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APPROXIMATELY LIPSCHITZ MAPPINGS AND SOBOLEV MAPPINGS BETWEEN METRIC SPACES

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ABSTRACT. We introduce a notion of Sobolev spaces between metric spaces which extends the Hajłasz' notion of Sobolev functions. We then compare this concept to Reshetnyack's definition.

1. INTRODUCTION

In recent years, there has been a number of works devoted to the notion of Sobolev functions and mappings on metric spaces. Let us in particular mention the definitions proposed by Piotr Hajłasz in [3] and by Yuri Reshetnyak in [7].

Definition. *Let* X *be a metric space equipped with a measure* μ*. We say that a function* $u : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ *belongs to* $M^{1,p}(X, \mathbb{R})$ *if* $u \in L^p(X)$ *and there exists a function* $w : X \to \overline{\mathbb{R}}$ *such that* $w \in L^p(X)$ *and*

$$|u(x) - u(x')| \leq (w(x) + w(x'))d(x, x') \tag{1}$$

for all $x, x' \in X$*.*

Remark. The usual definition states that the inequality (1) holds almost everywhere only; i.e. there exists a set $E \subset X$ of measure 0 such that (1) holds for all $x, x' \notin E$. However, one may always modify the function w on the set E so that the inequality holds everywhere (just set $w = \infty$ on E). This will be our point of view in this paper; it implies in particular that w is defined everywhere and we can no longer modify this function on a set of measure zero.

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The space $M^{1,p}(X,\mathbb{R})$ is called the Sobolev space in the sense of Hajłasz; it is a Banach space for the norm $\|u\|_{M^{1,p}} = \|u\|_{L^p} + \inf \|w\|_{L^p}$ where the infimum is taken over all functions $w \in L^p(X,\mathbb{R})$ satisfying (1).

Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a metric space Y, Yuri Reshetnyak proposes the following definition of the class $W^{1,p}(\Omega, Y)$ of Sobolev mappings.

Definition. A function $f : \Omega \to Y$ *belongs to* $W^{1,p}(\Omega, Y)$ *if there exists a function* $w \in L^p(\Omega)$ *such that for all* $y \in Y$ *the function* $\theta_y : \Omega \to \mathbb{R}$ *defined by* $\theta_y(x) = d(y, f(x))$ *satisfies*

a) $\theta_y \in W^{1,p}(\Omega,\mathbb{R})$,

b) $|\nabla\theta_y(x)| \leq w(x)$ *for a.e.* $x \in \Omega$.

A different approach to Sobolev spaces of maps with metric space targets has been proposed by N. Korevaar and R. Schoen, see [5]. We will not discuss here this approach.

My goal in this lecture is to extend Hajłasz definition to the case of mappings between two metric spaces and to show how this definition is compatible with Reshetnyak's definition under suitable hypothesis.

This work owes much to some discussions I had with P. Hajłasz, P. Koskela and S. Vodop'yanov.

2. Approximately Lipschitz Mappings

Throughout the paper, X and Y are arbitrary metric spaces and μ is a fixed Borel measure on X.

Definitions. i) The *difference quotient* of a mapping $f : X \to Y$ is the function $Q_f : X \times X \to \mathbb{R}$ defined by

$$Q_f(x_1,x_2) := \begin{cases} \frac{d(f(x_1),f(x_2))}{d(x_1,x_2)} & \text{if} \quad x_1 \neq x_2 \\ 0 & \text{if} \quad x_1 = x_2 \end{cases}$$

ii) We denote by $D[f]$ the set of all $\mu-$measurable functions $w : X \to \overline{\mathbb{R}}_+$ such that

$$Q_f(x_1,x_2) \leq w(x_1) + w(x_2)$$

for all $x_1, x_2 \in X$.

iii) The map f is *approximately Lipschitz* if there exists a function $w \in D[f]$ such that $w < \infty$ μ-a.e.

Remarks. 1) If a function $w \in D[f]$ is bounded on a subset $A \subset X$, then f is Lipschitz on the set A.

