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Mean Curvature and Asymptotic Volume of Small Balls

Dominique Hulin and Marc Troyanov

The aim of this note is to prove an asymptotic formula relating the mean curvature of a hypersurface at a given point and the volume of small balls centered at this point. For instance, consider a sphere S_R of radius R in Euclidean three-space $\mathbb{R}^3$. If p lies on S_R and $t > 0$ is small enough (i.e., $0 < t \leq R$), then

$$\frac{\operatorname{Vol}\left(B_p^+(t)\right)}{\operatorname{Vol}\left(B_p(t)\right)} = \frac{1}{2} - \frac{3}{16}\frac{1}{R}t,$$

where $B_p(t)$ is the ball of radius t centered at p and $B_p^+(t)$ is the portion of the ball lying inside the sphere S_R. Our goal is to show that, up to a negligible term, a similar formula holds for any hypersurface S in $\mathbb{R}^n$, the factor $1/R$ being replaced with the mean curvature of the hypersurface.

We briefly recall what mean curvature is. Let S be a hypersurface of class C^2 in Euclidean n-space $\mathbb{R}^n$, and assume that a unit normal vector field $N : S \to \mathbb{R}^n$ has been chosen (this is always possible locally). It is a basic fact that the normal acceleration $\langle c''(t), N(c(t))\rangle$ of a C^2-curve $c(t)$ on S depends only on its tangent vector $V(t) = c'(t)$: indeed,

$$0 = \frac{d}{dt}\big\langle c'(t), N\big(c(t)\big)\big\rangle = \big\langle c''(t), N\big(c(t)\big)\big\rangle + \left\langle c'(t), \frac{d}{dt}N\big(c(t)\big)\right\rangle,$$

so $\langle c''(t), N(c(t))\rangle = -\langle V, D_V N\rangle$ (with the notation $D_V N = d(N(c(t))/dt)$.

The *second fundamental form* of the hypersurface S at a point p of S is the bilinear form defined on the tangent space T_pS by

$$\mathrm{II}_p(v, w) = -\langle v, D_w N\rangle.$$

The previous calculation shows that $\mathrm{II}(c'(t), c'(t))$ is the normal acceleration of the curve. It is not difficult to check that II_p is a symmetric bilinear form. The *mean curvature* $H(p)$ of the hypersurface S at the point p is then defined as

$$H(p) = \frac{1}{n-1}\operatorname{Trace}(\mathrm{II}_p) = \frac{1}{n-1}\sum_{i=1}^{n-1}\mathrm{II}(v_i, v_i),$$

where $v_1, \ldots, v_{n-1}$ is any orthonormal basis of the tangent space T_pS. (Observe that replacing the normal field N by $-N$ changes the sign of the mean curvature.) The mean curvature and the second fundamental form of a surface in $\mathbb{R}^3$ are classical topics in geometry (see, for instance, [**1**, chap. 5]).

Let us now fix a ball $B_p(t)$ in $\mathbb{R}^n$ with radius $t > 0$ and center p on S. When t is small enough, the hypersurface S separates $B_p(t)$ into two connected components $B_p^+(t)$ and $B_p^-(t)$, with the convention that N_p points towards $B_p^+(t)$. We want to relate the mean curvature of S at p to the ratio of the volumes of $B_p^+(t)$ and $B_p(t)$.

It is clear that for small values of t the volume of $B_p^+(t)$ is roughly one-half the volume of the entire ball $B_p(t)$; in fact, we have

$$\mathrm{Vol}\left(B_p^+(t)\right) = \frac{1}{2}\alpha_n t^n + O(t^{n+1}),$$

where α_n denotes the volume of the unit ball in $\mathbb{R}^n$. We claim: *the next term of the Taylor expansion of this volume is given by*

$$\mathrm{Vol}\left(B_p^+(t)\right) = \frac{1}{2}\alpha_n t^n - \frac{1}{2}\left(\frac{n-1}{n+1}\alpha_{n-1}\right) H(p)\, t^{n+1} + O(t^{n+2}). \tag{1}$$

