A quickest detection problem with an observation cost

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Abstract

In the classical quickest detection problem, one must detect as quickly as possible when a Brownian motion without drift "changes" into a Brownian motion with positive drift. The change occurs at an unknown "disorder" time with exponential distribution. There is a penalty for declaring too early that the change has occurred, and a cost for late detection proportional to the time between occurrence of the change and the time when the change is declared. Here, we consider the case where there is also a cost for observing the process. This stochastic control problem can be formulated using either the notion of strong solution or of weak solution of the s.d.e. that defines the observation process. We show that the value function is the same in both cases, even though no optimal strategy exists in the strong formulation. We determine the optimal strategy in the weak formulation and show, using a form of the "principle of smooth fit" and under natural hypotheses on the parameters of the problem, that it is optimal to observe only when the posterior probability that the change has already occurred, given the observations, is larger than a threshold $A \geq 0$, and to declare that the disorder time has occurred when this posterior probability exceeds a threshold $B \geq A$. The constants A and B are determined explicitly from the parameters of the problem.

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1 Introduction

The classical quickest detection problem [15, Chapter 4.4] is as follows. An observers observes a stochastic process $X = (X_t)_{t\geq 0}$ that solves the stochastic differential equation (s.d.e.)

$$dX_t = r\, 1_{\{\theta \leq t\}}\, dt + \sigma\, dB_t. \tag{1.1}$$

Here, $r > 0$, $\sigma > 0$, $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, and θ is a nonnegative random variable that is independent of (B_t), sometimes called a "disorder time," or a "change point." The random variable θ is not observed directly, but only through its effect on the sample paths of X. When $t < \theta$, the observer is simply watching a Brownian motion, but when $t \geq \theta$, a drift (or signal) with intensity r appears. The observer seeks to detect as quickly as possible the appearance of this signal, while keeping sufficiently low the probability of a "false alarm", that is, declaring that the signal has appeared when, in fact, it has not. Typically, the distribution of θ is assumed known, and, given $\theta > 0$, even equal to an exponential distribution with known parameter $\lambda > 0$ (see [12] for many variations on this problem and for numerous references).

In this paper, we consider the situation where there is an *observation cost* $b \geq 0$ per unit time and the observer can choose to observe or not. When he does not observe, the process X is constant ($dX_t = 0$), and when he does observe, X satisfies (1.1). The objective is to detect the appearance of the signal as quickly as possible, while keeping low the probability of false alarm *and* the cost of observation. Therefore, the problem is no longer an optimal stopping problem but an *optimal stopping/control problem*, where the control $h = (h_t)_{t\geq 0}$ is a $[0, 1]$-valued process, where $h_t = 1$ means that observation occurs, and $h_t = 0$ means absence of observation. Therefore, the observation process is described by the stochastic differential

$$dX_t = r\, h_t 1_{\{\theta \leq t\}}\, dt + \sigma \sqrt{h_t}\, dB_t, \qquad X_0 = 0. \tag{1.2}$$

Note that when $h_t \in \{0, 1\}$, the square-root has no effect. However, it will be convenient during the resolution of the problem to consider also $h_t \in [0, 1]$, and since we are free to decide the formulation when $0 < h_t < 1$, we have chosen to use (1.2).

We assume that all objects are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Therefore, $B_t = B_t(\omega)$, $\theta = \theta(\omega)$, and $h_t = h_t(\omega)$. The assumption that h_t depends on ω ($h_t = h_t(\omega)$) does not create difficulties with definition of the ("h-controlled") process X via formula (1.2). However, we must define precisely what type of information the observer can use to decide to switch from one value of $h_t(\omega)$ to another.

It is reasonable to assume that the control function h_t depends on ω via the observation process: $h_t(\omega) = h_t(X(\omega))$. In this case, the s.d.e. (1.2) will take the form

$$dX_t = r h_t(X) 1_{\{\theta \leq t\}}\, dt + \sigma \sqrt{h_t(X)}\, dB_t, \tag{1.3}$$

and, inevitably, we have to explain how to formualte this s.d.e. and give a precise definition of the control $h = (h_t(X))_{t\geq 0}$.

These questions are considered in Section 2, where we give two precise but distinct formulations of the notion of a solution of equation (1.3), according to whether we interpret X as a *strong* or *weak* solution of (1.3). Then we derive some preliminary properties of the sufficient statistic π_t^h, which is the conditional probability, given the observations $(X_s,\ s \in [0,t])$, that $\theta \leq t$. In Section 3, we study the law of π_t^h, writing it, and the likelihood ratio $\varphi_t^h = \pi_t^h/(1-\pi_t^h)$, as solutions of diffusion equations in the filtration $\mathcal{F}^X$ of the observed process. In this section, we also establish, in the spirit of [4] and [11], a "verification lemma" (Lemma 3.7) that gives sufficient conditions for the optimality of a strategy.

In Section 4, we give the form of a candidate optimal strategy and associated candidate value function, and derive the ordinary differential equations with two free boundaries that characterize this function. These are completed by imposing boundary conditions that imply continuity and an appropriate degree of smoothness at the boundaries (see (4.10)–(4.14)). These equations are then solved completely, up to the resolution of a transcendental equation (see (4.26)). The form of the solution depends on the value of the observation cost b, and it turns out that there are three regimes: if b is large enough, then it is best never to observe, and to stop simply when the posterior probability π_t^h exceeds a certain threshold $B \in]0,1[$. For smaller positive values of b, there are two thresholds $0 < A < B < 1$ such that it is best not to observe when $\pi_t^h \leq A$, to observe when $\pi_t^h \in]A,B[$ and to declare an alarm when $\pi_t^h \geq B$. The candidate value function is given in Propositions 4.3 and 4.4, depending on the size of b. The third regime is when $b = 0$, which is the classical case of [15] and corresponds to $0 = A < B < 1$.

For small positive values of b, the candidate value function and optimal strategies are such that it is not clear whether an optimal strategy does indeed exist! In fact, in the strong formulation, no optimal strategy exists in general, but such an optimal strategy does exist in the weak formulation. It turns out, however, that the value function is the same in both formulations. We discuss this question at the end of Section 4.

In Section 5, we show that the candidate value function of Section 4 is indeed the value function in both the weak and strong formulations (Theorems 5.1 and 5.2). However, because of the absence of an optimal strategy in the strong formulation, it is not possible to conclude directly from a "verification lemma" (Lemma 3.7) that the candidate value function is indeed the value function in the strong formulation. Therefore, we use a different approach in Theorem 5.2: for $\varepsilon > 0$, we consider strategies that approximate the candidate optimal strategy but are defined via s.d.e.'s with sufficiently smooth coefficients. We then compute explicitly the cost associated with these strategies. This requires computing the expected time to hit a threshold, which, in turn, requires solving another o.d.e (given in (5.11)). We do this in Section 5, and in Proposition 5.7, we show by direct calculation that the expected costs of the approximately optimal strategies converge to the candidate value function, proving that this is indeed the value function in the strong formulation.

2 Stating the problem

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with a filtration $(\mathcal{F}_t)$ (satisfying the *usual hypotheses* [13]). Let θ be a random variable defined on Ω that is $\mathcal{F}_0$-measurable. We assume that there is $\pi_0 \in [0,1]$ and $\lambda > 0$ such that

$$P\{\theta = 0\} = \pi_0 \qquad \text{and} \qquad P\{\theta > x \mid \theta > 0\} = e^{-\lambda x}. \tag{2.1}$$

We let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$ such that for all $t \geq 0$, the process $(B_{s+t} - B_t,\ s \geq 0)$ is independent of $\mathcal{F}_t$. In particular, $(B_t)_{t\geq 0}$ is independent of θ.

Controls and stopping times

Definition 2.1. *A progressively measurable process $h = (h_t(\omega))_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with values in $[0,1]$ will be called a* stochastic control*.*

Let $C(\mathbb{R}_+, \mathbb{R})$ denote the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$.

Definition 2.2. *A* canonical control *$h = (h_t(x))_{t\geq 0}$ is a map $(t,x) \mapsto h_t(x)$ from $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R})$ to $[0,1]$ that is progressively measurable for the canonical filtration on $C(\mathbb{R}_+, \mathbb{R})$.*

A canonical stopping time *$\tau = \tau(x)$ is a random variable $\tau\colon C(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}_+$ that is a stopping time relative to the canonical filtration on $C(\mathbb{R}_+, \mathbb{R})$.*

Definition 2.3. *A stochastic control $h = (h_t(\omega))_{t\geq 0}$ is called an* admissible control *if it has the form $h_t(\omega) = h_t(X(\omega))$ for a canonical control $h_t(x)$ and the s.d.e.*

$$dX_t = r h_t(X) 1_{\{\theta \leq t\}}\, dt + \sigma \sqrt{h_t(X)}\, dB_t, \qquad X_0 = 0, \tag{2.2}$$

admits a strong solution in the sense of the next definition (Definition 2.4).

Definition 2.4. *Assume that a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is given a priori together with a random variable $\theta = \theta(\omega)$ which is $\mathcal{F}_0$-measurable and satisfies (2.1), and with a Brownian motion $B(\omega) = (B_t(\omega))_{t\geq 0}$ such that $B_t(\omega)$ is $\mathcal{F}_t$-measurable, for all $t \geq 0$.*

A strong solution *of the s.d.e. (2.2) is a continuous stochastic process $X = (X_t(\omega))_{t\geq 0}$ that satisfies (2.2) and is such that $X_t(\omega)$ is $\mathcal{F}_t$-measurable, for all $t \geq 0$.*

One may consider also the case where (2.2) has a weak solution.

Definition 2.5. *We assume that a canonical control $h = (h_t(x))_{t\geq 0}$ and the law of θ in (2.1) are given a priori. A* weak solution *of the s.d.e. (2.2) is a system of the following objects:*

– *a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ (which is not given a priori);*
– *a Brownian motion $B = (B_t)_{t\geq 0}$ such that B_t is $\mathcal{F}_t$-measurable, for all $t \geq 0$;*

– *an* $\mathcal{F}_0$*-measurable random variable* θ *with the law specified in* (2.1)*;*

– *an* $(\mathcal{F}_t)_{t\geq 0}$*-adapted process* $X = (X_t)_{t\geq 0}$ *which satisfies the s.d.e.* (2.2)*, that is, for all* $t \geq 0$

$$X_t = \int_0^t r h_s(X)\, 1_{\{\theta\leq s\}}\, ds + \int_0^t \sigma\sqrt{h_s(X)}\, dB_s. \tag{2.3}$$

Definition 2.6. *For the case of strong solutions, a* strategy *is a pair* (h, τ)*, where* $h = (h_t(X(\omega)))_{t\geq 0}$*,* $\tau = \tau(X(\omega))$ *for some canonical control* $(h_t(x))_{t\geq 0}$ *and canonical stopping time* $\tau(x)$*.*

For the case of weak solutions, (h, τ, X) *is called a* control system*.*

Cost

Definition 2.7. *The* cost *associated with a strategy* (h, τ) *or a control system* (h, τ, X) *is*

$$C(h, \tau) = C(h, \tau, X) = 1_{\{\tau(X)<\theta\}} + a(\tau(X) - \theta)1_{\{\tau(X)\geq\theta\}} + b\int_0^{\tau(X)} h_t(X)\, dt, \tag{2.4}$$

where $a > 0$*, so as to penalize late detection of the alarm time* θ*, and* $b \geq 0$*. Since the case* $b = 0$ *is covered in* [15, Chapter 4.4]*, we will focus on the case* $b > 0$*.*

Objective

Our first objective is to find the value

$$V = \inf_{(h,\tau)} E(C(h, \tau)),$$

where the infimum is over all strategies, and to find an *optimal* strategy (h^*, τ^*) that achieves this infimum, or at least, to find a strategy that is within ε of this infimum ($\varepsilon > 0$). A second objective is to find the value

$$V^{\mathrm{w}} = \inf_{(h,\tau,X)} E(C(h, \tau, X)),$$

where the infimum is over all control systems, and an optimal control system (h^*, τ^*, X^*). Clearly, $V^{\mathrm{w}} \leq V$.

Dependence on π_0

The quantities V and V^{w} are in fact functions of the number $\pi_0 = P\{\theta = 0\}$, which we denote $\tilde{g}(\pi_0)$ and $\tilde{g}^{\mathrm{w}}(\pi_0)$:

$$\tilde{g}(\pi_0) = \inf_{(h,\tau)} E(C(h, \tau)), \tag{2.5}$$

$$\tilde{g}^{\mathrm{w}}(\pi_0) = \inf_{(h,\tau,X)} E(C(h, \tau, X)). \tag{2.6}$$

Clearly, $\tilde{g}^{\mathrm{w}} \le \tilde{g}$. The following simple lemma provides important information about the form of these two functions.