2) If $f : X \to Y$ is approximately Lipschitz and $g : Y \to Z$ is k-Lipschitz, then $g \circ f : X \to Z$ is also approximately Lipschitz. In fact

$$w \in D[f] \quad \Rightarrow \quad kw \in D[g \circ f].$$

Theorem 1. *Let $f : X \to Y$ be a mapping between two metric spaces. Suppose that X is locally compact and separable and that μ is a Radon measure on X. Then the following conditions are equivalent*

a) *f is approximately Lipschitz.*

b) *There exists a monotone sequence of compact subsets $X_0 \subset X_1 \subset ... \subset X$ such that*

b.1) $f|_{X_k}$ *is Lipschitz and*

b.2) $\mu(X \setminus \bigcup_{k=1}^{\infty} X_k) = 0.$

Proof. (a) $\Rightarrow$ (b) By Lusin's theorem (see e.g. [8]), there exists a sequence of compact subsets $X_0 \subset X_1 \subset ... \subset X$ such that $\mu(X \setminus \bigcup_{k=1}^{\infty} X_k) = 0$ and w is continuous on each X_k. In particular w is bounded on X_k and hence $f|_{X_k}$ is Lipschitz.

(b) $\Rightarrow$ (a) Let $L_k :=\mathrm{Lip}(f|_{X_k})$ be the Lipschitz constant of $f|_{X_k}$. Observe that $\{L_k\}$ is a monotone sequence since $X_k \subset X_{k+1}$. Define the function $w : X \to \overline{\mathbb{R}}$ by

$$w(x) = \begin{cases} \min\{L_k : x \in X_k\} & \text{if } x \in \cup_k X_k \ , \\ \infty & \text{else.} \end{cases}$$

It is clear that $w < \infty$ a.e. and $Q_f(x_1, x_2) \leq \max\{w(x_1), w(x_2)\} \leq w(x_1) + w(x_2)$. $\square$

Corollary 1. *Assume that X is locally compact and separable and μ is Radon. Then $f : X \to Y$ is approximately Lipschitz if and only if $f|_K$ is approximately Lipschitz for all compact subsets $K \subset X$.*

$\square$

Corollary 2. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Then $f : \Omega \to \mathbb{R}^m$ is approximately Lipschitz if and only if it is approximately differentiable almost everywhere.*

Proof. Let $X_k \subset \Omega$ be as in Theorem 1. By Kirzbaum's Theorem, there exists a global Lipschitz map $f_k : \mathbb{R}^n \to \mathbb{R}^m$ such that f and f_k coincide on X_k. From Rademacher's theorem, we can find measurable subsets $A_k \subset X_k \setminus X_{k-1}$ such that $\mathrm{Vol}(X_k \setminus (A_k \cup X_{k-1})) = 0$ and f_k is differentiable at each point of A_k; we can also assume without lost of generality that $A_k \subset \mathbb{R}^n$ has density 1 at each of its point.
We now define $(df)_x := df_k(x)$ if $x \in A_k$. It is not difficult to verify that $(df)_x$ is the approximate differential of f at x. This concludes the proof since $\mathrm{Vol}(\nleq \setminus (\cup A_k)) = 0$.

$\square$

Remarks 3) As a consequence of this corollary, the change of variable formula holds for any approximately Lipschitz mapping between Riemannian manifolds satisfying Lusin's property. The same is true for the area and coarea formulas.

4) This discussion generalizes to other spaces such as rectifiable sets or subriemannian manifolds.

