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The Kelvin–Nevanlinna–Royden criterion for p-parabolicity

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Abstract. We generalize the so called Kelvin–Nevanlinna–Royden criterion for the parabolicity of manifolds to the case of p-parabolicity for all $1 < p < \infty$.

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1. Introduction

Definition Let (M, g) be a connected Riemannian manifold, and $D \subset M$ a compact set. For $1 < p < \infty$, the p-capacity of D is defined by:

$$\mathrm{Cap}_p(D) := \inf \left\{ \int_M |\nabla u|^p : u \in C_0^1(M),\ u \geq 1 \text{ on } D \right\}$$

The manifold M is said to be p-parabolic if $\mathrm{Cap}_p(D) = 0$ for all compact subsets $D \subset M$ and p-hyperbolic otherwise. We refer to [11], and [9] for a discussion of p-capacity and [14] for a discussion of p-parabolicity.

Generalizing a result of H.L. Royden for surfaces, Terry Lyons and Denis Sullivan proposed the following criterion, which they called the Kelvin-Nevanlinna-Royden criterion, to check the $2-$hyperbolicity of an oriented manifold (see [10] and [13, th. 4] for the case of surfaces).

Theorem Let M be an oriented connected riemannian manifold without boundary. Then M is 2-hyperbolic if and only if there exists a vector field X on M such that

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– $X \in L^2$;
– $\operatorname{div}(X) \in L^1(M)$;
– $\int_M \operatorname{div}(X) > 0$.

The main goal of the present paper is to give the following generalization for p-hyperbolicity; along the way this will also give a new proof of the above theorem.

Theorem A. *The manifold M is p-hyperbolic if and only if there exists a vector field X on M such that*

– $X \in L^q$ *(where* $\frac{1}{p} + \frac{1}{q} = 1$*);*
– *div*$(X) \in L^1_{loc}$ *and* $(\mathit{div}(X))^- \in L^1(M)$*;*
– $0 < \int_M \mathit{div}(X) \leq \infty$.

Here $(\operatorname{div}(X))^-$ is the function $(\operatorname{div}(X))^- := \min\{(\operatorname{div}(X)), 0\}$.

If one looks at a vector field X as an $(n-1)-$form, then we are led to a cohomological criterion. Let us define the cohomology space $H^n_{\text{comp},q}(M)$ to be the space of n-forms with compact support modulo the differentials of all $(n-1)$-forms in L^q.

Theorem B. *The manifold M is p-hyperbolic if and only if $H^n_{comp,q}(M) = 0$ (where $\frac{1}{p} + \frac{1}{q} = 1$).*

We conclude the paper with some geometric and analytic applications of these criteria.

2. Some calculus on manifolds

Let (M, g) be an n-dimensional oriented riemannian manifold. The Hodge star operator is the isomorphism $* : \Lambda^k T^*M \to \Lambda^{(n-k)} T^*M$ defined by $\langle \alpha, \beta \rangle\, \omega = (-1)^{k(n-k)} (*\alpha) \wedge \beta$ where $\alpha \in \Lambda^k T^*M$ and $\beta \in \Lambda^k T^*M$. Here, ω is the volume form. The canonical isomorphisms between the tangent and cotangent bundle are denoted by $\flat : TM \to T^*M$ and $\natural : T^*M \to TM$. The gradient of a function f is the vector field $\nabla f := (df)^\natural$. The divergence of a vector field X is the function $\operatorname{div}(X) := *d(*X^\flat)$.

We note $L^1_{loc}(M, \Lambda^k)$ the space of locally integrable differential forms of degree k.

Let $\phi \in L^1_{loc}(M, \Lambda^{k-1})$ and suppose that there exist $\theta \in L^1_{loc}(M, \Lambda^k)$ with the following property: for each $\psi \in C^\infty_0(M, \Lambda^{n-k})$,

$$\int_M \theta \wedge \psi = (-1)^k \int_M \phi \wedge d\psi\,,$$

Then we say that θ is the *weak exterior differential* of ϕ (and write $d\phi = \theta$).

