METRICS OF CONSTANT CURVATURE ON A SPHERE WITH TWO CONICAL SINGULARITIES.

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§1 INTRODUCTION

DEFINITION : A (conformal) metric ds^2 on a Riemann surface S has a *conical singularity* of *order* β (β a real number > -1) at a point $p \in S$ if in some neighbourhood of p :

$$(1.1) \qquad ds^2 = e^{2u}|dz|^2,$$

where z is a coordinate of S defined in this neighbourhood and u is a function such that

$$(1.2) \qquad u(z) - \beta \cdot log|z - z(p)|$$

is continuous at p.

Observe that $|z|^{2\beta} \cdot |dz|^2$ is the metric of an Euclidian cone of total angle $\theta = 2\pi(\beta + 1)$, thus, if a surface has at some point a conical singularity of order β, then this surface admits at this point a " tangent cone " of angle $\theta = 2\pi(\beta + 1)$.

For instance, if one whishes to consider a polyhedron, an orbifold or a branched covering of dimension two from a Riemannian viewpoint, then, what is seen is a Riemannian surface with conical singularities.

If (S, ds^2) has at $p_1, p_2, \ldots, p_n$ conical singularities of order $\beta_1, \beta_2, \ldots \beta_n$, then ds^2 is said to *represent* the *divisor* $\boldsymbol{\beta} := \sum_{i=1}^{n} \beta_i p_i$. (The divisors we shall consider are real linear combinations of points of S with coefficients > -1).

The general problem of the theory of surfaces with conical singularities can be formulated in the following way : given $(S, \boldsymbol{\beta})$ a Riemann surface with a divisor, try to understand the set of conformal metrics representing $\boldsymbol{\beta}$ on S!

A first information is given by the Gauss-Bonnet formula :

$$(1.3) \qquad \frac{1}{2\pi} \iint_S K dA = \chi(S) + \sum_{i=1}^{n} \beta_i$$

where $\chi(S)$ is the Euler characteristic of S, the *area element* dA is defined from the local expression (1.1) by $dA = \frac{i}{2} e^{2u} dz \wedge d\bar{z}$, and the curvature K is given by

$$(1.4) \qquad K = -4e^{-2u} \frac{\partial^2 u}{\partial z \partial \bar{z}}.$$

We will only consider the case where the curvature is constant. So we state the following problem : Given $(S,\boldsymbol{\beta})$, describe the set $\mathcal{M}(S,\boldsymbol{\beta})$ of conformal metrics with constant curvature representing $\boldsymbol{\beta}$ on S. In particular, are there obstructions for $\mathcal{M}(S,\boldsymbol{\beta}) \neq \emptyset$?

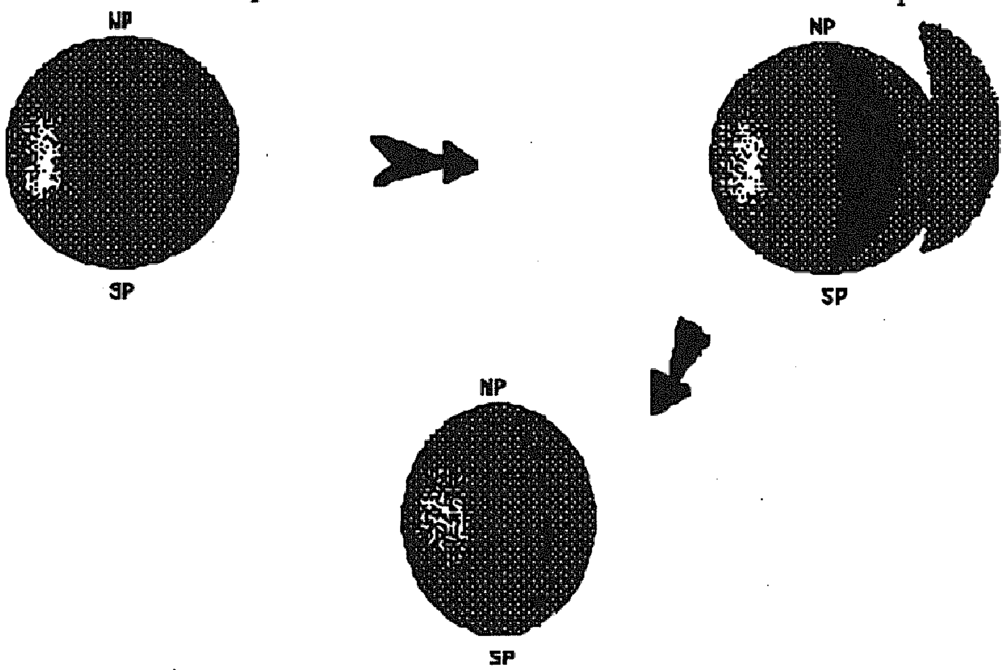
We have two goals in this paper : First we describe the relationships between metrics in $\mathcal{M}(S,\boldsymbol{\beta})$ and another structure on S (projective connections). Then we classify the metrics of constant curvature with two conical singularities on a sphere.