In particular, the mean curvature satisfies

$$H(p) = 2\left(\frac{n+1}{n-1}\frac{\alpha_n}{\alpha_{n-1}}\right)\lim_{t\to 0}\left[\frac{1}{t}\left(\frac{1}{2} - \frac{\mathrm{Vol}(B_p^+(t))}{\mathrm{Vol}(B_p(t))}\right)\right]. \tag{2}$$

To prove (1), we choose an orthonormal coordinate system centered at p such that $N_p = (0, \dots, 0, 1)$. The hypersurface is then locally defined as a graph

$$x_n = f(x_1, \dots, x_{n-1}),$$

where f is a smooth function satisfying $f(0) = 0$ and $\partial f/\partial x_i(0) = 0$ for $i = 1, \dots, n-1$. One then easily checks that the second fundamental form at $p = 0$ coincides with the Hessian of f at this point. Thus for $X = (x_1, \dots, x_{n-1})$ in $\mathbb{R}^{n-1}$, we have $f(X) = \frac{1}{2}\mathrm{II}(X, X) + o(\|X\|^2)$.

It is convenient to begin with an analogue of estimate (1) in which balls are replaced by cylinders. Namely, we set

$$C(t) = \left\{x = (X, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \max(\|X\|, |x_n|) \le t\right\}.$$

If $\rho > 0$ is small enough, then f is well defined on $\{X \in \mathbb{R}^n : \|X\| \le \rho\}$ and we consider the intersection $C^+(t) = \{x = (X, x_n) \in C(t) : x_n \ge f(X)\}$ of the cylinder $C(t)$ with the epigraph of f, where $0 \le t \le \rho$. We then have

$$\mathrm{Vol}\left(C^+(t)\right) = \frac{1}{2}\mathrm{Vol}\left(C(t)\right) - \int_{B^{n-1}(t)} f(X)\, dX,$$

where $B^{n-1}(t) = \{X \in \mathbb{R}^{n-1} : \|X\| \le t\}$. Now observe that for $i, j = 1, \dots, n-1$ such that $i \ne j$ we have

$$\int_{B^{n-1}(t)} x_i x_j \, dX = 0,$$

while for $i = 1, \dots, n-1$

$$\int_{B^{n-1}(t)} x_i^2 \, dX = \frac{1}{n-1}\int_{B^{n-1}(t)} \|X\|^2 \, dX = \left(\frac{1}{n+1}\alpha_{n-1}\right) t^{n+1}.$$

Combining these three identities, we obtain

$$\operatorname{Vol}\left(C^+(t)\right) = \frac{1}{2}\operatorname{Vol}\left(C(t)\right) - \frac{1}{2}\left(\frac{n-1}{n+1}\alpha_{n-1}\right) H(p)\, t^{n+1} + O(t^{n+2}), \tag{3}$$

since $f(X) = \frac{1}{2}\mathrm{II}(X, X) + o(\|X\|^2)$ and $H(p) = \mathrm{Trace}(\mathrm{II}_p)/(n-1)$.

We still have to verify that replacing cylinders with balls does not affect the significant term in our Taylor expansion, i.e., we need to show that

$$\left(\operatorname{Vol}\left(C^+(t)\right) - \frac{1}{2}\operatorname{Vol}\left(C(t)\right)\right) - \left(\operatorname{Vol}\left(B^+(t)\right) - \frac{1}{2}\operatorname{Vol}\left(B(t)\right)\right)$$

is at most of the order t^{n+2}. We in fact obtain a better estimate. For any c in $\mathbb{R}$ set

$$A_c(t) = \left\{x = (X, x_n) \in C(t) \setminus B(t) : x_n \geq ct^2\right\};$$

in particular,

$$\operatorname{Vol}(A_0(t)) = \frac{1}{2}\operatorname{Vol}\left(C(t)\right) - \frac{1}{2}\operatorname{Vol}\left(B(t)\right).$$

The volume of $A_{-c}(t) \setminus A_c(t)$ is of the order t^{n+4}, since this set is contained in a cylinder of height $2ct^2$ whose base is an annulus of outer radius t and inner radius roughly $t - c^2t^3/2$. More precisely,

$$A_{-c}(t) \setminus A_c(t) \subset \left\{x = (X, x_n) \,\middle|\, \sqrt{t^2 - c^2t^4} \leq \|X\| \leq t,\ |x_n| \leq ct^2\right\};$$

hence

$$0 < \operatorname{Vol} A_{-c}(t) - \operatorname{Vol} A_c(t) \leq 2\alpha_{n-1}ct^2\left(t^{n-1} - (\sqrt{t^2 - c^2t^4})^{n-1}\right) = O(t^{n+4}).$$