Lemma 2.1. *The functions $\tilde{g}$ and $\tilde{g}^{\mathrm{w}}$ are concave.*

Proof. By the law of total probability,

$$\begin{aligned} E(C(h,\tau)) &= \pi_0 E\Big(a\tau + b\int_0^{\tau(X)} h_t(X)\,dt \,\Big|\, \theta = 0\Big) \\ &\quad +(1-\pi_0)E\Big(1_{\{\tau(X)<\theta\}} + a(\tau(X)-\theta)1_{\{\tau(X)>\theta\}} \\ &\qquad + b\int_0^{\tau(X)} h_t(X)\,dt \,\Big|\, \theta > 0\Big). \end{aligned}$$

The first expectation does not depend on π_0, and the second does not either, since the conditional distribution of θ given that $\theta > 0$ does not depend on π_0. Therefore, $\pi_0 \mapsto E(C(h,\tau))$ is an affine function of π_0, and $\tilde{g}$, being the infimum of affine functions, is concave. The same argument applies to $\tilde{g}^{\mathrm{w}}$. □

Sufficient statistic

Let $\mathcal{F}^X = (\mathcal{F}_t^X)$ be the natural filtration of the observed process X, augmented with P-null sets. Let (π_t^h) be the optional projection of $(1_{\{\theta \le t\}},\ t \ge 0)$ onto this filtration, so that for all t, $\pi_t^h = P\{\theta \le t \mid X_s,\ s \le t\}$ a.s. The next several lemmas are identical both for strategies and for control systems, so we state them only for strategies.

Lemma 2.2. *With the above notation,*

$$E(C(h,\tau)) = E\left(1 - \pi_\tau^h + a\int_0^\tau \pi_s^h\,ds + b\int_0^\tau h_s\,ds\right). \tag{2.7}$$

Proof. Note that $E(1_{\{\tau<\theta\}}) = E(1-\pi_\tau^h)$ and

$$\begin{aligned} E\left((\tau-\theta)1_{\{\tau>\theta\}}\right) &= E\left(\int_0^\infty 1_{\{\theta<s\}}1_{\{s<\tau\}}ds\right) = \int_0^\infty E\left(\pi_s^h 1_{\{s<\tau\}}\right)\,ds \\ &= E\left(\int_0^\tau \pi_s^h\,ds\right). \end{aligned}$$

This proves the lemma. □

According to Lemma 2.2, the expected cost associated to a strategy (h,τ) is the expectation of an adapted functional of the *posterior probability* process (π_t^h). Therefore, it will be natural to express controls as functionals of (π_t^h). We proceed with the analysis of this process.

3 Semimartingale characteristics of (π_t^h) and a verification lemma

For $0 \le u < t$, let $\mu_{u,t}$ be the conditional distribution, given that $\theta = u$, of X restricted to $[0,t]$, and let μ_t be the unconditional distribution of X restricted to $[0,t]$.

Lemma 3.1. *The Radon–Nikodym derivative of $\mu_{u,t}$ with respect to $\mu_{t,t}$ is*

$$\frac{d\mu_{u,t}}{d\mu_{t,t}} = \exp\left(\int_u^t \frac{r}{\sigma^2}\,dX_s - \frac{1}{2}\int_u^t \frac{r^2}{\sigma^2}\,h_s(X)\,ds\right). \tag{3.1}$$

Proof. Recall Girsanov's theorem [10, thm. 8.6.6. p. 166]: let

$$\begin{aligned} dZ_t &= \sigma(Z_t)\,dB_t, \\ d\tilde{Z}_t &= \gamma_t\,dt + \sigma(\tilde{Z}_t)\,dB_t, \end{aligned}$$

and suppose that under P, the process (B_t) is a standard Brownian motion. Define $\tilde{P}$ by

$$\frac{d\tilde{P}}{dP} = \exp\left(-\int_0^t \frac{\gamma_s}{\sigma(\tilde{Z}_s)}\,dB_s - \frac{1}{2}\int_0^t \left(\frac{\gamma_s}{\sigma(\tilde{Z}_s)}\right)^2 ds\right).$$

If $E_P(\frac{d\tilde{P}}{dP}) = 1$, then the law of $(\tilde{Z}_t)$ under $\tilde{P}$ is the same as the law of (Z_t) under P.

If $\theta = u$, then the law of $(X_s,\ s \le t)$ is the same as that of $(Y_s,\ s \le t)$, where

$$dY_s = rh_s(Y)1_{\{u<s\}}\,ds + \sigma\sqrt{h_s(Y)}\,dB_s, \qquad 0 < s < t. \tag{3.2}$$

If $\theta = t$, then the law of $(X_s,\ s \le t)$ is the same as that of $(Z_s,\ s \le t)$, where

$$dZ_s = \sigma\sqrt{h_s(Z)}\,dB_s, \qquad 0 < s < t.$$

Therefore, for $A \in \mathcal{B}(C([0,t],\mathbb{R}))$,

$$\mu_{u,t}(A) = P\{Y_\cdot \in A\} = E_P(1_A(Y_\cdot)) = E_{\tilde{P}}\left(1_A(Y_\cdot)\frac{dP}{d\tilde{P}}\right),$$

where $\tilde{P}$ is defined by

$$\begin{aligned} \frac{d\tilde{P}}{dP} &= \exp\left(-\int_u^t \frac{rh_s(Y)}{\sigma\sqrt{h_s(Y)}}\,dB_s - \frac{1}{2}\int_u^t \left(\frac{rh_s(Y)}{\sigma\sqrt{h_s(Y)}}\right)^2 ds\right) \\ &= \exp\left(-\int_u^t \frac{r}{\sigma^2}\sigma\sqrt{h_s(Y)}\,dB_s - \frac{1}{2}\int_u^t \left(\frac{r}{\sigma}\right)^2 h_s(Y)\,ds\right). \end{aligned}$$

Note in particular that Novikov's condition [10] is satisfied. Using (3.2), we see that this can be written

$$\frac{d\tilde{P}}{dP} = \exp\left(-\int_u^t \frac{r}{\sigma^2}\, dY_s + \frac{1}{2}\int_u^t \left(\frac{r}{\sigma}\right)^2 h_s(Y)\, ds\right).$$

Therefore, by Girsanov's theorem,

$$\begin{aligned} \mu_{u,t}(A) &= E_P\left(1_A(Z.)\exp\left(\int_u^t \frac{r}{\sigma^2}\, dZ_s - \frac{1}{2}\int_u^t \left(\frac{r}{\sigma}\right)^2 h_s(Z)\, ds\right)\right) \\ &= \int_A \mu_{t,t}(d\omega)\exp\left(\int_u^t \frac{r}{\sigma^2}\, dX_s - \frac{1}{2}\int_u^t \left(\frac{r}{\sigma}\right)^2 h_s(X)\, ds\right). \end{aligned}$$

This proves Lemma 3.1. □

Let F_θ denote the probability distribution function of θ, so that

$$F_\theta(x) = \begin{cases} 0, & \text{if } x < 0, \\ \pi_0 + (1-\pi_0)(1-e^{-\lambda x}), & \text{if } x \geq 0. \end{cases}$$

Lemma 3.2. *We have*

$$\pi_t^h = \int_{0-}^t \frac{d\mu_{u,t}}{d\mu_t}\, F_\theta(du) = \frac{d\mu_{t,t}}{d\mu_t}\int_{0-}^t \frac{d\mu_{u,t}}{d\mu_{t,t}}\, F_\theta(du)$$

(note that the $0-$ *accounts for the discontinuity of* F_θ *at* 0*).*

Proof. The notation $\frac{d\mu_{u,t}}{d\mu_{t,t}}$ now refers to the right-hand side of (3.1), which is continuous in u. For the first equality in the lemma, it suffices to show that for all $B \in \mathcal{B}(C([0,t],\mathbb{R}))$,

$$E\left(1_{\{X|_{[0,t]}\in B\}}\int_{0-}^t \frac{d\mu_{u,t}}{d\mu_t}\, F_\theta(du)\right) = E\left(1_{\{X|_{[0,t]}\in B\}}1_{\{\theta\leq t\}}\right).$$

To see this, observe that

$$\begin{aligned} &\int_{\{X|_{[0,t]}\in B\}} dP(\omega)\int_{0-}^t \frac{d\mu_{u,t}}{d\mu_t}(\omega)\, F_\theta(du) \\ &\quad = \int_{0-}^t F_\theta(du)\int_{\{X|_{[0,t]}\in B\}} dP(\omega)\,\frac{d\mu_{u,t}(\omega)}{d\mu_t} \\ &\quad = \int_{0-}^t F_\theta(du)\,\mu_{u,t}\left\{X|_{[0,t]}\in B\right\} \\ &\quad = P\left\{\theta\leq t,\ X|_{[0,t]}\in B\right\}. \end{aligned}$$

This proves the first equality. The second is a consequence of the chain rule for Radon–Nikodym derivatives. □

Lemma 3.3. *We have*

$$1-\pi_t^h=\frac{d\mu_{t,t}}{d\mu_t}\int_t^\infty F_\theta(du)=(1-\pi_0)e^{-\lambda t}\,\frac{d\mu_{t,t}}{d\mu_t}.$$

Proof. As in Lemma 3.2, one checks that

$$P\{\theta>t\mid X_s,\ s\le t\}=\int_t^{+\infty}\frac{d\mu_{u,t}}{d\mu_t}\,F_\theta(du).$$

Since $\frac{d\mu_{u,t}}{d\mu_{t,t}}=1$ when $u>t$, the right-hand side is equal to

$$\int_t^{+\infty}\frac{d\mu_{u,t}}{d\mu_{t,t}}\,\frac{d\mu_{t,t}}{d\mu_t}\,F_\theta(du)=\frac{d\mu_{t,t}}{d\mu_t}\int_t^{+\infty}F_\theta(du).$$

This proves the first equality in the statement of the lemma. The second equality is a consequence of the fact that for $u>0$, $F_\theta(du)=(1-\pi_0)\lambda e^{-\lambda u}\,du$. □

Set

$$\varphi_t^h=\frac{\pi_t^h}{1-\pi_t^h}$$

and let

$$Z_{u,t}=\int_u^t\frac{r}{\sigma^2}\,dX_s-\frac{1}{2}\int_u^t\frac{r^2}{\sigma^2}\,h_s\,ds.$$

Use Lemmas 3.1, 3.2 and 3.3 to see that

$$\begin{aligned}
\varphi_t^h &= \frac{e^{\lambda t}}{1-\pi_0}\int_{0-}^t\exp\left(Z_{u,t}\right)\,F_\theta(du)\\
&= \frac{e^{\lambda t}}{1-\pi_0}\exp\left(Z_{0,t}\right)\int_{0-}^t\exp\left(-Z_{0,u}\right)\,F_\theta(du)\\
&= \frac{e^{\lambda t}}{1-\pi_0}\exp\left(Z_{0,t}\right)\left(\pi_0+(1-\pi_0)\int_0^t\exp\left(-Z_{0,u}\right)\lambda e^{-\lambda u}\,du\right).
\end{aligned}\tag{3.3}$$

Lemma 3.4. *The following s.d.e. is satisfied:*

$$d\varphi_t^h=\lambda(1+\varphi_t^h)\,dt+\frac{r}{\sigma^2}\,\varphi_t^h\,dX_t.\tag{3.4}$$

Proof. Observe from (1.2) that the quadratic variation of X_t is $d\langle X\rangle_t = \sigma^2 h_t\, dt$, so we can apply Itô's formula and (3.3) to get

$$\begin{aligned} d\varphi_t^h &= \lambda \varphi_t^h\, dt \\ &\quad + \frac{e^{\lambda t}}{1-\pi_0} \exp(Z_{0,t}) \left(\frac{r}{\sigma^2}\, dX_t - \frac{r^2 h_t}{2\sigma^2}\, dt + \frac{1}{2}(\frac{r}{\sigma^2})^2 \cdot \sigma^2 h_t\, dt \right) \int_{0-}^t \exp(-Z_{0,u})\, F_\theta(du) \\ &\quad + e^{\lambda t} \exp(Z_{0,t}) \exp\left(-Z_{0,t}\right) \lambda e^{-\lambda t} dt \\ &= \lambda(1+\varphi_t^h)\, dt + \frac{r}{\sigma^2}\, \varphi_t^h\, dX_t. \end{aligned}$$

□

Lemma 3.5. *The process $X = (X_t)_{t\geq 0}$ has the stochastic differential*

$$dX_t = r h_t(X) \pi_t\, dt + \sigma \sqrt{h_t}\, d\bar{B}_t,$$

where $(\bar{B}_t)$ is a standard Brownian motion.