Note that a metric in $\mathcal{M}(S,\boldsymbol{\beta})$ is analytic outside the singularities (since it has constant curvature).

§2 Explicit examples

Example 1 A *bigon* of angle φ is the region on a standard sphere bounded by two geodesics joining the Northpole (NP) to the Southpole (SP) and forming an angle φ. If one removes from the sphere S^2 a bigon of angle φ and glue isometrically together the two sides of the remaining bigon, one obtains a sphere with a metric of curvature $K \equiv +1$ and two conical singularities of order $\beta = -\frac{\varphi}{2\pi} \in]-1,0]$ (the divisor is $\boldsymbol{\beta} = -\frac{\varphi}{2\pi} \cdot NP - \frac{\varphi}{2\pi} \cdot SP$).

This is the standard recipe to make an american football out of a european one.



Example 1' If one cuts a sphere along a geodesic joining NP to SP and glue to the boundary of this cutted sphere a bigon of angle φ, one obtains a sphere with a metric of curvature $K \equiv +1$ representing the divisor $\boldsymbol{\beta} = \beta \cdot NP + \beta \cdot SP$, where $\beta = \frac{\varphi}{2\pi} \in]0,1]$.

Example 1'' By glueing n bigons of angle $\varphi_1, \varphi_2, \ldots \varphi_n$ using the same method as in the example 1' , one may construct a sphere with a metric of curvature $K \equiv +1$ representing the divisor $\boldsymbol{\beta} = \beta \cdot NP + \beta \cdot SP$, where $\beta = \sum_{i=1}^{n} \frac{\varphi_i}{2\pi}$ can take any positive value.

Observe that in the above examples, the two singularities are antipodal.

Example 2 Take a branched covering $f : S^2 \to S^2$ of degree d with two branching points $p, q \in S^2$ of order $\beta = d-1$. Then the pullback by f of the standard metric on S^2 is a metric with curvature $K \equiv +1$ representing the 'ramification divisor' $\boldsymbol{\beta} = \beta \cdot p' + \beta \cdot q'$ (where $p' = f^{-1}(p), q' = f^{-1}(q)$).

In this example, the two singularities are not necesseraly antipodal. The distance $d(p,q) = d(p',q') \in]0, 2\pi]$ is, with β, the only isometry invariant of the surface. (Observe however that β has to be an integer.)

Our purpose is to prove that these are the only examples of metric on a sphere with constant curvature and two conical singularities, more precisely, we have :

THEOREM I. *Let (S, ds^2) be a Riemannian surface with two conical singularities homeomorphic to the sphere. Assume that ds^2 has constant curvature K, then :*

(1) $K > 0$;

(2) The order of both singularities are equal;

(3) If the order β is an integer , then (S, ds^2) is isometric to a ramified covering of degree $d = \beta + 1$ of the sphere with constant curvature K;

(4) If β is not an integer, then the two singularities are antipodal (i.e. the distance between them is $\frac{\pi}{\sqrt{K}}$ and they are conjugate points).

