The stochastic wave equation

by

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Abstract. These notes give an overview of recent results concerning the non-linear stochastic wave equation in spatial dimensions $d \geq 1$, in the case where the driving noise is Gaussian, spatially homogeneous and white in time. We mainly address issues of existence, uniqueness and Hölder-Sobolev regularity. We also present an extension of Walsh's theory of stochastic integration with respect to martingale measures that is useful for spatial dimensions $d \geq 3$.

Key words and phrases. Stochastic partial differential equations, sample path regularity, spatially homogeneous random noise, wave equation.

MSC 2000 Subject Classifications. Primary 60H15; Secondary 60J45, 35R60, 35L05.

[1]Partially supported by the Swiss National Foundation for Scientific Research.

1 Introduction

The stochastic wave equation is one of the fundamental stochastic partial differential equations (s.p.d.e.'s), of hyperbolic type. The behavior of its solutions is significantly different from those of solutions to other s.p.d.e.'s, such as the stochastic heat equation. In this introductory section, we present two real-world examples that can motivate the study of this equation, even though in neither case is the mathematical technology sufficiently developed to answer the main questions of interest. It is however pleasant to have such examples in order to motivate the development of rigorous mathematics.

Example 1: The motion of a strand of DNA

A DNA molecule can be view as a long elastic string, whose length is essentially infinitely long compared to its diameter. We can describe the position of the string by using a parameterization defined on $\mathbb{R}_+ \times [0,1]$ with values in $\mathbb{R}^3$:

$$\vec{u}(t,x) = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \\ u_3(t,x) \end{pmatrix}.$$

Here, $\vec{u}(t,x)$ is the position at time t of the point labelled x on the string, where $x \in [0,1]$ represents the distance from this point to one extremity of the string if the string were straightened out. The unit of length is chosen so that the entire string has length 1.

A DNA molecule typically "floats" in a fluid, so it is constantly in motion, just as a particle of pollen floating in a fluid moves according to Brownian motion. The motion of the particle can be described by Newton's law of motion, which equates the sum of forces acting on the string with the product of the mass and the acceleration. Let $\mu = 1$ be the mass of the string per unit length. The acceleration at position x along the string, at time t, is

$$\frac{\partial^2 \vec{u}}{\partial t^2}(t,x),$$

and the forces acting on the string are mainly of three kinds: elastic forces $\vec{F}_1$, which include torsion forces, friction due to viscosity of the fluid $\vec{F}_2$, and random impulses $\vec{F}_3$ due the the impacts on the string of the fluid's molecules. Newton's equation of motion can therefore be written

$$1 \cdot \frac{\partial^2 \vec{u}}{\partial t^2} = \vec{F}_1 - \vec{F}_2 + \vec{F}_3.$$

This is a rather complicated system of three stochastic partial differential equations, and it is not even clear how to write down the torsion forces or the friction term.

Elastic forces are generally related to the second derivative in the spatial variable, and the molecular forces are reasonably modelled by a stochastic noise term.

The simplest 1-dimensional equation related to this problem, in which one only considers vertical displacement and forgets about torsion, is the following one, in which $u(t,x)$ is now scalar valued:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) - \int_0^1 k(x,y)\, u(t,y)\, dy + \dot{F}(t,x), \tag{1.1}$$

where the first term on the right hand side represents the elastic forces, the second term is a (non-local) friction term, and the third term $\dot{F}(t,y)$ is a Gaussian noise, with spatial correlation $k(\cdot,\cdot)$, that is,

$$E(\dot{F}(t,x)\, \dot{F}(s,y)) = \delta_0(t-s)\, k(x,y),$$

where δ_0 denotes the Dirac delta function. The function $k(\cdot,\cdot)$ is the same in the friction term and in the correlation.

Why is the motion of a DNA strand of biological interest? When a DNA strand moves around and two normally distant parts of the string get close enough together, it can happen that a biological event occurs: for instance, an enzyme may be released. Therefore, some biological events are related to the motion of the DNA string. Some mathematical results for equation (1.1) can be found in [18]. Some of the biological motivation can be found in [7].

Example 2: The internal structure of the sun

The study of the internal structure of the sun is an active area of research. One important international project is known as Project SOHO [8]. Its objective was to use measurements of the motion of the sun's surface to obtain information about the internal structure of the sun. Indeed, the sun's surface moves in a rather complex manner: at any given time, any point on the surface is typically moving towards or away from the center. There are also waves going around the surface, as well as shock waves propagating through the sun itself, which cause the surface to pulsate.

A question of interest to solar geophysicists is to determine the origin of these shock waves. One school of thought is that they are due to turbulence, but the location and intensities of the shocks are unknown, so a probabilistic model can be considered.

A model that was proposed by P. Stark of U.C. Berkeley is that the main source of shocks is located in a spherical zone inside the sun, which is assumed to be a ball of radius R. Assuming that the shocks are randomly located on this sphere, the equation for the pressure variations throughout the sun would be

$$\frac{\partial^2 u}{\partial t^2}(t,x) = c^2(x)\, \rho_0(x) \left(\vec{\nabla} \cdot \left(\frac{1}{\rho_0(x)}\, \vec{\nabla} u \right) - \vec{\nabla} \cdot \vec{F}(t,x) \right), \tag{1.2}$$

where $x \in B(0, R)$, the ball centered at the origin with radius R, $c^2(x)$ is the speed of wave propagation at position x, $\rho_0(x)$ is the density at position x and $\vec{F}(t, x)$ models the shock that originates at time t and position x.

A model for $\vec{F}$ that corresponds to the description of the situation would be 3-dimensional Gaussian noise concentrated on the sphere $\partial B(0, r)$, where $0 < r < R$. A possible choice of the spatial correlation for the components of $\vec{F}$ would be

$$\delta(t - s)\ f(x \cdot y),$$

where $x \cdot y$ denotes the Euclidean inner product. A problem of interest is to estimate r from the available observations of the sun's surface. Some mathematical results relevant to this problem are developed in [3].

2 The stochastic wave equation

Equation (1.2) is a wave equation in vector form. The (simpler) real-valued stochastic wave equation, that we will be studying in these notes, reads as follows:

$$\begin{cases} \left(\frac{\partial^2 u}{\partial t^2} - \Delta u\right)(t, x) = \sigma(t, x, u(t, x))\, \dot{F}(t, x) + b(t, x, u(t, x)), & (t, x) \in [0, t] \times \mathbb{R}^d \\ u(0, x) = v_0(x), \qquad \dfrac{\partial u}{\partial t}(0, x) = \tilde{v}_0(x), & \end{cases} \tag{2.1}$$

where $\dot{F}(t, x)$ is a Gaussian noise, which we take to be space-time white noise for the moment, and $\sigma, b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are functions that satisfy standard properties, such as being Lipschitz in the third variable. The term Δu denotes the Laplacian of u in the x-variables.

Mild solutions of the stochastic wave equation

It is necessary to specify the notion of solution to (2.1) that we are considering. We will mainly be interested in the notion of *mild solution*, which is the following integral form of (2.1):

$$\begin{aligned} u(t, x) &= \int_{[0,t]\times\mathbb{R}^d} G(t - s, x - y)\, [\sigma(s, y, u(s, y))\, \dot{F}(s, y) + b(s, y, u(s, y))]\, ds\, dy \\ &\qquad + \left(\frac{d}{dt}\, G(t) * v_0\right)(x) + (G(t) * \tilde{v}_0)(x). \end{aligned} \tag{2.2}$$

In this equation, $G(t - s, x - y)$ is the Green's function of (2.1), which we discuss next, and $*$ denotes convolution in the x-variables. For the term involving $\dot{F}(s, y)$, a notion of stochastic integral is needed, that we will discuss later on.

Green's function of a p.d.e.

We consider first the case of an equation with constant coefficients. Let L be a partial differential operator with constant coefficients. A basic example is the wave operator

$$Lf = \frac{\partial^2 f}{\partial t^2} - \Delta f.$$

Then there is a (Schwartz) distribution $G \in \mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^d)$ such that the solution of the p.d.e.

$$L\,u = \varphi, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d),$$

is

$$u = G \underset{(t,x)}{*} \varphi$$

where $\underset{(t,x)}{*}$ denotes convolution in the (t,x)-variables. We recall that $\mathcal{S}(\mathbb{R}^d)$ denotes the space of smooth test functions with rapid decrease, and $\mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^d)$ denotes the space of tempered distributions [13].

When G is a function, this convolution can be written

$$u(t,x) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} G(t-s,\, x-y)\, \varphi(s,y)\, ds\, dy.$$

We note that this is the solution with vanishing initial conditions.

In the case of an operator with non-constant coefficients, such as

$$Lf = \frac{\partial^2 f}{\partial t^2} + 2c(t,x)\,\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial x^2} \qquad (d=1),$$

the Green's function has the form $G(t,x\,;\,s,y)$ and the solution of

$$L\,u = \varphi$$

is given by the expression

$$u(t,x) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} G(t,x\,;\,s,y)\, \varphi(s,y)\, ds\, dy.$$

Example 2.1 *The heat equation.* The partial differential operator is

$$L\,u = \frac{\partial u}{\partial t} - \Delta u, \qquad d \geq 1,$$

and the Green's function is

$$G(t,x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right).$$

This function is smooth except for a singularity at $(0,0)$.

