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**Integral Equations and Operator Theory**

# CAPACITIES IN METRIC SPACES

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We discuss the potential theory related to the variational capacity and the Sobolev capacity on metric measure spaces. We prove our results in the axiomatic framework of [17].

# 1 Introduction

Various notions of Sobolev spaces on a metric measure spaces $(X, d, \mu)$ have been introduced and studied in recent years (see e.g. [4], [18], [23] and [41]). Good sources of information are the books [20] and [21].
The basic idea of all these constructions is to associate to a given function $u : X \to \mathbb{R}$ a collection $D[u]$ of measurable functions $g : X \to \overline{\mathbb{R}}_+$ which control the variation of $u$. The *Dirichlet p-energy* of the function $u$ is then defined as

$$\mathcal{E}_p(u) := \inf \left\{ \int_X g^p d\mu \,\middle|\, g \in D[u] \right\},$$

and the Sobolev space is $W^{1,p}(X) := \{u \in L^p(X) \mid \mathcal{E}_p(u) < \infty\}$.
We say that $g$ is a *pseudo-gradient* of $u$ if $g \in D[u]$ and the correspondence $u \to D[u]$ (i.e. the way pseudo-gradients are defined) is called a $D-$*structure.* The theory of $D-$structures is axiomatically developped in [17].
In the present paper, we introduce two notions of capacities on a metric measure space $X$ equipped with a $D$-structure. First the *Sobolev capacity* of an arbitrary subset $F \subset X$ is defined to be

$$C_p(F) := \inf\{ \|u\|_{W^{1,p}} \mid u \in \mathcal{B}_p(F) \},$$

where $\mathcal{B}_p(F) := \{ u \in W^{1,p}(X) | \, u \geq 1 \text{ near } F \text{ and } u \geq 0 \; a.e.\}$. We also define the *variational capacity* of a bounded subset $F \subset X$ as

$$\mathrm{Cap_p}(F) := \inf\{ \mathcal{E}_p(u) \mid u \in \mathcal{A}_p(F) \},$$

where $\mathcal{A}_p(F) := \{u \mid u \geq 1 \text{ near } F, \text{ and } u \text{ vanishes at } \infty\}$ (see Definitions 2.9 and 2.10).
In [7], G. Choquet presents an axiomatic theory of capacities, this theory encompasses capacities on Euclidean spaces associated to functionnals of the type $\Phi(x, u, |\nabla u|)$ of which the previously mentionned two capacities are special cases.
Such capacities have been studied in the 1960's, mainly motivated by the theory of PDE's (see e.g. [36]) and the theory of quasi-conformal mappings in higher dimension (Reshetnyak [40], Gehring [14]). See also [38, 37, 39, 15, 22] for further development.
The goal of this paper is to develop some basic topics in the potential theory related to these notions of capacity. Specifically, we discuss the following subjects:

- Polar sets. A set $S \subset X$ is said to be *p-polar* if it has locally zero variational $p$-capacity; these are the negligible sets of the theory. We show that for good spaces, polar sets can be described from the Sobolev capacity (see Propositions 3.6 and3.7). We also prove that a set is $p-$polar if and only if it is the set of poles of a Sobolev function (Proposition 3.9).

- Quasi-continuity. A Lusin type theorem is given (Theorem 4.2): it says that every Sobolev function $u$ has a $p-$quasi-continuous representative (i.e. a representative which is continuous except for a $p-$polar set).

- Embedding theorem. We discuss some embedding of the Sobolev spaces into the space of bounded measurable functions and into the space of bounded continuous functions (see §6).

- Choquet property. The Choquet property has been proved in [30] for the Sobolev capacity (in this paper, the proof is written for the Hajlasz Sobolev space, but it is in fact axiomatic). In section 8, we prove the Choquet property for the variational capacity.

- Extremal functions. Finally we prove the existence and uniqueness of an extremal function for the variational $p$-capacity of an arbitrary $p-$fat subset $F \in \mathcal{K}$ (for $1 < p < \infty$), see Theorem 10.1. (A subset $F$ is called *p–fat* if it supports a probability measure which is absolutely continuous with respect to $p-$capacity.) This fact is also true for the Sobolev capacity. Such results are well-known for compact subsets of a bounded Euclidean domain. In our abstract setting, the proof is more delicate since the Sobolev space may not be a uniformly convex Banach space.

We prove all our results in the axiomatic framework of [17]; they are thus not restricted to a particular construction of Sobolev space on metric space.

## 2 A review of Axiomatic Sobolev Spaces

In this section, we give a brief summary of the axiomatic theory of Sobolev spaces developed in [17], we refer to that paper for more details and for the proofs of all the results stated here.

## 2.1 The basic setting

An MM-space is a metric space $(X, d)$ equipped with a Borel regular outer measure $\mu$ such that $0 < \mu(B) < \infty$ for any ball $B \subset X$ of positive radius.
In order to define the notion of local Lebesgue space $L^p_{loc}(X)$, we introduce the following concept :

**Definition 2.1** A *local Borel ring* in the MM-space $(X, d, \mu)$ is a Boolean ring $\mathcal{K}$ of bounded Borel subsets of $X$ satisfying the following three conditions:

K1) $\bigcup_{A \in \mathcal{K}} A = X$;

K2) if $A \in \mathcal{K}$ and $A' \subset A$ is a Borel subset, then $A' \in \mathcal{K}$;

K3) for every $A \in \mathcal{K}$ there exists a finite sequence of open balls $B_1, B_2, ...B_m \in \mathcal{K}$
such that $A \subset \cup_{i=1}^m B_i$ and $\mu(B_i \cap B_{i+1}) > 0$ for $1 \leq i < m$.

A subset $A \subset X$ is called a $\mathcal{K}$-*set* if $A \in \mathcal{K}$.
A local Borel ring is always contained between the ring of all bounded Borel subsets of $X$ and the ring of all relatively compact subsets if $X$.
In the sequel, $X$ will always be an MM-space with metric $d$, measure $\mu$ and a local Borel ring $\mathcal{K}$.

**Definition 2.2** We say that the space $X$ is a $\sigma\mathcal{K}$, or that it is a $\mathcal{K}$-*countable space*, if $X$ is a countable union of open $\mathcal{K}$-sets.

**Definition 2.3** For $1 \leq p < \infty$, the space $L^p_{loc}(X) = L^p_{loc}(X, \mathcal{K}, \mu)$ is the space of measurable functions on $X$ which are $p$-integrable on every $\mathcal{K}$-set. It is a Frechet space for the family of semi-norms $\left\{ \|u\|_{L^p(K)} : K \in \mathcal{K} \right\}$.

**Notations** The notation $A \subset\subset \Omega$ (or $A \Subset \Omega$) means that there exists a closed $\mathcal{K}$-set $K$ such that $A \subset K \subset \Omega$ (in particular $A \Subset X$ if and only if $A$ is contained in a closed $\mathcal{K}$-set).
If $\Omega \subset X$ is open, we denote by $\mathcal{K}|_\Omega$ the set of all Borel sets $A$ such that $A \Subset \Omega$. It is a Boolean ring which we call the *trace* of $\mathcal{K}$ on $\Omega$. This ring satisfies conditions (K1) and (K2) above. If condition (K3) also holds, then we say that $\Omega$ is $\mathcal{K}$-*connected.*
We denote by $C(X)$ the space of all continuous functions $u : X \to \mathbb{R}$ and by $C_0(X) \subset C(X)$ the subspace of continuous functions whose support is contained in a $\mathcal{K}$-set. If $\Omega \subset X$ is an open subset, then $C_0(\Omega)$ is the set of continuous functions $u : \Omega \to \mathbb{R}$ such that $\text{supp}(u) \Subset \Omega$.
For any function $u \in C_0(\Omega)$, there exists an extension $\tilde{u} \in C_0(X)$ which vanishes on $X \setminus \Omega$ and such that $\tilde{u} = u$ on $\Omega$.
The space of bounded continuous functions on an open set $\Omega \subset X$ is denoted by $C_b(\Omega) = C(\Omega) \cap L^\infty(\Omega)$. It is a Banach space for the sup norm.
We conclude this section with a few more technical definitions:

**Definition 2.4** A subset $F$ of an MM-space $X$ is *strongly bounded* if there exists a pair of open sets $\Omega_1 \subset \Omega_2 \subset X$ such that $\Omega_2 \in \mathcal{K}$, $\mu(X \setminus \Omega_2) > 0$, $\text{dist}(\Omega_1, X \setminus \Omega_2) > 0$ and $F \subset \Omega_1$.

**Definition 2.5** An MM-space $X$ is strongly $\mathcal{K}$-coverable if there exist two countable families of open $\mathcal{K}$-sets $\{U_i\}$ and $\{V_i\}$ such that $V_i \neq X$ for all $i$ and

1) $X = \cup U_i$;
2) $U_i \subset V_i$ for all $i$;
3) $\text{dist}(U_i, X \setminus V_i) > 0$ and
4) $\mu(V_i \setminus U_i) > 0$.

Observe that if $F \subset U_i$ for some $i$, then it is a strongly bounded set.
It is clear that every strongly $\mathcal{K}$-coverable metric space is also $\mathcal{K}$-countable.

## 2.2 $D$-structure on an MM space

Let $X = (X, d, \mathcal{K}, \mu)$ be an MM space with a local Borel ring and fix $1 \leq p < \infty$.

**Definition 2.6 a)** A *$D$-structure* on $X$ is structure which associate to each function $u \in L^p_{loc}(X)$ a collection $D[u]$ of measurable functions $g : X \to \mathbb{R}_+ \cup \{\infty\}$ (called the *pseudo-gradient* of $u$). The correspondence $u \to D[u]$ is supposed to satisfy Axioms A1–A5 below.
**b)** A measure metric space equipped with a $D$-structure is called an *MMD-space.*

**Axiom A1 (Non triviality)** *If $u : X \to \mathbb{R}$ is non negative and $k$-Lipschitz, then the function*

$$g := k\, \chi_{\text{supp}(u)} = \begin{cases} k & \text{on} \quad \text{supp}(u) \\ 0 & \text{on} \quad X \setminus \text{supp}(u) \end{cases}$$

*belongs to $D[u]$.*

**Axiom A2** *If $g_1 \in D[u_1]$, $g_2 \in D[u_2]$ and $g \geq |\alpha| g_1 + |\beta| g_2$ almost everywhere, then $g \in D[\alpha u_1 + \beta u_2]$.*

**Axiom A3** *Let $u \in L^p_{loc}(X)$. If $g \in D[u]$, then for any bounded Lipschitz function $\varphi : X \to \mathbb{R}$ the function $h(x) = (\sup|\varphi| g(x) + Lip(\varphi)|u(x)|)$ belongs to $D[\varphi u]$.*

**Axiom A4** *Let $u := \max\{u_1, u_2\}$ and $v := \min\{u_1, u_2\}$ where $u_1, u_2 \in L^p_{loc}(X)$. If $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$, then $g := \max\{g_1, g_2\} \in D[u] \cap D[v]$.*

**Axiom A5** *Let $\{u_i\}$ and $\{g_i\}$ be two sequences of functions such that $g_i \in D[u_i]$ for all $i$. Assume that $u_i \to u$ in $L^p_{loc}$ topology and $(g_i - g) \to 0$ in $L^p$ topology, then $g \in D[u]$.*

**Definition 2.7** The $D$-structure is said to be *non degenerate* if it also satisfies the following additional axiom:

**Axiom A6** *Let $\{u_i\} \subset \mathcal{L}^{1,p}(X)$ be a sequence of functions such that $\mathcal{E}_p(u_i) \to 0$. Then for any metric ball $B \in \mathcal{K}$ there exists a sequence of constants $a_i = a_i(B)$ such that $\|u_i - a_i\|_{L^p(B)} \to 0$.*

The last Axiom is related to the existence of Poincaré inequalities as shown by the next two propositions.