We can define by the same way a notion of weak divergence of a vector field or weak gradient of a function. For instance a locally integrable vector field ξ on M is the *weak gradient* of the function $f : M \to \mathbb{R}$ if and only if for any smooth vector fields η with compact support one has

$$\int_M \langle \xi, \eta \rangle = - \int_M f \operatorname{div}(\eta) \,.$$

In the sequel, differentiations are usually meant in the weak sense, however the following theorem allows us to regularize differential forms in a way that preserves some of their properties:

Theorem 1. *Let (M, g) be a Riemannian manifold. There exists a sequence of operators $R_i^k : L^1_{loc}(M, \Lambda^k) \to C^\infty(M, \Lambda^k)$ satisfying the following four conditions:*

1) *$dR_i^k \alpha = R_i^k d\alpha$ for all $\alpha \in L^1_{loc}(M, \Lambda^k)$ such that $d\alpha \in L^1_{loc}(M, \Lambda^{k+1})$*
2) *If α has compact support, then so does $R_i^k \alpha$;*
3) *$R_i^k : L^p(M, \Lambda^k) \to L^p(M, \Lambda^k)$ is a bounded operator for any $1 \leq p < \infty$ and $\left\| R_i^k \right\|_{L^p \to L^p} \leq 1 + 1/i$;*
4) *For any form $\alpha \in L^p(M, \Lambda^k)$ and any $1 \leq p < \infty$ the sequence $R_i^k \alpha$ converges to α in L^p topology.*

These operators dR_i^k are constructed by De Rham. Properties (1) and (2) are stated in [2, th. 12 page 68]; properties (3) and (4) are proven in [4]. $\square$

3. The p-Laplacian

The p-Laplacian of a C^2 function f is the function $\Delta_p f = \operatorname{div}(|\nabla f|^{p-2} \nabla f)$, it can equivalently be defined by

$$\Delta_p f \, \omega = d \left(* |df|^{p-2} df \right) \;;$$

it is the Euler-Lagrange operator associated with the functional $\int_M |df|^p$.

The p-Laplacian has also a generalized interpretation: we say that a function $f \in L^1_{loc}(M)$ is a weak solution to the equation

$$\Delta_p f + h = 0 \tag{1}$$

if f has a weak gradient $\nabla f \in L^1_{loc}$ such that

$$\int_M \left\langle |\nabla f|^{p-2} \, \nabla f, \nabla \psi \right\rangle = \int_M h \, \psi$$

for all $\psi \in C_0^\infty(M)$.

Equations such as $h = -\Delta_p f$ are usually meant in the weak sense; the same holds for inequalities; For instance $-\Delta_p f \geq 0$ means that

$$\int_M \left\langle |\nabla f|^{p-2} \nabla f, \nabla \psi \right\rangle \geq 0$$

for all $\psi \in C_0^\infty(M)$ such that $\psi \geq 0$. Such functions are called *supersolution* to the equation $-\Delta_p f = 0$, and they are called p-*superharmonic* if they are furthermore lower-semicontinuous.

Example The function $f(x) = a - b\,\|x\|^q$ on $\mathbb{R}^n$ (where $b > 0$ and $q = p/(p-1)$) satisfies $-\Delta_p f = n(bq)^{p-1} > 0$ and is thus p-superharmonic.

On p-hyperbolic manifolds, one can solve the p-Laplace equation in the Dirichlet space $\mathcal{L}^{1,p}(M)$:

Theorem 2. *Suppose that M is a p-hyperbolic manifold ($1 < p < \infty$) and that $h \in L^\infty(M)$ has compact support. Then (1) has a weak solution $f \in \mathcal{L}^{1,p}(M)$. Moreover f is of class $C^{1,\alpha}$ on each compact set (where $\alpha \in (0,1)$ may depend on the compact set).*

This theorem is proven in [15]. $\square$

Remark By contrast, the above result is false on a p-parabolic manifold, see Sect. 6.4 below.

4. A divergence criterion

In this section, we will prove Theorem A. We begin with a result about vector fields on p-hyperbolic manifolds.