Example 2.2 *The wave equation.* The partial differential operator is

$$Lu = \frac{\partial^2 u}{\partial t^2} - \Delta u.$$

The form of the Green's function depends on the dimension d. We refer to [16] for $d \in \{1, 2, 3\}$ and to [6] for $d > 3$. For $d = 1$, it is

$$G(t, x) = \frac{1}{2}\, 1_{\{|x|<t\}},$$

which is a bounded but discontinuous function. For $d = 2$, it is

$$G(t, x) = \frac{1}{\sqrt{2\pi}}\, \frac{1}{\sqrt{t^2 - |x|^2}}\, 1_{\{|x|<t\}}.$$

This function is unbounded and discontinuous. For $d = 3$, the "Green's function" is

$$G(t, dx) = \frac{1}{4\pi}\, \frac{\sigma_t(dx)}{t}\,,$$

where σ_t is uniform measure on $\partial B(0, t)$, with total mass $4\pi t^2$. In particular, $G(t, \mathbb{R}^3) = t$. This Green's function is in fact *not* a function, but a measure. Its convolution with a test function φ is given by

$$\begin{aligned} (G * \varphi)(t, x) &= \frac{1}{4\pi} \int_0^t ds \int_{\partial B(0,s)} \varphi(t - s,\, x - y)\, \frac{\sigma_s(dy)}{s} \\ &= \frac{1}{4\pi} \int_0^t ds\; s \int_{\partial B(0,1)} \varphi(t - s,\, x - sy)\, \sigma_1(dy). \end{aligned}$$

Of course, the meaning of an expression such as

$$\int_{[0,t]\times\mathbb{R}^d} G(t - s,\, x - y)\, F(ds, dy)$$

where G is a measure and F is a Gaussian noise, is now unclear: it is certainly outside of Walsh's theory of stochastic integration [9].

In dimensions greater than 3, the Green's function of the wave equation becomes even more irregular. For $d \geq 4$, set

$$N(d) = \begin{cases} \frac{d-3}{2} & \text{if } d \text{ is odd,} \\ \\ \frac{d-2}{2} & \text{if } d \text{ is even.} \end{cases}$$

For d even, set

$$\sigma_t^d\,(dx) = \frac{1}{\sqrt{t^2 - |x|^2}}\, 1_{\{|x|\,<t\}}\, dx,$$

and for d odd, let $\sigma_t^d\,(dx)$ be the uniform surface measure on $\partial B(0,t)$ with total mass t^{d-1}. Then for d odd, $G(t,x)$ can formally be written

$$G(t,x) = c_d \left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{N(d)} \left(\frac{\sigma_s^d}{s}\right)\, ds,$$

that is, for d odd,

$$(G * \varphi)(t,x) = c_d \int_0^t ds \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{N(d)} \left(\int_{\mathbb{R}^d} \varphi(t-s, x-y)\, \frac{\sigma_r^d(dy)}{r}\right)\Bigg|_{r=s}$$

while for d even,

$$(G * \varphi)(t,x) = c_d \int_0^t ds \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{N(d)} \left(\int_{B(0,r)} \varphi(t-s, x-y) \frac{dy}{\sqrt{r^2 - |y|^2}}\right)\Bigg|_{r=s}.$$

The meaning of $\int_{[0,t]} G(t-s, x-y)\, F(ds, dy)$ is even less clear in these cases!

The case of spatial dimension one

Existence and uniqueness of the solution to the stochastic wave equation in spatial dimension 1 is covered in [17, Exercise 3.7 p.323]. It is a good exercise that we leave to the reader.

Problem 1. Establish existence and uniqueness of the solution to the non-linear wave equation on $[0,t] \times \mathbb{R}$, driven by space-time white noise :

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \sigma(u(t,x))\, \dot{W}(t,x)$$

with initial conditions

$$u(0,\cdot) = \frac{\partial u}{\partial x}\,(0,\cdot) \equiv 0.$$

The solution uses the following standard steps, which also appear in the study of the semilinear stochastic heat equation (see [17] and [9]):

- define the Picard iteration scheme;
- establish L^2-convergence using Gronwall's lemma;
- show existence of higher moments of the solution, using Burkholder's inequality

$$E(|M_t|^p) \leq c_p\, E(\langle M \rangle_t^{p/2}); \tag{2.3}$$

- establish Hölder continuity of the solution.

It is also a good exercise to do the following calculation.

Problem 2. Let G be the Green's function of the wave equation, as defined in Example 2.2. For $d = 1$ and $d = 2$, check that

$$u(t,x) = \int_0^t ds \int_{\mathbb{R}^d} dy\, G(t-s,\, x-y)\, \varphi(s,y)$$

satisfies

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) = \varphi(t,x).$$

Space-time white noise in dimension $d = 2$

Having solved the non-linear stochastic wave equation driven by space-time white noise in dimension $d = 1$, it is tempting to attempt the same thing in dimension $d = 2$. We are going to show that there is a fundamental obstacle to doing this.

To this end, consider the *linear case*, that is, $\sigma \equiv 1$ and $b \equiv 0$. The mild solution given in (2.2) is not an equation in this case, but a formula:

$$\begin{aligned} u(t,x) &= \int_{[0,t]\times\mathbb{R}^2} G(t-s,\, x-y)\, W(ds,dy) \\ &= \int_{[0,t]\times\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{(t-s)^2 - |y-x|^2}}\, 1_{\{|y-x|<t-s\}}\, W(ds,dy), \end{aligned}$$

where $W(ds,dy)$ is space-time white noise.

The first issue is whether this stochastic integral well-defined. For this, we would need (see [9, Exercise 5.5]) to have

$$\int_0^t ds \int_{\mathbb{R}^2} dy\, G^2(t-s,\, x-y) < +\infty.$$

The integral is equal to

$$\begin{aligned} \int_0^t ds \int_{|y-x|<t-s} \frac{dy}{(t-s)^2 + |y-x|^2} &= \int_0^t dr \int_{|z|<r} \frac{dz}{r^2 - |z|^2} \\ &= \int_0^t dr \int_0^r d\rho\, \frac{2\pi\rho}{r^2 - \rho^2} \\ &= \pi \int_0^t dr\, \ln(r^2 - \rho^2)\Big|_r^0 \\ &= +\infty. \end{aligned}$$

In particular, there is *no* mild solution to the wave equation (2.2) when $d = 2$.

There have been some attemps at overcoming this problem [11], but as yet, there is no fully satisfactory approach to studying non-linear forms of the stochastic wave or heat equations driven by space-time white noise in dimensions $d \geq 2$.

A different tack is to consider spatially homogeneous noise, which we introduce now.

3 Spatially homogeneous Gaussian noise

Let Γ be a non-negative and non-negative definite tempered measure on $\mathbb{R}^d$, so that $\Gamma(dx) \geq 0$,

$$\int_{\mathbb{R}^d} \Gamma(dx)\,(\varphi * \tilde{\varphi})(x) \geq 0, \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^d),$$

where $\tilde{\varphi}(x) \stackrel{\text{def}}{=} \varphi(-x)$, and there exists $r > 0$ such that

$$\int_{\mathbb{R}^d} \Gamma(dx)\, \frac{1}{(1+|x|^2)^r} < \infty.$$

According to the Bochner-Schwartz theorem [13], there is a nonnegative measure μ on $\mathbb{R}^d$ whose Fourier transform is Γ: we write $\Gamma = \mathcal{F}\mu$. By definition, this means that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \Gamma(dx)\,\varphi(x) = \int_{\mathbb{R}^d} \mu(d\eta)\; \mathcal{F}\varphi(\eta)\,.$$

We recall that the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is

$$\mathcal{F}\varphi(\eta) = \int_{\mathbb{R}^d} \exp(-i\,\eta \cdot x)\; \varphi(x)\,dx,$$

where $\eta \cdot x$ denotes the Euclidean inner product. The measure μ is called the *spectral measure*.

Definition 3.1 A *spatially homogeneous Gaussian noise that is white in time* is an $L^2(\Omega, \mathcal{F}, P)-$valued mean zero Gaussian process

$$\Big(F(\varphi),\ \varphi \in C_0^\infty(\mathbb{R}^{1+d})\Big)$$

such that

$$E(F(\varphi)\,F(\psi)) = J(\varphi, \psi),$$

where

$$J(\varphi, \psi) \stackrel{\text{def}}{=} \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx)\,(\varphi(s,\cdot) * \tilde{\psi}(s,\cdot))(x).$$

In the case where the covariance measure Γ has a density, so that $\Gamma(dx) = f(x)\,dx$, then it is immediate to check that $J(\varphi, \psi)$ can be written as follows:

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\; \varphi(s,x)\, f(x-y)\, \psi(s,y).$$

Using the fact that the Fourier transform of a convolution is the product of the Fourier transforms, this can also be written

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \mu(d\eta)\, \mathcal{F}\varphi(s)(\eta)\, \overline{\mathcal{F}\psi(s)(\eta)}.$$

Informally, one often writes

$$E(\dot{F}(t,x)\dot{F}(s,y)) = \delta_0(t-s)\, f(x-y).$$

Example 3.2 (a) If $\Gamma(dx) = \delta_0(x)$, where δ_0 denotes the Dirac delta function, then the associated spatially homogeneous Gaussian noise is simply space-time white noise.

(b) Fix $0 < \beta < d$ and let

$$\Gamma_\beta(dx) = \frac{dx}{|x|^\beta}.$$

One can check [15, Chapter 5] that $\Gamma_\beta = \mathcal{F}\mu_\beta$, with

$$\mu_\beta(d\eta) = c_{d,\beta}\,\frac{d\eta}{|\eta|^{d-\beta}}\,.$$

We point out that if $\beta \uparrow d$, then the spatially homogeneous Gaussain noise F_β with the covariance measure Γ_β converges weakly to space-time white noise. Indeed, the spectral measure μ_β converges weakly to a multiple of Lebesgue measure on $\mathbb{R}^d$, which is the spectral measure of space-time white noise, since $\mathcal{F}(d\eta) = \delta_0$.

Extension of $F(\varphi)$ to a worthy martingale measure

From the spatially homogenenous Gaussian noise, we are going to construct a worthy martingale measure $M = (M_t(A),\, t \geq 0,\, A \in \mathcal{B}_b(\mathbb{R}^d))$, where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the family of bounded Borel subsets of $\mathbb{R}^d$. For this, if $A \in \mathcal{B}_b(\mathbb{R}^d)$, we set

$$M_t(A) \stackrel{\text{def}}{=} \lim_{n\to\infty} F(\varphi_n),$$

where the limit is in $L^2(\Omega, \mathcal{F}, P)$, $\varphi_n \in C_0^\infty(\mathbb{R}^{d+1})$ and $\varphi_n \downarrow 1_{[0,t]\times A}$.