**Proposition 2.1** *A $D$ structure on $X$ is non degenerate if and only if for any pair of measurable subsets $Q \subset A \subset X$ such that $A \in \mathcal{K}$ and $\mu(Q) > 0$, the inequality*

$$\|u - u_Q\|_{L^p(A)} \leq C_{A,Q} \, \|g\|_{L^p(X)}$$

*holds for any $u \in \mathcal{L}^{1,p}(X)$ and $g \in D[u]$.*
*Here the constant $C_{A,Q}$ depends on $p$, $A$ and $Q$ only, and $u_Q := \frac{1}{\mu(Q)} \int_Q u \, d\mu$ is the average value of $u$ on $Q$.*

□

**Proposition 2.2** *Assume that axiom A6 holds and let $A \subset X$ be a measurable $\mathcal{K}$-sets such that $\mu(A) > 0$ and $\mu(X \setminus A) > 0$. Then there exists a constant $C_A$ depending on $p$ and $A$ only for which the inequality*

$$\|u\|_{L^p(A)} \leq C_A \, \|g\|_{L^p(X)}$$

*holds for any $u \in \mathcal{L}^{1,p}(X)$ such that $\operatorname{supp}(u) \subset A$ and $g \in D[u]$.*

□

## 2.3 The Dirichlet space $\mathcal{L}^{1,p}(X)$

**Definition 2.8 i)** The *$p$–Dirichlet energy* of a function $u$ is defined to be

$$\mathcal{E}_p(u) = \inf \left\{ \int_X g^p d\mu : g \in D\left[u\right] \right\}$$

**ii)** The *$p$-Dirichlet space* is the space $\mathcal{L}^{1,p}(X)$ of functions $u \in L^p_{loc}(X)$ with finite $p$-energy.

The Dirichlet space $\mathcal{L}^{1,p}(X)$ is equipped with a locally convex topology defined as follow: one says that a sequence $\{u_i\}$ converges to some function $u \in \mathcal{L}^{1,p}(X)$ if $\mathcal{E}_p(u - u_i) \to 0$ and $\|u - u_i\|_{L^p(A)} \to 0$ for all $A \in \mathcal{K}$.
It is also convenient to introduce a norm on $\mathcal{L}^{1,p}(X)$: to define this norm, we fix a set $Q \in \mathcal{K}$ such that $\mu(Q) > 0$ and we set

$$\|u\|^p_{\mathcal{L}^{1,p}(X,Q)} := \left( \int_Q |u|^p d\mu + \mathcal{E}_p(u) \right)^{1/p}. \tag{1}$$

**Theorem 2.3** *This norm turns $\mathcal{L}^{1,p}(X)$ into a Banach space. Furthermore the locally convex topology on $\mathcal{L}^{1,p}(X)$ defined above and the topology defined by this norm coincide; in particular the Banach space structure is independent of the choice of $Q \in \mathcal{K}$.*

□

The next definition will be our notion of Dirichlet functions vanishing at the boundary of an open subset $\Omega \subset X$:

**Definition 2.9** $\mathcal{L}_0^{1,p}(\Omega)$ is the closure of $C_0(\Omega) \cap \mathcal{L}^{1,p}(X)$ in $\mathcal{L}^{1,p}(X)$ for the norm (1).

## 2.4 The variational capacity

Let $\Omega \subset X$ be an open subset. Recall that $C_0(\Omega)$ is the set of continuous functions $u : \Omega \to \mathbb{R}$ such that $\mathrm{supp}(u) \Subset \Omega$, i.e. $\mathrm{supp}(u)$ is a closed $\mathcal{K}$-subset of $\Omega$.

**Definition 2.10** The *variational p-capacity* of a pair $F \subset \Omega \subset X$ (where $\Omega$ is open and $F$ is arbitrary) is defined as

$$\mathrm{Cap_p}(F, \Omega) := \inf \{ \mathcal{E}_p(u) \,|\, u \in \mathcal{A}_p(F, \Omega) \},$$

where the set of admissible functions is defined by

$$\mathcal{A}_p(F, \Omega) := \{ u \in \mathcal{L}_0^{1,p}(\Omega) \big| \ u \geq 1 \text{ on a neighbourhood of } F \text{ and } u \geq 0 \text{ a.e.} \}.$$

$\mathcal{A}_p(F, \Omega) = \emptyset$, then we set $\mathrm{Cap_p}(F, \Omega) = \infty$. If $\Omega = X$, we simply write $\mathrm{Cap_p}(F, X) = \mathrm{Cap_p}(F)$.

**Remarks 1.** The space $\mathcal{L}_0^{1,p}(\Omega)$ may depend on the ambient space $X \supset \Omega$, however we will avoid any heavier notation such as $\mathcal{L}_0^{1,p}(\Omega, X)$.
**2.** By definition capacity is decreasing with respect to the domain $\Omega$ : if $\Omega_1 \subset \Omega_2$, then $\mathrm{Cap_p}(F, \Omega_1) \geq \mathrm{Cap_p}(F, \Omega_2)$.

**Proposition 2.4** *The variational p-capacity* $\mathrm{Cap_p}(\ )$ *satisfies the following properties:*

*i)* $\mathrm{Cap_p}(\ )$ *is an outer measure;*

*ii) for any subset $F \subset X$ we have $\mathrm{Cap_p}(F) = \inf\{\mathrm{Cap_p}(U) : U \supset F \text{ open }\}$;*

*iii) If $X \supset K_1 \supset K_2 \supset K_3 ...$ is a decreasing sequence of compact sets, then*

$$\lim_{i \to \infty} \mathrm{Cap_p}(K_i) = \mathrm{Cap_p}\left(\cap_{i=1}^{\infty} K_i\right).$$

□

**Definition 2.11** The MMD space $X$ is said to be $p$-parabolic if $\mathrm{Cap_p}(K, X) = 0$ for all K-set $Q \in \mathcal{K}$ and $p$-hyperbolic otherwise.

**Theorem 2.5** *$X$ is $p-$hyperbolic if and only if one of the following equivalent condition holds.*

*1)* $1 \notin \mathcal{L}_0^{1,p}(X)$*;*

*2)* $\mathcal{L}_0^{1,p}(X)$ *is a Banach space for the norm* $\|u\| := (\mathcal{E}_p(u))^{1/p}$ ;

*3)* $\mathrm{Cap_p}(Q) > 0$ *where* $Q \subset X$ *is an arbitrary* $\mathcal{K}$*-set such that* $\mu(Q) > 0$*;*

*4) there exists a constant* $C$ *such that for any* $u \in \mathcal{L}_0^{1,p}(X)$ *we have* $\|u\|_{L^p(Q)} \leq C\ (\mathcal{E}_p(u))^{1/p}$.

□

# 3 Sobolev $p$-capacity and Polar sets

The Sobolev spaces associated to an MMD space $X$ is defined as

$$W^{1,p}(X) := \mathcal{L}^{1,p}(X) \cap L^p(X)\,;$$

it is a Banach space with norm

$$\|u\|_{W^{1,p}(X)} = \left(\int_X |u|^p d\mu + \mathcal{E}_p(u)\right)^{1/p}$$

(see [17, Th. 1.5]).

**Definition 3.1** *The* Sobolev $p$-capacity *of a pair* $F \subset \Omega$ *(where* $\Omega \subset X$ *is open and* $F$ *is arbitrary) is defined by*

$$C_p(F, \Omega) = \inf\left\{\, \|u\|^p_{W^{1,p}} \middle|\, u \in W^{1,p}(\Omega),\ u \geq 1 \ \text{near } F \ \text{and}\ u \geq 0 \ a.e.\right\}.$$

The Sobolev $p$-capacity $\mathrm{C}_p(F, X)$ with respect to $X$ is simply denoted by $\mathrm{C}_p(F)$, it satisfies the same basic properties as the variational $p$-capacity :

**Proposition 3.1** *i) The Sobolev p-capacity is an outer measure;*

*ii) for any subset* $F \subset X$ *we have* $C_p(F) = \inf\{C_p(U) : U \supset F \text{ open }\}$*;*

*iii) If* $X \supset K_1 \supset K_2 \supset K_3 ...$ *is a decreasing sequence of compact sets, then*

$$\lim_{i\to\infty} C_p(K_i) = C_p\left(\bigcap_{i=1}^{\infty} K_i\right).$$

**Proof** Use the same type of arguments as in the proof of Proposition 2.4 (see §3 in [17]). □

**Proposition 3.2** *For any function $u \in W^{1,p}(X)$, let $P_u := \{x \in X \mid \lim_{y\to x} u(y) = \infty\}$ be the set of poles of $u$. Then $C_p(P_u) = 0$.*

**Proof** For any $k \geq 1$ the function $u_k(x) := \frac{1}{k}\min(k, u(x))$ is an admissible function for the Sobolev $p-$capacity of the set $P_u$. Using the axioms A1, A2 and A4, we can check that $\|u_k\|_{W^{1,p}(X)} \leq \frac{1}{k}\|u\|_{W^{1,p}(X)}$. Hence $\|u_k\|_{W^{1,p}(X)} \to 0$ as $k \to \infty$ and thus the Sobolev $p-$capacity of the set $P_u$ is zero.

□

**Proposition 3.3** *For any set $A \subset X$, $C_p(A) = 0$ if and only if for any $\varepsilon > 0$ there exists a nonnegative function $u \in W^{1,p}(X)$ such that $\lim_{y\to x} u(y) = \infty$ for any $x \in A$ and $\|u\|_{W^{1,p}(X)} \leq \varepsilon$.*

**Proof** Suppose that $C_p(A) = 0$. By definition of the Sobolev $p$-capacity there exists a sequence of nonnegative functions $u_n$ such that $\|u_n\|_{W^{1,p}(X)} \leq 2^{-n}\varepsilon$ and $u_n = 1$ in some neighbourhood of $A$. Then $u = \sum_n u_n$ belongs to $W^{1,p}(X)$ and $\lim_{y\to x} u(y) = \infty$. Furthermore, we clearly have $\|u\|_{W^{1,p}(X)} \leq \varepsilon$.
The converse direction follows from the previous proposition.

□

**Definition 3.2 a)** A set $S \subset X$ is *p-polar* (or *p-null*) if for any pair of open $\mathcal{K}$-sets $\Omega_1 \subset \Omega_2 \neq X$ such that $\operatorname{dist}(\Omega_1, X \setminus \Omega_2) > 0$, we have $\operatorname{Cap_p}(S \cap \Omega_1, \Omega_2) = 0$.
**b)** A property is said to hold *p-quasi-everywhere* if it holds everywhere except on a $p$-polar set.

In the rest of this section, we compare $p$-polar sets and sets of Sobolev $p$-capacity zero, we show in particular that in good cases, the $p-$polar sets and the sets of Sobolev $p$-capacity zero are the same.
We begin with a technical lemma which is used in some cut-off arguments.

