Hölder continuity of solutions of SPDEs with reflection

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Abstract: In this paper, we obtain the Hölder continuity of the solutions of SPDEs with reflection, which have singular drifts (random measures).

Key words: parabolic obstacle problem; stochastic partial differential equations with reflection; random measure; Garsia Lemma.

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1 Introduction and framework

Consider the following stochastic partial differential equation (SPDE) with reflection:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x) + \dot{\eta}(t,x); \\ u(t,x) \geq 0 \\ u(0,\cdot) = u_0; \\ u(t,0) = u(t,1) = 0. \end{cases} \tag{1.1}$$

Here $\dot{W}$ denotes the space-time white noise defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, $\mathcal{F}_t = \sigma(W(s,x) : x \in [0,1], 0 \leq s \leq t)$; u_0 is a non-negative continuous function on $[0,1]$, which vanishes at 0 and 1; $\eta(x,t)$ is a random measure which is a part of the solution pair (u,η); $\frac{\partial^2}{\partial x^2}$ denotes the Laplacian operator on $[0,1]$ equipped with the Dirichlet boundary condition. The coefficients f and σ are measurable mappings from $\mathbb{R}$ into $\mathbb{R}$. The following definition is taken from [DP1], [NP].

Definition 1.1. *A pair (u,η) is said to be a solution of equation (1.1) if*
(i) u is a continuous random field on $\mathbb{R}_+ \times [0,1]$, $u(t,x)$ is $\mathcal{F}_t$ measurable and $u(t,x) \geq 0$ a.s.
(ii) η is a random measure on $\mathbb{R}_+ \times (0,1)$ such that
(a) $\eta(\{t\} \times (0,1)) = 0$, for all $t \geq 0$.
(b) $\int_0^t \int_0^1 x(1-x)\eta(ds,dx) < \infty$, for all $t \geq 0$.
(c) η is adapted in the sense that for any measurable mapping ψ :

$$\int_0^t \int_0^1 \psi(s,x)\eta(ds,dx) \text{ is } \mathcal{F}_t\text{-measurable.}$$

(iii) (u,η) solves the parabolic SPDE in the following sense ($(\cdot,\cdot)$ denotes the scalar product in

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$L^2[0,1]$*):* $\forall t \in \mathbb{R}_+, \phi \in C_0^2([0,1])$ *with* $\phi(0) = \phi(1) = 0$,

$$
\begin{aligned}
&(u(t), \phi) - \int_0^t (u(s), \phi'')ds - \int_0^t (f(u(s)), \phi)ds \\
= \quad &(u_0, \phi) + \int_0^t \int_0^1 \phi(x)\sigma(u(s,x))W(ds,dx) + \int_0^t \int_0^1 \phi(x)\eta(ds,dx) \ a.s,
\end{aligned}
$$

where $u(t) := u(t, \cdot)$.
(iv) $\int_Q u \, d\eta = 0$, *where* $Q = \mathbb{R}_+ \times (0,1)$.

This equation was first studied by Nualart and Pardoux in [NP] (PTRF 1992) when $\sigma(\cdot) = 1$, and by Donati-Martin and Pardoux in [DP1] (in PTRF 1993) for a general diffusion coefficient σ without obtaining the uniqueness and by T. Xu and T. Zhang in [XZ] for general σ with also the proof of the uniqueness. Various properties of the solution of equation (1.1) were studied later in [DMZ], [DP2], [HP], [ZA] and [Z]. SPDEs with reflection can also be used to model the evolution of random interfaces near a hard wall. It was proved by T. Funaki and S. Olla in [FO] that the fluctuations of a $\nabla\phi$ interface model near a hard wall converge in law to the stationary solution of a SPDE with reflection. We also mention that the random contact set $\{(t,x); u(t,x) = 0\}$ of the solution u was investigated in [DMZ] when $\sigma = 1$. For stochastic Cahn-Hilliard equations with reflection, see [ZA].