Proposition 1. *If M is p-hyperbolic then there exists a vector field X on M such that*

- *$X \in L^q$ (where $\frac{1}{p} + \frac{1}{q} = 1$) and div$(X) \in L^1_{loc}$;*
- *div$(X) \geq 0$ and div$(X) \not\equiv 0$;*
- *div(X) has compact support.*

Proof. Choose a non negative smooth function $h : M \to \mathbb{R}$ with compact support such that $h > 0$ somewhere. Let $f \in \mathcal{L}^{1,p}(M)$ be a solution of (1) (known to exist by Theorem 2) and set $X := -|\nabla f|^{p-2} \nabla f$. Then $\mathrm{div}(X) = -\Delta_p f = h$ has the desired properties and $X \in L^q$ since

$$|X|^q = |\nabla f|^{q(p-1)} = |\nabla f|^p$$

is integrable. $\square$

Proof of Theorem A

It follows from Proposition 1 that if M is p-hyperbolic then there exists a vector field $X \in L^q(M)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) such that $\mathrm{div}(X) \not\equiv 0$ and $\mathrm{div}(X) \geq 0$ (in particular $(\mathrm{div}(X))^- \equiv 0$).

To prove the converse direction, let $h = \mathrm{div}(X)$. By hypothesis, there exists a bounded subset $D \subset M$ of positive measure such that

$$\gamma := \inf_D h > 0 \qquad \text{and} \qquad \int_D h > \left| \int_M h^- \right|$$

Choose a number $0 < c < 1$ be such that

$$0 \leq -\int_M h^- \leq c \cdot \int_D h \, ,$$

and a function $v \in C_0^1(M)$ such that

i) $0 \leq v \leq 1$;
ii) $v \equiv 1$ on D.
iii) $\displaystyle\int_M |\nabla v|^p \leq \mathrm{Cap}_p(D) + \varepsilon$,

where $\varepsilon > 0$ is an arbitrary fixed number.

We then have $0 \geq \displaystyle\int_M v\, h^- \geq -c \cdot \int_D v\, h$ and thus

$$\begin{aligned} (1-c) \cdot \int_D v\, h \ &\leq \int_D v\, h + \int_M v\, h^- \\ &\leq \int_D v\, h + \int_M v\, h^- + \int_{(M \setminus D)} v\, h^+ = \int_M v\, h \, . \end{aligned}$$

But $\int_D v\, h \geq \gamma \cdot \mathrm{Vol}(D)$, hence an integration by parts with Hölder inequality gives

$$\begin{aligned} \gamma(1-c) \cdot \mathrm{Vol}(D) &\leq \int_M v\, \mathrm{div}(X) = -\int_M \langle \nabla v, X \rangle \\ &\leq \|X\|_q \, \|\nabla v\|_p \, . \end{aligned}$$

Since $\|\nabla v\|_p \leq (\mathrm{Cap}_p(D) + \varepsilon)^{1/p}$ and ε is arbitrary, we conclude that

$$0 < \mathrm{Vol}(D) \leq \frac{\|X\|_q}{\gamma(1-c)} \cdot (\mathrm{Cap}_p(D))^{1/p} \, ,$$

hence M is p-hyperbolic. $\square$

Remark 1 The divergence of the vector field X in the statement of Theorem A is meant in the weak sense. However, using the identification between vector fields and $(n-1)$-forms on M, we can apply Theorem 1 to X. In other words, we may assume that the vector field X of Theorem A is smooth.

Remark 2 By a theorem of Gaffney, we know that if X is a vector field on a *complete* riemannian manifold M such that $X \in L^1$ and $\operatorname{div}(X) \in L^1$, then $\int_M \operatorname{div}(X) = 0$ (see [3]). In some sense one can say that a complete manifold is always p-parabolic for $p = \infty$.

5. A cohomological criterion

Let M be a connected oriented n-dimensional non compact manifold. Then $H^n(M) = 0$ whereas $H^n_{\mathrm{comp}}(M) = \mathbb{R}$ where $H^n_{\mathrm{comp}}(M)$, the cohomology with compact support, is the space of smooth n-forms with compact support modulo the differentials of smooth $(n-1)$-forms with compact support.

Proposition 2. *Let M be an oriented connected manifold.*

Then $H^n_{comp}(M)$ is of dimension 1 and integration defines an isomorphism

$$H^n_{comp}(M) \to \mathbb{R}$$
$$\alpha \to \int_M \alpha$$

A good reference for this and other results on cohomology with compact support is the book of Bott and Tu [1]. □

We can define other cohomologies to capture some intermediate behaviour between the De Rham cohomology (which considers all forms) and the cohomology with compact support. The first space of this kind is the $L_{r,q}$ cohomology space $H^n_{r,q}(M)$, this is the space of n-forms $\alpha \in L^r$ modulo the differentials of $(n-1)$-forms $\beta \in L^q$. We refer to [12] and [6] for a discussion of this notion.