3. A review of Hajłasz Sobolev space $M^{1,p}(X,\mathbb{R})$

In [3], P. Hajłasz has introduced and studied the following notion of Sobolev space

$$M^{1,p}(X,\mathbb{R}) := \{u \in L^p(X,\mathbb{R}) \,|\, D[u] \cap L^p(X,\mathbb{R}) \neq \emptyset\} \ .$$

He has shown in particular that if $X = \Omega \subset \mathbb{R}^n$ is an extension domain, then $M^{1,p}(\Omega,\mathbb{R})$ coincides with the classical Sobolev space $W^{1,p}(\Omega,\mathbb{R})$. He has also proved both the following Poincaré inequality (see [3]):

Lemma 1. *If $u \in M^{1,p}(X,\mathbb{R})$ and $A \subset X$ is a bounded set with finite measure, then*

$$\int_A |u - u_A|^p d\mu \leq (2 \operatorname{diam} A)^p \int_A w^p d\mu$$

for all $w \in D[u]$, where $u_A = \frac{1}{\mu(A)} \int_A u\, d\mu$.

and the following embedding theorem:

Theorem 2. *Suppose that μ is a locally s-regular measure (meaning that for all point $x \in X$ there exists a radius $R_x > 0$ such that $\mu(B(x,r)) \geq cr^s$ for all $0 \leq r \leq R_x$). Then, if $p > s$, each function $u \in M^{1,p}(X,\mathbb{R})$ is continuous after modification on a subset of measure zero.*

See [3, Theorem 6] and [6, Theorem 9.4].

□

We will need some definitions.

Definitions. i) A measure μ on a metric space X is *doubling* if there exists a constant $\mathrm{C_{doub}}$ such that for every ball $B(x,r) \subset X$ we have

$$\mu(B(x,2r)) \leq \mathrm{C_{doub}}\ \mu(B(x,r))\,.$$

ii) A pair of functions $u, w : M \to \mathbb{R}$ satisfies a *(1,q)-Poincaré inequality* with constants C and λ if

$$(2) \quad \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq C\,r \ \left(\frac{1}{\mu(B(x,\lambda r))} \int_{B(x,\lambda r)} w^q d\mu \right)^{1/q}$$

for every ball $B(x,r) \subset M$.

iii) The Riemannian manifold M *supports a* $(1,q)$*-Poincaré inequality* if there are constants C and λ such that for every locally Lipschitz functions $u : M \to \mathbb{R}$, the pair $(u, |\nabla u|)$ satisfies the inequality (2) with constants C and λ.

Theorem 3. *Suppose that the measure μ is doubling and assume that the pair of functions $u \in L^p_{loc}(X)$ and $w \in L^p(X)$ satisfies a $(1,q)$-Poincaré inequality for some constants C, λ and some $1 \leq q < p$. Let us define a function $\hat{w}$ by*

$$\hat{w}(x) := C \sup_{r>0} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} w^q d\mu \right)^{1/q},$$

then $\hat{w} \in D[u] \cap L^p(X, \mathbb{R})$. In particular, if $u \in L^p(X)$, then $u \in M^{1,p}(X, \mathbb{R})$.

Proof. See Theorem 3.1 and 3.2 in [4].

□

In the case where $X = M$ is a Riemannian manifold (with its canonical volume measure), the work of P. Hajłasz, P. Koskela, Mac Manus and B. Franchi has lead to the following criterion for the coincidence of $M^{1,p}(M, \mathbb{R})$ with the usual Sobolev space $W^{1,p}(M, \mathbb{R})$.

Theorem 4. *If the Riemannian manifold M supports a $(1,q)$-Poincaré inequality for some $1 \leq q < p$, and the volume measure is doubling, then $M^{1,p}(M, \mathbb{R}) = W^{1,p}(M, \mathbb{R})$. Furthermore if the pair of functions $u \in L^p_{loc}(X, \mathbb{R})$ and $w \in L^p(X, \mathbb{R})$ satisfies a $(1,p)$-Poincaré inequality with constants C and λ, then $u \in \mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}(M, \mathbb{R})$ and $|\nabla u| \leq Cw$ a.e.*

Proof. This follows from Corollary 3 and Theorems 10 in [1].