We assume throughout the paper that the mappings

$$f, \sigma : \mathbb{R} \to \mathbb{R}$$

are Lipschitz continuous:

$$|f(u) - f(v)| + |\sigma(u) - \sigma(v)| \leq C|u - v|. \tag{1.2}$$

The purpose of this paper is to obtain the Hölder continuity of the solution of the SPDE with reflection. This is not trivial because of the singularity introduced by the random measure term in the equation. When the noise is additive (i.e. $\sigma = 1$), the question of Hölder continuity was investigated in [DMZ], where the Hölder continuity with respect to the space variable was obtained. Regarding the time variable, the authors in [DMZ] only established the following lower bound:

$$u(t,x) - u(s,x) \geq -\gamma(t-s)^\alpha, \quad t \geq s \geq 0.$$

In this paper, we obtain the Hölder continuity of the solution of the SPDE with reflection with respect to both the time and the space variables for general equations with multiplicative noise. Our method is a careful refinement of the approach in [DMZ]. The idea is to consider the penalized approximating equations and prove uniform moment estimates for the solutions of the approximating equations. For regularity of solutions of other type SPDEs, we refer the reader also to [DS], [SV].

Let $G_t(x,y)$ be the heat kernel associated with the Laplacian operator $\frac{\partial^2}{\partial x^2}$ on $[0,1]$ equipped with the Dirichlet boundary condition. If we let $v(t,x) = \int_0^1 G_t(x,y)u_0(y)dy$, then $u(t,x) - v(t,x)$ will solve a similar SPDE with reflection as (1.1), but with initial data (function) 0. Since the Hölder continuity of $v(t,x)$ is well understood, the study of the regularity of $u(t,x)$ is reduced to the study of $u(t,x) - v(t,x)$. Therefore, without loss of generality, in the paper we will assume $u_0 = 0$ in (1.1).

The organization of this paper is as follows: In Section 2, we prepare some results on the penalized approximating equations. In Section 3, we establish the Hölder continuity.

2 The approximating equations

For $\varepsilon > 0$, set

$$g_\varepsilon(u) = \frac{\arctan([u \wedge 0]^2)}{\varepsilon} \tag{2.3}$$

Consider the following penalized SPDEs:

$$\begin{aligned} \frac{\partial u^\varepsilon(t,x)}{\partial t} &= \frac{\partial^2 u^\varepsilon(t,x)}{\partial x^2} + f(u^\varepsilon(t,x)) \\ &\quad + g_\varepsilon(u^\varepsilon(t,x)) + \sigma(u^\varepsilon(t,x))\dot{W}(t,x) &(2.4)\\ u^\varepsilon(0,\cdot) &= 0 &(2.5)\\ u^\varepsilon(t,0) &= u^\varepsilon(t,1) = 0, &(2.6) \end{aligned}$$

or equivalently in the mild form:

$$\begin{aligned} u^\varepsilon(x,t) &= \int_0^t \int_0^1 G_{t-s}(x,y) f(u^\varepsilon(s,y))dsdy \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y) g_\varepsilon(u^\varepsilon(s,y))dsdy + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u^\varepsilon(s,y))W(ds,dy) &(2.7) \end{aligned}$$

Here, $G_t(x,y)$ is the heat kernel. Fix $T > 0$ and let $Q_T = [0,T] \times [0,1]$. For $v \in C(Q_T)$, set $||v||_\infty = \max_{0\leq s\leq T,\, 0\leq x\leq 1} |v(s,x)|$. It was shown in [DP1] (seel also [DMZ]) that for $p \geq 1$,

$$\lim_{\varepsilon\to 0} E[||u^\varepsilon - u||_\infty^p] = 0, \tag{2.8}$$

where u is the solution to equation (1.1).

Notice that the function $g_\varepsilon(u) = \frac{u^-}{\varepsilon}$ was used in [DP2]. Our choice of g_ε does not change the limit of u^ε, but makes g_ε differentiable.

Set

$$\begin{aligned} N^\varepsilon(t,x) &= \int_0^t \int_0^1 G_{t-s}(x,y) f(u^\varepsilon(s,y))dsdy \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u^\varepsilon(s,y))W(ds,dy) &(2.9) \end{aligned}$$

Let $v^\varepsilon(t,x) = u^\varepsilon(t,x) - N^\varepsilon(t,x)$. Then it is easy to see that v^ε satisfies the following random PDE:

$$\begin{aligned} \frac{\partial v^\varepsilon(t,x)}{\partial t} &= \frac{\partial^2 v^\varepsilon(t,x)}{\partial x^2} + g_\varepsilon(v^\varepsilon(t,x) + N^\varepsilon(t,x)) &(2.10)\\ v^\varepsilon(0,\cdot) &= 0 &(2.11)\\ v^\varepsilon(t,0) &= v^\varepsilon(t,1) = 0. &(2.12) \end{aligned}$$