One checks [2] that $(M_t(A),\, t \geq 0,\, A \in \mathcal{B}_b(\mathbb{R}^d))$ is a worthy martingale measure in the sense of Walsh; its covariation measure Q is given by

$$Q(A \times B \times]s,t]) = (t-s)\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, 1_A(x)\, f(x-y)\, 1_B(y),$$

and its dominating measure is $K \equiv Q$. The key relationship between F and M is that

$$F(\varphi) = \int_{\mathbb{R}_+\times\mathbb{R}^d} \varphi(t,x)\, M(dt,dx),$$

where the stochastic integral on the right-hand side is Walsh's martingale measure stochastic integral.

The underlying filtration $(\mathcal{F}_t,\ t \geq 0)$ associated with this martingale measure is given by

$$\mathcal{F}_t = \sigma\left(M_s(A),\ s \leq t,\ A \in \mathcal{B}_b(\mathbb{R}^d)\right) \vee \mathcal{N}, \qquad t \geq 0,$$

where $\mathcal{N}$ is the σ-field generated by all P-null sets.

4 The wave equation in spatial dimension 2

We shall consider the following form of the stochastic wave equation in spatial dimension $d = 2$:

$$\left(\frac{\partial^2 u}{\partial t^2} - \Delta u\right)(t,x) = \sigma(u(t,x))\,\dot{F}(t,x), \qquad (t,x) \in]0,T] \times \mathbb{R}^2, \tag{4.1}$$

with vanishing initial conditions. By a solution to (4.1), we mean a jointly measurable adapted process $(u(t,x))$ that satisfies the associated integral equation

$$u(t,x) = \int_{[0,t]\times\mathbb{R}^2} G(t-s, x-y)\,\sigma(u(s,y))\;M(ds,dy), \tag{4.2}$$

where M is the worthy martingale measure associated with $\dot{F}$.

The linear equation

A first step is to examine the linear equation, which corresponds to the case where $\sigma \equiv 1$:

$$\left(\frac{\partial^2 u}{\partial t^2} - \Delta u\right)(t,x) = \dot{F}(t,x), \tag{4.3}$$

with vanishing initial conditions. The mild solution should be

$$u(t,x) = \int_{[0,t]\times\mathbb{R}^2} G(t-s, x-y)\;M(ds,dy).$$

We know that the stochastic integral on the right-hand side is not defined for space-time white noise, so let us determine for which spatially homogeneous Gaussian noises it is well-defined. This is the case if

$$\int_0^t ds \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dz\; G(t-s,\,x-y)\,f(y-z)\;G(t-s,\,x-z) < +\infty,$$

or, equivalently, if

$$\int_0^t ds \int_{\mathbb{R}^2} \mu(d\eta)\;|\mathcal{F}G(s)(\eta)|^2 < +\infty. \tag{4.4}$$

Calculation of $\mathcal{F}G$

In principle, the Green's function of a p.d.e. solves the same p.d.e. with $\delta_{(0,0)}(t,x) = \delta_0(t)\,\delta_0(x)$ as right-hand side :

$$\frac{\partial^2 G}{\partial t^2} - \Delta G = \delta_0(t)\,\delta_0(x). \tag{4.5}$$

For fixed $t > 0$, the right-hand side vanishes. We shall take the Fourier transform in x on both sides of this equation, but first, we observe that since

$$\mathcal{F}G(t)(\xi) = \hat{G}(t)(\xi) = \int_{\mathbb{R}^2} e^{i\,\xi\cdot x}\, G(t,x)\, dx,$$

it is clear that

$$\mathcal{F}\left(\frac{\partial^2 G(t)}{\partial t}\right)(\xi) = \frac{\partial^2 \hat{G}(t)}{\partial t^2}\,(\xi),$$

and, using integration by parts, that

$$\begin{aligned} \mathcal{F}(\Delta G(t)) &= \int_{\mathbb{R}^2} e^{i\,\xi\cdot x}\, \Delta G(t,x)\, dx \\ &= \int_{\mathbb{R}^2} \Delta(e^{i\,\xi\cdot x})\, G(t,x)\, dx \\ &= -|\xi|^2\, \mathcal{F}G(t)\,(\xi). \end{aligned}$$

Therefore, we deduce from (4.5) that for $t > 0$,

$$\frac{\partial^2 \hat{G}(t)}{\partial t^2}\,(\xi) + |\xi|^2\, \hat{G}(t)\,(\xi) = \delta_0(t).$$

For fixed ξ, the solution to the associated homogeneous ordinary differential equation in t is

$$\hat{G}(t)(\xi) = a(\xi)\,\frac{\sin(t|\xi|)}{|\xi|} + b(\xi)\,\frac{\cos(t|\xi|)}{|\xi|}\,.$$

The solution that we seek [16, Chapter I, Section 4] is the one such that $\hat{G}(0)(\xi) = 0$ and $\frac{d\hat{G}(0)}{dt}(\xi) = 1$, so we conclude that for $t \geq 0$ and $\xi \in \mathbb{R}^2$,

$$\mathcal{F}G(t)\,(\xi) = \frac{\sin(t|\xi|)}{|\xi|}. \tag{4.6}$$

This formula is in fact valid in all dimensions $d \geq 1$.

Condition on the spectral measure

Condition (4.4) for existence of a mild solution on $[0, T]$ to the linear wave equation (4.3) becomes

$$\int_0^T ds \int_{\mathbb{R}^2} \mu(d\eta)\, \frac{\sin^2(s|\eta|)}{|\eta|^2} < +\infty.$$

Using Fubini's theorem, one can evaluate the ds-integral explicitly, or simply check that

$$\frac{c_1}{1+|\eta|^2} \leq \int_0^T ds\, \frac{\sin^2(s|\eta|)}{|\eta|^2} \leq \frac{c_2}{1+|\eta|^2}\,,$$

so condition (4.4) on the spectral measure becomes

$$\int_{\mathbb{R}^2} \mu(d\eta)\, \frac{1}{1+|\eta|^2} < +\infty. \tag{4.7}$$

Example 4.1 Consider the case where $f(u) = |x|^{-\beta}$, $0 < \beta < d$. In this case, $\mu(d\eta) = c_{d,\beta}|\eta|^{\beta-d}\, dy$, so one checks immediately that condition (4.7) holds if and only if $\beta < 2$. Therefore, the spatially homogeneous Gaussian noise is defined for $0 < \beta < d$, but a mild solution of the linear stochastic wave equation (4.3) exists if and only if $0 < \beta < 2$.

Reformulating (4.7) *in terms of the covariance measure*

Condition (4.7) on the spectral measure can be reformulated as a condition on the covariance measure Γ. It is shown in [10] that in dimension $d = 2$, (4.7) is equivalent to

$$\int_{|x|\leq 1} \Gamma(dx)\, \ln\left(\frac{1}{|x|}\right) < +\infty,$$

while in dimensions $d \geq 3$, (4.7) is equivalent to

$$\int_{|x|\, \leq 1} \Gamma(dx)\, \frac{1}{|x|^{d-2}} < +\infty.$$

In dimension $d = 1$, condition (4.7) is satisfied for any non-negative measure μ such that $\Gamma = \mathcal{F}\mu$ is also a non-negative measure.

The non-linear wave equation in dimension $d = 2$

We consider equation (4.1). The following theorem is the main result on existence and uniqueness.

Theorem 4.2 *Assume* $d = 2$*. Suppose that* σ *is a Lipschitz continuous function and that condition* (4.7) *holds. Then there exists a unique solution* $(u(t,x),\ t \geq 0,\ x \in \mathbb{R}^2)$ *of* (4.1) *and for all* $p \geq 1$*, this solution satisfies*

$$\sup_{0\leq t\leq T}\ \sup_{x\in\mathbb{R}^d}\ E\left(|u(t,x)|^p\right) < \infty.$$

Proof. This proof follows a classical Picard iteration scheme. We set $u_0(t,x) = 0$, and, by induction, for $n \geq 0$,

$$u_{n+1}(t,x) = \int_{[0,t]\times\mathbb{R}^2} G(t-s,\, x-y)\, \sigma(u_n(s,y))\, M(ds,dy).$$

Before establishing convergence of this scheme, we first check that for $p \geq 2$,

$$\sup_{n\geq 0}\ \sup_{0\leq s\leq T}\ \sup_{x\in\mathbb{R}^2}\ E\left(|u_n(s,x)|^p\right) < +\infty.$$

We apply Burkholder's inequality (2.3) and use the explicit form of the quadratic variation of the stochastic integral [9, Theorem 5.26] to see that

$$\begin{aligned} E\left(|u_{n+1}(t,x)|^p\right) \;\leq\; & cE\Big(\Big(\int_0^t ds \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dz\, G(t-s,\, x-y)\, \sigma(u_n(s,y)) \\ & \times f(y-z)\, G(t-s,\, x-z)\, \sigma(u_n(s,z))\Big)^{\frac{p}{2}}\Big). \end{aligned}$$

Since $G \geq 0$ and $f \geq 0$, we apply Hölder's inequality in the form

$$\left|\int f\, d\mu\right|^p \leq \left(\int 1\, d\mu\right)^{p/q} \left(\int |f|^p\, d\mu\right), \qquad \text{where } \frac{p}{q} = p-1 \tag{4.8}$$

and μ is a non-negative measure, to see that $E\left(|u_{n+1}(t,x)|^p\right)$ is bounded above by

$$\begin{aligned} & c\Big(\int_0^t ds \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dz\, G(t-s,\, x-y)\, f(y-z)\, G(t-s,\, x-z)\Big)^{\frac{p}{2}-1} \\ & \times \int_0^t ds \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dz\, G(t-s,\, x-y)\, f(y-z)\, G(t-s,\, x-z) \\ & \qquad \times E\left(|\sigma(u_n(s,y))\, \sigma(u_n(s,z))|^{\frac{p}{2}}\right). \end{aligned}$$