□

4. The class $L^p(X,Y)$

To define the notion of a $p-$integrable mapping $f : X \to Y$, we need to choose a base point $y_0 \in Y$. Let us denote by $\theta_{y_0} = \theta^f_{y_0} : X \to \mathbb{R}$ the function $\theta_{y_0}(x) = d(y_0, f(x))$. We now define :

$$L^p_{y_0}(X,Y) = \{ f : X \to Y \mid f \text{ is measurable and } \theta_{y_0} \in L^p(X, \mathbb{R}) \}.$$

Proposition 1. *The space $L^p_{y_0}(X,Y)$ is independent from the choice of the base point y_0 if and only if $\mu(X) < \infty$.*

The proof is easy.

□

In general, the base point y_0 has been fixed and we denote the space $L^p_{y_0}(X,Y)$ simply by $L^p(X,Y)$.

If two mappings which coincide $\mu-$almost everywhere are identified, then $L^p(X,Y)$ becomes a metric space for the distance

$$d_p(f,g) = \left(\int_X d(f(x), g(x))^p d\mu(x) \right)^{1/p}.$$

5. Sobolev mappings along Hajłasz's definition

It is straightforward to extend Hajłasz's definition to the case of mappings. It is useful to define first the *Dirichlet space* of mappings with finite p-energy:

$$\mathcal{L}^{1,p}(X,Y) = \{ f : X \to Y \mid D[f] \cap L^p(X,\mathbb{R}) \neq \emptyset \}.$$

In other words, a map $f \in \mathcal{L}^{1,p}(X,Y)$ if and only if there exists a function $w \in L^p(X,\mathbb{R})$ and a subset $F \subset X$ with $\mu(F) = 0$ such that

$$Q_f(x_1,x_2) \leq w(x_1) + w(x_2)$$

for all $x_1, x_2 \in X \setminus F$. The Sobolev space of mappings $X \to Y$ is then defined as

$$M^{1,p}(X,Y) := L^p(X,Y) \cap \mathcal{L}^{1,p}(X,Y).$$

When Y is a separable metric space, an alternative description of the space $\mathcal{L}^{1,p}(X,Y)$ can be given. First, for any map $f : X \to Y$ and any point $y \in Y$ we denote by $\theta_y : X \to \mathbb{R}$ the function $\theta_y(x) := d(y, f(x))$. It is the composition of f with the Lipschitz function $\rho_y : Y \to \mathbb{R}$ given by $\rho_y(z) := d(y,z)$. In particular, by Remark 2 above, we know that

$$w \in D[f] \quad \Rightarrow \quad w \in D[\theta_y] \qquad \text{for all } y \in Y$$

since ρ_y has Lipschitz constant 1.

Conversely, if Y is separable, then $D[f] = \bigcap_{y \in Y} D[\theta_y]$. In fact we have more precisely :

Lemma 2. *If $T \subset Y$ is a dense countable subset, then $D[f] = \bigcap_{t \in T} D[\theta_t]$.*

Proof The discussion above imply $D[f] \subset \bigcap_{t\in T} D[\theta_t]$. Conversely, let $w \in \bigcap_{t \in T} D[\theta_t]$, then

$$|\theta_t(x) - \theta_t(x')| \leq (w(x) + w(x'))d(x,x')$$

for all $x, x' \in X \setminus E_t$ where $E_t \subset X$ is a subset of measure zero. Let us set $E := \bigcup_{t \in T} E_t$; then $\mu(E) = 0$ and we have

$$|d(f(x),t) - d(f(x'),t)| \leq (w(x) + w(x'))d(x,x')$$

for all $x, x' \in X \setminus E$ and all $t \in T$. Since $T \subset Y$ is dense, we can choose t arbitrarily close to $f(x)$; hence $d(f(x), f(x')) \leq (w(x) + w(x'))d(x,x')$ for all $x, x' \in X \setminus E$. This implies $w \in D[f]$.