**Lemma 3.4** *Let $\Omega_1 \subset \Omega_2 \subset X$ be a pair of open sets such that $\Omega_2 \neq X$ and $\delta := \operatorname{dist}(\Omega_1, X \setminus \Omega_2) > 0$. Then for any subset $S \subset \Omega_1$ and every $\varepsilon > 0$, there exists a function $\varphi = \varphi_\varepsilon \in W^{1,p}(X)$ with support in a closed subset of $\Omega_2$, such that $\varphi \geq 1$ in a neighbourhood of $S$ and*

$$\|\varphi\|_{W^{1,p}(X)} \leq 2\left(1 + \frac{3}{\delta}\right)(C_p(S) + \varepsilon)^{1/p} . \tag{2}$$

**Proof** Let us set $\sigma(x) := \operatorname{dist}(x, X \setminus \Omega_2)$ and

$$\psi(x) = \begin{cases} 1 & \text{if } \sigma(x) \geq \frac{2}{3}\delta, \\ (\frac{3}{\delta}\sigma(x) - 1) & \text{if } \frac{1}{3}\delta \leq \sigma(x) \leq \frac{2}{3}\delta, \\ 0 & \text{if } \sigma(x) \leq \frac{1}{3}\delta. \end{cases}$$

Then $\psi : X \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant $\frac{3}{\delta}$, with support in a closed subset of $\Omega_2$ and such that $\psi \equiv 1$ in a neighbourhood of $\Omega_1$.

By definition, we can find for any $\varepsilon > 0$ two nonnegative functions $u, g : X \to \mathbb{R}$ such that $g \in D[u]$, $u \geq 1$ in a neighbourhood of $S$ and $\|u\|^p_{L^p(X)} + \|g\|^p_{L^p(X)} \leq C_p(S) + \varepsilon$. Let us set $\varphi := \psi u$; it is clear that $\mathrm{supp}(\varphi)$ is a closed subset of $\Omega_2$ and $\varphi \geq 1$ in a neighbourhood of $S$. From axiom A3 we know that $h := g + \frac{3}{\delta}|u| \in D[\varphi]$, hence

$$\begin{aligned}
\|\varphi\|_{W^{1,p}(X)} &\leq \|\varphi + h\|_{L^p(X)} \leq \|\varphi\|_{L^p(X)} + \|h\|_{L^p(X)} \\
&\leq \|u\|_{L^p(X)} + \|g + \frac{3}{\delta}u\|_{L^p(X)} \leq (1 + \frac{3}{\delta}) \left(\|u\|_{L^p(X)} + \|g\|_{L^p(X)}\right) \\
&\leq (1 + \frac{3}{\delta})\, 2\, (\|u\|^p_{L^p(X)} + \|g\|^p_{L^p(X)})^{1/p} \\
&\leq (1 + \frac{3}{\delta}) 2 (C_p(S) + \varepsilon)^{1/p} .
\end{aligned}$$

□

**Corollary 3.5** *Let $\Omega_1$, $\Omega_2$ and $S$ be as in the lemma. If $C(X) \cap W^{1,p}(X)$ is dense in $W^{1,p}(X)$ and $\Omega_2$ is a $\mathcal{K}$-set, then*

$$\mathrm{Cap_p}(S, \Omega_2) \leq 2^p \left(1 + \frac{3}{\delta}\right)^p C_p(S).$$

*In particular if $C_p(S) = 0$, then $S$ is $p$-polar.*

**Proof** Because $C(X) \cap W^{1,p}(X)$ is dense in $W^{1,p}(X)$, the function $\varphi$ constructed in the previous lemma belongs to $\mathcal{L}^{1,p}_0(\Omega_2)$. The proof follows then from the inequality (2) as $\varepsilon$ is arbitrarily small.

□

**Proposition 3.6** *Suppose that $C(X) \cap W^{1,p}(X)$ is dense in $W^{1,p}(X)$. Then a strongly bounded set $S \subset X$ is $p$−polar if and only if $C_p(S) = 0$.*

Recall that $S \subset X$ is strongly bounded if $S \subset \Omega_1 \subset \Omega_2 \subset X$ where $\Omega_1$ and $\Omega_2$ are open $\mathcal{K}$-sets such that $\mu(X \setminus \Omega_2) > 0$ and $\mathrm{dist}(\Omega_1, X \setminus \Omega_2) > 0$.

**Proof** By the previous Corollary, we already know that if $C_p(S) = 0$ then $S$ is $p$−polar. Assume conversely that $S$ is $p$-polar, we then have $\mathrm{Cap_p}(S \cap \Omega_1, \Omega_2) = 0$. This means that for every $\varepsilon > 0$ there exists a function $u \in \mathcal{L}^{1,p}_0(\Omega)$ such that $u \geq 1$ on a neighborhood of $S$ and $\mathcal{E}_p(u) \leq \varepsilon$.

Recall that any function $u \in \mathcal{L}^{1,p}_0(\Omega)$ is globally defined on $X$ and vanishes on $X \setminus \Omega$. Since $\mu(X \setminus \Omega_2) > 0$, we have from Proposition 2.2

$$\int_X |u|^p d\mu = \int_{\Omega_2} |u|^p d\mu \leq C\mathcal{E}_p(u)$$

where $C = C(\Omega_2, p)$. Thus $u$ is an admissible function for the Sobolev $p$-capacity $C_p(S)$ and $\|u\|_{W^{1,p}(X)} \leq ((1 + C)\varepsilon)^{1/p}$, therefore $C_p(S) = 0$ since $\varepsilon$ is arbitrary.

□

**Proposition 3.7** *Suppose that $C(X) \cap W^{1,p}(X)$ is dense in $W^{1,p}(X)$ and that $X$ is strongly $\mathcal{K}$-coverable. Then*

*i) If a set $S \subset X$ is p-polar then $\mathrm{Cap_p}(S, X) = 0$;*

*ii) A set $S \subset X$ is p-polar if and only if $C_p(S) = 0$.*

Recall that $X$ is strongly $\mathcal{K}$-coverable if there exist two countable families of open $\mathcal{K}$-sets $\{U_i\}$, $\{V_i\}$ such that: $X = \cup U_i$, $U_i \subset V_i$ for all $i$, $\mathrm{dist}(U_i, X \setminus V_i) > 0$ and $\mu(V_i \setminus U_i) > 0$.
**Proof** (i) Let $S_j := S \cap U_j$, then $\mathrm{Cap_p}(S_j, X) \leq \mathrm{Cap_p}(S_j, V_j) = 0$ since $S$ is $p$-polar. Thus, by countable subadditivity of the variational $p$-capacity, we have $\mathrm{Cap_p}(S, X) = \mathrm{Cap_p}(\cup S_j, X) \leq \sum_j \mathrm{Cap_p}(S_j, X) = 0$.
(ii) We already know from Corollary 3.5 that if $C_p(S) = 0$, then $S$ is $p$-polar. Conversely, if $S \subset X$ is an arbitrary $p$-polar set, then we consider the decomposition $S = \cup S_j$ where $S_j = S \cap U_j$. We know by Proposition 3.6 that $C_p(S_j) = 0$ and thus, by countable subadditivity of the Sobolev $p$-capacity, $C_p(S) = C_p(\cup S_j) \leq \sum C_p(S_j) = 0$. □

**Lemma 3.8** *Suppose that $X$ is a strongly $\mathcal{K}$-coverable metric space such that $C(X) \cap W^{1,p}(X)$ is dense in $W^{1,p}(X)$. Then any p-polar subset of $X$ has $\mu$-measure zero.*

**Proof** This follows from the trivial estimate $\mu(F) \leq C_p(F)$. □

**Remark** The converse of assertion (i) in Proposition 3.7 is not true in general. Indeed, suppose that $X$ is $p-$parabolic, then $\mathrm{Cap_p}(S, X) = 0$ for any subset $S$, yet no set of positive measure is $p$-polar. However, one may ask the following
**Question** Suppose that $X$ is strongly $\mathcal{K}$-coverable and $p-$hyperbolic. Do we have $\mathrm{Cap_p}(S, X) = 0 \Leftrightarrow S$ $p$-polar ?
Our final result explains the terminology: a set is $p-$polar if it is the set of poles for some function $u \in W^{1,p}(X)$.

**Proposition 3.9** *Suppose that $C(X) \cap W^{1,p}(X)$ is dense in $W^{1,p}(X)$ and that $X$ is strongly $\mathcal{K}$-coverable. A set $A \in \mathcal{K}$ is p-polar if and only if for any $\varepsilon > 0$ there exists a nonnegative function $u \in W^{1,p}(X)$ such that $\lim_{y \to x} u(y) = \infty$ for any $x \in A$ and $\|u\|_{W^{1,p}(X)} \leq \varepsilon$.*

**Proof** Follows from Propositions 3.3 and 3.7. □

# 4 Egorov type theorems and quasi-continuity

In this section, we prove Egorov and Lusin type theorems for the Dirichlet space $\mathcal{L}^{1,p}(X)$ with the topology induced by the norm (1):

$$\|u\|^p_{\mathcal{L}^{1,p}(X,Q)} := \left( \int_Q |u|^p d\mu + \mathcal{E}_p(u) \right)^{1/p},$$

where $Q$ is a fixed $\mathcal{K}$-set such that $\mu(Q) > 0$. Recall that, by Theorem 2.3, this norm is complete and the corresponding Banach space structure is independent of the choice of $Q$. It will be important throughout this section to keep in mind that a Cauchy sequence in the Dirichlet space $\mathcal{L}^{1,p}(X)$ converges in $W^{1,p}(\Omega)$ for any open $\mathcal{K}$-set $\Omega \subset X$; this follows from Theorem 2.3 and the floating Poincaré inequality.

**Theorem 4.1** *Let $\{u_i\} \subset \mathcal{L}^{1,p}(X) \cap C(X)$ be a Cauchy sequence in $\mathcal{L}^{1,p}(X)$. Then for any open set $\Omega \in \mathcal{K}$ there exists a subsequence $\{u_{i'}\}$ of $\{u_i\}$ and a sequence of open subsets $\Omega \supset U_1 \supset U_2 \supset U_3 \supset \ldots$ such that $\lim_{\nu\to\infty} C_p(U_\nu, \Omega) = 0$ and $\{u_{i'}\}$ converges uniformly in $\Omega\setminus U_\nu$ for all $\nu$. In particular $\{u_{i'}\}$ converges pointwise in the complement of the set of zero Sobolev p-capacity $S := \cap_{j=1}^{\infty} U_j$.*

**Proof** We know that $\{u_i\}$ converges in $W^{1,p}(\Omega)$ for any $\mathcal{K}$-set $\Omega \subset X$, we can thus find a subsequence (which we still denote $\{u_i\}$), such that

$$\sum_{i=1}^{\infty} 2^{ip} \|u_i - u_{i+1}\|^p_{W^{1,p}(\Omega)} < \infty \,. \tag{3}$$

For any $i \in \mathbb{N}$, we set $E_i := \{x \in \Omega : |u_i(x) - u_{i+1}(x)| > 2^{-i}\}$ and $U_j := \cup_{i=j}^{\infty} E_i$. Since the functions $u_i$ are continuous by hypothesis, the sets $E_i$ and $U_j$ are open; in particular $2^i|u_i - u_{i+1}|$ is admissible for the Sobolev $p$-capacity of $E_i$ in $\Omega$, hence

$$C_p(E_i, \Omega) \le 2^{ip} \|u_i - u_{i+1}\|^p_{W^{1,p}(\Omega)} \,.$$

By countable subadditivity of the Sobolev $p$-capacity, we obtain

$$C_p(U_j, \Omega) \le \sum_{i=j}^{\infty} C_p(E_i, \Omega) \le \sum_{i=j}^{\infty} 2^{ip} \|u_i - u_{i+1}\|^p_{W^{1,p}(\Omega)} \,,$$

and from the convergence of the sum (3) we conclude that

$$C_p(S, \Omega) \le \lim_{j\to\infty} C_p(U_j, \Omega) = 0$$

(where $S = \cap_{j=1}^{\infty} U_j$) and $\{u_i\}$ converges pointwise in $\Omega \setminus U$. Moreover we have for any $x \in \Omega\setminus U_\nu$ and all $k > j \ge \nu$

$$|u_j(x) - u_k(x)| \le \sum_{i=j}^{k-1} |u_i(x) - u_{i+1}(x)| \le \sum_{i=j}^{k-1} 2^{-i} \le 2^{1-j}.$$

This implies that $\{u_j\}$ converges uniformly in $\Omega \setminus U_\nu$ .