Furthermore, any two such surfaces are isometric if and only if they have the same curvature, the same order at the singularities and the same distance between the two singular points.

This theorem states that if S is a sphere with a divisor $\boldsymbol{\beta} = \beta_1 p_1 + \beta_2 p_2$, then $\mathcal{M}(S, \boldsymbol{\beta})$ is a segment if $\beta_1 = \beta_2 \in \mathbf{N}$, is reduced to a point if $\beta_1 = \beta_2 \notin \mathbf{N}$ and is empty if $\beta_1 \neq \beta_2$. In particular among the metrics representing some divisor $\boldsymbol{\beta} = \beta_1 p_1 + \beta_2 p_2$ on the sphere S^2, none has constant curvature. In [T], W.Thurston raises the following question :

What is the best pinching constant for a metric representing $\boldsymbol{\beta} = \beta_1 p_1 + \beta_2 p_2$ on S^2 ?

(i.e. what is the largest δ such that there exists a metric representing $\boldsymbol{\beta}$ with $\delta \leq K \leq 1$).

Theorem I will be a consequence of

THEOREM II. *If ds^2 is a conformal metric on $\mathbf{C} \cup \infty$ with constant curvature and conical singularities at $z = 0$ and $z = \infty$, then, there exists $K > 0$, $\mu \in [0, \infty[$, $\beta \in]-1, \infty[$, such that either β is an integer or $\mu = 0$, and, up to a change of coordinate ($z \to pz, p \in \mathbf{C}$ a constant), we have*

$$ds^2 = (2+2\beta)^2 \frac{|z|^{2\beta}|dz|^2}{(|1+\mu z^{\beta+1}|^2 + K|z|^{2\beta+2})^2}. \tag{2.1}$$

Let us derive theorem I from theorem II : By the uniformisation theorem, (S, ds^2) is conformally equivalent to $\mathbf{C} \cup \infty$; one may of course assume that the singularities of (S, ds^2) correspond to $z = 0$, and $z = \infty$. Theorem II implies therefore that (S, ds^2) is isometric to $\mathbf{C} \cup \infty$ equipped with the metric (2.1).

Using (1.4), we check that K in (2.1) is the curvature, this proves (1).
It is obvious from (2.1) that $z = 0$ is a conical singularity of order β. To see what happens at $z = \infty$, consider the coordinate $\zeta := 1/z$. A computation gives us the following expression for the metric in this coordinate :

$$ds^2 = (2+2\beta)^2 \frac{|\zeta|^{2\beta}|d\zeta|^2}{(|\mu + \zeta^{\beta+1}|^2 + K)^2}, \tag{2.2}$$

so that it is apparent that at $\zeta = 0$ (i.e. $z = \infty$), the metric also has a conical singularity of order β, we have thus shown (2).
To prove (3), suppose $\beta \in \mathbf{N}$ and consider the map

$$w = f(z) := \frac{z^{\beta+1}}{1 + \mu z^{\beta+1}}.$$

This map is a ramified covering $f : \mathbf{C} \cup \infty \to \mathbf{C} \cup \infty$ with ramification divisor $\boldsymbol{\beta} = \beta \cdot 0 + \beta \cdot \infty$ (f has singularities of order β over $w = 0$ and $w = \frac{1}{\mu}$). A straightforward computation shows that the metric (2.1) is the pullback under f of the standard metric on $\mathbf{C} \cup \infty$ with constant curvature $K > 0$:

$$ds^2 = f^* \left(\frac{4|dw|^2}{(1 + K|w|^2)^2} \right). \tag{2.3}$$

To show (4), observe that the distance between $z = 0$, and $z = \infty$ in the metric (2.1) is $d(0, \infty) = \frac{2}{\sqrt{K}} \cdot Arctg(\sqrt{K}/\mu)$[1] ; thus, if $\beta \notin \mathbf{N}$, then $\mu = 0$ and therefore $d(0, \infty) = \frac{\pi}{\sqrt{K}}$. Furtheremore, the lines $\{re^{i\varphi}, 0 \leq r \leq \infty\}$ form a family of geodesic segments joining the two singularities and depending on φ, hence these two points are conjugate.
Finally, the last assertion in the theorem trivially follows from the fact that β, K and μ are the only ingredients in the expression (2.1) for the metric. □