□

Theorem 5. *Let $T \subset Y$ be a countable dense subset. Then the following conditions are equivalent for a mapping $f : X \to Y$:*

a) $f \in \mathcal{L}^{1,p}(X,Y)$;

b) *There exists a function $w \in L^p(X,\mathbb{R})$ such that for any Lipschitz function*
$h : Y \to \mathbb{R}$ we have
b.1) $h \circ f \in \mathcal{L}^{1,p}(X,\mathbb{R})$ *and*

b.2) $(kw) \in D[h \circ f]$ *(where k is the Lipschitz constant of h);*

c) $\left(\bigcap_{t \in T} D[\theta_t] \right) \cap L^p(X, \mathbb{R}) \neq \emptyset.$

Proof. (a)⇒(b) follows immediately from Remark 2.
(b)⇒(c) is obvious and (c)⇒(a) follows from the previous lemma.

□

We finally have the following embedding theorem :

Theorem 6. *Assume that Y is a proper metric space and that μ is a locally s-regular measure on X. Then, if $p > s$, each mapping $f \in M^{1,p}(M, Y)$ has a continuous representative.*

The proof will use the result of the appendix.

Proof. Let $\varphi : Y \to \ell_\infty$ be the mapping defined in Theorem 9, and let $F = \varphi \circ f : X \to \ell_\infty$. For each $\nu \in \mathbb{N}$, the function $F_\nu = \varphi_\nu \circ f : X \to \mathbb{R}$ belongs to $M^{1,p}(M, \mathbb{R})$. Thus, by Theorem 2, we can find a subset $E_\nu \subset X$ of measure zero and a function $\tilde{F}_\nu : X \to \mathbb{R}$ such that $\tilde{F}_\nu$ is continuous on X and coincides with F_ν on $X \setminus E_\nu$.
We define $\tilde{F} : X \to \ell_\infty$ by $\tilde{F}(x) := (\tilde{F}_\nu)_\nu$; then F and $\tilde{F}$ coincide on $X \setminus E$ where $E := \bigcup_\nu E_\nu \subset X$, is a set of measure zero. Since each component $\tilde{F}_\nu$ of $\tilde{F}$ is continuous, assertion (e) of Theorem 9 implies that $\tilde{F}$ itself is continuous. Since $\varphi(Y) \subset \ell_\infty$ is closed by Theorem 9b), the image of $\tilde{F}$ is contained in $\varphi(Y)$, i.e. $\tilde{F}(X) \subset \varphi(Y) \subset \ell_\infty$. But φ is injective, so there exists a mapping $\tilde{f} : X \to Y$ such that $\tilde{F} = \varphi \circ \tilde{f}$. It follows that $\tilde{f}$ is continuous and coincides with f on $X \setminus E$.

□

6. On Y. Reshetnyak's Sobolev spaces

Following Y. Reshetnyak [7], we define another Sobolev space of mappings from a Riemannian manifold M to a metric space Y:

Definition. Let M be a Riemannian manifold and Y be a metric space. A map $f : M \to Y$ belongs to $R^{1,p}(M, Y)$ if and only if the following 3 conditions are satisfied:

1) $f \in L^p(M, Y)$;
2) for all $y \in Y$ we have $\theta_y \in \mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}(M, \mathbb{R})$ (= classic local Sobolev space);
3) there exists $w \in L^p(M, \mathbb{R})$ such that $|\nabla \theta_y(x)| \leq w(x)$ for all $y \in Y$ and almost all $x \in M$.

Theorem 7. *Assume that M supports a $(1, q)$-Poincaré inequality for some $1 \leq q < p$, that the volume measure is doubling, and that Y is separable then*

$$M^{1,p}(M, Y) = R^{1,p}(M, Y).$$

Proof. If $f \in M^{1,p}(M,Y)$, then $f \in L^p(M,Y)$ and there exists a function $w \in L^p(M,\mathbb{R})$ such that $w \in D[f]$. By Remark 2, $w \in D[\theta_y]$ for all $y \in Y$ and by Lemma 2, the pair (θ_y, w) satisfies a $(1,p)$-Poincaré inequality. By Theorem 4 we know that $\theta_y \in \mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}(M,\mathbb{R})$ and $|\nabla\theta_y| \leq Cw$ a.e., thus $f \in R^{1,p}(M,Y)$.