Lemma 2.1. *For any $\alpha < 1$, $\varepsilon > 0$, there exists a random variable $C_\varepsilon(\omega)$ such that*

$$\begin{aligned} |N^\varepsilon(t,x) - N^\varepsilon(s,y)| &\leq C_\varepsilon(\omega)[|t-s|^{\frac{\alpha}{4}} + |x-y|^{\frac{\alpha}{2}}] &(2.13)\\ &\qquad (t,x),(s,y) \in Q_T. \end{aligned}$$

Moreover, for any $p \geq 1$ it holds that

$$\sup_\varepsilon E[C_\varepsilon^p] < \infty.$$

Proof. First of all, we recall the following property of the heat kernel $G_t(x,y)$ from [BMS]: for $s,t \in [0,T]$ with $s \le t$ and $x,y \in [0,1]$,

$$\int_0^t \int_0^1 (G_{t-r}(x,z) - G_{t-r}(y,z))^2 drdz \le C|x-y|, \tag{2.14}$$

$$\int_0^s \int_0^1 (G_{t-r}(y,z) - G_{s-r}(y,z))^2 drdz \le C|t-s|^{\frac{1}{2}}, \tag{2.15}$$

$$\int_s^t \int_0^1 G_{t-r}(y,z)^2 drdz \le C|t-s|^{\frac{1}{2}}. \tag{2.16}$$

We claim that for any $p \ge 2$,

$$\begin{aligned} E[|N^\varepsilon(t,x) - N^\varepsilon(s,y)|^p] \quad &\le \quad C_p[|t-s|^{\frac{p}{4}} + |x-y|^{\frac{p}{2}}], \\ &\qquad (t,x),(s,y) \in Q_T, \end{aligned} \tag{2.17}$$

where C_p is a constant independent of ε. Set

$$I_1^\varepsilon(t,x) \quad = \quad \int_0^t \int_0^1 G_{t-s}(x,y) f(u^\varepsilon(s,y)) dsdy. \tag{2.18}$$

$$I_2^\varepsilon(t,x) \quad = \quad \int_0^t \int_0^1 G_{t-s}(x,y) \sigma(u^\varepsilon(s,y)) W(ds,dy). \tag{2.19}$$

It suffices to show that both terms I_1^ε and I_2^ε satisfy (2.17). The term $I_2^\varepsilon(t,x)$ is the more complicated of the two, so we only consider it. By Burkholder's inequality and Hölder's inequality, for $p \ge 2$, $s \le t \le T$, we have

$$\begin{aligned} & E[|I_2^\varepsilon(t,x) - I_2^\varepsilon(t,y)|^p] \\ \le \quad & C_p E\Big[\Big(\int_0^t \int_0^1 |G_{t-r}(x,z) - G_{t-r}(y,z)|^2 |\sigma(u^\varepsilon(r,z))|^2 drdz\Big)^{\frac{p}{2}}\Big] \\ \le \quad & C_p \left(\int_0^t \int_0^1 |G_{t-r}(x,z) - G_{t-r}(y,z)|^2 drdz\right)^{\frac{p}{2}-1} \\ & \quad \times \int_0^t \int_0^1 |G_{t-r}(x,z) - G_{t-r}(y,z)|^2 E[|\sigma(u^\varepsilon(r,z))|^p] drdz \\ \le \quad & C_p (1 + \sup_\varepsilon \sup_{0 \le t \le T, x \in [0,1]} E[|u^\varepsilon(t,x)|^p]) \left(\int_0^t \int_0^1 |G_{t-r}(x,z) - G_{t-r}(y,z)|^2 |drdz\right)^{\frac{p}{2}} \\ \le \quad & C|x-y|^{\frac{p}{2}}, \end{aligned} \tag{2.20}$$

where (2.14) and the fact that σ is Lipschitz continuous were used. Similarly, in view of (2.15), (2.16) it follows that