Proof. Observe that

$$dX_t - r h_t \pi_t\, dt = (r h_t 1_{\{\theta < t\}} - r h_t \pi_t)\, dt + \sigma \sqrt{h_t}\, dB_t,$$

and the right-hand side has mean zero (given $X|_{[0,t]}$) and quadratic variation $\sigma^2 h_t\, dt$. Further, the left-hand side is adapted to $\mathcal{F}^X$, so that the right-hand side is too, and has mean zero. In particular, it is the differential of a local $\mathcal{F}^X$-martingale with quadratic variation $\sigma\sqrt{h_t}\, dt$. According to [8, Chapter 3, Theorem 4.2], this term is equal to $\sigma\sqrt{h_t}$ times a standard Brownian motion increment. We note for future reference that $(\bar{B}_t)$ need not be $\mathcal{F}^X$-adapted, but the martingale

$$M_t = \sigma \int_0^t \sqrt{h_s}\, d\bar{B}_t = X_t - \int_0^t r h_s(X) \pi_s ds \tag{3.5}$$

is clearly $\mathcal{F}^X$-adapted. □

Lemma 3.6. *Set $\rho = \frac{r}{\sigma}$. Then*

$$d\pi_t^h = \lambda(1-\pi_t^h)\, dt + \frac{r}{\sigma^2} \pi_t^h (1-\pi_t^h)\, dX_t - \frac{r^2}{\sigma^2} (\pi_t^h)^2 (1-\pi_t^h) h_t\, dt \tag{3.6}$$

and

$$d\pi_t^h = \lambda(1-\pi_t^h)\, dt + \rho \pi_t^h (1-\pi_t^h) \sqrt{h_t}\, d\bar{B}_t. \tag{3.7}$$

Proof. Note that $\pi_t^h = \varphi_t^h(1+\varphi_t^h)^{-1} = f(\varphi_t^h)$, where $f(x) = x(1+x)^{-1}$. Since $f'(x) = (1+x)^{-2}$ and $f''(x) = -2(1+x)^{-3}$, Itô's formula and Lemma 3.4 yield

$$\begin{aligned} d\pi_t^h &= f'(\varphi_t^h)\, d\varphi_t^h + \frac{1}{2} f''(\varphi_t^h)\, d\langle \varphi^h \rangle_t \\ &= \frac{1}{(1+\varphi_t^h)^2} \left(\lambda(1+\varphi_t^h)\, dt + \frac{r}{\sigma^2} \varphi_t^h\, dX_t \right) + \frac{1}{2} \frac{-2}{(1+\varphi_t^h)^3} \left(\frac{r}{\sigma^2}\, \varphi_t^h \right)^2 \sigma^2 h_t\, dt. \end{aligned}$$

Recall that $1 + \varphi_t^h = \frac{1}{1-\pi_t}$ to see that this is equal to

$$\lambda(1-\pi_t^h)\, dt + \frac{r}{\sigma^2} \pi_t^h (1-\pi_t^h)\, dX_t - \frac{r^2}{\sigma^2} (\pi_t^h)^2 (1-\pi_t^h) h_t\, dt,$$

which establishes (3.6). By Lemma 3.5, this is equal to

$$\lambda(1-\pi_t^h) dt + \frac{r}{\sigma^2} \pi_t^h (1-\pi_t^h) \left(r\, h_t \pi_t^h\, dt + \sigma \sqrt{h_t}\, d\bar{B}_t \right) - \frac{r^2}{\sigma^2} (\pi^h)_t^2 (1-\pi_t^h) h_t\, dt,$$

which simplifies to

$$\lambda(1-\pi_t^h)\, dt + \frac{r}{\sigma} \pi_t^h (1-\pi_t^h) \sqrt{h_t}\, d\bar{B}_t.$$

This establishes (3.7). □

Strategies expressed in terms of (π_t^h)

According to (3.7), (π_t^h) is a diffusion process, and therefore an optimal canonical control will typically be expressed as a function of π_t^h, that is, we will mainly be interested in controls $h_t(X)$ of the form $h_t(X) = h(t, \pi_t^h)$, where $h\colon \mathbb{R}_+ \times [0,1] \to [0,1]$ is measurable and given. We explain here how to describe the observation process and the admissible control (h_t) associated with such a function h.

Consider the s.d.e.

$$\begin{aligned} dp_t = \lambda(1-p_t) dt &+ \frac{r}{\sigma^2} p_t (1-p_t) \left(r h(t,p_t) 1_{\{\theta \le t\}}\, dt + \sigma \sqrt{h(t,p_t)}\, dB_t \right) \\ &- \frac{r^2}{\sigma^2} (p_t)^2 (1-p_t) h(t,p_t)\, dt, \end{aligned} \tag{3.8}$$

with $p_0 = P\{\theta = 0\}$. Assume that h is such that (3.8) has a strong solution (that is, an $(\mathcal{F}_t)$-adapted solution). Then we define the observation process by $X_0 = 0$ and

$$dX_t = r h(t,p_t) 1_{\{\theta \le t\}}\, dt + \sigma \sqrt{h(t,p_t)}\, dB_t. \tag{3.9}$$

This process is adapted to $(\mathcal{F}_t)$, and by (3.8),

$$dp_t = \lambda(1-p_t)dt + \frac{r}{\sigma^2}p_t(1-p_t)\,dX_t - \frac{r^2}{\sigma^2}(p_t)^2(1-p_t)h(t,p_t)\,dt. \tag{3.10}$$

Let $q_t = p_t/(1-p_t)$. Applying Itô's formula, we find that

$$dq_t = \lambda(1+q_t)dt + \frac{r}{\sigma^2}q_t\,dX_t. \tag{3.11}$$

According to [14, Chapter IX, (2.3)], the solution of this linear s.d.e. is

$$\begin{aligned} q_t = \exp&\left(\frac{r}{\sigma^2}X_t + \lambda t - \frac{1}{2}\frac{r^2}{\sigma^4}\langle X\rangle_t\right) \\ &\times \left[q_0 + \int_0^t \exp\left(-\frac{r}{\sigma^2}X_s - \lambda s + \frac{1}{2}\frac{r^2}{\sigma^4}\langle X\rangle_s\right)\lambda\,ds\right]. \end{aligned}$$

In particular, q_t, and therefore p_t, is a function of $X|_{[0,t]}$ and we can write $p_t = \hat{h}_t(X)$, where $(t,x) \mapsto \hat{h}_t(x)$ from $\mathbb{R}_+ \times C(\mathbb{R}_+,\mathbb{R})$ is progressively measurable. Looking back to (3.9), we see that (X_t) is a strong solution of the s.d.e.

$$dX_t = rh_t(X)1_{\{\theta\le t\}}\,dt + \sigma\sqrt{h_t(X)}\,dB_t, \tag{3.12}$$

where $h_t(x) = h(t,\hat{h}_t(x))$. Therefore, (h_t) is an admissible control.

Comparing (3.11) and (3.4), we conclude that $q_t = \varphi_t^h$ and therefore

$$p_t = \pi_t^h = P\{\theta \le t \mid \mathcal{F}_t^X\}. \tag{3.13}$$

This means that the control $h_t(X)$ is indeed equal to $h(t,\pi_t^h)$.

We note that as in (3.7), there is a Brownian motion $(\bar{B}_t)$ such that

$$dp_t = \lambda(1-p_t)\,dt + \rho p_t(1-p_t)\sqrt{h(t,p_t)}\,d\bar{B}_t. \tag{3.14}$$

If τ is a stopping time defined using π_t^h, for instance,

$$\tau = \inf\{t \ge 0 : \pi_t^h \in S\} \tag{3.15}$$

for some Borel set $S \subset [0,1]$, then

$$\tau = \inf\{t \ge 0 : \hat{h}_t(X) \in S\},$$

so $\tau = \tau(X)$ is a canonical stopping time. In particular, $((h_t(x)),\tau(x))$ is a strategy.

The above discussion shows that if (3.8) has a strong solution, then we can construct a strategy $((h_t), \tau)$ for which (2.1) or (3.12) admits a strong solution (X_t), such that $p_t = \pi_t^h$ and the expected cost $E(C((h_t), \tau))$ is given by (2.7).

In the case where (3.8) admits a weak solution, we would similarly conclude that (2.2) or (3.12) admits a weak solution, and considering τ as in (3.15), we would conclude that $((h_t), \tau, X)$ is a control system with the same expected cost.

Verification lemma

For $\pi \in [0, 1]$, let E_π denote expectation in the case where $\pi_0 = \pi$. Recall that we have defined

$$\tilde{g}(\pi) = \inf_{(h,\tau)} E_\pi(C(h, \tau)), \qquad \tilde{g}^{\mathrm{w}}(\pi_0) = \inf_{(h,\tau,X)} E(C(h, \tau, X)).$$

By Lemma 2.1, $\tilde{g}$ is concave, and by Lemma 2.2,

$$\tilde{g}(\pi) = \inf_{(h,\tau)} E_\pi \left(1 - \pi_\tau^h + a \int_0^\tau \pi_s^h \, ds + b \int_0^\tau h_s \, ds\right),$$

with a similar properties for $\tilde{g}^{\mathrm{w}}$. According to [4, Theorem 3.67], we expect to be able to characterize each of these two functions as a function g^* with certain properties concerning martingales and submartingales. The next lemma gives conditions that will allow us to show that a function g^* is equal to $\tilde{g}$ (resp. $\tilde{g}^{\mathrm{w}}$) and check that a strategy $((h_t^*), \tau^*)$ (resp. a control system $((h_t^*), \tau^*, X^*)$) is optimal.

Lemma 3.7. *(Verification Lemma)*

Suppose that g^* is a bounded continuous function defined on $[0, 1]$ such that $0 \le g^*(x) \le 1 - x$, $x \in [0, 1]$.

(1) *Suppose that for any $\pi \in [0, 1]$, the following property holds:*

(a) *for any strategy $((h_t), \tau)$ (resp. for any control system (h, τ, X)), the process (Y_t) is an $\mathcal{F}^X$-submartingale under P_π, where*

$$Y_t = g^*(\pi_t^h) + a \int_0^t \pi_s^h \, ds + b \int_0^t h_s \, ds. \tag{3.16}$$

Then $g^* \le \tilde{g}$ (resp. $g^* \le \tilde{g}^{\mathrm{w}}$).

(2) *Suppose that for any $\pi \in [0, 1]$, in addition to (a), the following three properties hold:*

(b) *for the strategy $((h_t^*), \tau^*)$ (resp. the control system $((h_t^*), \tau^*, X^*)$), the process $(Y_{t \wedge \tau^*}^*)$ is an $\mathcal{F}^X$-martingale under P_π, where*

$$Y_t^* = g^*(\pi_t^{h^*}) + a \int_0^t \pi_s^{h^*} \, ds + b \int_0^t h_s^* \, ds;$$

(c) $E_\pi(\tau^*) < +\infty$;

(d) $g^*(\pi_{\tau^*}^{h^*}) = 1 - \pi_{\tau^*}^{h^*}$.

Then $g^* = \tilde{g}$ *and* $((h_t^*), \tau^*)$ *is an optimal strategy (resp.* $g^* = \tilde{g}^{\mathrm{w}}$ *and* $((h_t^*), \tau^*, X^*)$ *is an optimal control system).*

Proof. We first establish (1). Let $((h_t), \tau)$ be a strategy. If $E(\tau) = +\infty$, then $E(C(h,\tau)) = +\infty$. Indeed, by (2.4), $E(C(h,\tau)) \geq a\ E(\tau\, 1_{\{\tau>\theta\}}) - aE(\theta)$. Since

$$E(\tau) = E(\tau\, 1_{\{\tau>\theta\}}) + E(\tau\, 1_{\{\tau\leq\theta\}})$$

and the second term is no greater than $E(\theta) < +\infty$, we conclude that $E(\tau\, 1_{\{\tau>\theta\}}) = +\infty$ and so $E(C(h,\tau)) = +\infty$.

Therefore, in the definition of $\tilde{g}$, we can restrict the infimum to those strategies for which $E(\tau) < +\infty$. Since $1 - x \geq g^*(x)$, Lemma 2.2 implies that

$$E_\pi(C(h,\tau)) \geq E(Y_\tau).$$

Since (Y_t) is a submartingale by (a) and $t \wedge \tau$ is a bounded stopping time, $E_\pi(Y_{t\wedge\tau}) \geq E_\pi(Y_0) = g^*(\pi)$. By Fatou's lemma in the form $E(\limsup Y_{n\wedge\tau}) \geq \limsup E(Y_{n\wedge\tau})$ (cf. [1, Chapter 1], which applies since $E(\tau) < +\infty$, we see that

$$E_\pi(Y_\tau) \geq \limsup_{t\to\infty} E_\pi(Y_{t\wedge\tau}) \geq g^*(\pi).$$

We conclude that $E_\pi(C(h,\tau)) \geq g^*(\pi)$ for all strategies $((h_t), \tau)$, and therefore $\tilde{g} \geq g^*$. The proof for $\tilde{g}^{\mathrm{w}}$ is identical and is omitted.

We now establish (2) for $\tilde{g}$. It suffices to show that $g^*(\pi) = E_\pi(Y_{\tau^*}^*)$. Indeed, this will complete the proof, since by (d) and Lemma 2.2,

$$\begin{aligned} g^*(\pi) &= E_\pi(Y_{\tau^*}^*) = E_\pi\left(g^*(\pi_{\tau^*}^{h^*}) + a\int_0^{\tau^*} \pi_s^{h^*}\,ds + b\int_0^{\tau^*} h_s^*\,ds\right) \\ &= E_\pi(C(h^*,\tau^*)) \geq \tilde{g}(\pi). \end{aligned}$$

Since we have already proved that $\tilde{g} \geq g^*$, this shows that $g^*(\pi) = \tilde{g}(\pi)$.