Conversely, if $f \in R^{1,p}(M,Y)$, then $f \in L^p(M,Y)$ and $\theta_t \in \mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}(M,\mathbb{R})$ for all $t \in T$ where $T \subset Y$ is some fixed dense subset. Furthermore there exists a function $w \in L^p(M,\mathbb{R})$ such that $|\nabla\theta_t| \leq w$ a.e. for all t. Because the manifold M supports a $(1,q)$-Poincaré inequality, the pair (θ_t, w) satisfies a Poincaré inequality and therefore the function $\hat{w}$ defined in Theorem 3 belongs to $\left(\bigcap_t D[\theta_t]\right) \cap L^p(M,\mathbb{R})$. Applying Theorem 5c, we conclude that $f \in \mathcal{L}^{1,p}(M,Y)$.

□

We finally have the following embedding theorem.

Theorem 8. *Assume that Y is a proper metric spaces. If $p > n = \dim(M)$, then every mapping $f \in R^{1,p}(M,Y)$ has a continuous representative.*

Proof. We argue as in the proof of Theorem 6 using the classical Sobolev's embedding theorem $\mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}(M,\mathbb{R}) \subset C^0(M,\mathbb{R})$ instead of Theorem 2.

□

This result was also obtained by Y. Reshetnyak in [7, Theorem 6.2].

Appendix A. Embedding a metric space into ℓ_∞

Recall that ℓ_∞ is the space of bounded infinite sequences $u = (u_1, u_2, ...)$ $(u_\nu \in \mathbb{R})$. It is a Banach space for the norm $\|u\|_\infty := \sup_{1\leq\nu<\infty} |u_\nu|$.

Theorem 9. *If Y is a proper metric space, then there exists a map $\varphi : Y \to \ell_\infty$, $\varphi = (\varphi_1, \varphi_2, ...)$ with the following properties:*

a) *φ is an isometric embedding;*
b) *$\varphi(Y) \subset \ell_\infty$ is closed;*
c) *a subset $A \subset Y$ is bounded if and only if $\varphi_\nu(A) \subset \mathbb{R}$ is bounded for some ν;*
d) *a sequence $(y_i) \subset Y$ converges if and only if for each ν the sequence $(\varphi_\nu(y_i)) \subset \mathbb{R}$ converges (and $y_i \to y \in Y \Leftrightarrow \varphi_\nu(y_i) \to \varphi_\nu(y) \in \mathbb{R}$ for all ν);*
e) *for any metric space X, a map $f : X \to Y$ is continuous if and only if for all $\nu \in \mathbb{N}$, the "coordinate function" $\varphi_\nu \circ f : X \to \mathbb{R}$ is continuous.*

Recall that a metric space is *proper* if every closed bounded subset is compact.

Remark. The meaning of (d) and (e) is that the strong and weak topologies on ℓ_∞ coincide on locally compact subsets. This is false for general subsets: the Kronecker sequence $\{\delta_j\} \subset \ell_\infty$ (defined by $\delta_{j\nu} = 1$ if $j = \nu$ and 0 else) is an example of non converging sequence all of whose coordinate converge to 0.

Lemma 3. *If* (Y, d) *is a proper metric space, then it is separable*

Proof. Fix $y_0 \in Y$. For all $k \in \mathbb{N}$, the closed ball $\overline{B}_{(y_0,k)} \subset Y$ is compact, hence totally bounded. Let $T_k \subset \overline{B}_{(y_0,k)}$ be a finite subset with codiameter $\leq \frac{1}{k}$ (i.e. T_k is a $\frac{1}{k}$-net). Then $T := \bigcup_{k \in \mathbb{N}} T_k$ is a countable dense subset in Y. □

Proof of the Theorem. Let us fix a base point $y_0 \in Y$ and choose a dense countable subset $T = \{t_\nu\}_{\nu \in \mathbb{N}} \subset Y$. Then define

$$\varphi_\nu(y) := d(y, t_\nu) - d(y_0, t_\nu).$$

We will check that the map $y \to \varphi(y) = (\varphi_1(y), \varphi_2(y), \ldots)$ has the desired properties.