We apply the Cauchy-Schwartz inequality to the expectation and use the Lipschitz property of σ to bound this by

$$\begin{aligned} & C\left(\int_0^t ds \int_{\mathbb{R}^2} \mu(d\eta)\, |\mathcal{F}G(t-s)(\eta)|^2\right)^{\frac{p}{2}-1} \\ & \times \int_0^t ds \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dz\, G(t-s,\, x-y)\, f(y-z)\, G(t-s,\, x-z) \\ & \qquad \times \left(E\left(1+|u_n(s,y)|^p\right)\right)^{1/2} \left(E\left(1+|u_n(s,z)|^p\right)\right)^{1/2}. \end{aligned}$$

Let

$$J(t) = \int_0^t ds \int_{\mathbb{R}^2} \mu(d\eta)\, |\mathcal{F}G(t-s)(\eta)|^2 \leq C \int_{\mathbb{R}^2} \mu(d\eta)\, \frac{1}{1+|\eta|^2}.$$

Then

$$\begin{aligned} E\left(|u_{n+1}(t,x)|^p\right) \;\leq\; & C\,(J(t))^{\frac{p}{2}-1} \int_0^t ds \left(1 + \sup_{y\in\mathbb{R}^2} E\left(|u_n(s,y)|^p\right)\right) \\ & \qquad \times \int_{\mathbb{R}^2} \mu(d\eta)\, |\mathcal{F}G(t-s)(\eta)|^2 \\ \leq\; & \tilde{C} \int_0^t ds \left(1 + \sup_{y\in\mathbb{R}^2} E(|u_n(s,y)|^p)\right). \end{aligned}$$

Therefore, if we set

$$M_n(t) = \sup_{x\in\mathbb{R}^2} E\left(|u_n(t,x)|^p\right),$$

then

$$M_{n+1}(t) \leq \tilde{C}\int_0^t ds\ \left(1 + M_n(s)\right).$$

Using Gronwall's lemma, we conclude that

$$\sup_{n\in\mathbb{N}}\ \sup_{0\leq t\leq T} M_n(t) < +\infty.$$

We now check L^2-convergence of the Picard iteration scheme. By the same reasoning as above, we show that

$$\sup_{x\in\mathbb{R}^2} E\left(|u_{n+1}(t,x) - u_n(t,x)|^p\right) \leq C\int_0^t ds\ \sup_{y\in\mathbb{R}^2} E\left(|u_n(s,y) - u_{n-1}(s,y)|^p\right).$$

Gronwall's lemma shows that $(u_n(t,x),\ n\geq 1)$ converges in $L^2(\Omega,\ \mathcal{F},\ P)$, uniformly in $x\in\mathbb{R}^2$.

Uniqueness of the solution follows in a standard way: see [9, Proof of Theorem 6.4]. □

Hölder-continuity $(d=2)$

In order to establish Hölder continuity of the solution to the stochastic wave equation in spatial dimension 2, we first recall the *Kolmogorov continuity theorem.* It is a good idea to compare this statement with the equivalent one in [9, Theorem 4.3].

Theorem 4.3 (The Kolmogorov Continuity Theorem). *Suppose that there is* $q>0$, $\rho\in\,]\frac{d}{q},1[$ *and* $C>0$ *such that for all* $x,y\in\mathbb{R}^d$,

$$E\left(|u(t,x)-u(t,y)|^q\right) \leq C\,|x-y|^{\rho q}. \tag{4.9}$$

Then $x\mapsto u(t,x)$ *has a* $\tilde{\rho}$*-Hölder continuous version, for any* $\tilde{\rho}\in\,]0,\ \rho-\frac{d}{q}[$.

In order to use the statement of this theorem to establish $(\rho-\varepsilon)$-Hölder continuity, for any $\varepsilon>0$, it is necessary to obtain estimates on arbitrarily high moments of increments, that is, to establish (4.9) for arbitrarily large q.

L^q-moments of increments

From the integral equation (4.2), we see that

$$u(t,x)-u(s,y) = \iint \left(G(t-r,\ x-z) - G(s-r,\ y-z)\right)\ \sigma(u(r,z))\,M(dr,dz),$$

and so, by Burkholder's inequality (2.3),

$$
\begin{aligned}
&E\left(|u(t,x)-u(s,y)|^p\right)\\
&\quad\le C\,E\Bigg(\bigg|\int_0^t dr\int_{\mathbb{R}^2}dz\int_{\mathbb{R}^2}dv\ \left(G(t-r,\,x-z)-G(s-r,\,y-z)\right)\ f(z-v)\\
&\qquad\qquad \times\left(G(t-r,\,x-v)-G(s-r,\,y-v)\right)\sigma(u(r,z))\,\sigma(u(r,v))\bigg|^{p/2}\Bigg)\\
&\quad\le C\left(\int dr\int dz\int dv\,|G()-G()|\,f(\ )\,|G()-G()|\right)^{\frac{p}{2}-1}\\
&\qquad\qquad\times\int dr\int dz\int dv\ |G()-G()|\,f(\ )\,|G()-G()|\\
&\qquad\qquad\qquad\times E\left(|\sigma(u(r,z))|^{p/2}\ |\sigma(u(r,v))|^{p/2}\right),
\end{aligned}
$$

where the omitted variables are easily filled in. The Lipschitz property of σ implies a bound of the type "linear growth", and so, using also the Cauchy-Schwartz inequality, we see that the expectation is bounded by

$$
C\sup_{r\le T,\,z\in\mathbb{R}^2}\left(1+E(|u(r,z)|^p)\right).
$$

Define

$$
\begin{aligned}
J(t,\,x\,;\,s,y)\ &=\ \int_0^t dr\int_{\mathbb{R}^2}dz\int_{\mathbb{R}^2}dv\,|G(t-r,\,x-z)-G(s-r,\,y-z)|\ f(z-v)\\
&\qquad\qquad\times|G(t-r,\,x-v)-G(s-r,\,y-v)|.
\end{aligned}
$$

We have shown that

$$
E\left(|u(t,x)-u(s,y)|^p\right)\le\left(J(t,\,x\,;\,s,y)\right)^{p/2}.
$$

Therefore, we will get Hölder-continuity provided, for some $\gamma>0$ and $\rho>0$, we can establish an estimate of the type

$$
J(t,x\,;\,s,y)\le c(|t-s|^{\gamma}+|x-y|^{\rho}).
$$

Indeed, this will establish $\frac{\gamma_1}{2}$-Hölder continuity in time, and $\frac{\rho_1}{2}$-Hölder continuity in in space, for all $\gamma_1\in\,]0,\gamma[$ and $\rho_1\in\,]0,\rho[$.

Analysis of $J(t,x\,;\,s,y)$

If there were no absolute values around the increments of G, then we could use the Fourier transform to rewrite $J(t,x\,;\,s,y)$, in the case $x=y$ and $s>t$, for instance, as

$$
\begin{aligned}
J(t,x\,;\,s,x)=\int_0^s dr\int_{\mathbb{R}^2}\mu(d\eta)\,|\mathcal{F}G(t-r)(\eta)-\mathcal{F}G(s-r)(\eta)|^2&\\
+\int_s^t dr\int_{\mathbb{R}^2}\mu(d\eta)\,|\mathcal{F}G(t-r)(\eta)|^2.&
\end{aligned}
$$

We could then analyse this using the specific form of $\mathcal{F}G$ in (4.6). However, the presence of the absolute values makes this approach inoperable. By a direct analysis of $J(t,x\,;\,s,x)$, Sanz-Solé and Sarrá [12] have established to following results. If

$$\int_{\mathbb{R}^2} \mu(d\eta)\, \frac{1}{(1+|\eta|^2)^a} < \infty, \qquad \text{for some } \ a \in]0,1[,$$

then $t \mapsto u(t,x)$ is γ_1-Hölder continuous, for

$$\gamma_1 \in \left]0,\ \frac{1}{2}\ \wedge (1-a)\right[,$$

and $x \mapsto u(t,x)$ is γ_2-Hölder continuous, for $\gamma_2 \in]0,\ 1-a[$.

When $\mu(d\eta) = |\eta|^{-\beta}\, d\eta$, these intervals become

$$\gamma_1 \in \left]0,\ \frac{1}{2}\ \wedge\ \frac{2-\beta}{2}\right[\qquad \text{and} \qquad \gamma_2 \in \left]0,\ \frac{2-\beta}{2}\right[.$$

We note that the best possible interval for γ_1 is in fact $]0, \frac{2-\beta}{2}[$ (see [5, Chapter 5]).

5 A function-valued stochastic integral

Because the Green's function in spatial dimension 3 is a measure and not a function, the study of the wave equation in this dimension requires different methods than those used in dimensions 1 and 2. In particular, we will use a function-valued stochastic integral, developed in [4].

Our first objective is to define a stochastic integral of the form

$$\int_{[0,t]\times\mathbb{R}^d} G(s,\, x-y)\; Z(s,y)\; M(ds,dy),$$

where $G(s,\cdot)$ is the Green's function of the wave equation (see Example 2.2) and $Z(s,y)$ is a random field that plays the role of $\sigma(u(s,y))$.

We shall assume for the moment that $d \geq 1$ and that the following conditions are satisfied.

Hypotheses

(H1) For $0 \leq s \leq T$, $Z(s,\cdot) \in L^2(\mathbb{R}^d)$ a.s., $Z(s,\cdot)$ is $\mathcal{F}_s-$measurable, and $s \mapsto Z(s,\cdot)$ from $\mathbb{R}_+ \to L^2(\mathbb{R}^d)$ is continuous.