In order to check that $g^*(\pi) = E_\pi(Y_{\tau^*}^*)$, note that $0 \leq Y_t^* \leq 1+(a+b)t$ and $E_\pi(\tau^*) < +\infty$ by (c). Therefore, $(Y_{t\wedge\tau^*}^*)$, which is a martingale by (b), is uniformly integrable. By the Optional Sampling Theorem [3], $E(Y_\tau^*) = E(Y_0^*) = g^*(\pi)$. This completes the proof for $\tilde{g}$. The proof for $\tilde{g}^{\mathrm{w}}$ is identical and is omitted. □

4 A candidate for the value function

We now seek analytical conditions on a function g^* that will guarantee the properties of Lemma 3.7. Consider the process (Y_t) defined in (3.16) (we write g instead of g^* to simplify the notation). By Itô's formula and Lemma 3.6,

$$\begin{aligned} dY_t &= g'(\pi_t^h)\, d\pi_t^h + \frac{1}{2} g''(\pi_t^h)\, d\langle \pi^h \rangle_t + a\pi_t^h\, dt + bh_t\, dt \\ &= \left[\lambda g'(\pi_t^h)(1-\pi_t^h) + \frac{1}{2} g''(\pi_t^h)(\rho \pi_t^h (1-\pi_t^h))^2 h_t + a\pi_t^h + bh_t \right] dt \\ &\qquad + g'(\pi_t^h) \frac{r}{\sigma} \pi_t^h (1-\pi_t^h) \sqrt{h_t}\, d\bar{B}_t. \end{aligned} \tag{4.1}$$

Therefore, (Y_t) will be a submartingale if the term in brackets is nonnegative, for any value of h_t. Since this term is an affine function of h_t, this is equivalent to this term being nonnegative for $h_t = 0$ and $h_t = 1$, that is, for all $x \in [0,1]$,

$$\lambda g'(x)(1-x) + ax \geq 0 \tag{4.2}$$

and

$$\lambda g'(x)(1-x) + \frac{1}{2} g''(x)(\rho x(1-x))^2 + ax + b \geq 0. \tag{4.3}$$

Intuition and smooth fit

We can imagine that the optimal strategy, in either the strong or the weak formulation, is of the following form: do not observe if π_t^h is small, declare the alarm if π_t^h is close to 1, and observe otherwise. More precisely, we postulate that there are two constants $0 \leq A \leq B \leq 1$ such that on $[0, A]$, it is optimal *not* to observe, on $]A, B[$ it is optimal to observe without declaring an alarm, and on $[B, 1]$, it is optimal to stop and declare the alarm. That is,

$$h_t^* = 1_{\{\pi_t^{h^*} > A\}} \qquad \text{and} \qquad \tau^* = \inf\{t \geq 0 : \pi_t^{h^*} \geq B\}. \tag{4.4}$$

In order to satisfy condition (b) of Lemma 3.7, we need

$$\lambda g'(x)(1-x) + ax = 0, \qquad x \in]0, A], \tag{4.5}$$

and

$$\lambda g'(x)(1-x) + ax + \frac{1}{2} g''(x) \rho^2 x^2 (1-x)^2 + b = 0, \qquad x \in]A, B[. \tag{4.6}$$

In order to satisfy condition (d) of Lemma 3.7, we need

$$g(x) = 1 - x, \qquad x \in [B, 1]. \tag{4.7}$$

In order to find an expression for g, it is natural to solve first the differential equations (4.5) and (4.6) separately, that is, to seek two functions g_1 and g_2 such that

$$\lambda g_1'(x)(1-x) + ax = 0, \qquad 0 < x < A, \tag{4.8}$$

and

$$\lambda g_2'(x)(1-x) + ax + \frac{1}{2} g_2''(x)\rho^2 x^2 (1-x)^2 + b = 0, \qquad A < x < B. \tag{4.9}$$

Three constants of integration will appear, one for g_1 and two for g_2. These constants can then be determined by "pasting together" g_1 and g_2, that is, requiring equalities such as

$$g_1(A) = g_2(A) \tag{4.10}$$

and, by (4.7),

$$g_2(B) = 1 - B. \tag{4.11}$$

These two equalities are referred to as "continuous fit" [12]. As in most problems of optimal stopping or control, they are not sufficient to determine the five unknown constants, namely, the three constants of integration and the two "free boundaries" A and B. For this, it is necessary to use a version of the "principle of smooth fit" (see [12]). In particular, one can postulate that

$$g_2'(B) = -1 \tag{4.12}$$

and

$$g_1'(A) = g_2'(A). \tag{4.13}$$

We need one more equation in addition to (4.10)-(4.13), since there are five unknown constants. Since we want to apply Itô's formula, it is natural to want g to be twice differentiable at A. This gives one more equation:

$$g_1''(A) = g_2''(A). \tag{4.14}$$

Solving the equations

We seek functions g_1 and g_2 defined on $[0, 1]$ satisfying (4.8)–(4.14). Set

$$f_1(x) = g_1'(x), \qquad f_2(x) = g_2'(x).$$

The value of A

For $0 < x < A$, differentiate (4.8) to get

$$-\lambda f_1(x) + \lambda f_1'(x)(1-x) + a = 0,$$

that is,

$$f_1'(x) = \frac{\lambda f_1(x) - a}{\lambda(1-x)}\,. \tag{4.15}$$

From (4.9), we get

$$f_2'(x) = \frac{-ax - b - \lambda f_2(x)(1-x)}{\frac{1}{2}\rho^2 x^2 (1-x)^2}. \tag{4.16}$$

By (4.14), if we plug $x = A$ into (4.15), (4.16), we get

$$-\lambda(aA+b) - \lambda^2 f_2(A)(1-A) = -\frac{a}{2}\rho^2\, A^2(1-A) + \frac{\lambda\rho^2}{2} A^2(1-A) f_1(A).$$

Since $f_2(A) = f_1(A)$ by (4.13), we solve for $f_1(A)$:

$$f_1(A) = \frac{\frac{a\rho^2}{2}A^2(1-A) - \lambda(aA+b)}{(1-A)(\lambda^2 + \frac{\lambda\rho^2}{2}A^2)}. \tag{4.17}$$

Plugging (4.17) into (4.8) gives an equation for A, whose solution is

$$A = \sqrt{\frac{2\lambda b}{a\rho^2}}\,. \tag{4.18}$$

For the observation region $]A, B[$ to be non-empty, we must have $A < 1$, but further, since we want g_1 to be concave by Lemma 2.1, we also must have

$$f_1(A) = g_1'(A) > -1. \tag{4.19}$$

From (4.8),

$$g_1'(x) = -\frac{a}{\lambda}\frac{x}{1-x}, \tag{4.20}$$

so (4.8) and (4.19) give

$$-\frac{a}{\lambda}\frac{A}{1-A} > -1,$$

or equivalently,

$$A < \frac{\lambda}{a+\lambda}.$$

With (4.18), we conclude that the observation region $]A, B[$ is not empty if

$$b < \frac{\lambda a \rho^2}{2(a+\lambda)^2}. \tag{4.21}$$

Determining $f_2(x)$

For $A < x < B$, equation (4.9) becomes

$$\lambda f_2(x)(1-x) + ax + \frac{1}{2} f_2'(x)\rho^2 x^2 (1-x)^2 + b = 0. \tag{4.22}$$

A solution of the homogeneous equation

$$\lambda f(x)(1-x) + \frac{1}{2} f'(x)\rho^2 x^2 (1-x)^2 = 0$$

is

$$f(x) = \left(\frac{1-x}{x}\right)^{\alpha} e^{\alpha/x}, \qquad \text{where } \alpha = \frac{2\lambda}{\rho^2}. \tag{4.23}$$

Therefore, the solution of the inhomogeneous equation (4.22) is

$$f_2(x) = K_1 f(x) + f(x) \int_A^x \frac{-2}{\rho^2} \frac{ay+b}{y^2(1-y)^2} \frac{1}{f(y)}\, dy. \tag{4.24}$$

From (4.13) and (4.8), we conclude that

$$K_1 = -\frac{a}{\lambda} \frac{A}{1-A} \frac{1}{f(A)}. \tag{4.25}$$

Formulas (4.25) and (4.24) together determine $f_2(x)$.

Remark 4.1. In the case where $b = 0$, then $A = 0$ by (4.18), and we must have $K_1 = 0$ in order that $f_2(x)$ be bounded. This recovers the case discussed in [15, Chapter 4.4]. Therefore, we consider the case $b > 0$.

Determining B

Observe that

$$\lim_{x\to 1} f(x) = 0 \quad \text{and} \quad \lim_{x\to 1} f_2(x) = -\infty.$$

Indeed, the first equality is obvious and the second holds because for x near 1,

$$f(x) \sim (1-x)^{\alpha},$$

and, using l'Hopital's rule,

$$\begin{aligned} f_2(x) &\sim \frac{-2}{\rho^2}(1-x)^{\alpha} \int_A^x \frac{(a+b)e^{-\alpha}}{(1-y)^{2+\alpha}}\, dy \sim -(1-x)^{1+\alpha}(1-x)^{-2-\alpha} \\ &\sim -(1-x)^{-1}. \end{aligned}$$

Therefore, if (4.21) holds, then $f_2(A) = K_1 f(A) = -\frac{a}{\lambda}\frac{A}{1-A} > -1$, so there is $B \in]A, 1[$ such that

$$f_2(B) = -1. \tag{4.26}$$

With this choice of B, (4.12) is satisfied. The next lemma shows that in fact, there is only one solution to (4.26).

Lemma 4.2. *The function f_2 defined in* (4.24) *is strictly decreasing on $[A, 1[$, and therefore, there is a unique $B \in]A, 1[$ satisfying* (4.26)*.*

Proof. By (4.22),

$$f_2'(x) = \frac{2\lambda}{\rho^2}\frac{1}{x^2(1-x)}\,(\psi(x) - f_2(x)), \tag{4.27}$$

where

$$\psi(x) = -\frac{ax+b}{\lambda(1-x)}.$$

Therefore, $f_2'(x) < 0$ if and only if $\psi(x) < f_2(x)$. In fact, we will see in (4.51) (see also (4.32)) that

$$f_2(x) > -\frac{a}{\lambda}\frac{x}{1-x} > \psi(x), \qquad x \in]A, 1[.$$

We conclude that $f_2'(x) < 0$ for $x \in]A, 1[$, and this proves the lemma. □

Determining $g_2(x)$

Because $g_2'(x) = f_2(x)$, $g_2(x)$ can be written

$$g_2(x) = \int_A^x f_2(y)\,dy + K_2. \tag{4.28}$$

From (4.11), we see that

$$K_2 = 1 - B - \int_A^B f_2(y)\,dy, \tag{4.29}$$

so that

$$g_2(x) = \int_B^x f_2(y)\,dy + 1 - B. \tag{4.30}$$

Determining $g_1(x)$

Because $g_1'(x) = f_1(x)$, $g_1(x)$ can be written

$$g_1(x) = \int_A^x f_1(y)\,dy + K_3, \tag{4.31}$$

where $f_1(x)$ is determined from (4.8):

$$f_1(x) = -\frac{a}{\lambda}\frac{x}{1-x}. \tag{4.32}$$

From (4.10), (4.28) and (4.31), we get

$$K_3 = K_2. \tag{4.33}$$

We can perform the integration in (4.31) to get

$$g_1(x) = \frac{a}{\lambda}(x + \ln(1-x) - A - \ln(1-A)) + K_2, \tag{4.34}$$

with K_2 determined by (4.29).

We have now found two functions g_1 and g_2 that solve (4.8)–(4.14). In order to ensure that this solves our optimal control problem, slightly more is needed: in particular, we need the inequalities (4.2) and (4.3) for all $x \in [0,1]$. Set

$$L_1 g(x) = \lambda\, g'(x)(1-x) + ax, \tag{4.35}$$
$$L_2 g(x) = \lambda\, g'(x)(1-x) + ax + \frac{1}{2} g''(x)\rho^2 x^2 (1-x)^2 + b. \tag{4.36}$$

Proposition 4.3. *(Candidate value function) Suppose that* $0 < b < \lambda a \rho^2/(2(a+\lambda)^2)$*. Define* $g(x)$ *on* $[0,1]$ *by*

$$g(x) = \begin{cases} g_1(x) & \text{if } 0 \le x \le A, \\ g_2(x) & \text{if } A \le x \le B, \\ 1-x & \text{if } B \le x \le 1, \end{cases} \tag{4.37}$$

where A *is defined in* (4.18) *and* B *is defined in* (4.26)*. Then* g *is strictly concave in* $[0,B]$*, and*

$$0 \le g(x) \le 1-x, \qquad 0 \le x \le 1, \tag{4.38}$$
$$L_1 g(x) = 0, \qquad 0 \le x \le A, \tag{4.39}$$
$$L_2 g(x) = 0, \qquad A \le x < B. \tag{4.40}$$

Furthermore,

$$L_2 g(x) \ge 0, \qquad 0 \le x \le A, \tag{4.41}$$
$$L_1 g(x) \ge 0, \qquad A \le x \le B, \tag{4.42}$$
$$L_1 g(x) \ge 0, \qquad B \le x \le 1, \tag{4.43}$$
$$L_2 g(x) \ge 0, \qquad B \le x \le 1. \tag{4.44}$$

Proof. Properties (4.39) and (4.40) follow from the construction of g_1 and g_2 (see (4.8) and (4.9)). The strict concavity of g_1 and g_2 (hence of g on $[0, B]$) follow from (4.32) and Lemma 4.2. This concavity property and (4.26) imply $g(x) \leq 1-x$, $0 \leq x \leq 1$. Finally, since $g_1'(x) = f_1(x) < 0$ for $0 < x \leq A$ and $g_2'(x) = f_2(x) \leq 0$ for $A \leq x \leq B$, g is non-decreasing on $[0, B]$, therefore nonnegative on $[0, B]$ since $g_2(B) = 1 - B \geq 0$. This proves (4.38).