$$
\begin{aligned}
& E[|I_2^\varepsilon(t,y) - I_2^\varepsilon(s,y)|^p] \\
\leq\ & C_p E\Big[\Big(\int_0^s \int_0^1 |G_{t-r}(y,z) - G_{s-r}(y,z)|^2 |\sigma(u^\varepsilon(r,z))|^2 drdz\Big)^{\frac{p}{2}} \\
& \quad + C_p E\Big[\Big(\int_s^t \int_0^1 |G_{t-r}(y,z)|^2 |\sigma(u^\varepsilon(r,z))|^2 drdz\Big)^{\frac{p}{2}} \\
\leq\ & C_p \left(\int_0^s \int_0^1 |G_{t-r}(y,z) - G_{s-r}(y,z)|^2 drdz\right)^{\frac{p}{2}-1} \\
& \quad \times \int_0^s \int_0^1 |G_{t-r}(y,z) - G_{s-r}(y,z)|^2 E[|\sigma(u^\varepsilon(r,z))|^p] drdz \\
& \quad + C_p \left(\int_s^t \int_0^1 |G_{t-r}(y,z)|^2 drdz\right)^{\frac{p}{2}-1} \int_s^t \int_0^1 |G_{t-r}(y,z)|^2 E[|\sigma(u^\varepsilon(r,z))|^p] drdz \\
\leq\ & C_p |t-s|^{\frac{p}{4}}. \qquad (2.21)
\end{aligned}
$$

Putting together (2.20) and (2.21), we prove (2.17) for I_2^ε. Since the constant C_p in (2.17) is independent of ε, applying a version of the Garsia's lemma proved in [DKN](Proposition A.1 and Corollary A.3 in [DKN]), it follows that for any $p > 8$, there exists a random variable $\eta_{p,\varepsilon}(\omega)$ such that

$$|N^\varepsilon(t,x) - N^\varepsilon(s,y)| \ \leq\ \eta_{p,\varepsilon}(\omega)(|t-s|^{\frac{1}{4}-\frac{2}{p}} + |x-y|^{\frac{1}{2}-\frac{4}{p}}), \qquad (2.22)$$

and

$$\sup_\varepsilon E[\eta_{p,\varepsilon}^p] \ <\ \infty. \qquad (2.23)$$

Since p can be chosen to be arbitrarily large, the Lemma follows. □

3 Hölder continuity

Recall the following lemma from [DMZ].

Lemma 3.1. *Let $V \in C^{1,2}(Q_T)$ and $\psi, F \in C(Q_T)$ with $\psi \leq 0$. Suppose that V solves the equation*

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{1}{2}\frac{\partial^2 V}{\partial x^2} + \psi V + \psi F, \qquad (3.24) \\
V_0(x) &= 0, \qquad (3.25)
\end{aligned}
$$

with Dirichlet or Neumann boundary conditions. Then the following estimate holds:

$$||V||_\infty \leq ||F||_\infty$$

We need the following lemma in the proof of the main result.

Lemma 3.2. *Let $f \in C^{\alpha,\beta}(Q_T)$ satisfying*

$$|f(t,x) - f(s,y)| \leq C_f(|t-s|^\alpha + |x-y|^\beta). \qquad (3.26)$$

Then, for $\rho_1 > 0, \rho_2 > 0$, there exists $f^{\rho_1,\rho_2} \in C^\infty(Q_T)$ such that

$$\begin{aligned} &||f^{\rho_1,\rho_2} - f||_\infty \le C_{\alpha,\beta} C_f(\rho_1^\alpha + \rho_2^\beta), \\ &||\frac{\partial f^{\rho_1,\rho_2}}{\partial t}||_\infty \le C_{\alpha,\beta} C_f \frac{1}{\rho_1^{1-\alpha}}, \\ &||\frac{\partial f^{\rho_1,\rho_2}}{\partial x}||_\infty \le C_{\alpha,\beta} C_f \frac{1}{\rho_2^{1-\beta}}, \end{aligned} \tag{3.27}$$

where $C_{\alpha,\beta}$ is a constant only depending on α, β.