□

A consequence of the previous result is the following Lusin type theorem for $p-$capacities in $\mathcal{K}$-countable metric spaces. We first need the following

**Definition 4.1** A function $v : X \to \mathbb{R}$ is *$p-$quasi-continuous* if for every $\varepsilon > 0$, we can find a subset $S \subset X$ such that $C_p(S) < \varepsilon$ and $v$ is continuous on $X \setminus S$.

**Remark** Without loss of generality, we can suppose that the set $S$ is open.

**Theorem 4.2** *Suppose that* $X$ *is* $\mathcal{K}$*-countable. For each* $u \in \overline{C(X) \cap \mathcal{L}^{1,p}(X)}$ *there is a function* $v \in \mathcal{L}^{1,p}(X)$ *such that*

*i)* $u = v$ *almost everywhere on* $X$ *and*

*ii)* $v$ *is p-quasi-continuous.*

The function $v$ is called a *p−quasi-continuous representative* of $u$. Note in particular that every function $u \in \mathcal{L}_0^{1,p}(X)$ has a $p$−quasi-continuous representative (since continuous functions are dense in $\mathcal{L}_0^{1,p}(X)$ by definition).

**Proof** By hypothesis, there exists a sequence of open $\mathcal{K}$-sets $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset ... \subset X$ such that $X = \cup\Omega_j$. Choose $\varepsilon > 0$ and a function $u \in \overline{C(X) \cap \mathcal{L}^{1,p}(X)}$; there exists a sequence $\{u_i\} \in \mathcal{L}^{1,p}(X) \cap C(X)$ which converges to $u$ in $\mathcal{L}^{1,p}(X)$.

In particular $\{u_i\}$ converges in $W^{1,p}(\Omega_j)$ for any $j$, hence the previous theorem (applied to $X = \Omega_j$) tells us that for any $j$ there exist a subsequence $\{u_{i,j}\}$ of $\{u_i\}$ which converges to a function $v_j = \lim_{i\to\infty} u_{i,j}$ in $W^{1,p}(\Omega_j)$-norm, and a subset $F_j \subset \Omega_j$ such that $C_p(F_j, \Omega_j) < 2^{-j}\varepsilon$ and $\{u_{i,j}\}$ converges uniformly toward $v_j$ in $\Omega_j \setminus F_j$; in particular $v_j$ is continuous in that set.

Choose $i_j$ such that $\|u_{i_j,j} - v_j\|_{W^{1,p}(\Omega_j)} < 1/j$ and let us note to simplify $w_j := u_{i_j,j}$. Because $\Omega_j \subset \Omega_{j+1}$, the sequence $w_j$ converges in $\mathcal{L}^{1,p}(X)$-topology to a function $v$ which coincides with $v_j$ in $\Omega_j$ for any $j$. In particular, $v_j = u$ almost everywhere in $\Omega_j$, and thus $v = u$ almost everywhere in $X$. Since $v = v_j$ on $\Omega_j$ and is therefore continuous in $\Omega_j \setminus F_j$ for all $j$, it is a $p$−quasi-continuous function.

□

Theorem 4.1 has a version for sequences of quasi-continuous functions in the space $\mathcal{L}_0^{1,p}(X)$: every Cauchy sequence of quasi-continuous functions in $\mathcal{L}_0^{1,p}(X)$ contains a subsequence which converges uniformly outside of set of arbitrary small $p$−capacity:

**Proposition 4.3** *Let* $\{u_i\} \subset \mathcal{L}_0^{1,p}(X)$ *be a Cauchy sequence of p-quasi-continuous functions. Then for any open set* $\Omega \in \mathcal{K}$ *and any* $\varepsilon > 0$*, there exists a subsequence* $\{u_{i'}\}$ *of* $\{u_i\}$ *which converges uniformly in* $\Omega \setminus F_\varepsilon$*, where* $F_\varepsilon \subset \Omega$ *is a subset such that* $C_p(F_\varepsilon, \Omega) \leq 2\varepsilon$.

**Proof** We know that, given an arbitrary open $\mathcal{K}$-set $\Omega \subset X$, the sequence $\{u_i\}$ converges in $W^{1,p}(\Omega)$ to a function $u := \lim_{i\to\infty} u_i$. Since continuous functions are dense in $\mathcal{L}_0^{1,p}(X)$, there exists for any $i$ a sequence of continuous functions $v_{i,j} \in \mathcal{L}_0^{1,p}(X)$ which converges in $W^{1,p}(\Omega)$ to $u_i$. By Theorem 4.1 there exists for each $i$ a sequence of open subsets $\Omega \supset U_{i,1} \supset U_{i,2} \supset U_{i,3} \supset ...$ such that $\lim_{s\to\infty} C_p(U_{i,s}, \Omega) = 0$ and a subsequence of the sequence $\{v_{i,j}\}$ (which we still denote $\{v_{i,j}\}$) which converges uniformly toward $v_i = \lim_{j\to\infty} v_{i,j}$ in $\Omega \setminus U_{i,s}$ for any $s$. Given $\varepsilon$ and $i$, we can therefore find $j_i$ such that $C_p(U_{i,j_i}, \Omega) < 2^{-i}\varepsilon$ and $\sup_{x\in\Omega\setminus U_{i,j_i}} |v_i - v_{i,j_i}| < 2^{-i}\varepsilon$.

Because $u_i$ and $v_i$ are both $p$-quasi-continuous and $\|u_i - v_i\|_{W^{1,p}(\Omega)} = 0$ , there exists a subset $F_i \subset \Omega$ such that $C_p(F_i, \Omega) = 0$ and $u_i = v_i$ in $\Omega \setminus F_i$.

By construction the sequence of continuous functions $w_i := v_{i,j_i}$ converges in $W^{1,p}(\Omega)$ to $u$. By Theorem 4.1 again, there exists a set $U_\varepsilon$ such that $C_p(U_\varepsilon, \Omega) < \varepsilon$ and a subsequence $\{w_{i'}\}$ that converges uniformly on $\Omega \setminus U_\varepsilon$ to $w := \lim_{i\to\infty} w_i$; clearly $w = u$ almost everywhere on $\Omega \setminus U_0$.
Set $F_\varepsilon := U_\varepsilon \cup (\bigcup_i U_{i,j_i}) \cup (\bigcup_i F_i)$; by construction and countable subadditivity of Sobolev $p$-capacity we have

$$C_p(F_\varepsilon, \Omega) \leq C_p(U_\varepsilon, \Omega) + \sum_{i=1}^{\infty} C_p(U_{i,j_i}, \Omega) + \sum_{i=1}^{\infty} C_p(F_i, \Omega) \leq 2\varepsilon.$$

Because $u_{i'} = v_{i'}$ on $\Omega \setminus F_\varepsilon$ we obtain finally

$$\sup_{x\in\Omega\setminus F_\varepsilon} |u_{i'} - w| \leq \sup_{x\in\Omega\setminus F_\varepsilon} |v_{i'} - w_{i'}| + \sup_{x\in\Omega\setminus F_\varepsilon} |w - w_{i'}| \leq 2^{-i'}\varepsilon + \sup_{x\in\Omega\setminus F_\varepsilon} |w - w_{i'}|.$$

Thus $\{u_{i'}\}$ converges uniformly to $w$ in $\Omega \setminus F_\varepsilon$.

□

If $X$ is $\mathcal{K}-$countable, then we can globalize the previous result:

**Corollary 4.4** *Assume that $X$ is $\mathcal{K}-$countable. Let $\{u_i\} \subset \mathcal{L}_0^{1,p}(X)$ be a Cauchy sequence of p-quasi-continuous functions. Then for any $\varepsilon > 0$, there exists a subsequence $\{u_{i'}\}$ of $\{u_i\}$ which converges uniformly in $X \setminus F_\varepsilon$, where $F_\varepsilon \subset X$ is a subset such that $C_p(F_\varepsilon, X) \leq \varepsilon$.*

The proof follows from previous proposition and countable subadditivity of the Sobolev capacity.

□

**Remark** The proofs shows that the last two results also hold for Cauchy sequences in $\overline{C(x) \cap \mathcal{L}^{1,p}(X)}$.
Recall that a continuous function $f : X \to \mathbb{R}$ is said to be monotone (in the sense of Lebesgue) if its restriction to any compact set assumes its extremal values at the boundary of that set. For $p-$quasi-continuous we have a corresponding notion:

**Definition 4.2** A function $u : X \to \mathbb{R}$ is *$p-$quasi-monotone* if for every domain $D \subset X$ and every subdomain $D_1 \Subset D$ the inequalities

$$\inf_{(D\setminus D_1)} u \leq \inf_{D_1} u \leq \sup_{D_1} u \leq \sup_{(D\setminus D_1)} u.$$

hold $p-$quasieverywhere (i.e. on the complement of a $p$-polar subset of $D$).

# 5 The Sobolev capacity of a point

In this section, we study a metric relation between the Sobolev capacity of a point and the measure $\mu$.

Recall that if $B(x_0, 2R) \in \mathcal{K}$ and $\mu(X \setminus B(x_0, 2R)) > 0$, then there exists a constant $C_{x_0,R}$ such that the inequality

$$\|u\|_{L^p(B(x_0,2R))} \leq C_{x_0,R}\, \mathcal{E}_p(u)^{1/p} \tag{4}$$

holds for any $u \in \mathcal{L}^{1,p}(X)$ such that $\operatorname{supp}(u) \subset B(x_0, 2R)$ (Proposition 2.2).

**Theorem 5.1** *Suppose that $B(x_0, 2R) \in \mathcal{K}$ and $\mu(X \setminus B(x_0, 2R)) > 0$. Then we have for all $x \in B(x_0, R)$, all $0 < r < R$ and any $1 \leq p < \infty$.*

$$\mu(B(x,r)) \geq \left(\frac{r^p}{2^p(1 + C^p_{x_0,R})}\right) C_p(\{x\}).$$

**Proof** Let us define the function $u_r$ by

$$u_r(z)) = \begin{cases} 1 & \text{if } \ z \in B(x, r/2) \\ \frac{2}{r}\,(r - d(x, x_0)) & \text{if } \ z \in B(x, r) \setminus B(x, r/2) \\ 0 & \text{if } \ z \notin B(x, r), \end{cases}$$

it is clearly a Lipschitz function with $\operatorname{Lip}(u_r) \leq \frac{2}{r}$. We have $\operatorname{supp}(u_r) \subset B(x, r) \subset B(x_0, 2R)$ (because $x \in B(x_0, R)$ and $r < R$).
By Axiom A1, $u_r \in \mathcal{L}^{1,p}(X)$ and a pseudo-gradient $g \in D[u_r]$ is given by $g(z) = 2/r$ if $z \in B(x, r)$ and $g(z) = 0$ for all other $z$. Therefore

$$\mathcal{E}_p(u_r) \leq \frac{2^p}{r^p}\int_{B(x,r)} d\mu = 2^p\frac{\mu(B(x,r))}{r^p}.$$

Using the inequality (4) above, we obtain $\|u_r\|^p_{L^p(X)} \leq C^p_{x_0,R}\,\mathcal{E}_p(u_r)$ for some constant $C_{x_0,R}$, thus

$$\|u_r\|^p_{W^{1,p}(X)} \leq (1 + C^p_{x_0,R})\mathcal{E}_p(u_r) \leq (1 + C^p_{x_0,r})2^p\frac{\mu(B(x,r))}{r^p}.$$

Since the function $u_r$ is an admissible function for the Sobolev $p$-capacity of the point $\{x\}$, we have

$$C_p(\{x\}) \leq \|u_r\|^p_{W^{1,p}(X)} \leq (1 + C^p_{x_0,R})2^p\left(\frac{\mu(B(x,r)}{r^p}\right).$$

□

# 6 On embeddings of $W^{1,p}(X)$

In this section we discuss embedding theorems of Sobolev spaces into the space of bounded or continuous functions.