Note that (4.43) implies (4.44), and on $[B, 1]$, (4.43) becomes

$$-\lambda(1 - x) + ax \geq 0,$$

that is,

$$x \geq \frac{\lambda}{a + \lambda}.$$

Therefore, (4.43) will hold provided we show that

$$B \geq \frac{\lambda}{a + \lambda}. \tag{4.45}$$

To see this, note from (4.32) that

$$f_1\left(\frac{\lambda}{a + \lambda}\right) = -\frac{a}{\lambda} \cdot \frac{\frac{\lambda}{a+\lambda}}{1 - \frac{\lambda}{a+\lambda}} = -1, \tag{4.46}$$

We shall show that

$$f_2(x) \geq f_1(x), \qquad \text{for } x \geq A. \tag{4.47}$$

Then, (4.47) and (4.46) imply that

$$f_2\left(\frac{\lambda}{a + \lambda}\right) \geq -1, \qquad \text{that is,} \qquad B \geq \frac{\lambda}{a + \lambda}$$

(since $f_2(x) < -1$ for $x > B$, by (4.26) and Lemma 4.2), proving (4.45).

It remains to prove (4.47). Set $h(x) = f_2(x) - f_1(x)$. From (4.8) and (4.9), we see that for $x > A$,

$$\lambda h(x)(1 - x) + \frac{1}{2}h'(x)\rho^2 x^2 (1 - x)^2 + b + \frac{1}{2}f_1'(x)\rho^2 x^2 (1 - x)^2 = 0. \tag{4.48}$$

By (4.20),

$$f_1'(x) = -\frac{a}{\lambda}\frac{1}{(1 - x)^2}, \tag{4.49}$$

so (4.48) becomes

$$\lambda h(x)(1 - x) + \frac{1}{2}h'(x)\rho^2 x^2 (1 - x)^2 + b - \frac{a\rho^2}{2\lambda}x^2 = 0. \tag{4.50}$$

Recall from (4.18) that $b - \frac{a\rho^2}{2\lambda}x^2 < 0$ for $x > A$. We note that $h(A) = h'(A) = 0$ by (4.13) and (4.14), and from (4.50), the following holds: for $x > A$, it is not possible to have simultaneously $h(x) < 0$ and $h'(x) < 0$. Since $h(A) = 0$, this implies that for $x > A$, $h(x)$ cannot be negative (since otherwise, there would be $y \in]A, x[$ with $h(y) < 0$ and $h'(y) < 0$), therefore $h(x) > 0$ for $x > A$, that is,

$$f_2(x) > f_1(x) \qquad \text{for } x > A. \tag{4.51}$$

This proves (4.47). Therefore, (4.43) is proved.

To check (4.41), we use (4.8), to see that for $0 \le x \le A$,

$$L_2 g(x) = \frac{1}{2} g_1''(x) \rho^2 x^2 (1-x)^2 + b,$$

and from (4.49),

$$g_1''(x) = -\frac{a}{\lambda} \frac{1}{(1-x)^2},$$

therefore,

$$L_2 g(x) = -\frac{a}{2\lambda} \rho^2 x^2 + b, \qquad x \le A,$$

and the right-hand side is nonnegative for $x \le A$ by (4.18). This proves (4.41).

Finally, (4.42) is a consequence of (4.47), since (4.47) implies that

$$L_1 g_2(x) \ge L_1 g_1(x) = 0.$$

□

Case where $b \ge \frac{\lambda a \rho^2}{2(a+\lambda)^2}$

In this case, we postulate that the observation region $]A, B[$ is empty (i.e. $B = A$), so we seek $g_1(x)$ such that

$$\lambda g_1'(x)(1-x) + ax = 0, \qquad 0 \le x < B, \tag{4.52}$$

$$g_1(B) = 1 - B \tag{4.53}$$

$$g_1'(B) = -1. \tag{4.54}$$

From (4.52), we see that

$$g_1'(x) = -\frac{a}{\lambda}\frac{x}{1-x} = \frac{a}{\lambda}\left(1 - \frac{1}{1-x}\right), \tag{4.55}$$

so for some constant K to be determined,

$$g_1(x) = K + \frac{a}{\lambda}x + \frac{a}{\lambda}\ln(1-x). \tag{4.56}$$

From (4.55) and (4.54), we see that

$$\frac{a}{\lambda}\left(1 - \frac{1}{1-B}\right) = -1,$$

that is,

$$B = \frac{\lambda}{a+\lambda}. \tag{4.57}$$

From (4.53) and (4.56), we obtain

$$\begin{aligned} K &= 1 - B - \frac{a}{\lambda}B - \frac{a}{\lambda}\ln(1-B) \\ &= -\frac{a}{\lambda}\ln\left(\frac{a}{a+\lambda}\right), \end{aligned}$$

Therefore,

$$g_1(x) = \frac{a}{\lambda}x + \frac{a}{\lambda}\left(\ln(1-x) - \ln\left(\frac{a}{a+\lambda}\right)\right). \tag{4.58}$$

We note that $g_1'(x)$ is decreasing, $g_1'(0) = 0$ and $g_1'(B) = -1$, so $1 - x \geq g_1(x)$ for $0 \leq x \leq B$, by (4.53). Since $1 - x \geq a/(a+\lambda) = 1 - B$ for $x \leq B$, $g_1(x) \geq 0$ for $0 \leq x \leq B$.

Proposition 4.4. *(Candidate value function) Suppose that* $b \geq \lambda a\rho^2/(2(a+\lambda)^2)$*. Define* $g_1(x)$ *as in* (4.58) *and* $g(x)$ *on* $[0,1]$ *by*

$$g(x) = \begin{cases} g_1(x) & \text{if } 0 \leq x \leq B, \\ 1-x & \text{if } B \leq x \leq 1, \end{cases} \tag{4.59}$$

where B *is defined in* (4.57)*. Then* g *is strictly concave on* $[0,B]$,

$$0 \leq g(x) \leq 1-x, \qquad 0 \leq x \leq 1, \tag{4.60}$$

$$L_1 g(x) = 0, \qquad x \in [0,B], \tag{4.61}$$

and furthermore,

$$L_1 g(x) \geq 0, \qquad \text{for } B \leq x \leq 1, \tag{4.62}$$

$$L_2 g(x) \geq 0, \qquad \text{for } 0 \leq x \leq 1. \tag{4.63}$$

Proof. Property (4.61) follows from (4.52), and the strict concavity of g_1, hence of g, on $[0, B]$ and (4.60) are established just after (4.58).

Note that for $B \leq x \leq 1$,

$$\begin{aligned} L_1 g(x) \geq 0 &\iff -\lambda(1-x) + ax \geq 0 \\ &\iff x \geq \frac{\lambda}{a+\lambda} = B, \end{aligned}$$

and this is indeed that case, so (4.62) holds.

For $B \leq x \leq 1$, $L_2 g(x) = L_1 g(x) + b$, and both of these terms are nonnegative, so $L_2 g(x) \geq 0$ for these x, proving part of (4.63).

For $0 < x < B$,

$$L_2 g(x) = L_2 g_1(x) = L_1 g_1(x) + \frac{1}{2} g_1''(x) \rho^2 x^2 (1-x)^2 + b.$$

Since $L_1 g_1(x) = 0$,

$$\begin{aligned} L_2 g(x) &= \frac{1}{2} \frac{a}{\lambda} \frac{-1}{(1-x)^2} \rho^2 x^2 (1-x)^2 + b \\ &= -\frac{a\rho^2 x^2}{2\lambda} + b, \end{aligned}$$

so

$$L_2 g(x) \geq 0 \iff \frac{a\rho^2 x^2}{2\lambda} \leq b \iff x \leq \sqrt{\frac{2\lambda b}{a\rho^2}}.$$

This will hold for $x \leq B$ provided it holds for $x = B$. Now

$$B \leq \sqrt{\frac{2\lambda b}{a\rho^2}} \iff \left(\frac{\lambda}{a+\lambda}\right)^2 \leq \frac{2\lambda b}{a\rho^2} \iff b \geq \frac{\lambda a \rho^2}{2(a+\lambda)^2},$$

which is the assumption of this Case. This proves (4.63). □

Comments on the optimal strategy

In the case where $b \geq \lambda a \rho^2 / (2(a+\lambda)^2)$, the observation region is empty, the candidate optimal control is $h_t^* \equiv 0$ (with this control, (2.2) obviously has a strong solution), and the candidate optimal stopping time is

$$\tau^* = \inf\{t \geq 0 : \pi_t^* \geq B\}, \tag{4.64}$$

where (π_t^*) is defined by

$$d\pi_t^* = \lambda(1-\pi_t^*)dt, \qquad \pi_0^* = \pi_0 \tag{4.65}$$

(so $\pi_t^* = \pi_t^{h^*}$, where $(\pi_t^{h^*})$ is defined in (3.6) with h there replaced by h^*). It is straightforward to check that (h^*, τ^*) is indeed an optimal strategy (both in the weak and strong formulations), and we do this in Section 5 in the proof of Theorem 5.1.

On the other hand, in the case where $b < \lambda a\rho^2/(2(a+\lambda)^2)$, the optimal strategy should take the form mentioned in (4.4):

$$h_t^* = 1_{\{\pi_t^* > A\}} \qquad \text{and} \qquad \tau^* = \inf\{t \geq 0 : \pi_t^* \geq B\}, \tag{4.66}$$

where the law of (π_t^*) should be determined by the diffusion equation

$$d\pi_t^* = \lambda(1-\pi_t^*)dt + \rho\pi_t^*(1-\pi_t^*)1_{\{\pi_t^* > A\}}d\bar{B}_t, \tag{4.67}$$

or, looking back to (3.8) and (3.6),

$$\begin{aligned} d\pi_t^* = {} & \lambda(1-\pi_t^*)dt + \frac{r}{\sigma^2}\pi_t^*(1-\pi_t^*)1_{\{\pi_t^* > A\}}(r1_{\{\theta \leq t\}}dt + \sigma dB_t) \\ & - \frac{r^2}{\sigma^2}(\pi_t^*)^2(1-\pi_t^*)1_{\{\pi_t^* > A\}}\, dt. \end{aligned} \tag{4.68}$$

Because of the irregularity of $p \mapsto 1_{\{p > A\}}$, equations such as (4.67) and (4.68) do not have a strong solution in general (see for instance [2, 16, 7]), but according to the theory developed in [5, Chapter 5, § 24], they do have a weak solution (such that the process (π_t^*) spends an amount of time at A that has positive Lebesgue measure). Therefore, from the discussion in (3.8)–(3.15), we expect (4.66) to determine an optimal control system in the weak formulation of our problem, but there will be no optimal strategy in the strong formulation! This means that we will be able to use the Verification Lemma 3.7 to prove, in Section 5, that the function g defined in Proposition 4.3 is equal to the value function $\tilde{g}^{\mathrm{w}}$, but a different approach via ε-optimal strategies will be used to show that g is equal to $\tilde{g}$.

5 The value function

Formulas (4.37) and (4.59) provide candidates, denoted g, for the value functions $\tilde{g}$ and $\tilde{g}^{\mathrm{w}}$ defined respectively in (2.5) and (2.6). The objective of this section is to prove that indeed, these two value functions are equal, and equal to g.

Theorem 5.1. *(a)* Case where $0 < b < \lambda a\rho^2/(2(a+\lambda)^2)$. *Define A by* (4.18)*, let f be as in* (4.23)*, K_1 as in* (4.25)*, f_2 as in* (4.24)*, B as in* (4.26)*, K_2 as in* (4.29)*, g_1 as in* (4.34) *and g_2 as in* (4.30)*. Then the function g defined in* (4.37) *is equal to the value function $\tilde{g}^{\mathrm{w}}$ defined in* (2.6)*. Further, the control system associated to $h(t,p) = 1_{\{p > A\}}$ and to τ^* in* (4.66) *is optimal.*

(b) Case where $b \geq \lambda a\rho^2/(2(a+\lambda)^2)$. *Define* B *by* (4.57) *and* g_1 *by* (4.58). *Then the function* g *defined in* (4.59) *is equal to the value function* $\tilde{g}^{\mathrm{w}}$ *defined in* (2.6).

Theorem 5.2. *In both cases of Theorem 5.1, the two value functions* $\tilde{g}$ *(strong formulation) and* $\tilde{g}^{\mathrm{w}}$ *(weak formulation), defined respectively in* (2.5) *and* (2.6), *are equal (and equal to the function* g *of Theorem 5.1).*

Remark 5.3. It is interesting to observe how the value function $\tilde{g}$ and the thresholds A and B depend on the observation cost b: we write $\tilde{g}(x,b)$, $A(b)$ and $B(b)$ to indicate this dependence.