Because $A_{-c}(t) \subset A_0(t) \subset A_c(t)$, we thus obtain

$$\operatorname{Vol}(A_c(t)) = \operatorname{Vol}(A_0(t)) + O(t^{n+4}) = \frac{1}{2}\operatorname{Vol}(C(t)) - \frac{1}{2}\operatorname{Vol}(B(t)) + O(t^{n+4})$$

for all c. The same estimate also holds for $\operatorname{Vol} A_{-c}(t)$.

If we now choose the positive constants c and $\rho > 0$ so that $|f(X)| \leq c\,\|X\|^2$ whenever $\|X\| < \rho$, then we have for $t \leq \rho$:

$$A_{-c}(t) \subset C^+(t) \setminus B^+(t) \subset A_c(t).$$

In this way we arrive at

$$\operatorname{Vol}\left(A_{-c}(t)\right) \leq \operatorname{Vol}\left(C^+(t)\right) - \operatorname{Vol}\left(B^+(t)\right) \leq \operatorname{Vol}\left(A_c(t)\right),$$

and the previous estimates implies that

$$\operatorname{Vol}\left(C^+(t)\right) - \operatorname{Vol}\left(B^+(t)\right) = \frac{1}{2}\operatorname{Vol}\left(C(t)\right) - \frac{1}{2}\operatorname{Vol}\left(B(t)\right) + O(t^{n+4}).$$

It is now clear that the estimate (1) follows from (3). ■

The notions of the second fundamental form and the mean curvature both extend without substantial modification to the setting of hypersurfaces in Riemannian manifolds (see, for instance, [**2**, pp. 132–142]). We conclude this note by showing that the Taylor expansion (1) still holds in this context.

Let S be a hypersurface in a Riemannian manifold (M^n, g), and let N be a (local) unit normal vector field defined in a neighbourhood of the point p on S. The exponential map $\phi = \exp_p : U \subset T_pM \to M$ is well defined in a neighbourhood U of 0 in T_pM. Moreover, when $t > 0$ is small enough, ϕ is a diffeomorphism from the Euclidean ball $\widetilde{B}_0(t)$ of radius t and center 0 in T_pM to the Riemannian ball $B_p(t)$ in M. Denote by $\widetilde{S}$ the Euclidean hypersurface $(\phi|_U)^{-1}(S)$ and by $\widetilde{N}$ the vector field $(\phi^{-1})_*N$.

From the fact that the 1-jets of the pull-back metric ϕ^*g and of the Euclidean metric g_p on T_pM coincide at the origin (see [**2**, Proposition 5.11, p. 78]), we infer that:

(i) the mean curvature $H(p)$ of S at p, and $\widetilde{H}(0)$ of $\widetilde{S}$ at the origin (with respect to the normal $\widetilde{N}$) are equal;

(ii) $\mathrm{Vol}_{\mathrm{E}}(\widetilde{B}_0(t)) = \mathrm{Vol}_g(B_p(t)) + O(t^{n+2})$ (and similarly for $\widetilde{B}_0^+(t)$ and $B_p^+(t)$).

It follows that the formula (1) is still valid in the Riemannian setting without any correction. ■

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Timing Is Everything: The French Connection

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1. INTRODUCTION. Marin Mersenne (1588–1648) is remembered chiefly as a one-man scientific clearing house—the Internet of his day. During the first half of the seventeenth century, when mathematical physics was taking its first tentative steps guided by Descartes and Galileo, "Mersenne did more to propagate emerging new sciences of acoustics, pneumatics, and ballistics than anyone else of his time" [**4**, p. 20]. But Mersenne was more than just a go-between and publicist. He also performed "careful experiments" [**5**, p. 186] to test some of the new theories. Galileo's parabolic model for the ballistic trajectory of a particle in a resistanceless medium was one of the "hot" new theories of the time. This model, which is featured in nearly every elementary calculus textbook, implies that the ascent time and descent time