Proof. First we extend the definition of f to $\mathbb{R}^2$ by setting

$$\bar{f}(s,y) = f(p(s,y)), \tag{3.28}$$

where $p(s,y)$ denotes the point in Q_T that is nearest (in Euclidean norm) to (s,y). In particular, $\bar{f}(s,y) = f(s,y)$ if $(s,y) \in Q_T$. Then it is easy to see that $\bar{f}$ satisfies (3.26) with the same constant C_f. Denote by $P_u(x,y)$, $u > 0$ the Gaussian heat kernel:

$$P_u(x,y) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{(x-y)^2}{2u}}.$$

For $\rho_1 > 0, \rho_2 > 0$, define f^{ρ_1,ρ_2} by

$$f^{\rho_1,\rho_2}(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}} P_{\rho_1^2}(t,s) P_{\rho_2^2}(x,y) \bar{f}(s,y) ds dy. \tag{3.29}$$

We will show that f^{ρ_1,ρ_2} has the required properties. Since $P_u(\cdot,\cdot)$ is a probability density, we have

$$\begin{aligned} & |f^{\rho_1,\rho_2}(t,x) - f(t,x)| \\ = \ & |\int_{\mathbb{R}} \int_{\mathbb{R}} P_{\rho_1^2}(t,s) P_{\rho_2^2}(x,y) [\bar{f}(s,y) - \bar{f}(t,x)] ds dy| \\ \le \ & C_f \int_{\mathbb{R}} \int_{\mathbb{R}} P_{\rho_1^2}(t,s) P_{\rho_2^2}(x,y) [|t-s|^\alpha + |x-y|^\beta] ds dy \\ = \ & C_f [\int_{\mathbb{R}} P_{\rho_1^2}(t,s) |t-s|^\alpha ds + \int_{\mathbb{R}} P_{\rho_2^2}(x,y) |x-y|^\beta dy] \\ \le \ & C_{\alpha,\beta} C_f [(\rho_1^2)^{\frac{\alpha}{2}} + (\rho_2^2)^{\frac{\beta}{2}}] \\ = \ & C_{\alpha,\beta} C_f [(\rho_1)^\alpha + (\rho_2)^\beta], \end{aligned} \tag{3.30}$$

which is what we need. Note that

$$\int_{\mathbb{R}} P_{\rho_1^2}(t,s)(t-s) ds = 0$$

Differentiating f^{ρ_1,ρ_2} with respect to t we get

$$\begin{aligned} & |\frac{\partial f^{\rho_1,\rho_2}}{\partial t}(t,x)| = |\frac{1}{\rho_1^2} \int_{\mathbb{R}} \int_{\mathbb{R}} P_{\rho_1^2}(t,s)(t-s) P_{\rho_2^2}(x,y) \bar{f}(s,y) ds dy| \\ = \ & |\frac{1}{\rho_1^2} \int_{\mathbb{R}} \int_{\mathbb{R}} P_{\rho_1^2}(t,s)(t-s) P_{\rho_2^2}(x,y) [\bar{f}(s,y) - \bar{f}(t,y)] ds dy| \\ \le \ & C_f \frac{1}{\rho_1^2} \int_{\mathbb{R}} \int_{\mathbb{R}} P_{\rho_1^2}(t,s) |t-s|^{1+\alpha} P_{\rho_2^2}(x,y) ds dy \\ = \ & C_f \frac{1}{\rho_1^2} \int_{\mathbb{R}} P_{\rho_1^2}(t,s) |t-s|^{1+\alpha} ds = C_f C_\alpha \frac{1}{\rho_1^2} (\rho_1^2)^{\frac{1+\alpha}{2}} \\ = \ & C_f C_{\alpha,\beta} \frac{1}{\rho_1^{1-\alpha}}. \end{aligned} \tag{3.31}$$

Similar calculations yield the estimate for $||\frac{\partial f^{\rho_1,\rho_2}}{\partial x}||_\infty$. The proof is complete. $\square$

The following is the main result of the paper.

Theorem 3.1. *Let u be the solution of the SPDE with reflection (1.1)with $u_0 = 0$, and fix $T > 0$. Then for any $\alpha < 1$ and $p > 1$, we have the following moment estimate:*

$$E[|u(t,x) - u(s,y)|^p] \le C_p[|t-s|^{\alpha\frac{p}{4}} + |x-y|^{\alpha\frac{p}{2}}], \quad (t,x),(s,y) \in [0,T] \times [0,1]. \tag{3.32}$$

In particular, u admits a version that is Hölder $(\frac{1}{4}-, \frac{1}{2}-)$ on $[0,T] \times [0,1]$.

Proof. Fix any $\alpha < 1$ and let $N^\varepsilon(t,x)$ be defined as in Section 2. For $\rho_1 > 0$ and $\rho_2 > 0$, define the smooth function $N^{\varepsilon,\rho_1,\rho_2}(t,x)$ as $f^{\rho_1,\rho_2}(t,x)$ in (3.29) replacing f by N^ε.