§3 Projective connection on a Riemann Surface

Definition A projective connection η on a Riemann Surface S is a rule which associates to each local uniformizer z on S a meromorphic quadratic differential

$$\eta(z) = \phi(z) dz^2 \tag{3.1}$$

defined in the domain of z , in such a way that if $w = w(z)$ is another local uniformizer, then, we have in the overlap of the domain of z with that of w :

$$\eta(w) = \eta(z) + \{z, w\} dw^2, \tag{3.2}$$

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[1] In computing with the metric (2.1), use : $\frac{d}{dt} \frac{2}{\sqrt{K}} Arctg\left(\frac{\sqrt{K} t^{\beta+1}}{1+\mu t^{\beta+1}}\right) = \frac{(2+2\beta)t^\beta}{(1+\mu t^{\beta+1})^2 + K t^{2\beta+2}}$.

where { , } denotes the Schwarzian derivative :

$$\{f, w\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2, \tag{3.3}$$

(Some facts concerning this differential operator are collected in the appendix).

Formula (3.2) means that if $\eta(w) = \psi(w)dw^2$, then $\psi(w(z)) = \phi(z)\left(\frac{dz}{dw}\right)^2 + \{z, w\}$. The projective connection η is *regular* at $p \in S$ if η is holomorphic at this point, it has a *regular singularity of weight* c if

$$\eta(z) = \left(\frac{c}{z^2} + \frac{d}{z} + \phi_1(z)\right) dz^2, \qquad \phi_1 \quad \textit{holomorphic}, \tag{3.4}$$

where z is a uniformizer at p (such that $z(p) = 0$).

LEMMA. *This definition of the weight is independant of the choice of uniformizer.*

Proof: Suppose that η is given by (3.4) and that w is another uniformizer at p (such that $w(p) = 0$). We have $\eta(w) = \psi(w)dw^2$, with $\psi(w(z)) = \left(\frac{c}{z^2} + \frac{d}{z} + \phi_1(z)\right)\cdot\left(\frac{dz}{dw}\right)^2 + \{z, w\}$. However, $\frac{dz}{dw} = \left(\frac{z}{w} + zg(z)\right)$ with g holomorphic, so $\psi(w) = \frac{c}{w^2} + h(w)$, h having at most a simple pole □

The connection is said to be *compatible* with the divisor $\boldsymbol{\beta} := \sum_{i=1}^{n} \beta_i p_i$ if it is regular in $S - \{p_i\}$ and has, for each i, a regular singularity of wheight $c_i = -\frac{1}{2}\beta_i(\beta_i + 2)$ at p_i. Observe that if $\beta_i > -1$, then $c_i < \frac{1}{2}$. Projective connections compatible with integral divisors have been studied by Mandelbau [M] .

Example If f is a meromorphic function on S, then a projective connection is defined on S by

$$\eta(z) = \{f, z\}dz^2. \tag{3.5}$$

It is compatible with the ramification divisor of $f : S \to \mathbf{CP}^1$.

LEMMA. *If $ds^2 = e^{2u}|dz|^2$ is a (conformal) metric of constant curvature on S representing the divisor $\boldsymbol{\beta}$ then*

$$\eta(z) = 2\left(\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial u}{\partial z}\right)^2\right)\cdot dz^2 \tag{3.6}$$

defines a projective connection compatible with the divisor $\boldsymbol{\beta}$.