(H2) For all $s \geq 0$,

$$\int_0^T ds \sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\eta)\, |\mathcal{F}G(s)(\xi-\eta)|^2 < +\infty.$$

We note that $\mathcal{F}G(s)(\xi-\eta)$ is given in (4.6), so that (H2) is a condition on the spectral measure μ, while (H1) is a condition on Z.

Fix $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\psi \geq 0$, supp $\psi \subset B(0,1)$ and

$$\int_{\mathbb{R}^d} \psi(x)\, dx = 1.$$

For $n \geq 1$, set

$$\psi_n(x) = n^d\, \psi(n\, x).$$

In particular, $\psi_n \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$, and $\mathcal{F}\,\psi_n(\xi) = \mathcal{F}\,\psi(\xi/n)$, so that $|\mathcal{F}\,\psi_n(\xi)| \leq 1$. Define

$$G_n(s,\cdot) = G(s) * \psi_n,$$

so that G_n is a C_0^∞-function. Then

$$v_{G_n,Z}(t,x) \stackrel{\text{def}}{=} \int_{[0,t]\times\mathbb{R}^d} G_n(s,\, x-y)\; Z(s,y)\; M(ds,dy)$$

is well-defined as a Walsh-stochastic integral, and

$$E\left(\|v_{G_n,Z}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\right) = I_{G_n,\,Z}, \tag{5.1}$$

where

$$\begin{aligned} I_{G_n,Z} &= \int_{\mathbb{R}^d} dx\, E\left((v_{G_n,Z}(t,x))^2\right) \\ &= \int_{\mathbb{R}^d} dx \int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\; G_n(s,\, x-y) Z(s,y)\; f(y-z) \\ &\qquad\qquad \times\, G_n(s, x-z)\; Z(s,z). \end{aligned}$$

Using the fact that the Fourier transform of a convolution (respectively product) is the product (resp. convolution) of the Fourier transforms, one easily checks that

$$I_{G_n,Z} = \int_0^t ds \int_{\mathbb{R}^d} d\xi\; E\left(|\mathcal{F}Z(s,\cdot)(\xi)|^2\right) \int_{\mathbb{R}^d} \mu(d\eta)\, |\mathcal{F}G_n(s,\cdot)\,(\xi-\eta)|^2.$$

We note that:

(a) the following inequality holds:

$$I_{G_n,Z} \leq \tilde{I}_{G_n,Z}, \tag{5.2}$$

where

$$\tilde{I}_{G_n,Z} \stackrel{\text{def}}{=} \int_0^t ds\; E\left(\|Z(s,\cdot)\|^2_{L^2(\mathbb{R}^d)}\right) \sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\eta)\, |\mathcal{F}G_n(s,\cdot)(\xi-\eta)|^2; \tag{5.3}$$

(b) the equality (5.1) plays the role of an isometry property;

(c) by elementary properties of convolution and Fourier transform,

$$\tilde{I}_{G_n,Z} \leq \tilde{I}_{G,Z} < +\infty,$$

by (H2) and (H1).

In addition, one checks that the stochastic integral

$$v_{G,Z}(t) \stackrel{\text{def}}{=} \lim_{n\to\infty} v_{G_n,Z}$$

exists, in the sense that

$$E\left(\|v_{G,Z}\,(t) - v_{G_n,Z}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\right) \longrightarrow 0,$$

and

$$E\left(\|v_{G,Z}(t)\|^2_{L^2(\mathbb{R}^d)}\right) = I_{G,Z} \leq \tilde{I}_{G,Z}\,.$$

We use the following notation for the stochastic integral that we have just defined:

$$v_{G,Z}(t) = \int_{[0,t]\times\mathbb{R}^d} G(s,\cdot - y)\; Z(s,y)\; M(ds,dy).$$

For t fixed, $v_{G,Z}(t) \in L^2(\mathbb{R}^d)$ is a square-integrable function that is defined almost-everywhere.

The definition of the stochastic integral requires in particular that hypothesis (H2) be satisfied. In the case where

$$\Gamma(dx) = k_\beta(x)\,dx, \qquad \text{with } k_\beta(x) = |x|^{-\beta}, \quad \beta > 0, \tag{5.4}$$

this condition becomes

$$\int_0^T ds \sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} d\eta\; |\eta|^{\beta-d}\, \frac{\sin^2(s|\xi-\eta|)}{|\xi-\eta|^2} < +\infty.$$

One checks [4] that this is the case if and only if $0 < \beta < 2$.

6 The wave equation in spatial dimension $d \geq 1$

We consider the following stochastic wave equation in spatial dimension $d \geq 1$, driven by spatially homogeneous Gaussian noise $\dot{F}(t,x)$ as defined in Section 3:

$$\begin{cases} \left(\frac{\partial^2 u}{\partial t^2} - \Delta u\right)(t,x) = \sigma(x,\, u(t,x))\,\dot{F}(t,x), & t \in \,]0,T],\; x \in \mathbb{R}^d, \\ u(0,x) = v_0(x), \qquad \frac{\partial u}{\partial t}\,(0,x) = \tilde{v}_0(x), & \end{cases} \tag{6.1}$$

where $v_0 \in L^2(\mathbb{R}^d)$ and $\tilde{v}_0 \in H^{-1}(\mathbb{R}^d)$. By definition, $H^{-1}(\mathbb{R}^d)$ is the set of square-integrable functions $\tilde{v}_0$ such that

$$\|\tilde{v}_0\|^2_{H^{-1}(\mathbb{R}^d)} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} d\xi \, \frac{1}{1+|\xi|^2} \, |\mathcal{F}\,\tilde{v}_0(\xi)|^2 < +\infty.$$

We shall restrict ourselves, though this is not really necessary (see [4]) to the case where $\Gamma(dx)$ is as in (5.4), and $0 < \beta < 2$.

The past-light cone property

Consider a bounded domain $D \subset \mathbb{R}^d$. A fundamental property of the wave equation is that $u(T,x)$, $x \in D$, only depends on $v_0|_{K^D}$ and $\tilde{v}_0|_{K^D}$, where

$$K^D = \{\, y \in \mathbb{R}^d : \, d(y,D) \leq T \}$$

and $d(y,D)$ denotes the distance from y to the set D, and on the noise $\dot{F}(s,y)$ for $y \in K^{D(s)}$, $0 \leq s \leq T$, where

$$K^D(s) = \Big\{ y \in \mathbb{R}^d : \; d(y,D) \leq T - s \Big\}.$$

Therefore, the solution $u(t,x)$ in D is unchanged if we take the s.p.d.e.

$$\left(\frac{\partial^2 u}{\partial t^2} - \Delta u \right)(t,x) = 1_{K^d(t)}(x)\, \sigma(x, u(t,x))\, \dot{F}(t,x).$$

We shall make the following linear growth and Lipschitz continuity assumptions on the function σ.

Assumptions

(a) $|\sigma(x,u)| \leq c(1+|u|)\, 1_{K^D(x)}$, for all $x \in \mathbb{R}^d$ and $u \in \mathbb{R}$;

(b) $|\sigma(x,u) - \sigma(x,v)| \leq c\,|u-v|$, for all $x \in \mathbb{R}^d$ and $u, v \in \mathbb{R}$.

Definition 6.1 An adapted and mean-square continuous $L^2(\mathbb{R}^d)$-valued process $(u(t),\ 0 \leq t \leq T)$ is *a solution of* (6.1) *in* D if for all $t \in]0,T]$,

$$\begin{aligned} u(t)\, 1_{K^D(t)} \;&=\; 1_{K^D(t)} \Bigg(\frac{d}{dt} G(t) * v_0 + G(t) * \tilde{v}_0 \\ &\qquad + \int_{[0,t]\times\mathbb{R}^d} G(t-s,\, \cdot - y)\, \sigma(y, u(s,y))\, M(ds,dy) \Bigg). \end{aligned}$$

Theorem 6.2 *Let* $d \geq 1$. *Suppose* $0 < \beta < 2 \wedge d$ *and that the assumptions above on* σ *are satisfied. Then* (6.1) *has a unique solution* $(u(t),\ 0 \leq t \leq T)$ *in* D.

Proof. We use a Picard iteration scheme. Set

$$u_0(t,x) = \frac{d}{dt}\, G(t) * v_0 + G(t) * \tilde{v}_0.$$

We first check that $u_0(t) \in L^2(\mathbb{R}^d)$. Indeed,

$$\begin{aligned}
\left\| \frac{d}{dt}\, G(t) * v_0 \right\|_{L^2(\mathbb{R}^d)} &= \left\| \mathcal{F}\left(\frac{d}{dt}\, G(t)\right) \cdot \mathcal{F} v_0 \right\|_{L^2(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} d\xi \left| |\xi| \, \frac{\cos(t|\xi|)}{|\xi|} \cdot \mathcal{F} v_0(\xi) \right|^2 \\
&\leq \| v_0 \|_{L^2(\mathbb{R}^d)} \,,
\end{aligned}$$

and, similarly,

$$\| G(t) * \tilde{v}_0 \|_{L^2(\mathbb{R}^d)} \leq \| \tilde{v}_0 \|_{H^{-1}(\mathbb{R}^d)} \,.$$

One checks in a similar way that $t \mapsto u_0(t)$ from $[0,T]$ into $L^2(\mathbb{R}^d)$ is continuous.