From (2.4) and (2.5), $b \mapsto \tilde{g}(x,b)$ is nondecreasing. For $b=0$, $g(\cdot,0)$ is the value function obtained in [15, Chapter 4.4, Theorem 9]. As b increases from 0 to $b_c = \lambda a\rho^2/(2(a+b)^2)$, $B(b)$ decreases from $B(0)$ to $B(b_c)$, and $A(b)$ increases from 0 to $A(b_c) = \lambda/(a+\lambda) = B(b_c)$ (see (4.18) for the first equality and the second follows from the lines preceding (4.26) since $f_2(A(b_c)) = -1$). For $b \geq b_c$, $\tilde{g}(\cdot,b) = \tilde{g}(\cdot,b_c)$ since there is no dependence on b.

Theorem 5.1 will be proved in two steps. We begin by showing that $g \leq \tilde{g}^{\mathrm{w}}$.

Lemma 5.4. *In both cases (a) and (b) of Theorem 5.1, the inequality* $g \leq \tilde{g}^{\mathrm{w}}$ *holds.*

Proof. We are going to use part (1) of Lemma 3.7. Suppose first that we are in Case (a) of Theorem 5.1. By construction, and in particular by (4.10), (4.13) and (4.14), g is C^2 on $[0,B[$, and C^1 on $[0,1]$ by (4.12) and (4.7):

$$g'(B-) = g'(B+) = -1. \tag{5.1}$$

By (4.38), $0 \leq g(x) \leq 1-x$. Let $((h_t), \tau, X)$ be a control system and set

$$Y_t = g(\pi_t^h) + a\int_0^t \pi_s^h\, ds + b\int_0^t h_s\, ds. \tag{5.2}$$

We now apply Itô's formula, in the form given in [12, Section 3.5]:

$$\begin{aligned} Y_t = Y_0 &+ \int_0^t g'(\pi_s^h)\, d\pi_s^h + \int_0^t (a\pi_s^h + bh_s)\, ds + \frac{1}{2}\int_0^t g''(\pi_s^h)\, d\langle \pi^h\rangle_s \\ &+ \frac{1}{2}\left(g'(B+) - g'(B-)\right) L_t^B, \end{aligned} \tag{5.3}$$

where L_t^B is the local time of (π_s^h) at B. By (5.1), the factor $g'(B+) - g'(B-)$ vanishes, so as in (4.1), we find that

$$Y_t = Y_0 + \int_0^t g'(\pi_s^h)\frac{r}{\sigma}\,\pi_s^h(1-\pi_s^h)\sqrt{h_s}\, d\bar{B}_s + \int_0^t \Phi(\pi_s^h, h_s)\, ds, \tag{5.4}$$

where

$$\Phi(x,\eta) = L_1 g(x) + \eta \left[\frac{1}{2} g''(x)(\rho x(1-x))^2 + b \right], \qquad \eta \in [0,1],$$

and L_1 is defined in (4.35). We note that by construction and by Proposition 4.3,

$$\begin{aligned} \Phi(x,0) &= L_1 g(x) \geq 0, \qquad \text{for all } x \in [0,1], \\ \Phi(x,1) &= L_2 g(x) \geq 0, \qquad \text{for all } x \in [0,1] \setminus \{B\}, \end{aligned}$$

where L_2 is defined in (4.36), and since $\eta \mapsto \Phi(x,\eta)$ is an affine function, we conclude that $\Phi(x,\eta) \geq 0$, for all $\eta \in [0,1]$. Since g' is bounded on $[0,1]$, the stochastic integral in (5.4) is an $\mathcal{F}^X$-martingale (recall (3.5)), and therefore (Y_t) is an $\mathcal{F}^X$-submartingale. The conclusion now follows from part (1) of Lemma 3.7.

Now suppose that we are in Case (b) of Theorem 5.1. By construction, g is C^2 on $[0,B[$, and C^1 on $[0,1]$ by (4.53) and (4.54):

$$g'(B-) = g'(B+) = -1.$$

By (4.60), $0 \leq g(x) \leq 1-x$, for all $x \in [0,1]$. Let $((h_t),\tau,X)$ be a control system and define Y_t as in (5.2). Applying Itô's formula, we obtain (5.3), and this leads again to (5.4). Using this time Proposition 4.4, we see that $\Phi(x,\eta) \geq 0$, for all $\eta \in [0,1]$. Therefore, we conclude, as before, that (Y_t) is an $\mathcal{F}^X$-submartingale, and the conclusion follows from part (1) of Lemma 3.7. □

We now prove Theorem 5.1.

Proof of Theorem 5.1. We begin with Case (b). As mentioned in (4.64) and (4.65), the candidate optimal control system is (h^*,τ^*,X^*), where $h^*_t \equiv 0$, $X^* \equiv 0$ and

$$\tau^* = \inf\{t \geq 0 : \pi^*_t \geq B\},$$

where (π^*_t) is defined in (4.65). Clearly, (h^*,τ^*,X^*) is a control system, and so it suffices to check properties (b), (c) and (d) of Lemma 3.7. By (4.61),

$$dY^*_t = L_1 g(\pi^*_t)\, dt = 0 \qquad \text{for } t < \tau^*.$$

Therefore, $(Y^*_{t\wedge\tau^*})$ is a (constant and deterministic) martingale, proving (b).

Further, since (π^*_t) is deterministic, we solve (4.65) to find that

$$\tau^* = \begin{cases} \frac{1}{\lambda} \ln\left(\frac{1-\pi_0}{1-B}\right) & \text{if } \pi_0 < B, \\ 0 & \text{if } \pi_0 \geq B, \end{cases} \tag{5.5}$$

so (c) holds. Finally, if $\pi_0 < B$, then

$$g(\pi^*_{\tau^*}) = g(B) = 1 - B = 1 - \pi^*_{\tau^*}$$

by (4.53), and if $\pi_0 \geq B$, then

$$g(\pi^*_{\tau^*}) = g(\pi^*_0) = 1 - \pi^*_0 = 1 - \pi^*_{\tau^*}$$

by (4.59). This proves Case (b) of Theorem 5.1.

We now consider Case (a). We have seen in Lemma 5.4 that $g \leq \tilde{g}^{\mathrm{w}}$. In order to establish the converse inequality, consider (h^*_t) and τ^* defined in (4.66), and the associated control system $((h^*_t), \tau^*, X^*)$ constructed as in (3.8)–(3.15) using the function $h(t,p) = 1_{\{p>A\}}$, and π^*_t defined as a weak solution of (4.68). Then for $t \leq \tau^*$,

$$\begin{aligned} dY^*_t &= \begin{cases} L_1 g(\pi^*_t) & \text{if } \pi^*_t < A, \\ L_2 g(\pi^*_t) & \text{if } \pi^*_t \in [A, B[, \end{cases} \\ &= 0 \end{aligned}$$

by (4.39) and (4.40). Therefore, $(Y^*_{t\wedge\tau^*})$ is an $\mathcal{F}^X$-martingale. According to Lemma 5.5 below, $E_\pi(\tau^*) < \infty$, and $g(\pi^{h^*}_{\tau^*}) = g(B) = 1 - B$ by (4.66) and (4.37). This proves properties (b), (c), and (d) of Lemma 3.7 and concludes the proof that $g = \tilde{g}^{\mathrm{w}}$ and $((h^*_t), \tau^*, X^*)$ is an optimal control system, since we already verified (a) of Lemma 3.7 during the proof of Lemma 5.4. □

Lemma 5.5. *Suppose that we are in Case (a) of Theorem 5.1. Let τ^* be defined as in* (4.66). *Then for all $\pi \in [0,1]$, $E_\pi(\tau^*) < \infty$.*

Proof. If $\pi \in [B,1]$, then $E_\pi(\tau^*) = 0$, and if $\pi \in [0, A[$, then (π_t) reaches A at the deterministic time $\lambda^{-1} \ln((1-\pi)/(1-A))$ (see (5.5)), so the problem reduces to considering $\pi \in [A, B[$.

Recall from (3.14) and (4.67) that (π^*_t) solves, in the terminology of [5, Chapter 5, § 24], an s.d.e. with *delayed* reflection at the boundary point A, and this process is associated to a diffusion (ξ_t) with *instantaneous* reflection at the boundary:

$$d\xi_t = \lambda(1-\xi_t)dt + \rho\, \xi_t(1-\xi_t)\, d\bar{B}_t + d\zeta_t, \tag{5.6}$$

where (ζ_t) is an nondecreasing process that increases at those points where $\xi_t = A$. As explained in [5], (π^*_t) the same law as ξ_{τ_t}, where τ_t is defined by the relation

$$t = \tau_t + \frac{1}{1-A}\, \zeta_{\tau_t}.$$

Therefore, τ^* has the same law as $T = \inf\{t \in \mathbb{R}_+ : \xi_{\tau_t} = B\}$. Letting $\sigma = \inf\{s \in \mathbb{R}_+ : \xi_s = B\}$, we see that $\sigma = \tau_T$. Further, according to Lemma 5 in [5, Chapter 5, § 23], $E_\pi(\sigma) = V_0(y) - V_0(\pi)$, where $V_0(A) = 0$ and for $y \in]A, 1[$,

$$\lambda(1-y)V_0'(y) + \frac{1}{2}\rho^2 y^2 (1-y)^2 V_0''(y) = 1.$$

An explicit expression for V_0 can be obtained by using (4.24) with $K_1 = 0$ and $a = b = 0$, and then integrating from A to y. In particular, $E_\pi(\sigma) < +\infty$.

Notice that

$$T = \tau_T + \frac{1}{1-A}\,\zeta_{\tau_T} = \sigma + \frac{1}{1-A}\,\zeta_\sigma.$$

Therefore, it suffices to show that $E_\pi(\zeta_\sigma) < +\infty$. By (5.6),

$$\xi_{t\wedge\sigma} = \xi_0 + \int_0^{t\wedge\sigma} \lambda(1-\xi_s)ds + \int_0^{t\wedge\sigma} \rho\,\xi_s(1-\xi_s)\,d\bar{B}_s + \zeta_{t\wedge\sigma}. \tag{5.7}$$

The stochastic integral is an L^2-bounded martingale, since

$$E_\pi\left(\int_0^{t\wedge\sigma} \rho^2\,\xi_s^2(1-\xi_s)^2\,ds\right) \leq \rho^2\,E_\pi(\sigma) < +\infty.$$

Therefore, the optional sampling theorem can be applied and, since the ds-integral in (5.7) is nonnegative, we find that

$$B = E_\pi(\xi_\sigma) \geq \pi + E_\pi(\zeta_\sigma),$$

so $E_\pi(\zeta_\sigma) < +\infty$, as was to be proved. □

For the remainder of this section, we put ourselves in Case (a) of Theorem 5.1. Since we have observed just after (2.6) that $\tilde{g} \geq \tilde{g}^{\mathrm{w}}$, and $\tilde{g}^{\mathrm{w}} = g$ by Theorem 5.1(a), in order to prove Theorem 5.2, it suffices to establish the inequality $g \geq \tilde{g}$. For $\varepsilon > 0$, we are going to define an admissible control h^ε, and a strategy $(h^\varepsilon, \tau^\varepsilon)$, with associated cost $\tilde{g}_\varepsilon = E(C(h^\varepsilon, \tau^\varepsilon))$, and we shall show that $\tilde{g}_\varepsilon \to g$ as $\varepsilon \downarrow 0$. From the definition of $\tilde{g}$ in (2.5), this will establish that $g \geq \tilde{g}$, and this will prove Theorem 5.1.

An almost optimal strategy

Define the function

$$h^{(\varepsilon)}(x) = \frac{x-A}{\varepsilon} 1_{]A,A+\varepsilon[}(x) + 1_{[A+\varepsilon,\infty[}(x).$$

Consider the s.d.e.

$$\begin{aligned} dp_t^\varepsilon = \lambda(1-p_t^\varepsilon)dt + \frac{r}{\sigma^2}p_t^\varepsilon(1-p_t^\varepsilon)\left(rh^{(\varepsilon)}(p_t^\varepsilon)1_{\{\theta<t\}}dt + \sigma\sqrt{h^{(\varepsilon)}(p_t^\varepsilon)}\,dB_t\right) \\ - \frac{r^2}{\sigma^2}(p_t^\varepsilon)^2(1-p_t^\varepsilon)h^{(\varepsilon)}(p_t^\varepsilon)dt, \end{aligned} \tag{5.8}$$

with $p_0^\varepsilon = \pi_0$. According to [6, Theorem 3.2 p.168], this s.d.e. has a unique strong solution $(p_t^\varepsilon,\ t \geq 0)$, since $\sqrt{h^{(\varepsilon)}}$ is Hölder-continuous with exponent 1/2.

Set

$$\tau^\varepsilon = \begin{cases} \inf\{t \geq 0 : p_t^\varepsilon \geq B\} & \text{if } \{\cdots\} \neq \emptyset, \\ +\infty & \text{otherwise,} \end{cases} \tag{5.9}$$

Using (3.8)–(3.15), we associate to $(h^{(\varepsilon)}, \tau^\varepsilon)$ a strategy $((h_t^\varepsilon), \tau^\varepsilon)$.