Proof : Three things have to be checked : (A) The quadratic differential η, given by (3.6) is holomorphic outside of the support of $\boldsymbol{\beta}$, (B) η is a projective connection (i.e. (3.2) is verified) , (C) If p is a conical singularity of order β, then η has at p a regular singularity of weight $c = -\frac{1}{2}\beta(\beta + 2)$.

To check (A), one applies $\frac{\partial}{\partial z}$ to the equation (1.4) :

$$0 = \frac{\partial K}{\partial z} = -4e^{-2u}\left(\frac{\partial^3 u}{\partial z^2 \partial \bar{z}} - 2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z \partial \bar{z}}\right)$$
$$= -4e^{-2u}\frac{\partial}{\partial \bar{z}}\left(\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial u}{\partial z}\right)^2\right)$$

(B) If w and z are two coordinates in some part of S, then $ds^2 = e^{2u}|dz|^2 = e^{2v}|dw|^2$ with $v = u + log|dz/dw|$; therefore we have :

$$\eta(w) = 2\left(\frac{\partial^2(u + log|dz/dw|)}{\partial w^2} - \left(\frac{\partial(u + log|dz/dw|)}{\partial w}\right)^2\right)\cdot dw^2$$
$$= \eta(z) + \{z, w\}dw^2.$$

(C) Suppose that p is a conical singularity of order β; then we can write $ds^2 = e^{2u}|dz|^2$, where $u(z) = \beta log|z| + u_1(z)$ with u_1 continuous and verifying the elliptic equation

$$\frac{\partial^2 u_1}{\partial z \partial \bar{z}} = -4K|z|^{2\beta}e^{2u_1}.$$

Assume first that $\beta > 0$, using the classical elliptic regularity theory,we can show that u_1 is of class C^2 at $z = 0$.
We have

$$2\left(\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial u}{\partial z}\right)^2\right) = 2\left(\frac{\partial^2 u_1}{\partial z^2} - \left(\frac{\partial u_1}{\partial z}\right)^2\right) - 2\frac{\beta}{z}\frac{\partial u_1}{\partial z} - \frac{\beta(\beta+2)}{2z^2},$$

this proves (C) in this case.
If $-1 < \beta < 0$, then u_1 might not be C^2 and the computation above might not work. However, we may lift the metric to a local branched cover, to this aim, set $z = w^m$ ($m \in \mathbf{N}$), then the metric can be lifted in the $w - plane$: $ds'^2 = e^{2u'}|dw|^2$, with $u' = u + log|dz/dw| = u + (m-1)log|w| + log\, m$, and one sees that ds'^2 has at $w = 0$ a conical singularity of order $\beta' = m(\beta+1) - 1$. Choosing m large enough, we can have $\beta' > 0$, so that $\eta(w) = 2\left(\frac{\partial^2 u'}{\partial w^2} - \left(\frac{\partial u'}{\partial w}\right)^2\right)\cdot dw^2$ has a regular singularity of weight $-\frac{1}{2}\beta'(\beta'+2)$ at $w = 0$. Now we have

$$\eta(z) = \eta(w) - \{z, w\}dw^2$$
$$= \eta(w) - \frac{1-m^2}{2w^2}dw^2$$
$$= \left(-\frac{\beta'(\beta'+2)}{2w^2} - \frac{1-m^2}{2w^2} + \psi_1(w)\right)dw^2$$
$$= -\frac{1}{2}(\beta(\beta+2)m^2)\frac{dw^2}{w^2} + \psi_1(w)dw^2,$$

with ψ_1 having at most a simple pole. Now (C) follows from $\frac{dw}{w} = \frac{dz}{mz}$ $\square$

If a projective connection η is related to a metric ds^2 according to (3.6), one says that ds^2 is *associated* to η.

Let us introduce the following sets defined on a Riemann surface with divisor $(S,\boldsymbol{\beta})$:

$\mathcal{PC}(S,\boldsymbol{\beta})$ = set of projective connections compatible with $\boldsymbol{\beta}$ on S.