**Proposition 6.1** *Suppose that we have a bounded embedding $W^{1,p}(X) \subset L^\infty(X)$. Then we have $C_p(\{x\}) \geq \frac{1}{\nu^p}$ for all $x \in X$ where $\nu$ is the norm of the embedding $W^{1,p}(X) \subset L^\infty(X)$.*

**Proof** If $u \in W^{1,p}(X)$ is an admissible function for the Sobolev $p$-capacity of the point $\{x\}$, then the truncated function $Tu(x) := \max\{0; \min\{1; u\}\}$ is also admissible for the Sobolev $p$-capacity of $\{x\}$. The claim follows then from the inequality $1 = \|Tu\|_{L^\infty(X)} \leq \nu\|Tu\|_{W^{1,p}(X)}$ and the definition of the Sobolev $p$-capacity. □

**Corollary 6.2** *Suppose that we have a bounded embedding $W^{1,p}(X) \subset L^\infty(X)$. If $B(x_0, 2R) \in \mathcal{K}$ and $\mu(X \setminus B(x_0, 2R)) > 0$, then there exists a constant $\kappa$ such that $\mu(B(x,r)) \geq \kappa r^p$ for any $x \in B(x_0, R)$ and any $0 \leq r < R$.*

**Proof** Define $\kappa$ by $1/\kappa = (2\nu)^p(1+C^p_{x_0,R})$ where $\nu$ is the norm of the embedding $W^{1,p}(X) \subset L^\infty(X)$ and $C_{x_0,R}$ is the constant in inequality (4). The result follows then from Proposition 6.1 and Theorem 5.1. □

We have the following result in the converse direction:

**Proposition 6.3** *Assume that $C_p(\{x\}) \geq \gamma > 0$ for all $x \in X$, then every continuous function in $W^{1,p}(X)$ is bounded.*

**Proof** We need to prove that for any function $u \in C(X) \cap W^{1,p}(X)$ we have

$$\|u\|_{W^{1,p}(X)} \leq 1 \quad \Longrightarrow \quad \|u\|_{L^\infty(X)} \leq \gamma^{-1/p}. \tag{5}$$

This can be proved by contradiction, indeed assume that $\|u\|_{W^{1,p}(X)} \leq 1$ and $\|u\|_{L^\infty(X)} > \gamma^{-1/p}$, then there exists $\lambda > 1$ and $x_0$ such that $|u(x_0)| \geq \lambda^2\gamma^{-1/p}$. We may assume w.l.o.g. that $u(x_0) > 0$. By continuity $v := \left(\frac{\gamma^{1/p}}{\lambda}u\right) > 1$ in a neighbourhood of $x_0$, hence it is an admissible function for the capacity $C_p(\{x_0\})$. We thus have

$$\gamma \leq C_p(\{x_0\}) \leq \|v\|^p_{W^{1,p}(X)} = \frac{\gamma}{\lambda^p}\|u\|^p_{W^{1,p}(X)} < \gamma\,.$$

This contradiction implies (5) and the Proposition follows. □

**Corollary 6.4** *Assume that $C_p(\{x\}) \geq \gamma > 0$ for all $x \in X$, then*

*a) $C(X) \cap W^{1,p}(X)$ is complete for the norm $\|\ \ \|_{W^{1,p}(X)}$;*

*b) If continuous functions are dense in $W^{1,p}(X)$, then $W^{1,p}(X) \subset C_b(X)$.*

**Proof** (a) By condition (5), we know that if $\{u_i\} \subset C(X) \cap W^{1,p}(X)$ is a Cauchy sequence (for the $W^{1,p}(X)$-norm) then it converges uniformly. The limit is thus a continuous function. (b) Follows from (a) and the previous Proposition. □

For the Hajłasz-Sobolev space $HW^{1,p}(X)$ (see [18] or [17] for the definition) , we also have the following result based on a volume estimate rather than a capacity estimate:

**Theorem 6.5** *Suppose that there exists a constant $\kappa > 0$ such that $\mu(B(x,r)) \geq \kappa r^s$ for any $x \in X$ and any $0 < r \leq D$ for some $D > 0$. If $p > s$, then any function $u \in HW^{1,p}(X)$ is locally Hölder continuous.*

**Proof** P. Hajłasz has proved in [18] that if $p > s$, then for any $\varphi \in HW^{1,p}(X)$ and almost all $x, y \in B(x_0, 3R)$ (where $R$ is small enough) the following (Morrey type) inequality holds:

$$|\varphi(x) - \varphi(y)| \leq C_1 \mathrm{diam}(B(x_0, 3R))\, \mu(B(x_0, 3R))^{-1/p}\, \|\varphi\|_{HW^{1,p}(X)} \tag{6}$$

combining this fact with the inequality

$$\mu(B(x_0, 3R))^{-1/p} \leq 2^{2/p} \kappa^{-1/p}\, \mathrm{diam}(B(x_0, 3R))^{-s/p},$$

we obtain

$$|\varphi(x) - \varphi(y)| \leq C_2 \mathrm{diam}(B(x_0, 3R))^{\,1-s/p}\, \|\varphi\|_{HW^{1,p}(X)}$$

from which the local Hölder continuity of $\varphi$ follows:

$$|\varphi(x) - \varphi(y)| \leq \left(C_3 \|\varphi\|_{HW^{1,p}(X)}\right) |y - x|^{1-s/p}.$$

(Here the constants $C_2$ and $C_3$ depends on the constants in the previous inequalities.)

□

# 7 Admissible functions for capacities

Recall that the set of admissible functions for the variational $p$−capacity of a set $F \subset X$ was defined as

$$\mathcal{A}_p(F, \Omega) := \left\{ u \in \mathcal{L}_0^{1,p}(\Omega) \middle| \, u \geq 1 \text{ on a neighbourhood of } F \text{ and } u \geq 0 \right\}.$$

Let us denote by $\mathcal{A}_p'(F, X)$ the closure of $\mathcal{A}_p(F, X))$ in $\mathcal{L}_0^{1,p}(X)$, it is a closed convex subset of $\mathcal{L}_0^{1,p}(X)$.

**Proposition 7.1** *Suppose that $X$ is $\mathcal{K}$-countable. Then for any function $u \in \mathcal{A}_p'(F, X)$ there exists a $p$-quasi-continuous representative $v$ such that $v = u$ almost everywhere and $v \geq 1$ $p$-quasi-everywhere on $F$.*

**Proof** By definition any function $u \in \mathcal{A}_p'(F, X)$ is the limit of a sequence of non negative functions $u_i \in \mathcal{L}_0^{1,p}(X)$ such that $u_i(x) \geq 1$ for any $x$ in some neighborhood of $F$. By Theorem 4.2 any function $u_i$ admits a non negative $p$-quasi-continous representative $v_i$ such that $u_i = v_i$ almost everywhere. We may assume that $v_i(x) \geq 1$ in some neighbourhood of $F$. By Corollary 4.4, we can find a subsequence (which we still denote $\{v_i\}$) which converges pointwise in the complement of a set $S$ of zero Sobolev $p$−capacity to a $p$-quasi-continuous function $v$ such that $v = u$ almost everywhere. Therefore $v(x) \geq 1$ on $F \setminus S$ and $v$ is the desired $p$-quasi-continuous representative of $u$.

□

The previous proposition motivates the following definition of a more "natural" admissible set for the variational $p-$capacity :

$$\mathcal{A}_p''(F,X) = \left\{u \in \mathcal{L}_0^{1,p}(X) \middle| \, 0 \le u(x) \le 1 \text{ for all } x \text{ and } \right. \\ \left. u = 1 \ p\text{-quasi-everywhere on } F\right\} .$$

(Given a subset $F \subset X$, the notation $\{u \in \mathcal{L}_0^{1,p}(X) | \, u = 1 \ p\text{-quasi-everywhere on } F\}$ means the set of those functions $u \in \mathcal{L}_0^{1,p}(X)$ which have a $p-$quasi-continuous representative $v$ such that $v = 1$ $p$-quasi-everywhere on $F$.)

**Proposition 7.2** *Suppose that $X$ is $\mathcal{K}$-countable, then $\mathcal{A}_p''(F,X)$ is convex and closed in $\mathcal{L}_0^{1,p}(X)$.*

**Proof** Convexity is clear. To prove closedness, consider a sequence $v_i \in \mathcal{A}_p''(F,X)$ which converges to some function $v \in \mathcal{L}_0^{1,p}(X)$. By Corollary 4.4, we can find a subsequence (which we still denote $\{v_i\}$) which converges pointwise in the complement of a set $S$ of zero Sobolev $p-$capacity to a $p$-quasi-continuous function $w$ such that $w = v$ almost everywhere. Therefore $w(x) = 1$ on $F \setminus S$ and thus $v \in \mathcal{A}_p''(F,X)$.

□

We define the *truncation operator* $T : \mathcal{L}_0^{1,p}(X) \to \mathcal{L}_0^{1,p}(X)$ by

$$Tu(x) = \begin{cases} 0 & \text{if } u(x) < 0 \ , \\ u(x) & \text{if } 0 \le u(x) \le 1 \ , \\ 1 & \text{if } u(x) > 1. \end{cases}$$

By Axiom A4, the operator $T$ does not increase the Dirichlet energy, therefore

**Proposition 7.3** *We have* $\mathrm{Cap_p}(F,X) = \inf\{\mathcal{E}_p(u) | \, u \in T(\mathcal{A}_p'(F,X))\}$.

□

Recall that a subset $F \subset X$ is *strongly bounded* if there exists a pair of open $\mathcal{K}$-sets $F \subset \Omega_1 \subset \Omega_2 \subset X$ such that $\mu(\Omega_2 \setminus X) > 0$ and $\operatorname{dist}(\Omega_1, X \setminus \Omega_2) > 0$.

**Proposition 7.4** *Suppose that $X$ is $p-$hyperbolic and $\mathcal{K}$-countable and that $C(X)$ is dense in $W^{1,p}(X)$. If $F \subset X$ is strongly bounded then*

$$T(\mathcal{A}_p'(F,X)) \subset \mathcal{A}_p''(F,X) \subset \mathcal{A}_p'(F,X).$$

**Proof** The inclusion $T(\mathcal{A}_p'(F,X)) \subset \mathcal{A}_p''(F,X)$ follows from Proposition 7.1.
To prove the inclusion $\mathcal{A}_p''(F,X) \subset \mathcal{A}_p'(F,X)$, we have to show that for any function $u \in \mathcal{A}_p''(F,X)$ and for any $\eta > 0$, we can find a function $\tilde{u} \in \mathcal{A}_p(F,X)$ such that $\|u - \tilde{u}\|^p_{\mathcal{L}^{1,p}(X,Q)} \le \eta$. Since $X$ is $p-$hyperbolic, we know that $(\mathcal{E}_p(\,))^{1/p}$ is equivalent to the norm $\| \, \|^p_{\mathcal{L}^{1,p}(X,Q)}$

in the space $\mathcal{L}^{1,p}(X)$ (see Theorem 2.5). It is therefore enough to construct a function $\tilde{u} \in \mathcal{A}_p(F,X)$ such that $(\mathcal{E}_p(u-\tilde{u}))$ is arbitrarily small.
Because $F$ is strongly bounded there exists a pair of open $\mathcal{K}$- sets $F \subset \Omega_1 \subset \Omega_2 \subset X$ such that $\Omega_2 \neq X$ and $\delta := \operatorname{dist}(\Omega_1, X \setminus \Omega_2) > 0$. Applying Theorem 4.1 and using the density of continuous functions in $\mathcal{L}_0^{1,p}(X)$, we know that for any $\varepsilon > 0$ there exists a continuous functions $v$ and an open sets $U \subset \Omega_2$ with the following properties:

i) $C_p(U, \Omega_2) < \varepsilon^p$;

ii) $(\mathcal{E}_p(u-v))^{1/p} < \varepsilon$;

iii) $|v(x) - u(x)| < \varepsilon/2$ for all $x \in \Omega_2 \setminus U$.