We now define the Picard iteration scheme. For $n \geq 0$, assume that $(u_n(t),\ 0 \leq t \leq T)$ has been defined, and satisfies (H1). Set

$$u_{n+1}(t) = 1_{K^D(t)}\left(u_0(t) + v_{n+1}(t)\right), \tag{6.2}$$

where

$$v_{n+1}(t) = \int_{[0,t]\times\mathbb{R}^d} G(t-s,\, \cdot - y)\, \sigma(y, u_n(s,y))\, M(ds,dy). \tag{6.3}$$

By induction, $Z_n(s,y) = \sigma(y,\, u_n(s,y))$ satisfies (H1). Indeed, this process is adapted, and since

$$\| \sigma(\cdot,\, u_n(s,\cdot)) - \sigma(\cdot, u_n(t,\cdot)) \|_{L^2(\mathbb{R}^d)} \leq C \| u_n(s,\cdot) - u_n(t,\cdot) \|_{L^2(\mathbb{R}^d)},$$

it follows that $s \mapsto u_n(s,\cdot)$ is mean-square continuous. One checks that u_{n+1} also satisfies (H1): this uses assumption (a).

Therefore, the stochastic integral (6.3) is well-defined. Let

$$\begin{aligned}
M_n(r) &= \sup_{0\leq t\leq r} E\left(\| u_{n+1}(t) - u_n(t) \|^2_{L^2(K^D(t))} \right) \\
&= \sup_{0\leq t\leq r} E\left(\| v_{n+1}(t) - v_n(t) \|^2_{L^2(K^D(t))} \right) \\
&= \sup_{0\leq t\leq r} E\Bigg(\Bigg\| \int_{[0,t]\times\mathbb{R}^d} G(t-s,\cdot - y) \\
&\qquad\qquad \times \left(\sigma(y,u_n(s,y)) - \sigma(y,u_{n-1}(s,y))\right)\, M(ds,dy) \Bigg\|^2_{L^2(K^D(t))} \Bigg) \\
&\leq \sup_{0\leq t\leq r} \int_0^t ds\, E\left(\| \sigma(\cdot,u_n(s,\cdot)) - \sigma(\cdot,u_{n-1}(s,\cdot)) \|^2_{L^2(K^D(t))} \right) J(t-s),
\end{aligned}$$

where

$$J(s) = \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} d\eta \, |\eta|^{\beta - d} \, \frac{\sin^2(s|\xi - \eta|)}{|\xi - \eta|^2} \, .$$

A direct calculation shows that

$$\sup_{0 \le s \le T} J(s) < +\infty,$$

since $0 < \beta < 2$, so

$$M_n(r) \le C \sup_{0 \le t \le r} \int_0^t ds \, E\left(\|u_n(s,\cdot) - u_{n-1}(s,\cdot)\|^2_{L^2(K^D(t))} \right),$$

that is,

$$M_n(r) \le C \int_0^r M_{n-1}(s) \, ds.$$

Because $M_0(T) < +\infty$, Gronwall's lemma implies that

$$\sum_{n=0}^{+\infty} (M_n(r))^{1/2} < +\infty.$$

Therefore, $(u_n(t,\cdot),\, n \in \mathbb{N})$ converges in $L^2(\Omega \times \mathbb{R}^d,\, dP \times dx)$, uniformly in $t \in [0,T]$, to a limit $u(t,\cdot)$. Since u_n satisfies (H1) and u_n converges uniformly in t to $u(t,\cdot)$, it follows that $u(t,\cdot)$ is a solution to (6.1): indeed, it suffices to pass to the limit in (6.2) and (6.3).

Uniqueness of the solution follows by a standard argument. □

7 Spatial regularity of the stochastic integral ($d = 3$)

We aim now to analyze spatial regularity of the solution to the 3-dimensional stochastic wave equation (6.1) driven by spatially homogeneous Gaussian noise, with covariance given by a Riesz kernel $f(x) = |x|^{-\beta}$, where $0 < \beta < 2$. For this, we shall first examine the regularity in the x-variable of the function-valued stochastic integral defined in Section 5 when $d = 3$.

We recall that studying regularity properties requires information on higher moments. With these, one can use the Kolmogorov continuity theorem (Theorem 4.3) or the Sobolev embedding theorem, which we now recall.

Theorem 7.1 (The Sobolev Embedding Theorem). *Suppose that $g \in W^{\rho,q}(\mathcal{O})$. Then $x \mapsto g(x)$ is $\tilde{\rho}$-Hölder continuous, for all $\tilde{\rho} \in]0, \rho - \frac{d}{q}[$.*

We recall [14] that the norm in the space $W^{p,q}(\mathcal{O})$ is defined by

$$\|g\|^q_{W^{p,q}(\mathcal{O})} = \|g\|^q_{L^q(\mathcal{O})} + \|g\|^q_{p,q,\mathcal{O}}\,, \tag{7.1}$$

where

$$\begin{aligned}
\|g\|^q_{L^q(\mathcal{O})} &= \int_{\mathcal{O}} |g(x)|^q\,dx \\
\|g\|^q_{p,q,\mathcal{O}} &= \int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \quad \frac{|g(x)-g(y)|^q}{|x-y|^{d+\rho q}}\,.
\end{aligned}$$

Our first objective is to determine conditions that ensure that

$$E\left(\|v_{G,Z}\|^q_{L^q(\mathcal{O})}\right) < +\infty.$$

For $\varepsilon > 0$, we let

$$\mathcal{O}^\varepsilon = \left\{x \in \mathbb{R}^3 : \exists\, z \in \mathcal{O} \ \text{ with } \ |x-z| \ < \varepsilon\right\}$$

denote the ε-enlargement of $\mathcal{O}$, and use the notation

$$v^t_{G,Z} = \int_{[0,t]\times\mathbb{R}^3} G(t-s,\,\cdot - y)\;Z(s,y)\;M(ds,dy).$$

An estimate in L^p-norm

Theorem 7.2 *Suppose $0 < \beta < 2$. Fix $T > 0$, $q \in [2,+\infty[$ and let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded domain. Suppose that*

$$\int_0^t ds\; E\left(\|Z(s)\|^q_{L^q(\mathcal{O}^{t-s})}\right) < +\infty.$$

Then

$$E\left(\|v^t_{G,Z}\|^q_{L^q(\mathcal{O})}\right) \le C \int_0^t ds\, E\left(\|Z(s)\|^q_{L^q(\mathcal{O}^{t-s})}\right).$$

Proof. We present the main ideas, omitting some technical issues that are handled in [5, Proposition 3.4]. First, we check inequality with G replaced by G_n :

$$\begin{aligned}
&E\left(\|v^t_{G_n,Z}\|^q_{L^q(\mathcal{O})}\right) \\
&\quad = \int_{\mathcal{O}} dx\, E\left(\Big|\int_{[0,t]\times\mathbb{R}^3} G_n(t-s,\,x-y)\;Z(s,y)\;M(ds,dy)\Big|^q\right) \\
&\quad \le \int_{\mathcal{O}} dx\, E\Big(\Big|\int_0^t ds \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dz\; G_n\,(t-s,\,x-y)\;Z(s,y)\;f(y-z) \\
&\qquad\qquad \times G_n(t-s,\,x-z)\,Z(s,z)\Big|^{q/2}\Big)\,.
\end{aligned}$$

Let

$$\mu_n(t,x) = \int_0^t ds \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dz\, G_n(t-s,\, x-y)\, f(y-z)\, G_n(t-s,\, x-z).$$

Assume that

$$\sup_{n,x,\ t\le T} \mu_n(t,x) < +\infty. \tag{7.2}$$

By Hölder's inequality, written in the form (4.8), we see, since $G_n \ge 0$, that

$$\begin{aligned} E\left(\|v^t_{G_n,Z}\|^q_{L^q(\mathcal{O})}\right) &\le \int_{\mathcal{O}} dx\, (\mu_n(t,x))^{\frac{q}{2}-1}\, E\biggl(\int_0^t ds \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dz\, G_n(t-s,\, x-y) \\ &\qquad \times f(y-z)\, G_n(t-s,\, x-z)\, |Z(s,y)|^{q/2}\, |Z(s,z)|^{q/2}\biggr) \\ &= I_{G_n\,,\,|Z\,1_{\mathcal{O}^{t-s+1/n}}|^{q/2}}. \end{aligned}$$

We apply (5.2), then (5.3), to bound this by

$$\begin{aligned} \tilde{I}_{G_n\,,\,|Z\,1_{\mathcal{O}^{t-s+1/n}}|^{q/2}} &= \int_0^t ds\, E\left(\|\ |Z(s)|^{q/2}\ 1_{\mathcal{O}^{t-s+1/n}}\|^2_{L^2(\mathbb{R}^3)}\right) \\ &\qquad \times \sup_{\xi\in\mathbb{R}^3} \int_{\mathbb{R}^3} \mu(d\eta)\, |\mathcal{F}G_n(s,\cdot)(\xi-\eta)|^2. \end{aligned}$$

Since $0<\beta<2$, the supremum over ξ is finite, therefore

$$E\left(\|v^t_{G_n,Z}\|^q_{L^q(\mathcal{O})}\right) \le C \int_0^t ds\, E\left(\|Z(s)\|^q_{L^q(\mathcal{O}^{t-s+1/n})}\right).$$

By Fatou's lemma,

$$\begin{aligned} E\left(\|v^t_{G,Z}\|^q_{L^q(\mathcal{O})}\right) &\le \liminf_{k\to\infty}\ E\left(\|v^t_{G_{n_k},Z}\|^q_{L^q(\mathcal{O})}\right) \\ &\le \liminf_{k\to\infty} \int_0^t ds\, E\left(\|Z(s)\|^q_{L^q(\mathcal{O}^{t-s+1/n_k})}\right) \\ &= \int_0^t ds\, E\left(\|Z(s)\|^q_{L^q(\mathcal{O}^{t-s})}\right). \end{aligned}$$

It remains to check that (7.2) holds. Since

$$|\mathcal{F}G_n(t-s)(\eta)|^2 \le |\mathcal{F}G(t-s)(\eta)|^2 = \frac{\sin^2((t-s)|\eta|)}{|\eta|^2},$$

it follows that for $t\in[0,T]$ and $x\in\mathbb{R}^3$,

$$\mu_n(t,x) \le \int_0^t ds \int_{\mathbb{R}^3} d\eta\, |\eta|^{\beta-3}\, \frac{\sin^2((t-s)|\eta|)}{|\eta|^2} \le C(T),$$

since $0 < \beta < 2$. This completes the proof. $\square$

An estimate in Sobolev norm

We consider here a spatially homogeneous Gaussian noise $\dot{F}(t,x)$, with covariance given by $f(x) = |x|^{-\beta}$, where $0 < \beta < 2$. We seek an estimate of the Sobolev norm of the stochastic integral $v^t_{G,Z}$. We recall that the Sobolev norm is defined in (7.1).