Let $v^{\varepsilon,\rho_1,\rho_2}$ be the solution of the following random PDE:

$$\frac{\partial v^{\varepsilon,\rho_1,\rho_2}(t,x)}{\partial t} = \frac{\partial^2 v^{\varepsilon,\rho_1,\rho_2}(t,x)}{\partial x^2} + g_\varepsilon(v^{\varepsilon,\rho_1,\rho_2}(t,x) + N^{\varepsilon,\rho_1,\rho_2}(t,x)) \tag{3.33}$$
$$v^{\varepsilon,\rho_1,\rho_2}(0,\cdot) = 0 \tag{3.34}$$
$$v^{\varepsilon,\rho_1,\rho_2}(t,0) = v^{\varepsilon,\rho_1,\rho_2}(t,1) = 0. \tag{3.35}$$

Since $g'_\varepsilon(u) \le 0$, applying Lemma 3.1 (or Lemma in [NP]) we conclude that

$$||v^{\varepsilon,\rho_1,\rho_2} - v^\varepsilon||_\infty \le ||N^{\varepsilon,\rho_1,\rho_2} - N^\varepsilon||_\infty. \tag{3.36}$$

In view of Lemma 3.2, it follows from (3.36) that

$$||v^{\varepsilon,\rho_1,\rho_2} - v^\varepsilon||_\infty \le C_{\alpha,\beta}C_\varepsilon(\omega)[\rho_1^{\frac{\alpha}{4}} + \rho_2^{\frac{\alpha}{2}}], \tag{3.37}$$

where $C_\varepsilon(\omega)$ is the random variable appeared in (2.13). Introduce the following random PDEs:

$$\frac{\partial m^{\varepsilon,\rho_1,\rho_2}(t,x)}{\partial t} = \frac{\partial^2 m^{\varepsilon,\rho_1,\rho_2}(t,x)}{\partial x^2} + g'_\varepsilon(v^{\varepsilon,\rho_1,\rho_2}(t,x) + N^{\varepsilon,\rho_1,\rho_2}(t,x))[m^{\varepsilon,\rho_1,\rho_2}(t,x) + \frac{\partial N^{\varepsilon,\rho_1,\rho_2}}{\partial t}(t,x)] \tag{3.38}$$
$$m^{\varepsilon,\rho_1,\rho_2}(0,\cdot) = 0 \tag{3.39}$$
$$m^{\varepsilon,\rho_1,\rho_2}(t,0) = m^{\varepsilon,\rho_1,\rho_2}(t,1) = 0. \tag{3.40}$$

$$\frac{\partial w^{\varepsilon,\rho_1,\rho_2}(t,x)}{\partial t} = \frac{\partial^2 w^{\varepsilon,\rho_1,\rho_2}(t,x)}{\partial x^2} + g'_\varepsilon(v^{\varepsilon,\rho_1,\rho_2}(t,x) + N^{\varepsilon,\rho_1,\rho_2}(t,x))[w^{\varepsilon,\rho_1,\rho_2}(t,x) + \frac{\partial N^{\varepsilon,\rho_1,\rho_2}}{\partial x}(t,x)] \tag{3.41}$$
$$w^{\varepsilon,\rho_1,\rho_2}(0,\cdot) = 0 \tag{3.42}$$
$$w^{\varepsilon,\rho_1,\rho_2}(t,0) = w^{\varepsilon,\rho_1,\rho_2}(t,1) = 0. \tag{3.43}$$

Formally differentiating $v^{\varepsilon,\rho_1,\rho_2}(t,x)$ we see that $m^{\varepsilon,\rho_1,\rho_2}(t,x) = \frac{\partial v^{\varepsilon,\rho_1,\rho_2}}{\partial t}(t,x)$ and $w^{\varepsilon,\rho_1,\rho_2}(t,x) = \frac{\partial v^{\varepsilon,\rho_1,\rho_2}}{\partial x}(t,x)$. Notice $g'_\varepsilon \le 0$, apply Lemma 3.1 and Lemma 3.2 to obtain

$$||m^{\varepsilon,\rho_1,\rho_2}||_\infty = ||\frac{\partial v^{\varepsilon,\rho_1,\rho_2}}{\partial t}||_\infty \le ||\frac{\partial N^{\varepsilon,\rho_1,\rho_2}}{\partial t}||_\infty \le C_\alpha C_\varepsilon(\omega)\frac{1}{\rho_1^{1-\frac{\alpha}{4}}}, \tag{3.44}$$