Another cohomology is the space $H^n_{\mathrm{comp},q}(M)$, of smooth n-forms with compact support modulo the differentials of $(n-1)$-forms in $L^q \cap C^\infty$.

Let us recall the statement of our second theorem.

Theorem B. *The manifold M is p-hyperbolic if and only if $H^n_{comp,q}(M) = 0$ (where $\frac{1}{p} + \frac{1}{q} = 1$).*

We have the immediate

Corollary 1. *Suppose that M is p-parabolic. Then $H^n_{r,q}(M) \neq 0$ for all $1 \leq r \leq \infty$.*

This is obvious since $H^n_{\mathrm{comp},q}(M) \subset H^n_{r,q}(M)$. □

The proof of the theorem will be based on the following capacity estimate.

Lemma 1. *Let M be a connected oriented riemannian manifold (perhaps with boundary) and suppose that there exist a smooth $(n-1)$-form $\beta \in L^q$ such that $*d\beta \geq 0$ and $\beta|_{\partial M} = 0$. Then for all compact subset $D \subset M \backslash \partial M$ we have*

$$\mathit{Vol}(D) \cdot \min_D (*d\beta) \leq \|\beta\|_{L^q} \left(\mathit{Cap}_p(D)\right)^{1/p}.$$

(Recall that $\lambda := *d\beta$ is the function defined by $d\beta = \lambda\omega$ where ω is the volume form).

Proof. Choose a smooth function v as in the proof of Theorem A (i.e. v has compact support, $v \equiv 1$ on D, $0 \leq v \leq 1$ and $\int |dv|^p \leq \mathrm{Cap}_p(M) + \epsilon$) and let $\gamma = \inf_D \lambda$. Then

$$\begin{aligned} \gamma\,\mathrm{Vol}(D) &= \gamma \int_D v\,\omega \leq \int_D v\,d\beta \leq \int_M v\,d\beta \\ &\overset{\text{Stokes}}{=} \int_{\partial M} v\,\beta - \int_M dv \wedge \beta = \int_M \beta \wedge dv \\ &\leq \|\beta\|_{L^q} \|dv\|_{L^p}. \end{aligned}$$

The lemma follows. $\square$

Proof of Theorem B

Suppose $H^n_{\mathrm{comp},q}(M) = 0$ for some q. Choose a function $\varphi \in C_0^\infty(M)$ such that $\varphi \geq 0$ and $\varphi \not\equiv 0$; then $[\varphi\omega] = 0 \in H^n_{\mathrm{comp},q}(M)$ where ω is the volume form of M. Hence there exists an $(n-1)$-form $\beta \in L^q$ such that $d\beta = \varphi\,\omega$. Lemma 1 implies that the p-capacity of some domain D is positive hence M is p-hyperbolic.

In the converse direction we need to show that if M is p-hyperbolic, then for any n-forms with compact support α, there exists an $(n-1)$-form $\gamma \in L^q$ such that $d\gamma = \alpha$.

Since M is p-hyperbolic, Proposition 1 (together with Remark 1) implies the existence of a smooth vector field $X \in L^q$ such that $div(X)$ has compact support and $\int div(X) \neq 0$. Let us set $\beta = *X^\flat$, then $\beta \in L^q$, $d\beta = *div(X)$ and $\int d\beta = \int div(X) \neq 0$. Thus we can find a number $c \in \mathbb{R}$ such that

$$\int (\alpha - c\,d\beta) = 0.$$

By Proposition 2, we can find an $(n-1)$-form θ with compact support such that $d\theta = \alpha - c\,d\beta$. Clearly the $(n-1)$-form $\gamma := \theta + c\beta$ is in L^q and satisfy $d\gamma = \alpha$. We conclude that $[\alpha] = 0 \in H^n_{comp,q}(M)$, and hence $H^n_{comp,q}(M) = 0$. $\square$

The previous theorem can be made more precise:

Theorem 3. *Let M be p-parabolic. Then $H^n_{comp,q}(M)$ is of dimension 1 and the integration $\alpha \to \int_M \alpha$ defines an isomorphism $H^n_{comp,q}(M) \cong \mathbb{R}$.*

Another way to express this is by saying that if M is p-parabolic. Then there exists an $(n-1)$-form $\beta \in L^q$ such that $d\beta$ has compact support and $\int_M d\beta \neq 0$; and $[d\beta]$ is a basis of $H^n_{\text{comp},q}(M)$ for any such form.