From these conditions and the continuity of $v$, we deduce that $v \geq (1-\varepsilon/2)$ on $(F \setminus U)$. Let us set $w := v/(1-\varepsilon)$, then $w(x) \geq 1$ in some neighbourhood of $(F \setminus U)$ and

$$(\mathcal{E}_p(w-v))^{1/p} = \left(\frac{\varepsilon}{1-\varepsilon}\right)(\mathcal{E}_p(v))^{1/p} \leq \left(\frac{\varepsilon}{1-\varepsilon}\right)\left((\mathcal{E}_p(u))^{1/p} + \epsilon\right) \leq \beta\,\varepsilon$$

where $\beta$ is some constant depending on $u$.
By Corollary 3.5 we have

$$\mathrm{Cap_p}(F \cap U, X) \leq \alpha^p\, C_p(F \cap U, \Omega_2) < \alpha^p\, \varepsilon^p$$

where $\alpha := 2\left(1 + \frac{3}{\delta}\right)$; and thus, by definition of the variational $p$-capacity, there exists $\varphi \in \mathcal{A}_p(F,X)$ such that $\varphi(x) \geq 1$ in some neighbourhood of $F \cap U$ and $\mathcal{E}_p(\varphi) < \alpha^p \varepsilon^p$.
Since $w$ and $\varphi$ are nonnegative the function $\tilde{u} := (w + \varphi) \geq 1$ on a neighbourhood of $F$, hence $\tilde{u} \in \mathcal{A}_p(F,X)$. On the other hand we have

$$(\mathcal{E}_p(\tilde{u}-u))^{1/p} \leq (\mathcal{E}_p(w-v))^{1/p} + (\mathcal{E}_p(v-u))^{1/p} + (\mathcal{E}_p(\varphi))^{1/p} < (1+\beta+\alpha)\,\varepsilon\,.$$

□

**Corollary 7.5** *Under the conditions of the previous Proposition, we have*

$$\mathrm{Cap_p}(F,X) := \inf\left\{\, \mathcal{E}_p(u) \,|\, u \in \mathcal{A}_p''(F,X) \,\right\}\,.$$

**Proof** This follows from the two previous Propositions.

□

The situation for the Sobolev capacity is similar; recall that $\mathrm{C}_p(F,\Omega)$ is the infimum of the Sobolev norm $\|u\|_{W^{1,p}}^p$ of all functions $u \in \mathcal{B}_p(F,X)$ where

$$\mathcal{B}_p(F,X) := \{u \in W^{1,p}(X) \big|\; u \geq 0 \text{ and } u \geq 1 \text{ on a neighborhood of } F\}\,.$$

If we define $\mathcal{B}_p'(F,X)$ to be the closure of $\mathcal{B}_p(F,X)$ in $W^{1,p}(X)$ and $\mathcal{B}_p''(F,X)$ to be the set of those functions $u \in W^{1,p}(X)$ such that $0 \leq u \leq 1$ and $u = 1$ $p$-quasi-everywhere on $F$. Then all results of this section hold for the sets $\mathcal{B}_p$, $\mathcal{B}_p'$ and $\mathcal{B}_p''$.
The equivalent result to Proposition 7.4 is more generally true for any subset $F \subset X$, i.e. without assumptions that $F$ is strongly bounded, because the Corollary 3.5 used in the proof to compare the variational and the Sobolev capacity is not needed.

# 8 The Choquet Property

The abstract notion of capacity was introduced by Gustave Choquet (see [8] or [9]):

**Definition 8.1** We say that a set function $\mathrm{Ch} : 2^X \to \mathbb{R}$ defined in $X$ is a *Choquet capacity* if it satisfies the following conditions:

i) Ch is monotone : $A \subset B \Rightarrow \mathrm{Ch}(A) \leq \mathrm{Ch}(B)$;

ii) If $X \supset K_1 \supset K_2 \supset \ldots.$ is a decreasing sequence of compact sets, then

$$\lim_{i\to\infty} \mathrm{Ch}(K_i) = \mathrm{Ch}(\cap_{i=1}^{\infty} K_i);$$

iii) If $A_1 \subset A_2 \subset \ldots. \subset X$ is an arbitrary increasing sequence of non empty sets, then

$$\lim_{i\to\infty} \mathrm{Ch}(A_i) = \mathrm{Ch}(\cup_{i=1}^{\infty} A_i)\,.$$

**Theorem 8.1** *If $1 < p < \infty$, then the Sobolev capacity $F \to C_p(F, X)$ is a Choquet capacity.*

**Proof** Because of Proposition 3.1, we only need to prove that condition (iii) holds. The proof is given in [30] in the case of capacity relative to Hajłasz Sobolev space; however, the same proof works for all capacities relative to any axiomatic Sobolev space. □

For variational capacities, the situation is more complex; we first define a local version of the Choquet condition :

**Definition 8.2** We say that a set function $\mathrm{Ch} : 2^X \to \mathbb{R}$ defined in $X$ is a *Choquet capacity relatively to strongly bounded subsets* if it satisfies the conditions (i) and (ii) above as well as

iii') If $A_1 \subset A_2 \subset \ldots. \subset X$ is an increasing sequence of non empty sets such that $A := \cup_{i=1}^{\infty} A_i$ is strongly bounded, then

$$\lim_{i\to\infty} \mathrm{Ch}(A_i) = \mathrm{Ch}(\cup_{i=1}^{\infty} A_i)\,.$$

**Theorem 8.2** *Suppose that $X$ is $\mathcal{K}$-countable and $C(X)$ is dense in $W^{1,p}(X)$. If $1 < p < \infty$, then the variational $p$-capacity $F \to \mathrm{Cap_p}(F, X)$ is a Choquet capacity relatively to strongly bounded subsets.*

For an arbitrary subset $F \subset X$, we define

$$P(F) := D[\mathcal{A}_p''(F, X)] = \{g \in L^p(X) : g \in D[u] \text{ for some } u \in \mathcal{A}_p''(F, X)\}\,.$$

Under the hypothesis of the theorem, we have

$$\mathrm{Cap_p}(F, X) = \inf_{g\in P(F)} \|g\|^p_{L^p(X)}\,.$$

for any strongly bounded subset $F \subset X$. This identity is proved in Corollary 7.5 for $p$-hyperbolic metric spaces $X$ and for $p$-parabolic spaces it is trivial.
We will need the following two lemmas.

**Lemma 8.3** *For any increasing sequence of arbitrary sets* $A_1 \subset A_2 \subset \ldots \subset X$ *we have*

$$\bigcap_m P(A_m) \subset P(\bigcup_m A_m).$$

We will use the lighter notation $\mathcal{A}''(F)$ for $\mathcal{A}''_p(F,X)$.
**Proof** Let $g \in \bigcap_m P(A_m)$. By definition of $P(A_m)$, there exists a function $u_m \in \mathcal{A}''(A_m)$ such that $g \in D[u_m]$. Let us set $v_m := \sup_{1 \le k \le m} u_k$. Then $g \in D[v_m]$ for all $m$ by axiom A4. Set $v := \sup_{1 \le m \le \infty} u_m = \lim_{m \to \infty} v_m$. Then $0 \le v(x) \le 1$ for all $x$ and $v(x) = 1$ $p$-quasi-everywhere on $\bigcup_m A_m$, hence by definition $v \in \mathcal{A}''(\cup A_m)$.
Since $v$ is bounded, it belongs to $L^p_{loc}(X)$. Furthermore, $v$ is the monotone limit of the sequence of non-negative functions $v_m$; hence, by Fatou lemma, $v_m$ converges to $v$ in $L^p_{loc}(X)$. Applying axiom A5, we conclude that $g \in D[u]$ and hence $g \in P(\cup_m A_m)$. □

**Lemma 8.4** *If $X$ is $\mathcal{K}$-countable and $p$-hyperbolic for some $1 < p < \infty$, then for any subset $F \subset X$, the set $P(F)$ is convex and closed in $L^p(X)$.*

**Proof** Let $f, g \in P(F)$. If $h = \lambda f + (1-\lambda)g$ for some $0 \le \lambda \le 1$, then, by definition of $P(F)$ there exist $u, v \in \mathcal{A}''_p(F,X)$ such that $f \in D[u]$ and $g \in D[v]$. By convexity, of $\mathcal{A}''_p(F,X)$ we have $\lambda u + (1-\lambda)v \in \mathcal{A}''_p(F,X)$, thus $h \in D[\lambda u + (1-\lambda)v] \subset P(F)$ by Axiom A2; this shows that $P(F)$ is convex.
To show that $P(F)$ is closed in $L^p(X)$, we need to prove that for any sequence $\{g_n\} \subset P(F)$ such that $g_n \to g_0 \in L^p(X)$, we have $g_0 \in P(F)$.
Since $X$ is $\mathcal{K}$-countable we can find an exhaustion of $X$ by open sets $\{U_m\}_{m \in \mathbb{N}} \subset \mathcal{K}$.
For each $n$, we have $g_n \in D[u_n]$ for some $u_n \in \mathcal{A}''_p(F,X)$ and by Theorem 2.5 (assertion 1), we know that, for each $m$, there exists a constant $C_m$ such that

$$\|u_n\|_{L^p(U_m)} \le C_m \mathcal{E}_p^{1/p}(u_n) \le C_m \|g_n\|_{L^p(X)}. \tag{7}$$

As $\|g_n\|_{L^p(X)}$ is bounded, this inequality implies that the sequence $\{u_n\}$ is bounded in $L^p(U_m)$ for all $m$; thus the pairs of functions $(u_n, g_n)$ is a bounded sequence in the direct product $S_m := L^p(U_m) \times L^p(X)$.
Since $S_m$ is a reflexive Banach space, we may assume (passing to a subsequence if necessary) that the sequence of restrictions $\{(u_n|_{U_m}, g_n)\}$ has a weak limit $(v_m, g_0) \in S_m$.
Using Mazur's lemma we can find for each $n \ge m$ a collection of numbers $\alpha^m_{n,1}, \alpha^m_{n,2}, \ldots \alpha^m_{n,n} \ge 0$ such that $\sum_{s=1}^n \alpha^m_{n,s} = 1$ and the sequence of convex combinations $w_{m,n} := \sum_{s=1}^n \alpha^m_{n,s} u_s$ converges strongly to $v_m$ in $L^p(U_m)$ and the sequence $f_{m,n} := \sum_{s=1}^n \alpha^m_{n,s} g_s$ converges strongly to $g_0 \in L^p(X)$ as $n \to \infty$ ($m$ being fixed). Let us observe that, since $g_s \in D[u_s]$ for all $s$, axiom A2 implies that $f_{m,n} \in D[w_{m,n}]$ for all $m, n$.
Let us choose for each $m$ a number $n_m \in \mathbb{N}$ such that $\|w_{n,m_n} - v_m\|_{L^p(U_m)} < 1/m$, and consider the diagonal subsequence $\tilde{w}_m := u_{m,n_m}$; it is clear that $\tilde{w}_m$ converges in $L^p_{loc}(X)$-topology to a function $v_0 \in L^p_{loc}(X)$ such that $v_0|_{U_m} = v_m$ for all $m$. Since $\tilde{f}_m := f_{m,n_m} \to g_0$ (in

$L^p(X))$, we conclude from axiom A5 that $g_0 \in D[v_0]$. By convexity we have $\tilde{w}_m \in \mathcal{A}_p''(F, X)$ for all $m$ and thus, by Proposition 7.2 $v_0 = \lim w_m \in \mathcal{A}_p''(F, X)$. We have proved that $g_0 \in D[v_0] \subset P(F)$.