We are now going to determine the cost of the strategy $(h^\varepsilon, \tau^\varepsilon)$, and we will see in Proposition 5.7 below that for ε small, this strategy is nearly optimal. Let

$$\tilde{g}_\varepsilon(\pi_0) = E(C(h^\varepsilon, \tau^\varepsilon)). \tag{5.10}$$

In order to determine the function $\tilde{g}_\varepsilon$, we will use the following lemma.

Lemma 5.6. *Suppose that we are in Case (a) of Theorem 5.1 and that we can find a continuous function g_ε on $[0,1]$ that is C^2 on $[0,1] \setminus \{A, B\}$, C^1 on $[0,1] \setminus \{B\}$ and such that*

$$Lg_\varepsilon(x) = -(ax + bh^{(\varepsilon)}(x)), \tag{5.11}$$

where $Lg_\varepsilon(x)$ is defined by

$$Lg_\varepsilon(x) = \lambda(1-x)g_\varepsilon'(x) + \frac{1}{2}\rho^2 x^2 (1-x)^2 h^{(\varepsilon)}(x) g_\varepsilon''(x), \tag{5.12}$$

and

$$g_\varepsilon(x) = 1 - x, \qquad \text{for } x \in [B, 1]. \tag{5.13}$$

If, in addition,

$$E_x(\tau_\varepsilon) < +\infty, \qquad \text{for all } x \in [0,1], \tag{5.14}$$

then $g_\varepsilon = \tilde{g}_\varepsilon$.

Proof. Suppose $\pi_0 \in [B, 1]$. Then $g_\varepsilon(\pi_0) = 1 - \pi_0$, and since $\tau^\varepsilon = 0$ a.s., (2.4) gives

$$\tilde{g}_\varepsilon(\pi_0) = E(C(h^\varepsilon, \tau^\varepsilon)) = P\{\theta > 0\} = 1 - \pi_0.$$

Therefore, by (5.13), $g_\varepsilon(\pi_0) = \tilde{g}_\varepsilon(\pi_0)$ in this case.

Now suppose that $\pi_0 \in [0, B[$. According to Lemma 2.2 and (3.13),

$$\tilde{g}_\varepsilon(p_0^\varepsilon) = E(C(h^\varepsilon, \tau^\varepsilon)) = E\left(1 - p_{\tau^\varepsilon}^\varepsilon + a \int_0^{\tau^\varepsilon} p_s^\varepsilon \, ds + b \int_0^{\tau^\varepsilon} h_s^\varepsilon \right).$$

Since $\tau^\varepsilon < +\infty$ a.s. by (5.14), $p_{\tau^\varepsilon}^\varepsilon = B$, and $1 - B = g_\varepsilon(B)$ by (5.13), so

$$E(C(h^\varepsilon, \tau^\varepsilon)) = E\left(g_\varepsilon(p_{\tau^\varepsilon}^\varepsilon) + a \int_0^{\tau^\varepsilon} p_s^\varepsilon \, ds + b \int_0^{\tau^\varepsilon} h^{(\varepsilon)}(p_s^\varepsilon) \, ds \right).$$

As in Lemmas 3.5 and 3.6, we see from (5.8) and (3.9) that

$$dp_t^\varepsilon = \lambda(1-p_t^\varepsilon)dt + \rho p_t^\varepsilon(1-p_t^\varepsilon)\sqrt{h^{(\varepsilon)}(p_t^\varepsilon)}\, d\bar{B}_t^\varepsilon, \tag{5.15}$$

where $(\bar{B}_t^\varepsilon)$ is an Brownian motion. From (5.11) and Itô's formula in the form given in [12, Section 3.5.3] and using the fact that $g_\varepsilon'(A-) = g_\varepsilon'(A+)$, we see that $(M_{t\wedge\tau^\varepsilon},\ t \geq 0)$ is an $\mathcal{F}^{X^\varepsilon}$-martingale, where

$$M_t = g_\varepsilon(p_t^\varepsilon) + a\int_0^t p_s^\varepsilon\, ds + b\int_0^t h^{(\varepsilon)}(p_s^\varepsilon)\, ds.$$

Since, by (3.13), $0 \leq p_s^\varepsilon \leq 1$, and $0 \leq h^{(\varepsilon)} \leq 1$ and g_ε is bounded, we see that $|M_t| \leq A + (a+b)t$, so $|M_{t\wedge\tau^\varepsilon}| \leq A + (a+b)\tau^\varepsilon$. Since $E_{\pi_0}(\tau^\varepsilon) < +\infty$ by (5.14), $(M_{t\wedge\tau^\varepsilon})$ is uniformly integrable, and so

$$\tilde{g}_\varepsilon(\pi_0) = E(C(h^\varepsilon, \tau^\varepsilon)) = E_{\pi_0}(M_{\tau^\varepsilon}) = E_{\pi_0}(M_0) = g_\varepsilon(p_0^\varepsilon) = g_\varepsilon(\pi_0).$$

Therefore, $g_\varepsilon(\pi_0) = \tilde{g}_\varepsilon(\pi_0)$ as claimed. This completes the proof of Lemma 5.6. □

Constructing g_ε

It remains to construct the function g_ε satisfying the assumptions of Lemma 5.6. Notice that on $]0, A[$, writing $\bar{g}_1$ instead of g_ε, equation (5.11) becomes

$$\lambda(1-x)\bar{g}_1'(x) + ax = 0, \tag{5.16}$$

and as in (4.56), the solution of this differential equation is

$$\bar{g}_1(x) = \frac{a}{\lambda}\,(x + \ln(1-x)) + K_1^\varepsilon, \tag{5.17}$$

where K_1^ε is a constant to be determined.

On $]A+\varepsilon, B[$, writing $\bar{g}_3$ instead of g_ε, equation (5.11) becomes

$$\lambda(1-x)\bar{g}_3'(x) + \frac{1}{2}\rho^2 x^2(1-x)^2\bar{g}_3''(x) + ax + b = 0, \tag{5.18}$$

which is the same equation as in (4.9), and as in (4.28), its solution is

$$\bar{g}_3(x) = \int_{A+\varepsilon}^x \bar{h}_3(y)\, dy + K_\varepsilon^3, \tag{5.19}$$

where

$$\bar{h}_3(x) = K_2^\varepsilon f(x) + f(x)\int_{A+\varepsilon}^x \frac{-2}{\rho^2}\frac{ay+b}{y^2(1-y)^2}\frac{1}{f(y)}\, dy, \tag{5.20}$$

and

$$f(x) = \left(\frac{1-x}{x}\right)^{\alpha} e^{\alpha/x}, \qquad \text{where } \alpha = \frac{2\lambda}{\rho^2}, \tag{5.21}$$

and K_2^ε, K_3^ε are constants to be determined.

Finally, on $]A, A+\varepsilon[$, writing $\bar{g}_2$ instead of g_ε, equation (5.11) becomes

$$\lambda(1-x)\bar{g}_2'(x) + \frac{1}{2}\rho^2 x^2 (1-x)^2 \frac{x-A}{\varepsilon} \bar{g}_2''(x) + ax + b\frac{x-A}{\varepsilon} = 0, \tag{5.22}$$

Let $\bar{h}_2(x) = \bar{g}_2'(x)$, so the associated homogeneous equation is

$$\lambda(1-x)\bar{f}_2(x) + \frac{1}{2}\rho^2 x^2 (1-x)^2 \frac{x-A}{\varepsilon} \bar{f}_2'(x) = 0, \tag{5.23}$$

whose solution is

$$\bar{f}_2^\varepsilon(x) = \psi_\varepsilon(x)(x-A)^{-\beta_\varepsilon}, \tag{5.24}$$

where

$$\beta_\varepsilon = \frac{1}{A^2(1-A)} \frac{2\lambda\varepsilon}{\rho^2},$$

and

$$\psi_\varepsilon(x) = x^{2\lambda\varepsilon(1+A)/(\rho A)^2} (1-x)^{2\lambda\varepsilon/(\rho^2(1-A))} \exp\left(-\frac{2\lambda\varepsilon}{A\rho^2}\frac{1}{x}\right).$$

Therefore,

$$\bar{h}_2(x) = K\bar{f}_2^\varepsilon(x) + \bar{f}_2^\varepsilon(x) \int_A^x \frac{-2\varepsilon}{\rho^2} \left(ay + b\,\frac{y-A}{\varepsilon}\right) \frac{1}{y^2(1-y)^2(y-A)} \frac{1}{\bar{f}_2^\varepsilon(y)}\, dy, \tag{5.25}$$

and if we want $\bar{h}_2$ to be bounded as $x \downarrow A$, then we must set $K = 0$ (notice that there is no integrability problem at $y = A$). We conclude that

$$\bar{h}_2(x) = \bar{f}_2^\varepsilon(x) \int_A^x \frac{-2\varepsilon}{\rho^2} \left(ay + b\,\frac{y-A}{\varepsilon}\right) \frac{1}{y^2(1-y)^2(y-A)} \frac{1}{\bar{f}_2^\varepsilon(y)}\, dy \tag{5.26}$$

and

$$\bar{g}_2(x) = \int_A^x \bar{h}_2(y)\, dy + K_4^\varepsilon, \tag{5.27}$$

where K_4^ε is a constant to be determined.

In order to determine the four constants $K_1^\varepsilon, \ldots, K_4^\varepsilon$, we shall impose the four equations

$$\bar{g}_3(B) = 1 - B \tag{5.28}$$
$$\bar{g}_3(A+\varepsilon) = \bar{g}_2(A+\varepsilon) \tag{5.29}$$
$$\bar{g}_3'(A+\varepsilon) = \bar{g}_2'(A+\varepsilon) \tag{5.30}$$
$$\bar{g}_1(A) = \bar{g}_2(A). \tag{5.31}$$

We note that (5.29) and (5.30), together with (5.18) and (5.22), imply that $\bar{g}_3''(A+\varepsilon) = \bar{g}_2''(A+\varepsilon)$, and (5.31), together with (5.16) and (5.22), implies that $\bar{g}_1'(A) = \bar{g}_2'(A)$.

From (5.28) and (5.19), we see that

$$K_3^\varepsilon = 1 - B - \int_{A+\varepsilon}^{B} \bar{h}_3(y)\, dy, \tag{5.32}$$

while (5.29), (5.19) and (5.27) imply that

$$K_3^\varepsilon = \int_{A}^{A+\varepsilon} \bar{h}_2(y)\, dy + K_4^\varepsilon. \tag{5.33}$$

Equality (5.30), (5.20) and (5.26) give the relation

$$K_2^\varepsilon = \frac{\bar{f}_2^\varepsilon(A+\varepsilon)}{f(A+\varepsilon)} \int_{A}^{A+\varepsilon} \frac{-2\varepsilon}{\rho^2} \left(ay + b\,\frac{y-A}{\varepsilon} \right) \frac{1}{y^2(1-y)^2(y-A)} \frac{1}{\bar{f}_2^\varepsilon(y)}\, dy, \tag{5.34}$$

while (5.31), (5.17) and (5.27) give

$$\frac{a}{\lambda}(A + \ln(1-A)) + K_1^\varepsilon = K_4^\varepsilon. \tag{5.35}$$

Therefore, (5.34) determines K_2^ε, (5.32) determines K_3^ε, then (5.33) determines K_4^ε and (5.35) determines K_1^ε.

Proposition 5.7. *For $\varepsilon > 0$, let $K_1^\varepsilon, \dots, K_4^\varepsilon$ be determined by* (5.32)–(5.35)*, define $\bar{g}_1(x)$ as in* (5.17)*, $\bar{g}_2(x)$ as in* (5.27)*, and $\bar{g}_3(x)$ as in* (5.19)*. Set*

$$g_\varepsilon(x) = \begin{cases} \bar{g}_1(x), & \text{if } 0 \le x \le A, \\ \bar{g}_2(x), & \text{if } A < x < A+\varepsilon, \\ \bar{g}_3(x), & \text{if } A+\varepsilon \le x < B, \\ 1-x, & \text{if } B \le x \le 1. \end{cases}$$

Then g_ε satisfies the assumptions of Lemma 5.6. Further, let g be as in Case (a) of Theorem 5.1. Then

$$\lim_{\varepsilon \downarrow 0} g_\varepsilon(x) = g(x), \qquad \text{for all } x \in [0,1].$$

Proof. By the comments that follow (5.31), g_ε is C^2 on $[0,1] \setminus \{A, B\}$, C^1 on $[0,1] \setminus \{B\}$ and continuous on $[0,1]$. For $x \in [B,1]$, $g_\varepsilon(x) = 1 - x = g(x)$, so we consider the case where $x \in [0, B[$.