Proof. By Proposition 2, $H^n_{\text{comp}}(M)$ is 1-dimensional. Since the canonical map $H^n_{\text{comp}}(M) \to H^n_{\text{comp},q}(M)$ is an epimorphism, and $H^n_{\text{comp},q}(M) \neq 0$ by Theorem B, $H^n_{\text{comp}}(M) \to H^n_{\text{comp},q}$ must be an isomorphism and the result follows from Proposition 2 $\square$

Corollary 2. *Suppose that there exists a smooth $(n-1)$-form $\beta \in L^q$ such that $d\beta$ has compact support and $\int_M d\beta \neq 0$. Then M is p-hyperbolic.*

Remark 3 By Lemma 1, we already know this corollary in the special case $d\beta \geq 0$.

Proof. Since, $\beta \in L^q$ we have $[d\beta] = 0 \in H^n_{\text{comp},q}(M)$. If M where p-parabolic the previous theorem would imply that $\int_M d\beta = 0$. $\square$

Remark 4 Throughout this section, we have only considered smooth forms. In applications however it is sometimes usefull to consider non smooth forms. For instance, using Theorem 1 one sees that Corollary 2 still holds without any smoothness assumtion.

6. Applications

6.1. Area growth

Let M be a complete oriented riemannian $n-$manifold and $D \subset\subset M$ be some compact subset. Let

$$D_t := \{x \in M \mid \text{dist}(x, D) \leq t\}$$

and denote by $a(t) = |\partial D_t|$ the $(n-1)$-dimensional Hausdorff measure of ∂D_t.

Proposition 3. *Suppose that*

$$\int^{\infty} \frac{dt}{a(t)^{1/(p-1)}} = \infty \tag{2}$$

then M is p-parabolic.

Remark This result is known (see e.g. §4.4 in [14]) and essentially goes back to the work of Ahlfors in the thirties. However, the proof below is new.

Proof. We will show that if M is p-hyperbolic, then the integral (2) converges. Indeed, assume that M is p-hyperbolic, then by Theorem B, we have $H^n_{\text{comp},q}(M) = 0$. Now let $h \geq 0$ be a smooth function with support in D_1 and such that $\int_M h\omega = 1$.

Since $[h\omega] = 0$ in $H^n_{\text{comp},q}(M)$, there exists an $(n-1)$-form $\beta \in L^q$ such that $d\beta = h\omega$. By Stokes formula $\int_{\partial D_t} \beta = 1$ for any $t \geq 1$.

Using Hölder inequality one obtains

$$1 = \int_{\partial D_t} \beta \leq \|\beta\|_{L_q(\partial D_t)} \mid \partial D_t \mid^{1/p},$$

i.e. $\|\beta\|^q_{L_q(\partial D_t)} \geq \mid \partial D_t \mid^{1/(1-p)}$. From this estimates follows

$$\int_1^\infty \frac{dt}{\mid \partial D_t \mid^{1/(p-1)}} \leq \int_M |\beta|^q < \infty.$$

□

6.2. *Manifold with warped cylindrical end*

A Riemannian manifold M will be said to have a warped cylindrical end if there exists a compact Riemannian manifold (N, g_N) and a compact subset $D \subset\subset M$ such that $M \setminus D = N \times_f [1, \infty)$ is the warped product of N and $[1, \infty)$ (i.e the direct product with the Riemannian metric $dt^2 + f^2(t)g_N$).

Proposition 4. *M is p-parabolic if and only if*

$$\int_1^\infty \frac{dt}{f(t)^{(n-1)/(p-1)}} = \infty.$$

(In particular, the euclidean space $\mathbb{R}^n$ is p-parabolic if and only if $p \geq n$.)