□

**Proof of Theorem 8.2** If the space $X$ is $p$-parabolic, then we know by Theorem 2.5 that $1 \in \mathcal{L}_0^{1,p}(X)$, therefore $\mathrm{Cap_p}(A, X) = 0$ for any set $A \subset X$ and the Theorem is trivial.
We thus assume $X$ to be $p$-hyperbolic. By Theorem 2.4 it is enough to prove the property (iii'), i.e. that $\lim_{i\to\infty} \mathrm{Cap_p}(A_m, X) = \mathrm{Cap_p}(\cup_{m=1}^{\infty} A_m, X)$ for any sequence of non empty sets $A_1 \subset A_2 \subset ...$ such that $A := \cup_m A_m$ is strongly bounded.
The inequality $\lim_{m\to\infty} \mathrm{Cap_p}(A_m, X) \leq \mathrm{Cap_p}(A, X)$ immediately follows from the monotonicity of the variational $p$-capacity; it thus only remains to prove the converse inequality : $\mathrm{Cap_p}(A, X) \leq \lim_{m\to\infty} \mathrm{Cap_p}(A_m, X)$. If $\lim_{m\to\infty} \mathrm{Cap_p}(A_m, X) = \infty$ there is nothing to prove and we may therefore assume $\lim_{m\to\infty} \mathrm{Cap_p}(A_m, X) < \infty$. Set $\gamma := \mathrm{Cap_p}(A, X)$, fix some $\epsilon > 0$ and define the set of functions

$$P_m := \{g \in P(A_m) \, : \, \|g\|_{L^p(X)}^p \leq \gamma + \epsilon\} \subset L^p(X) \, .$$

This set is clearly non empty since $\inf_{g\in P_m} \|g\|_{L^p(X)}^p = \mathrm{Cap_p}(A_m, X) \leq \gamma + \epsilon$. By the previous lemma, $P_m$ is a non empty closed convex subset of the reflexive Banach space $L^p(X)$. Therefore $P_m \supset P_{m+1} \supset \cdots$ is a nested sequence of non empty weakly compact subsets of $L^p(X)$ and $P := \cap P_m$ is thus non empty by Cantor's theorem.
By Lemma 8.3 we have $\cap_m P_m \subset P(A)$. Because $A$ is strongly bounded, we have by Corollary 7.5

$$\mathrm{Cap_p}(A, X) = \inf_{g\in P(A)} \|g\|_{L^p(X)}^p \leq \inf_{g\in\cap P_m} \|g\|_{L^p(X)}^p \leq \gamma + \epsilon \, .$$

Since $\epsilon$ is arbitrary, this inequality implies $\mathrm{Cap_p}(A, X) \leq \gamma = \lim_{m\to\infty} \mathrm{Cap_p}(A_m, X)$.

□

**Corollary 8.5** *Suppose that $F \subset X$ is a strongly bounded Borel set which is contained in a countable union of compact sets, then*

$$\mathrm{Cap_p}(F) = \sup\{\mathrm{Cap_p}(K) |\ K \subset F \ \text{ a compact subset}\} \, .$$

**Proof** This result follows from Theorem 8.2 and Choquet's capacitability theorem (see [8] or [9, theorem 9.3]).

□

**Remark.** The proof of the Choquet property for classical Sobolev spaces on Riemannian manifolds is much simpler (see e.g. the proof of Theorem 2(viii) in [11, chapter2] where the argument is given for $\mathbb{R}^n$). The classical proof uses the fact that if $w := \max(u, v)$, then $\nabla w = \nabla u$ a.e. on the set $\{u \geq v\}$. This fact is not guaranteed by our axioms and this is the main source of complication in the proof.

# 9 Fat sets

**Definition 9.1** *A Borel measure* $\tau$ *on* $X$ *is said to be* absolutely continuous with respect to p-capacity *if* $\tau(S) = 0$ *for all p-polar subsets* $S \subset X$.

For any Borel subset $F \subset X$ we denote by $\mathcal{M}_p(F)$ the set of all probability measures $\tau$ on $X$ which are absolutely continuous with respect to $p$-capacities and whose support is contained in $F$.

**Definition 9.2** A subset $F$ is said to be *p−fat* if it is a Borel subset and $\mathcal{M}_p(F) \neq \emptyset$.

For instance any measurable subset $F \subset \mathbb{R}^n$ such that $\mu(F) > 0$ is $p$−fat. On the other hand, a $p$−polar set is never $p$−fat.
In a Riemannian manifold $M$, any Borel subset $F \subset M$ is either $p$−polar or $p$−fat (we will give a proof of this fact in §11.1).
The next result gives us a geometric criterion to check if a set is $p$−polar or $p$−fat in the context of Hajłasz theory.
Let us recall first that a metric space $X$ is said to be *locally s-regular* if for each $x \in X$, there exists $c, R > 0$ such that

$$\frac{r^s}{c} \leq \mu(B(x,r) \leq cr^s$$

for all $0 \leq r \leq R$.

**Theorem 9.1** *Suppose that the space* $X$ *is locally s-regular and consider capacities with respect to Hajłasz-Sobolev space. If* $1 < p < s$, *then*

*i) If* $\mathcal{H}^{s-p}(F) = 0$, *then* $F$ *is p−polar;*

*ii) If* $F$ *contains a subset* $A$ *such that* $0 < \mathcal{H}^t(A) < \infty$ *for some* $t > (s-p)$, *then* $F$ *is p−fat and* $\tau := \frac{1}{\mathcal{H}^t(A)}\mathcal{H}^t \llcorner A$ *belongs to* $\mathcal{M}_p(F)$.

The proof of this theorem is given in [29, Theorems 4.13, 4.15].

□

# 10 The extremal function

We now prove the existence and uniqueness of an extremal function for the variational $p$-capacity of an arbitrary $p$−fat subset $F \in \mathcal{K}$.

**Theorem 10.1** *Let* $(X,d)$ *be a* $\mathcal{K}$*-countable measure metric space and* $F \subset X$ *be a p−fat subset* $(1 < p < \infty)$. *Then there exists a unique function* $u^* \in \mathcal{L}_0^{1,p}(X)$ *such that* $u^* = 1$ *p-quasi-everywhere on* $F$ *and* $\mathcal{E}_p(u^*) = \mathrm{Cap_p}(F,X)$. *Furthermore* $u^*$ *is p−quasi-monotone on* $X \setminus \overline{F}$ *and* $0 \leq u^*(x) \leq 1$ *for all* $x \in X$.

This extremal function $u^*$ is called the *capacitary function* or the *equilibrium potential* of the condenser $F \subset X$.
Recall that the notion of $p-$quasi-monotone function was defined in 4.2.
For the proof of this Theorem, we need the following Lemma. Recall that a Banach space $E$ is *uniformly convex* if for every $\epsilon > o$ there exists a $\delta > 0$ such that if $x, y \in E$ with $\|x\| = \|x\| = 1$

$$\|y - x\| \geq \epsilon \qquad \Rightarrow \qquad \|\frac{1}{2}(x+y)\| \leq (1-\delta).$$

**Lemma 10.2** *In any nonempty closed convex subset $A \subset E$ of a uniformly convex Banach space $E$, there exists a unique element $x^* \in A$ with minimal norm: $\|x^*\| = \inf_{x \in A} \|x\|$.*

The proof can be found in [17] or [25].

□

**Proof of Theorem 10.1** Let us choose a measure $\tau \in \mathcal{M}_p(F)$ and set $E := L^p(X, d\tau) \oplus L^p(X, d\mu)$. Then $E$ is a uniformly convex Banach space for the norm

$$\|(u,g)\|_E := \left(\int_X |u|^p d\tau + \int_X |g|^p d\mu\right)^{1/p}.$$

Let us set $A := \{(u,g) \in E \,|\, u \in T(\mathcal{A}'_p(F,X)) \text{ and } g \in D[u]\}$. Then $A$ is a convex closed subset of $E$, and thus, by Lemma 10.2, we know that there exists a unique element $(u^*, g^*) \in A$ which minimizes the norm. It is clear that $g^*$ is the minimal pseudo-gradient of $u^*$, i.e. that $\mathcal{E}_p(u^*) = \int_X |g^*|^p d\mu$.
We assert that $\mathcal{E}_p(u^*) = \mathrm{Cap_p}(F)$. Indeed, if $\mathcal{E}_p(u^*) > \mathrm{Cap_p}(F)$, then, by Proposition 7.3, one can find $(u,g) \in A$ such that $\int_X |g|^p d\mu < \int_X |g^*|^p d\mu$. Since $u, u^* \in T(\mathcal{A}'_p(F,X))$, we may assume that $u = u^* = 1$ $p$-quasi everywhere on $F$ (see Proposition 7.1) and thus $u = u^* = 1$ $\tau$-almost everywhere on $F$ because $\tau$ is absolutely continuous with respect to $p$-capacity. Therefore

$$\|(u,g)\|_E = \left(1 + \int_X |g|^p d\mu\right)^{1/p} < \left(1 + \int_X |g^*|^p d\mu\right)^{1/p} = \|(u^*, g^*)\|_E$$

which contradicts the minimality of $(u^*, g^*)$. The quasi-monotonicity of $u$ can be proved by a simple truncation argument.

□

## The case of condensers

We define a *condenser* in $X$ to be a pair of disjoint non empty sets $F_1, F_2 \in \mathcal{K}$. The variational $p$-capacity of such a condenser is defined by

$$\mathrm{Cap_p}(F_1, F_2, X) := \inf\{\mathcal{E}_p(u) \,|\, u \in \mathcal{A}_p(F_1, F_2, X)\}$$

where $\mathcal{A}_p(F_1, F_2, X)$ is the set of all functions $u \in \mathcal{L}^{1,p}(X)$ such that $u \geq 1$ on a neighbourhood of $F_1$ and $u \leq 0$ on a neighbourhood of $F_2$.

**Theorem 10.3** *Let $F_1, F_2 \subset X$ be any condenser in a $\mathcal{K}$-countable metric space $X$ such that either $F_1$ or $F_2$ is p−fat. Then there exists a unique function $u^* \in \mathcal{L}_0^{1,p}(X)$ such that $u^* = 1$ p-quasi-everywhere on $F_1$, $u^* = 0$ p-quasi-everywhere on $F_2$ and $\mathcal{E}_p(u^*) = \mathrm{Cap_p}(F_1, F_2, X)$. Furthermore $u$ is monotone and $0 \leq u \leq 1$.*

The proof is similar to that of Theorem 10.1 and we omit it.