(a) We have $|\varphi_\nu(y) - \varphi_\nu(y')| = |d(y, t_\nu) - d(y', t_\nu)| \leq d(y, y')$ for all ν, hence $\|\varphi(y) - \varphi(y')\| \leq d(y, y')$. Conversely if $\{t_{\nu_j}\}$ is a sequence converging to y', then $|\varphi_\nu(y) - \varphi_\nu(y')| = |d(y, t_\nu) - d(y', t_\nu)| \to d(y, y')$, hence $\|\varphi(y) - \varphi(y')\| \geq d(y, y')$.
We have proven that φ is an isometric embedding.

(b) $\varphi(Y) \subset \ell_\infty$ is closed because it is a proper metric space, hence each closed ball in ℓ_∞ intersects $\varphi(Y)$ along a compact set.

(c) Let $A \subset Y$ be a subset such that $\varphi_\nu(A) \subset \mathbb{R}$ is bounded for some ν, then there exists a constant $C > 0$ such that $|\varphi_\nu(y)| = |d(y, t_\nu) - d(y_0, t_\nu)| \leq C$ for all $y \in A$, hence A is contained in the ball of center y_0 and radius $r = C + d(y_0, t_\nu)$.

(d) Suppose that the sequence $\{\varphi_\nu(y_i)\}_{i \in \mathbb{N}} \subset \mathbb{R}$ converges for all ν. We will show that $\{y_i\}$ converges in Y.
By hypothesis, $\{\varphi_1(y_i)\}_{i \in \mathbb{N}} \subset \mathbb{R}$ is bounded, hence, by (c), we know that $\{y_i\} \subset Y$ is a bounded sequence. Since Y is proper, $\{y_i\}$ admits an accumulation point $\bar{y}$. For each $\varepsilon > 0$ there exists ν_ε such that $d(\bar{y}, t_{\nu_\varepsilon}) < \varepsilon$. Because $\varphi_{\nu_\varepsilon}$ is continuous and $\{\varphi_{\nu_\varepsilon}(y_i)\}_{i \in \mathbb{N}} \subset \mathbb{R}$ converges, we have $\lim_{i \to \infty} \varphi_{\nu_\varepsilon}(y_i) = \varphi_{\nu_\varepsilon}(\bar{y})$, thus, for $i \geq k$ large enough, we have $|\varphi_{\nu_\varepsilon}(y_i) - \varphi_{\nu_\varepsilon}(\bar{y})| = |d(y_i, t_{\nu_\varepsilon}) - d(\bar{y}, t_{\nu_\varepsilon})| < \varepsilon$, hence $d(y_i, t_{\nu_\varepsilon}) \leq d(\bar{y}, t_{\nu_\varepsilon}) + |d(y_i, t_{\nu_\varepsilon}) - d(\bar{y}, t_{\nu_\varepsilon})| \leq 2\varepsilon$ and finally $d(y_i, \bar{y}) \leq d(\bar{y}, t_{\nu_\varepsilon}) + d(y_i, t_{\nu_\varepsilon}) \leq 3\varepsilon$ for all $i \geq k$. This implies $y_i \to \bar{y}$.

(e) Assume that $f_\nu := \varphi_\nu \circ f : X \to \mathbb{R}$ is continuous for all ν and let $\{x_i\} \subset X$ be any sequence. If $x_i \to x$, then $\{\varphi_\nu(y_i)\} \subset \mathbb{R}$ converges for all ν where $y_i = f(x_i)$. By (d), the limit $y := \lim_{i \to \infty} y_i = \lim_{i \to \infty} f(x_i) \in Y$ exists; furthermore $\varphi(f(x)) = (\varphi_\nu(f(x))) = \lim_{i \to \infty} (\varphi_\nu(f(x_i))) = \lim_{i \to \infty} \varphi(f(x_i)) \in \ell_\infty$ and, since φ is an isometric embedding, we have $f(x) = \lim_{i \to \infty} f(x_i)$. This proves that f is continuous.

□

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