Proof. Suppose first that $\int_1^\infty f^{-\frac{n-1}{p-1}}(t)dt = \infty$. We have $\partial D_t = N \times \{t\}$ and its $(n-1)$-measure is $a(t) = |\partial D_t| = \text{const}\, f^{n-1}(t)$. Hence we have

$$\int_1^\infty \frac{dt}{a(t)^{1/(p-1)}}\, dt = \infty,$$

and Proposition 3 implies that M is p-parabolic.

Assume now that $\int_1^\infty f^{-\frac{n-1}{p-1}}(t)dt < \infty$. Let γ be the volume form of (N, g_N), and choose a smooth function $\psi : [0, \infty) \to \mathbb{R}$ such that

i) $0 \leq \psi(t) \leq 1$;
ii) $\psi \equiv 0$ for $t \leq 1$;
iii) $\psi \equiv 1$ for $t \geq 2$.

Let us define an $(n-1)$-form β by $\beta \equiv 0$ on D and $\beta = \psi(t)\,\gamma$ on $M \setminus D$ (= $N \times_f [1,\infty)$). Then $d\beta$ has compact support and $\int_M d\beta \neq 0$. Furthermore, since $\beta = \psi(t)\gamma$, we have $|\beta| = \psi(t)\, f(t)^{-(n-1)}$ and thus

$$\begin{aligned}\int_M |\beta|^q &= \int_{N\times[1,\infty)} \psi(t)^q f(t)^{(1-q)(n-1)} \gamma \wedge dt \\ &\leq \text{const} \int_{[1,\infty)} f(t)^{(1-q)(n-1)} dt \\ &\leq \text{ const} \int_1^{\infty} f(t)^{-(n-1)/(p-1)} dt < \infty\,.\end{aligned}$$

Hence $\beta \in L^q$ and we conclude by Corollary 2 that M is p-hyperbolic. $\square$

For a p-parabolic manifold M, we know by Theorem A that $\int_M divX = 0$ for any vector field $X \in L^q$ ($q = p/(p-1)$) provided $(divX)^- \in L^1$ (the latter condition is necessary to give a meaning to the integral $\int_M divX$ and may of course be replaced by $(divX)^+ \in L^1$).

Klauss Steffen has observed that for parabolic manifolds with warped cylindrical ends, one can prove that $\int divX$ is small on large balls without assuming $(divX)^-$ to be integrable. We owe him the following Proposition:

Proposition 5. *Let M be a Riemannian manifold with warped cylindrical end. Then M is p-parabolic if and only if*

$$\liminf_{t\to\infty} \int_{D_t} \mathit{div}(X) = 0. \tag{3}$$

for all C^1 vector field X on M such that $|X| \in L^q(M)$, $q = p/(p-1)$.

Proof. If M is p-hyperbolic, then from Proposition 1, Sect. 4 follows that there exists a vector field X on M violating the condition (3).

Now let us assume M p-parabolic and let $g = |X|$, then $g \in L^q(M) \cap C^0(M)$. On $M \setminus D = N \times [0,\infty)$, one may write $g(x) = g(y,t)$. From Hölder's inequality we have

$$\begin{aligned}&\int_1^{\infty} \left[\int_N g(y,t)\, f(t)^{n-1}\gamma\right]^q \cdot f(t)^{-\frac{n-1}{p-1}} dt \\ &\leq \int_1^{\infty} \left[\left(\int_N g^q(y,t)\gamma\right)^{1/q} \left(\int_N f(t)^{p(n-1)}\gamma\right)^{1/p}\right]^q \cdot f(t)^{-\frac{n-1}{p-1}} dt \\ &\leq (\text{Vol } N)^{q/p} \int_1^{\infty}\int_N g^q(y,t)\, f(t)^{n-1}\gamma \wedge dt \\ &\leq (\text{Vol } N)^{q/p} \int_M g^q(x) dx < \infty.\end{aligned}$$

Where $p = q/(q-1)$, and γ is the volume form of (N, g_N). Since M is p-parabolic, we have by the previous proposition $\int_1^\infty f(t)^{-\frac{n-1}{p-1}} dt = \infty$. Hence the inequality above implies

$$\liminf_{t\to\infty} \int_{\partial D_t} g(y,t)\gamma_t = 0$$

where $\gamma_t = f(t)^{n-1}\gamma$ is the volume form of the hypersurface ∂D_t. We conclude the proof from the divergence formula. $\square$

Question: For what kind of (more general) manifolds is this proposition true?

