*

$$P\{u(I\times J)\cap A \ne \emptyset\} \ge c\ \mathrm{Cap}_{d-\frac{2(k+1)}{2-\beta}}(A). \tag{43}$$

(2) *For any $t \in I$, there exists a positive constant $c = c(J,N,\beta,k,d,t)$ such that, for any Borel set $A \subset [-N,N]^d$,*

$$P\{u(\{t\}\times J)\cap A \ne \emptyset\} \ge c\ \mathrm{Cap}_{d-\frac{2k}{2-\beta}}(A). \tag{44}$$

(3) For any $x \in J$, there exists a positive constant $c = c(I, N, \beta, k, d, x)$ such that, for any Borel set $A \subset [-N, N]^d$,

$$P\{u\,(I \times \{x\}) \cap A \neq \emptyset\} \geq c \ \mathrm{Cap}_{d-\frac{2}{2-\beta}}(A). \tag{45}$$

Proof. The three statements follow from Theorem 2.1 and Proposition 4.3 applied respectively to the stochastic process U, $U(t) = \{u(t,x), x \in \mathbb{R}^k\}$ with $t \in I$, and $U(x) = \{u(t,x), t \in [t_0, T]\}$ with $x \in J$. Notice that by (40) and (41), the parameters γ and α in Theorem 2.1 are $\gamma = \frac{d(2-\beta)}{2}$, $\alpha = 2 - \beta$, and $m = k+1$, $m = k$, $m = 1$, respectively. □

Remark 4.2 Since the probability of visiting translates of a compact set A decreases to 0 as the distance of this translated set to the origin tends to infinity, it is not possible to replace $[-N, N]^d$ by $\mathbb{R}^d$ in the above theorem. In contrast, this will be possible in the upper bounds of the next theorem.

Theorem 4.5 *Assume $(\mathbf{H}_\beta)$. Let I, J be compact subsets of $[t_0, T]$ and $\mathbb{R}^k$, respectively, each with positive Lebesgue measure. Then:*

(1) There exists a positive constant $c = c(I, J, \beta, k, d)$ such that, for any Borel set $A \subset \mathbb{R}^d$,

$$P\{u(I \times J) \cap A \neq \emptyset\} \leq c \ \mathcal{H}_{d-\frac{2(k+1)}{2-\beta}}(A). \tag{46}$$

(2) For any $t \in I$, there exists a positive constant $c = c(J, \beta, k, d, t)$ such that, for any Borel set $A \subset \mathbb{R}^d$,

$$P\{u\,(\{t\} \times J) \cap A \neq \emptyset\} \leq c \ \mathcal{H}_{d-\frac{2k}{2-\beta}}(A). \tag{47}$$

(3) For any $x \in J$, there exists a positive constant $c = c(I, \beta, k, d, x)$ such that, for any Borel set $A \subset \mathbb{R}^d$,

$$P\{u\,(I \times \{x\}) \cap A \neq \emptyset\} \leq c \ \mathcal{H}_{d-\frac{2}{2-\beta}}(A). \tag{48}$$

Proof. We first note that if we replace d in the Hausdorff dimensions of the bounds by any $\gamma \in \,]0, d[$, then these statements would be a consequence of Theorem 2.4 applied respectively to the stochastic processes U, $U(t) = \{u(t,x), x \in \mathbb{R}^k\}$ with $t \in I$, and $U(x) = \{u(t,x), t \in [t_0, T]\}$ with $x \in J$. Indeed, assumption (1) of Theorem 2.4 is given in (42). Moreover, since U is a Gaussian process, the right-hand side of (33) yields the validity of hypothesis (2) of Theorem 2.4, with $\delta = \frac{2-\beta}{2}$.

The improvement to $\gamma = d$ is obtained by applying Theorem 2.6 to each of the stochastic processes mentioned before. Let us argue with the process U for the sake of illustration. From (33), we easily deduce that

$$E\left[\exp\left\{\frac{|u_i(s,y)-u_i(t,x)|}{(|s-t|+|x-y|)^{\frac{2-\beta}{2}}}\right\}\right] \le E\left[\exp(cX)\right] = C,$$

where X stands for a standard Normal random variable. Thus, when $m = k+1$, $\delta = \frac{2-\beta}{2}$, the left-hand side of (25) is bounded by a constant times the square of the volume of R_j^{ε}, that is, $C\varepsilon^{\frac{4(k+1)}{2-\beta}}$. Hence, the assumptions of Theorem 2.6 are satisfied.

The proof of the theorem is complete. □

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