□

# 11 The case of Riemannian Manifolds

## 11.1 Polar sets in Riemannian manifolds

From Proposition 3.7, we immediately have :

**Proposition 11.1** *A compact subset $S$ of a Riemannian manifold $M$ is p-polar if and only if $C_p(S) = 0$.*

□

In this section we give a proof of the following

**Theorem 11.2** *A Borel subset $F \subset M$ of a Riemannian manifold is either p−polar or p−fat.*

**Proof** Observe first that if $p > n$ (= dimension of $M$), then the only $p$-polar set is the empty set (see [37] or [22]), thus every measure on $M$ is absolutely continuous with respect to $p$-capacity and, therefore, any probability measure supported on a Borel set $F$ belongs to $\mathcal{M}_p(F)$. Thus every non empty Borel set is $p$-fat.
We may thus assume $p \leq n$. By Choquet's theorem, we know that if $F \subset M$ is a non $p$-polar subset, then it contains a compact subset $K$ such that $C_p(K) > 0$.
Since being $p$−fat is clearly a local property which is stable under diffeomorphisms, it is enough to prove this theorem for subset of Euclidean space.
For a compact subset $K \subset \mathbb{R}^n$, the Bessel capacitary measure $\sigma_{p,K}$, suitably renormalized, belongs to $\mathcal{M}_p(F)$. Let us be more specific.
We first recall some facts about Bessel potentials, basic references are [1], [38] and [46]. The Bessel kernel is defined by $G_\alpha := \mathcal{F}^{-1}((1 + |\xi^2|^{-\alpha/2})$ where $\mathcal{F}$ is the Fourier transform. The Bessel kernel has two important basic properties: first we have the convolution rule

$$G_\alpha * G_\beta = G_{\alpha+\beta}$$

and secondly, the Bessel potential inverts the operator $(I - \Delta)^{\alpha/2}$ (where $\Delta$ is the Laplacian), i.e.

$$v = (I - \Delta)^{\alpha/2} u \qquad \Leftrightarrow \qquad u = G_\alpha * v\,.$$

The Bessel potential space $B^{\alpha,p} = B^{\alpha,p}(\mathbb{R}^m)$ is defined by

$$B^{\alpha,p}(\mathbb{R}^m) := \{u |\, u = G_\alpha * v,\ v \in L^p(\mathbb{R}^m)\}\,,$$

and the norm in $B^{\alpha,p}$ is given by $\|u\|_{B^{\alpha,p}} = \|G_\alpha * v\|_{B^{\alpha,p}} := \|v\|_p$ (so that the operator $\mathcal{G}_\alpha : L^p(\mathbb{R}^m) \to B^{\alpha,p}(\mathbb{R}^m)$ defined by $\mathcal{G}_\alpha(v) = G_\alpha * v$ is an isometry).
The following important theorem of Calderon allows us to use Bessel spaces instead of Sobolev spaces in the study of $p$-polar subsets of $\mathbb{R}^n$.

**Theorem 11.3** *For $\alpha \in \mathbb{N}$ and $1 < p < \infty$, we have $W^{\alpha,p}(\mathbb{R}^m) \simeq B^{\alpha,p}(\mathbb{R}^m)$ with equivalent norms.*

The Bessel $p$-capacity of a compact subset $K \subset \mathbb{R}^n$ is defined as:

$$B_{(1,p)}(K) := \inf\left\{ \|u\|^p_{B^{1,p}} \;\middle|\; u \in A_k \right\} .$$

where $A_K := \{u \in C^1_0(\mathbb{R}^n) \ : \ u \geq 1 \text{ on } K\}$. This is a Choquet capacity and there are constants $c_1, c_2 > 0$ depending only on $p$ and $n$ such that

$$c_1 B_{(1,p)}(K) \leq C_p(K) \leq c_2 B_{(1,p)}(K) \tag{8}$$

for all compact subset $K \subset \mathbb{R}^n$, where $C_p(K)$ is the Sobolev $p$-capacity.
Using uniform convexity arguments, we obtain the following theorem (see [38] or [1, Theorems 2.2.7 and Proposition 6.3.13]) :

**Theorem 11.4** *Assume $1 < p < \infty$. Given a non polar compact subset $K \subset \mathbb{R}^n$, there is a unique measure $\tau$ with the following properties:*

*1) $\tau$ is a probability measure supported on $K$;*

*2) $u_K := \frac{1}{B_{(1,p)}(K)} G_\alpha * (G_\alpha * \tau)^{1/(p-1)} \in \overline{A}_K$;*

*3) $\|u_K\|^p_{(1,p)} = B_{(1,p)}(K)$;*

*4) $\tau$ is absolutely continuous with respect to Bessel capacity.*

($\overline{A}_K$ denotes the closure of $A_K$ in $B^{1,p}(\mathbb{R}^m)$).
The function $u_K$ is the extremal function for the Bessel capacity of $K$; the measure $\sigma_{p,K} = B_{(1,p)}(K)\,\tau$ is called the *Bessel capacitary measure* of $K$.
In view of this theorem and the inequalities (8), the proof of Theorem 11.2 is complete. □

## 11.2 Existence of extremal function

Let $(M, g)$ be a Riemannian manifold, recall that a function $u \in W^{1,p}_{loc}(M)$ is called weakly *$p$-harmonic* if $\Delta_p u = 0$ where $\Delta_p$ is the $p$-Laplacian defined by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$; the function $u$ is thus weakly $p$-harmonic if and only if

$$\int_M \langle |\nabla u|^{p-2}\, \nabla u, \nabla \psi \rangle = 0$$

for any $\psi \in W^{1,p}(M)$ (where $\nabla u$ is the weak gradient of $f$).

**Theorem 11.5** *Let $F$ be a compact non $p$-polar subset of the Riemannian manifold $M$. Then there exists a unique function $u^* \in \mathcal{L}_0^{1,p}(M)$ such that*

*a) $u^* = 1$ $p-$quasi-everywhere on $F$;*

*b) $0 \leq u^*(x) \leq 1$ for all $x \in M$ and $u^*$ is monotone;*

*c) $\int_M |\nabla u^*|^p = \mathrm{Cap_p}(F)$;*

*d) $u^*$ is weakly $p-$harmonic in the exterior domain $M \setminus \overline{F}$.*

*Furthermore, $u^* = 1$ $p-$quasi everywhere on $M$ if $M$ is $p-$parabolic and $0 < u^*(x) < 1$, $p-$quasi everywhere in $M \setminus \overline{F}$ if $M$ is $p-$hyperbolic.*

**Proof** Since every non $p-$polar set is $p-$fat in a Riemannian manifold, the existence of a function satisfying (a), (b) and (c) is a consequence of Theorem 10.1. Property (d) is clear since $\Delta_p$ is the Euler-Lagrange operator associated to the Dirichlet energy. □

**Remark** A generalization of condition (d) also holds in the case of subriemannian manifolds, see e.g. [6, Proposition 6.1].

## 11.3 Regularity in $M \setminus F$

The previous existence theorem is completed by the following regularity result:

**Theorem 11.6** *Let $u^* \in \mathcal{L}^{1,p}(M)$ be the p-capacitary function of $F \subset M$. Then for each relatively compact domain $\Omega \subset M \setminus F$, there exists $0 < \alpha < 1$ such that $u \in C^{0,\alpha}(\Omega)$.*

The famous theorem of De Giorgi, Nash and Moser gives conditions under which weak solutions to elliptic partial differential equations are Hölder continuous. We present here an alternative argument, due to De Giorgi's, which is well adapted to our situation. The argument is based on the following lemma (which is a Caccioppoli type inequality):

**Lemma 11.7** *Let $\Omega \subset M$ be an open subset and $u \in \mathcal{L}^{1,p}(\Omega)$ be a bounded weak solution to $\Delta_p u = 0$. Then for any pair of concentric balls $B(x_0,\rho) \subset B(x_0,R) \subset \Omega$ and any constant $k \in \mathbb{R}$ one has*

$$\int_{B(x_0,\rho)\cap\{u(x)\geq k\}} |\nabla u|^p \, dx \leq \frac{c}{(R-\rho)^p} \left( \int_{B(x_0,R)\cap\{u(x)\geq k\}} |u-k|^p \, dx \right).$$

**Proof** By assumption we have

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \, dx = 0 \tag{9}$$

for any test function $\varphi \in W_0^{1,p}(\Omega)$. Let us choose a function $\eta \in C_0^1(B(x_0,R))$ such that $\eta \equiv 1$ on $B(x_0,\rho)$ and $|\nabla \eta| \leq \frac{2}{(R-\rho)}$ and set

$$\varphi(x) := \max\{u(x)-k, 0\} \cdot \eta(x)^p.$$

Observe that

$$\nabla\varphi = \begin{cases} \eta^p \nabla u + p(u-k)\eta^{p-1}\nabla\eta & \text{on } B(x_0,R)\cap\{u(x)\geq k\} \\ 0 & \text{elsewhere.} \end{cases}$$

Using $\varphi$ as test function in (9) one obtains

$$\int_{E_R} \eta^p |\nabla u|^p = -p\int_{E_R} (u-k)\eta^{p-1}|\nabla u|^{p-2}\langle \nabla u, \nabla\eta\rangle,$$

where we have conveniently set $E_R := B(x_0,R)\cap\{u(x)\geq k\}$.
Using Hölder's inequality we then get

$$\int_{E_R} \eta^p|\nabla u|^p \leq p\left(\int_{E_R}\eta^p|\nabla u|^p\right)^{\frac{p-1}{p}}\left(\int_{E_R}|u-k|^p|\nabla\eta|^p\right)^{\frac{1}{p}}.$$

Raising this inequality to the power $p$ gives

$$\int_{E_R}\eta^p|\nabla u|^p \leq p^p\int_{E_R}|u-k|^p|\nabla\eta|^p.$$

Finally, it follows from our assumptions on $\eta$ that

$$\int_{B(x_0,\rho)\cap\{u(x)\geq k\}}|\nabla u|^p \leq \left(\frac{2p}{R-\rho}\right)^p\int_{E_R}|u-k|^p,$$

this proves the lemma. □

**Proof of Theorem 11.6** It is known that any bounded function $u\in\mathcal{L}^{1,p}(\Omega)$ satisfying the conclusion of Lemma 11.7 is locally Hölder continuous (see chapter 2, §2 in [33] or §2.3.4 in [35]). The proof of the Theorem follows. □

**Remarks** (1) Observe that for $p>n$, the above statement is a direct consequence of Sobolev embedding's theorem.

(2) For the special case $(n-1)<p\leq n$, a different proof is given in [32].

(3) In fact, it is known that $u^*$ is locally $C^{1,\alpha}$ in the exterior domain $M\setminus F$, see [34] or [42].

(4) The continuity of extremal functions is also known for the case of weighted Sobolev spaces in subriemannian geometry (see Theorem 4.4 in [5]). There are also proofs of Hölder continuity for some Carnot groups, see e.g. [2, 3].

(5) Using the results and techniques of [31], it should be possible to prove continuity of extremal functions for a wide class of axiomatic Sobolev space (perhaps assuming that $D$ is local and $\mu$ is doubling)

## 11.4 Boundary Regularity

A point $x \in \partial F$ is a *Wiener point* of $F$ if

$$\int_0^\delta \left( \frac{\mathrm{Cap}_p(\bar{B}_{x,t} \cap F; B_{x,\delta})}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t} = \infty$$

for some $\delta$. One also says that the set $F$ is $p$-thin at $x$ if $x$ is not a Wiener point. We easily verify that a point $x \in \partial F$ satisfying an interior cone condition is a Wiener point. One says that $F \subset M$ is *Wiener regular* if all points of $\partial F$ are Wiener points; examples of Wiener regular subsets are polyhedral and $C^1$domains.

**Theorem 11.8** *Let $u^*$ be the p-capacitary function of $F \subset M$. If $x_0 \in \partial F$ is a Wiener point then $\lim_{x \to x_0} u^*(x) = 1$. In particular, if $F$ is Wiener regular, then $u^*$ is everywhere continuous.*

See [26] or [35, Corollary 4.18].

□

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