and

$$
\begin{aligned}
||w^{\varepsilon,\rho_1,\rho_2}||_\infty &= ||\frac{\partial v^{\varepsilon,\rho_1,\rho_2}}{\partial x}||_\infty \\
&\leq ||\frac{\partial N^{\varepsilon,\rho_1,\rho_2}}{\partial x}||_\infty \leq C_\alpha C_\varepsilon(\omega)\frac{1}{\rho_2^{1-\frac{\alpha}{2}}}.
\end{aligned}
\tag{3.45}
$$

Setting $\rho_1 = |t-s|$, $\rho_2 = |x-y|$, it follows from (3.37), (3.44) and (3.45) that

$$
\begin{aligned}
& |v^\varepsilon(t,x) - v^\varepsilon(s,y)| \\
\leq\ & |v^\varepsilon(t,x) - v^{\varepsilon,\rho_1,\rho_2}(t,x)| + |v^{\varepsilon,\rho_1,\rho_2}(t,x) - v^{\varepsilon,\rho_1,\rho_2}(s,y)| \\
& \quad + |v^{\varepsilon,\rho_1,\rho_2}(s,y) - v^\varepsilon(s,y)| \\
\leq\ & 2||N^{\varepsilon,\rho_1,\rho_2} - N^\varepsilon||_\infty + ||\frac{\partial v^{\varepsilon,\rho_1,\rho_2}}{\partial t}||_\infty |t-s| + ||\frac{\partial v^{\varepsilon,\rho_1,\rho_2}}{\partial x}||_\infty |x-y| \\
\leq\ & 2C_\alpha C_\varepsilon(\omega)[\rho_1^{\frac{\alpha}{4}} + \rho_2^{\frac{\alpha}{2}}] + C_\alpha C_\varepsilon(\omega)\frac{1}{\rho_1^{1-\frac{\alpha}{4}}}|t-s| + C_{\alpha_0} C_\varepsilon(\omega)\frac{1}{\rho_2^{1-\frac{\alpha}{2}}}|x-y| \\
\leq\ & 3C_\alpha C_\varepsilon(\omega)[|t-s|^{\frac{\alpha}{4}} + |x-y|^{\frac{\alpha}{2}}].
\end{aligned}
\tag{3.46}
$$

Thus,

$$
\begin{aligned}
& |u^\varepsilon(t,x) - u^\varepsilon(s,y)| \\
\leq\ & |v^\varepsilon(t,x) - v^\varepsilon(s,y)| + |N^\varepsilon(t,x) - N^\varepsilon(s,y)| \\
\leq\ & (3C_\alpha + 1)C_\varepsilon(\omega)[|t-s|^{\frac{\alpha}{4}} + |x-y|^{\frac{\alpha}{2}}].
\end{aligned}
\tag{3.47}
$$

This yields that for $p \geq 1$,

$$
\begin{aligned}
& E[|u^\varepsilon(t,x) - u^\varepsilon(s,y)|^p] \\
\leq\ & CE[C_\varepsilon^p][|t-s|^{\frac{\alpha}{4}p} + |x-y|^{\frac{\alpha}{2}p}] \\
\leq\ & CM_p[|t-s|^{\frac{\alpha}{4}p} + |x-y|^{\frac{\alpha}{2}p}]
\end{aligned}
\tag{3.48}
$$

By Fatou Lemma, we obtain from (3.48) that

$$
E[|u(t,x) - u(s,y)|^p] \quad \leq \quad C_p[|t-s|^{\frac{\alpha}{4}p} + |x-y|^{\frac{\alpha}{2}p}] \tag{3.49}
$$

Applying a variant of Garsia's Lemma (Proposition A.1 and Corollary A.3 in [DKN]) we conclude that

$$
|u(t,x) - u(s,y)| \quad \leq \quad M_p(\omega)[|t-s|^{\frac{\alpha}{4}-\frac{2}{p}} + |x-y|^{\frac{\alpha}{2}-\frac{4}{p}}] \tag{3.50}
$$

Since p can be chosen to be arbitrarily large and α to be as close to 1 as one wants, we see that u is $(\frac{1}{4}-, \frac{1}{2}-)$ Hölder. The proof is complete.

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