Case 1: $x \in]A,B[$. We first check that $K_2^\varepsilon \to K_1$, where K_1 is defined in (4.25). We note that

$$K_2^\varepsilon = \frac{\psi_\varepsilon(A+\varepsilon)}{f(A+\varepsilon)}\, \varepsilon^{-\beta_\varepsilon} \int_A^{A+\varepsilon} \frac{-2\varepsilon}{\rho^2} \left(ay + b\, \frac{y-A}{\varepsilon} \right) \frac{1}{y^2(1-y)^2} \frac{1}{\psi_\varepsilon(y)}\, (y-A)^{\beta_\varepsilon - 1}\, dy.$$

Notice that $\psi_\varepsilon(A+\varepsilon) \to 1$ and $f(A+\varepsilon) \to f(A)$ as $\varepsilon \downarrow 0$. Set

$$\lambda_0 = \frac{1}{A^2(1-A)} \frac{2\lambda}{\rho^2}, \qquad \text{so that } \beta_\varepsilon = \lambda_0 \varepsilon.$$

Then

$$\begin{aligned} K_\varepsilon^2 &\sim \frac{1}{f(A)}\, \varepsilon^{1-\lambda_0\varepsilon} \int_A^{A+\varepsilon} \frac{-2}{\rho^2} \left(ay + b\, \frac{y-A}{\varepsilon} \right) \frac{1}{y^2(1-y)^2 \psi_\varepsilon(y)}\, (y-A)^{\lambda_0\varepsilon - 1}\, dy \\ &\sim \frac{1}{f(A)} \frac{-2}{\rho^2} \frac{1}{A^2(1-A)^2 \psi_\varepsilon(A)}\, \varepsilon^{1-\lambda_0\varepsilon} \int_A^{A+\varepsilon} \left[aA(y-A)^{\lambda_0\varepsilon - 1} + \frac{b}{\varepsilon}\, (y-A)^{\lambda_0\varepsilon} \right] dy, \end{aligned}$$

and the integral is equal to

$$aA \frac{\varepsilon^{\lambda_0\varepsilon}}{\lambda_0\varepsilon} + \frac{b}{\varepsilon} \frac{\varepsilon^{\lambda_0\varepsilon+1}}{\lambda_0\varepsilon + 1},$$

therefore,

$$\lim_{\varepsilon \downarrow 0} K_2^\varepsilon = \frac{1}{f(A)} \frac{-2}{\rho^2} \frac{1}{A^2(1-A)^2} \frac{aA}{\lambda_0} = -\frac{a}{\lambda} \frac{A}{1-A} \frac{1}{f(A)} = K_1,$$

as claimed.

This implies that for $y > A+\varepsilon$, $\bar{h}_3(y) \to f_2(y)$, where $\bar{h}_3$ and f_2 are respectively defined in (5.20) and (4.24). By Dominated Convergence, we deduce that $K_3^\varepsilon \to K_2$, and for $x \in]A,B[$ and for $\varepsilon \downarrow 0$ with $0 < \varepsilon < x - A$,

$$g_\varepsilon(x) = \bar{g}_3(x) \to g_2(x) = g(x),$$

where g_2 is defined in (4.30).

Case 2: $x \in [0,A]$. From (5.33), we see that $K_3^\varepsilon - K_4^\varepsilon \to 0$, therefore $K_4^\varepsilon \to K_2$ by the above, and from (5.35), we see that

$$K_1^\varepsilon \to K_2 - \frac{a}{\lambda}\, (A + \ln(1-A)).$$

We conclude from (5.17) and (4.34) that for $x \in [0,A]$, as $\varepsilon \downarrow 0$,

$$g_\varepsilon(x) = \bar{g}_1(x) \to g_1(x) = g(x).$$

This completes the proof of Proposition 5.7. □

The next lemma checks the condition (5.14).

Lemma 5.8. *Fix* $\varepsilon > 0$ *and let* τ^ε *be defined in* (5.9)*. Then for all* $x \in [0, 1]$*,* $E_x(\tau^\varepsilon) < \infty$*.*

Proof. We first seek a bounded function γ_ε defined on $[0, B]$ such that

$$L\gamma_\varepsilon = -1, \tag{5.36}$$

where L is the operator defined in (5.12).

For $0 < x < A$, (5.36) becomes

$$\lambda(1-x)\gamma_\varepsilon'(x) = -1, \tag{5.37}$$

so

$$\gamma_\varepsilon(x) = \frac{1}{\lambda}\ln(1-x) + D_1, \qquad 0 \le x \le A. \tag{5.38}$$

For $A < x < A + \varepsilon$, (5.36) becomes

$$\lambda(1-x)\gamma_\varepsilon'(x) + \frac{1}{2}\rho^2 x^2 (1-x)^2 \frac{x-A}{\varepsilon}\gamma_\varepsilon''(x) = -1, \tag{5.39}$$

and as in (5.22) and (5.25), the solution to this equation is

$$\gamma_\varepsilon(x) = \int_A^x h_4(y)\,dy + D_3, \qquad A < x < A + \varepsilon, \tag{5.40}$$

where

$$h_4(x) = D_2 \bar{f}_2^\varepsilon(x) + \bar{f}_2^\varepsilon(x)\int_A^x \frac{-2\varepsilon}{\rho^2}\frac{1}{y^2(1-y)^2(y-A)}\frac{1}{\bar{f}_2^\varepsilon(y)}\,dy, \tag{5.41}$$

and $\bar{f}_2^\varepsilon$ is defined in (5.24). Since we want h_4 and γ_ε to be bounded (as $x \downarrow A$), we set $D_2 = 0$.

For $A + \varepsilon < x < B$, (5.36) becomes

$$\lambda(1-x)\gamma_\varepsilon'(x) + \frac{1}{2}\rho^2 x^2 (1-x)^2 \gamma_\varepsilon''(x) = -1, \tag{5.42}$$

and as in (5.19), the solution of this equation is

$$\gamma_\varepsilon(x) = \int_{A+\varepsilon}^x h_5(y)\,dy + D_4, \tag{5.43}$$

where

$$h_5(x) = D_5 f(x) + f(x)\int_{A+\varepsilon}^x \frac{-2}{\rho^2 y^2 (1-y)^2}\frac{1}{f(y)}\,dy, \tag{5.44}$$

and $f(x)$ is defined in (5.21).

We must determine the constants $D_1, \ldots, D_5$. For this, we impose the following conditions:

(a) $\gamma_\varepsilon(B) = 0$,
(b) $\gamma_\varepsilon((A+\varepsilon)+) = \gamma_\varepsilon((A+\varepsilon)-)$,
(c) $\gamma'_\varepsilon((A+\varepsilon)+) = \gamma'_\varepsilon((A+\varepsilon)-)$,
(d) $\gamma_\varepsilon(A+) = \gamma_\varepsilon(A-)$.

We note that (b) and (c), together with (5.39) and (5.42), imply that

$$\gamma''_\varepsilon((A+\varepsilon)+) = \gamma''_\varepsilon((A+\varepsilon)-) \tag{5.45}$$

so γ_ε will be C^2 at $A+\varepsilon$. Also, (d) together with (5.37) and (5.39) implies that

$$\gamma'_\varepsilon(A+) = \gamma'_\varepsilon(A-), \tag{5.46}$$

so γ_ε will be C^1 at A.

From property (c), (5.44) and (5.41), we see that

$$D_5 f(A+\varepsilon) = \bar{f}_2^\varepsilon(A+\varepsilon) \int_A^{A+\varepsilon} \frac{-2\varepsilon}{\rho^2} \frac{1}{y^2(1-y)^2(y-A)} \frac{1}{\bar{f}_2^\varepsilon(y)}\, dy,$$

and this determines D_5 (and therefore h_5).

From (a) and (5.43), we find that

$$D_4 = \int_B^{A+\varepsilon} h_5(y)\, dy,$$

so that

$$\gamma_\varepsilon(x) = \int_B^x h_5(y)\, dy, \qquad \text{for } A+\varepsilon < x < B. \tag{5.47}$$

From (b), (5.47) and (5.40), we see that

$$\int_B^{A+\varepsilon} h_5(y)\, dy = \int_A^{A+\varepsilon} h_4(y)\, dy + D_3,$$

and this determines D_3.

Finally, from (d), (5.38) and (5.40), we see that

$$\frac{1}{\lambda} \ln(1-A) + D_1 = D_3,$$

and this now determines D_1.

With the choice of constants $D_1, \ldots, D_5$ above,we have determined a function $\gamma_\varepsilon : [0, B] \to \mathbb{R}$ which is C^1 on $[0, B]$ and C^2 on $[0, A]$ and $[A, B]$.

We now turn to the study of $E_x(\tau^\varepsilon)$. For $x \in [A, B]$, we note that while p_t^ε belongs to $[A, A + \varepsilon[$, the s.d.e. (5.15) is essentially that of the Cox-Ingersoll-Ross model [9, Theorem 6.2.3, Prop. 6.2.4], and in fact, $p_t^\varepsilon - A$ behaves like a BESQ-process [14, Chapter XI]. In particular, $p_t^\varepsilon \geq A$ since $x \geq A$, so (p_t^ε) never goes strictly below A (though it may hit A and A is instantaneously reflecting), and $A + \varepsilon$ is hit in finite time because p_t^ε is either recurrent or transient, depending on the values of λ and ρ.

We now apply Itô's formula to $\gamma_\varepsilon(p^\varepsilon_{t\wedge\tau^\varepsilon})$, since γ_ε is C^2 on $[A, B]$:

$$\begin{aligned}\gamma_\varepsilon(p^\varepsilon_{t\wedge\tau^\varepsilon}) &= \gamma_\varepsilon(p_0^\varepsilon) + \int_0^{t\wedge\tau^\varepsilon} \gamma'_\varepsilon(p_s^\varepsilon)\, dp_s^\varepsilon + \frac{1}{2}\int_0^{t\wedge\tau^\varepsilon} \gamma''_\varepsilon(p_s^\varepsilon)\, d\langle p^\varepsilon\rangle_s \\ &= \gamma_\varepsilon(p_0^\varepsilon) + \int_0^{t\wedge\tau^\varepsilon} \gamma'_\varepsilon(p_s^\varepsilon)\rho p_s^\varepsilon(1 - p_s^\varepsilon)\sqrt{h^{(\varepsilon)}(p_s^\varepsilon)}\, d\bar{B}_s^\varepsilon \\ &\quad + \int_0^{t\wedge\tau^\varepsilon} L\gamma_\varepsilon(p_s^\varepsilon)\, ds.\end{aligned}$$

According to (5.36), $L\gamma_\varepsilon(p_s^\varepsilon) = -1$ for $s < \tau^\varepsilon$, so, taking expectations, we find that

$$E_x(\gamma_\varepsilon(p^\varepsilon_{t\wedge\tau^\varepsilon})) = \gamma_\varepsilon(x) - E_x(t \wedge \tau^\varepsilon),$$

so

$$E_x(t \wedge \tau^\varepsilon) = -E_x(\gamma_\varepsilon(p^\varepsilon_{t\wedge\tau^\varepsilon})) + \gamma_\varepsilon(x).$$

The right-hand side is bounded, so $\sup_{t\in\mathbb{R}} E_x(t \wedge \tau^\varepsilon) < +\infty$. By the Monotone Convergence Theorem, $E_x(\tau^\varepsilon) < +\infty$ as claimed (and in fact, $E_x(\tau^\varepsilon) = \gamma_\varepsilon(x)$), $x \in [A, B]$.

For $x \in [0, A[$, we observe from (5.15) that p_t^ε is deterministic and increases at speed $\geq \lambda(1 - A)$ until reaching A. Thus A is hit in less than some τ_0 units of time, and so

$$E_x(\tau^\varepsilon) \leq \tau_0 + E_A(\tau^\varepsilon) < +\infty.$$

Finally, for $x \in [B, 1]$, $\tau_\varepsilon = 0$ P_x-a.s., so $E_x(\tau_\varepsilon) = 0$. This proves Lemma 5.8. □

Lemma 5.9. *The function g_ε defined in Proposition 5.7 is the cost associated with the strategy $(h^\varepsilon, \tau^\varepsilon)$, that is, for all $x \in [0, 1]$, $g_\varepsilon(x) = \tilde{g}_\varepsilon(x) = E(C(h^\varepsilon, \tau^\varepsilon))$ ($\tilde{g}_\varepsilon$ is defined in* (5.10)*).*

Proof. According to Proposition 5.7, g_ε satisfies the assumptions of Lemma 5.6, and according to Lemma 5.8, (5.14) holds. Therefore, by Lemma 5.6, $g_\varepsilon = \tilde{g}_\varepsilon$, and this proves Lemma 5.9. □

Proof of Theorem 5.2. In Case (a) of Theorem 5.1, in view of the considerations that follow the proof of Theorem 5.1, it remains only to prove that $\tilde{g} \leq g$. By definition of $\tilde{g}$ and Lemma 5.9, the inequality $\tilde{g} \leq g_\varepsilon$ holds. Since $g = \lim_{\varepsilon \downarrow 0} g_\varepsilon$ by Proposition 5.7, we conclude that $\tilde{g} \leq g$. This completes the proof of Theorem 5.2 in Case (a) of Theorem 5.1.

The statement of Theorem 5.2 in Case (b) of Theorem 5.1 follows from the fact that the optimal control system exhibited in the proof of Theorem 5.1 is (trivially) a strategy, which then is necessarily optimal. □

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