Theorem 7.3 *Suppose $0 < \beta < 2$. Fix $T > 0$, $q \in]3, +\infty[$, and let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded domain. Fix $\gamma \in]0,1[$, and suppose that*

$$\int_0^t ds\, E\left(\|Z(s)\|^q_{W^{\gamma,q}(\mathcal{O}^{t-s})}\right) < +\infty.$$

Consider

$$\rho \in \left]0,\, \gamma \wedge \left(\frac{2-\beta}{2} - \frac{3}{q}\right)\right[.$$

Then there exists $C < +\infty$ (depending on ρ but not on Z) such that

$$E\left(\|v^t_{G,Z}\|^q_{\rho,q,\mathcal{O}}\right) \leq C \int_0^t ds\, E\left(\|Z(s)\|^q_{W^{\rho,q}(\mathcal{O}^{t-s})}\right).$$

Remark 7.4 In the case of the heat equation, spatial regularity of the stochastic integral process, that is, of $x \mapsto v^t_{G,Z}(x)$, occurs because of regularity of the heat kernel G, even if Z is merely integrable. Here, the spatial regularity of $v^t_{G,Z}$ is due to the regularity of Z.

Proof of Theorem 7.3. The key quantity that we need to estimate is

$$E\left(\int_{\mathcal{O}} dx \int_{\mathcal{O}} dy\, \frac{|v^t_{G,Z}(x) - v^t_{G,Z}(y)|^q}{|x-y|^{3+\rho q}}\right).$$

Let $\bar{\rho} = \rho + \frac{3}{q}$, so that $3 + \rho\, q = \bar{\rho}\, q$. If we replace G by G_n, then the numerator above is equal to

$$\left|\int_0^t ds \int_{\mathbb{R}^3} (G_n(t-s,\, x-u) - G_n(t-s,\, y-u))\ Z(s,u)\ M(ds,du)\right|^q,$$

so by Burkholder's inequality (2.3),

$$\begin{aligned} E&\left(|v^t_{G_n,Z}(x) - v^t_{G_n,Z}(y)|^q\right) \\ &\leq C\, E\Big(\Big|\int_0^t ds \int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv\ Z(s,u)\ f(u-v)\ Z(s,v) \\ &\qquad \times (G_n(t-s,\, x-u) - G_n(t-s, y-u)) \\ &\qquad \times (G_n(t-s,\, x-v) - G_n(t-s,\, y-v))\Big|^{q/2}\Big). \end{aligned} \tag{7.3}$$

If we had G instead of G_n, and *if* G were smooth, then we would get a bound involving an exponent of $|x-y|$, even if Z were merely integrable.

Here we use a different idea: we shall pass the increments on the G_n over to the factors $Z\,f\,Z$ by changing variables. For instance, if there were only one factor involving increments of G_n, we could use the following calculation, where G_n is generically denoted g and $Z\,f\,Z$ is denoted ψ:

$$\begin{aligned}
&\int_{\mathbb{R}^3} du\,(g(x-u)-g(y-u))\,\psi(u)\\
&\qquad=\int_{\mathbb{R}^3} du\,g(x-u)\,\psi(u)-\int_{\mathbb{R}^3} du\,g(y-u)\,\psi(u)\\
&\qquad=\int_{\mathbb{R}^3} d\tilde{u}\,g(\tilde{u})\,\psi(x-\tilde{u})-\int_{\mathbb{R}^3} d\tilde{u}\,g(\tilde{u})\,\psi(y-\tilde{u})\Big)\\
&\qquad=\int_{\mathbb{R}^3} d\tilde{u}\,g(\tilde{u})\,(\psi(x-\tilde{u})-\psi(y-\tilde{u})).
\end{aligned}$$

Using this idea, it turns out that the integral on the right-hand side of (7.3) is equal to

$$\sum_{i=1}^{4} J_{i,n}^{t}\,(x,y),$$

where

$$J_{i,n}^{t}(x,y)=\int_0^t ds\int_{\mathbb{R}^3} du\int_{\mathbb{R}^3} dv\;G_n(s,u)\;G_n(s,v)\;h_i(t,s,x,y,u,v),$$

and

$$\begin{aligned}
h_1(t,s,x,y,u,v) &= f(y-x+v-u)\,(Z(t-s,\,x-u)-Z(t-s,\,y-u))\\
&\qquad\times(Z(t-s,\,x-v)-Z(t-s,y-v),\\[1ex]
h_2(t,s,x,y,u,v) &= Df(v-u,x-y)Z(t-s,x-u)\\
&\qquad\times(Z(t-s,x-v)-Z(t-s,y-v)),\\[1ex]
h_3(t,s,x,y,u,v) &= Df(v-u,y-x)Z(t-s,y-v)\\
&\qquad\times(Z(t-s,x-u)-Z(t-s,y-u)),\\[1ex]
h_4(t,s,x,y,u,v) &= -D^2 f\,(v-u,\,x-y)\;Z(t-s,\,x-u)\;Z(t-s,\,x-u),
\end{aligned}$$

and we use the notation

$$\begin{aligned}
Df(u,x)&=f(u+x)-f(u),\\
D^2 f(u,x)&=f(u-x)-2\,f(u)+f(u+x).
\end{aligned}$$

We can now estimate separately each of the four terms

$$T_n^i(t,\mathcal{O})=\int_{\mathcal{O}} dx\int_{\mathcal{O}} dy\,\frac{E(|J_{i,n}^t(x,y)|^{q/2})}{|x-y|^{\bar{\rho}q}}\,,\qquad i=1,\ldots,4.$$

The term $T_n^1(t,\mathcal{O})$. Set

$$\begin{aligned}\mu_n(x,y) &= \sup_{s\in[0,T]} \int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv\, G_n(s,u)\, G_n(s,v)\, f(y-x+v-u) \\ &= \sup_{s\in[0,T]} \int_{\mathbb{R}^3} \mu(d\eta)\, e^{i\,\eta\cdot(x-y)} |\mathcal{F}G_n(s)(\eta)|^2,\end{aligned}$$

so that

$$\sup_{n,x,y} \mu_n(x,y) < +\infty$$

since $\beta < 2$. Therefore, since $G_n(s,u) \geq 0$, by Hölder's inequality,

$$\begin{aligned}E\left(|J_{1,n}^t(x,y)|^{q/2}\right) \leq\ & (T\,\mu_n(x,y))^{\frac{q}{2}-1} \\ & \times E\biggl(\int_0^t ds \int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv\, G_n(s,u)\, G_n(s,v)\, f(y-x+v-u) \\ & \qquad \times |Z(t-s,\, x-u) - Z(t-s,\, y-u)|^{q/2} \\ & \qquad \times |Z(t-s,\, x-v) - Z(t-s,\, y-v)|^{q/2}\biggr).\end{aligned}$$

Apply the Cauchy-Schwarz inequality with respect to the measure

$$dP\, dx\, dy\, ds\, du\, dv\, G_n(s,u)\, G_n(s,v)\, f(y-x+v-u)$$

to see that

$$T_n^1(t,\mathcal{O}) \leq \left(T_n^{1,1}(t,\mathcal{O})\, T_n^{1,2}(t,\mathcal{O})\right)^{1/2},$$

where

$$\begin{aligned}T_n^{1,1}(t,\mathcal{O}) =\ & \int_0^t ds \int_{\mathcal{O}} dx \int_{\mathcal{O}} dy \int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv\, G_n(s,u)\, G_n(s,v)\, f(y-x+v-u) \\ & \times \frac{E\left(|Z(t-s,\, x-u) - Z(t-s,\, y-u)|^q\right)}{|x-y|^{\bar{\rho}q}},\end{aligned}$$

and there is an analogous expression for $T_n^{1,2}(t,\mathcal{O})$. We note that for $x \in \mathcal{O}$, when $G_n(s,u) > 0$ (resp. for $y \in \mathcal{O}$, when $G_n(s,v) > 0$), $x - u \in \mathcal{O}^{s(1+1/n)}$ (resp. $y - u \in \mathcal{O}^{s(1+1/n)}$), so

$$T_n^{1,1}(t,\mathcal{O}) \leq \int_0^t ds\, E\left(\|Z(t-s)\|^q_{\rho,q,\mathcal{O}^{s(1+1/n)}}\right) \sup_{n,x,y} \mu_n(x,y).$$

The same bound arises for the term $T_n^{1,2}(t,\mathcal{O})$, so this gives the desired estimate for this term.

We shall not discuss the terms $T_n^2(t,\mathcal{O})$ and $T_n^3(t,\mathcal{O})$ here: the interested reader may consult [5, Chapter 3], but we consider the term $T_n^4(t,\mathcal{O})$.

The term $T_n^4(t,\mathcal{O})$. In order to bound $T_n^4(t,\mathcal{O})$, we aim to bring the exponent $q/2$ inside the $ds\,du\,dv$ integral, in such a way that it only affects the Z factors but not f.