6.3. Graphs in $\mathbb{R}^{n+1}$ with non-negative mean curvature

Let $S \subset \mathbb{R}^{n+1}$ be a hypersurface which is an entire graph, i.e. S is of the form

$$S := \{(x,z) \in \mathbb{R}^{n+1} : z = f(x)\}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a function of class C^2.

We will denote by H the mean curvature of S. We also introduce the angle $\varphi(x)$ between the vertical direction and the normal direction of S at the point $(x, f(x)) \in S$. This is a measure of horizontality of the surface.

Theorem 4. *Suppose that $H \geq 0$ and that $\varphi \in L^q(\mathbb{R}^n)$ for some $q \leq n/(n-1)$. Then S is a minimal surface.*

In the case $n \leq 7$ or $\nabla f \in L^\infty$, then S is a horizontal hyperplane.

The assumption on φ is satisfied if e.g. $\varphi(x) \leq C\,|x|^{-s}$ outside a compact set for some $s > n-1$.

Proof. Let $p := q/(q-1)$, observe that $p \geq n$ hence $\mathbb{R}^n$ is p-parabolic. Now let

$$X := \frac{1}{n}\frac{\nabla f}{\sqrt{1+\|\nabla f\|^2}}\,,$$

then $\|X\| = \frac{1}{n}|\sin(\varphi(x))| \in L^q(\mathbb{R}^n)$ and $H = \mathrm{div}(X) \geq 0$.

By Theorem A, we must have $H \equiv 0$; indeed if $H \not\equiv 0$, then the vector field X satisfies (1) $X \in L^q$, (2) $\mathrm{div}X = H \geq 0$ and (3) $\mathrm{div}\,X > 0$ somewhere.

In particular $(div\,X)^- \equiv 0 \in L^1$ and from (2) and (3) follows that $\int_{\mathbb{R}^n} \mathrm{div}X > 0$. Thus Theorem A implies that $\mathbb{R}^n$ is p-hyperbolic, but this contradicts $p \geq n$.

Thus S is a minimal surface and by Bernstein theorem (see theorem 17.5 and 17.8 in [8]), we conclude that φ is constant if $n \leq 7$ or $\nabla f \in L^\infty$. This constant must be zero since $\varphi \in L^q(\mathbb{R}^n)$, hence S is a horizontal hyperplane. $\square$

6.4. The non Solvability of Δ_p

We now show that theorem 2 is false for parabolic manifolds.

Theorem 5. *Suppose that M is p-parabolic, and let $h \in L^1(M)$ be a function such that $\int_M h \neq 0$. Then the equation*

$$\Delta_p u + h = 0 \tag{4}$$

has no weak solution $u \in \mathcal{L}^{1,p}(M)$.

Proof. Assume that a solution f exists and define $X := -|\nabla f|^{p-2}\nabla f$. Then X satisfies the conditions of Theorem A and M is thus p-hyperbolic. □

In particular, (4) has no weak solution in $\mathcal{L}^{1,p}(\mathbb{R}^n)$ if $p \geq n$ (this result was anounced in [7]). When the manifold is the euclidean space, we may state more generally

Theorem 6. *Let $n \geq 2$, $p > 1$ and $1 \leq \frac{s}{p-1} \leq \frac{n}{n-1}$. If h is a function of class L^1 on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} h \neq 0$, then (4) has no weak solution in $\mathcal{L}^{1,s}(\mathbb{R}^n)$.*

Proof. Set $t = \frac{s}{p-1}$ and $t' = \frac{s}{s+1-p}$. Observe first that $t' \geq n$, (in particular $\mathbb{R}^n$ is t'-parabolic) and also that $\frac{1}{t} + \frac{1}{t'} = 1$.

Assume that a solution $f \in \mathcal{L}^{1,s}(\mathbb{R}^n)$ to (4) exists and define $X := -|\nabla f|^{p-2}\nabla f$. Then $\operatorname{div}(X) = h$ and $X \in L^t(\mathbb{R}^n)$. But this is impossible by Theorem A since $\mathbb{R}^n$ is t'-parabolic. □

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