Set

$$\mu_n(x,y) = \sup_{s\in[0,T]} \int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv\, G_n(s,u)\, G_n(s,v)\, \frac{|D^2 f(v-u,\, x-y)|}{|x-y|^{2\bar\rho}}\,. \tag{7.4}$$

We will show below that

$$\sup_{n\geq 1,\ x,y\in\mathcal{O}} \mu_n(x,y) \leq C < +\infty, \tag{7.5}$$

which will turn out to require quite an interesting calculation. Assuming this for the moment, let $p = q/2$. Then, by Hölder's inequality,

$$\begin{aligned} &\frac{E\left(|J_{4,n}^t(x,y)|^p\right)}{|x-y|^{2p\bar\rho}} \\ &\quad\leq \sup_{n,x,y} (\mu_n(x,y))^{p-1} \int_0^t ds \int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv\, G_n(s,u)\, G_n(s,v) \\ &\qquad\qquad \times \frac{|D^2 f(v-u,\, x-y)|}{|x-y|^{2\bar\rho}}\, E\left(|Z(t-s,\, x-u)|^p\, |Z(t-s,\, x-v)|^p\right). \end{aligned}$$

This quantity must be integrated over $\mathcal{O}\times\mathcal{O}$. We apply the Cauchy-Schwarz inequality to the measure $ds\,du\,dv(\cdots)dP$, and this leads to

$$T_n^4(t,\mathcal{O}) \leq \sup_{n,x,y} (\mu_n(x,y))^p \int_0^t ds\, E\left(\|Z(s)\|^q_{L^q(\mathcal{O}^{(t-s)(1+1/n)})}\right).$$

This is the desired bound for this term.

It remains to check (7.5). The main difficulty is to bound the second order difference $|D^2 f(v-u,\, x-y)|$. We explain the main issues below.

Bounding symmetric differences

Let $g:\mathbb{R}\to\mathbb{R}$. Suppose that we seek a bound on

$$D^2 g(x,h) = g(x-h) - 2g(x) + g(x+h).$$

Notice that if g is differentiable only once ($g\in C^1$), then the best that we can do is essentially to write

$$\begin{aligned} |D^2 g(x,h)| &\leq |g(x-h)-g(x)| + |g(x+h)-g(x)| \\ &\leq c\,h. \end{aligned}$$

On the other hand, if g is twice differentiable ($g\in C^2$), then we can do better:

$$|D^2 g(x,h)| \leq c\,h^2.$$

In the case of a Riesz kernel $f(x) = |x|^{-\beta}$, $x \in \mathbb{R}^3$, we can write

$$\begin{aligned} |D^2 f(u,x)| &= \Big| |u-x|^{-\beta} - 2|u|^{-\beta} + |u+x|^{-\beta} \Big| \\ &\leq C\,|f''(u)|\,|x|^2 \\ &= C\,|u|^{-\beta-2}\,|x|^2 . \end{aligned}$$

Taking into account the definition of $\mu_n(x,y)$ in (7.4), this inequality leads to the bound

$$\mu_n(x,y) \leq \sup_{s\in[0,T]} \left(\int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv\; G_n(s,u)\; G_n(s,v)\, |u-v|^{-(\beta+2)} \right) \frac{|x-y|^2}{|x-y|^{2\bar\rho}} .$$

However, the double integral converges to $+\infty$ as $n \to \infty$, since $\beta + 2 > 2$.

Since differentiating once does not necessarily give the best bound possible and differentiating twice gives a better exponent but with an infinite constant, it is natural to want to differentiate a fractional number of times, namely just under $2-\beta$ times. If we "differentiate α times" and all goes well, then this should give a bound of the form $\mu_n(x,y) \leq C|x-y|^\alpha$, for $\alpha \in]0, 2-\beta[$. We shall make this precise below.

Riesz potentials, their fractional integrals and Laplacians

Let $\alpha \overset{\text{def}}{=} 2\bar\rho$. We recall that

$$\rho < \frac{2-\beta}{2} - \frac{3}{q} \quad \text{and} \quad \bar\rho = \rho + \frac{3}{q}\,, \quad \text{so} \quad \alpha < 2 - \beta.$$

The *Riesz potential* of a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$(I_a\,\varphi)(x) = \frac{1}{\gamma(a)} \int_{\mathbb{R}^d} \frac{\varphi(y)}{|x-y|^{d-a}}\, dy, \qquad a \in]0, d[,$$

where $\gamma(a) = \pi^{d/2} 2^a \Gamma(a/2)/\Gamma(\frac{d-a}{2})$. Riesz potentials have many interesting properties (see [15]), of which we mention the following.

(1) $I_{a+b}(\varphi) = I_a(I_b\,\varphi)$ if $a + b \in]0, d[$. Further, I_a can be seen as a "fractional integral of order a", in the sense that

$$\mathcal{F}(I_a\,\varphi)(\xi) = \mathcal{F}\varphi(\xi)\,\mathcal{F}\left(\frac{1}{|\cdot|^{d-a}}\right)(\xi) = \frac{\mathcal{F}\varphi(\xi)}{|\xi|^a} .$$

(2) Our covariance function $k_\beta(x) = |x|^{-\beta}$ is a *Riesz kernel*. These kernels have the following property:

$$|x|^{-d+a+b} = \int_{\mathbb{R}^d} dz\, k_{d-b}(x-z)\, |z|^{-d+a} \tag{7.6}$$

$$= I_b\left(|\cdot|^{-d+a}\right). \tag{7.7}$$

This equality can be viewed as saying that $|z|^{-d+a}$ is "b^{th} derivative (or Laplacian)" of $|z|^{-d+a+b}$, in the sense that

$$(-\Delta)^{b/2}\left(|z|^{-d+a+b}\right) = |z|^{-d+a}.$$

Indeed, taking Fourier transforms, this equality becomes simply

$$|\xi|^b\,|\xi|^{-a-b} = |\xi|^{-a}.$$

Recall the notation

$$Df(u,y) = f(u+y) - f(u).$$

From (7.6), one can easily deduce (see [5, Lemma 2.6]) that

$$Dk_{d-a-b}(u,cx) = |c|^b \int_{\mathbb{R}^d} dw\, k_{d-a}(u-cw)\; Dk_{d-b}(w,x)$$

and

$$|D^2 k_{d-a-b}(u,x)| \leq |x|^b \int_{\mathbb{R}^d} dw\, k_{d-a}(u-|x|w)\; D^2 k_{d-b}\left(w, \frac{x}{\|x\|}\right).$$

Set $b = \alpha = 2\bar{\rho}$ and $a = 3-\alpha-\beta$, where $\alpha+\beta \in]0,2[$. Looking back to (7.4), these two relations lead to the following estimate:

$$\begin{aligned}
\mu_n(x,y) \;\leq\; & \sup_{s\in[0,T]} \frac{1}{|x-y|^\alpha} \int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv\; G_n(s,u)\; G_n(s,v)\, |x-y|^\alpha \\
& \qquad \times \int_{\mathbb{R}^3} dw\, k_{\alpha+\beta}(v-u-|y-x|w) \times \left|D^2 k_{3-\alpha}\left(w, \frac{x}{|x|}\right)\right| \\
\leq\; & \sup_{s\in[0,T]} \left(\sup_{x,y,w} \int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv\; G_n(s,u)\; G_n(s,v)\, k_{\alpha+\beta}(v-u-|y-x|w) \right) \\
& \qquad \times \sup_x \int dw \left|D^2 k_{3-\alpha}\left(w, \frac{x}{\|x\|}\right)\right|.
\end{aligned}$$

The double integral above is finite since $\alpha+\beta < 2$. Indeed, taking Fourier transforms, the shift $-|y-x|w$ introduces a factor $e^{i\eta\cdot|y-x|w}$, which is of no consequence. The second integral is finite (and does not depend on x). For this calculation, see [5, Lemma 2.6]. □

8 Hölder-continuity in the 3-d wave equation

We consider the stochastic wave equation (6.1) for $d=3$, driven by spatially homogeneous Gaussian noise with covariance $f(x) = |x|^{-\beta}$, where $0<\beta<2$.

The main idea for checking Hölder continuity of the solution is to go back to the Picard iteration scheme that was used to construct the solution, starting with a smooth function $u_0(t,x)$ (whose smoothness depends only on the regularity of the initial conditions), and then check that regularity is preserved at each iteration step and passes to the limit. The details are carried out in [5, Chapter 4]. The main result is the following.

Theorem 8.1 *Assume the following three properties:*

(a) the initial value v_0 is such that $v_0 \in C^2(\mathbb{R}^3)$ and Δv_0 is Hölder continuous with exponent γ_1;

(b) the initial velocity $\tilde{v}_0$ is Hölder continuous with exponent γ_2;

(c) the nonlinearities $\sigma, b : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous.

Then, for any $q \in [2, \infty[$ and

$$\alpha \in]0,\ \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} [, \tag{8.1}$$

there is $C > 0$ such that for all (t,x), $(s,y) \in [0,T] \times D$,

$$E\left(|u(t,x) - u(s,y)|^q\right) \leq C\left(|t-s| + |x-y|\right)^{\alpha q}.$$

In particular, $(t,x) \mapsto u(t,x)$ has a Hölder continuous version with exponent α.

We observe that the presence of $\gamma_1 \wedge \gamma_2$ in (8.1) can be interpreted by saying that the (ir)regularity of the initial conditions limits the possible regularity of the solution: there is no smoothing effect in the wave equation, contrary to the heat equation.

We note that this result is *sharp.* Indeed, if we consider the linear wave equation, in which we take $\sigma \equiv 1$ and $b \equiv 0$ in (6.1), with vanishing initial condition $v_0 \equiv \tilde{v}_0 \equiv 0$, then it is possible to show (see [5, Chapter 5] that

$$E\left(|u(t,x) - u(t,y)|^2\right) \geq c_1\ |x-y|^{2-\beta}$$

and

$$E\left(|u(t,x) - u(s,x)|^2\right) \geq c_2\ |t-s|^{2-\beta}.$$

This implies in particular that $t \mapsto u(t,x)$ and $x \mapsto u(t,x)$ are *not* γ-Hölder continuous, for $\gamma > \frac{2-\beta}{2}$.

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