$\mathcal{M}(S,\boldsymbol{\beta},\eta)$ = set of conformal metrics with constant curvature representing $\boldsymbol{\beta}$ on S and associated to η.

$\mathcal{M}(S,\boldsymbol{\beta},) = \cup_\eta \mathcal{M}(S,\boldsymbol{\beta},\eta)$ (= set of all conformal metrics with constant curvature representing $\boldsymbol{\beta}$ on S).

If one wants to describe the set $\mathcal{M}(S,\boldsymbol{\beta})$, one has first to understand $\mathcal{PC}(S,\boldsymbol{\beta})$, and then , for each $\eta \in \mathcal{PC}(S,\boldsymbol{\beta})$, one has to study $\mathcal{M}(S,\boldsymbol{\beta},\eta)$.

Two projective connections compatible with the same divisor $\boldsymbol{\beta}$ on a Riemann Surface S differ by a quadratic differential having at most a simple pole at p_i $(i = 1,\ldots,n)$; thus $\mathcal{PC}(S,\boldsymbol{\beta})$ is either empty or it is a finite dimensional affine space.

The set $\mathcal{M}(S,\boldsymbol{\beta})$ can be embedded in a linear space. To this purpose, choose a point $q \in S$ not in the support of $\boldsymbol{\beta}$ and denote by $Jet^\infty(q)$ the space of infinite jets at q of functions $h : S \to \mathbf{C}$, and choose also a uniformizer z at q. Then $ds^2 = e^{2u}|dz|^2$ at q and one may define : $j(ds^2) :=$ jet at q of e^{-u}, (= jet at q of $\frac{|dz|}{ds}$).

Since ds^2 is analytic, j defines an injective map (assuming S connected) $j : \mathcal{M}(S,\boldsymbol{\beta}) \to Jet^\infty(q)$.

The situation can be summarized by the following diagramm :

$$\begin{array}{ccccc} \mathcal{M}(S,\boldsymbol{\beta},\eta) & \xrightarrow{\;i\;} & \mathcal{M}(S,\boldsymbol{\beta}) & \xrightarrow{\;j\;} & Jet^\infty(q) \\ & & {\scriptstyle\pi}\big\downarrow & & \\ & & \mathcal{PC}(S,\boldsymbol{\beta}) & & \end{array}$$

Once a projective connection is fixed, a metric associated to this connection depends on four parameters only.[2]

Inddeed, one has the following

[2] A heuristical argument for this fact is the following observation : If ds^2 is a conformal metric of constant curvature K defined in a neighbourhood of a point $q \in \mathbf{C}\cup\infty$ and $g \in SL_2\mathbf{C}$ is a Möbius transformation sufficiently close to the identity, then g^*ds^2 is also a conformal metric of constant curvature K defined in a neighbourhood of q. Hence, the local 6 dimensional Lie group $SL_2\mathbf{C}$ acts on the set of germs at q of conformal metrics with constant curvature K associated to a fixed projective connection. The stabiliser of a metric for this action is the 3 dimensional local subgroup of its isometries , thus, one should except a metric associated with a given projective connection to depend on $6 - 3 + 1 = 4$ real parameters only (the additional parameter beeing the curvature).

Proposition 1. *There exists a linear map $Jet^{\infty}(q) \to \mathbf{R}^4$ such that the restriction of this map to $\mathcal{M}(S,\boldsymbol{\beta},\eta)$ is injective.*

Proof: A projective connection is given at q by $\eta(z) = \phi(z)dz^2$, where ϕ is a holomorphic function, and can thus be written as

$$\phi(z) = -2\sum_{\nu=0}^{\infty} b_\nu z^\nu. \tag{3.7}$$

Suppose $ds^2 = e^{2u}|dz|^2 \in \mathcal{M}(S,\boldsymbol{\beta},\eta)$, and set $h = e^{-u}$. Then we have from (3.6)

$$\frac{\partial^2 h}{\partial z^2} = -\frac{1}{2}\phi(z)\cdot h. \tag{3.8}$$

The function $h = h(z,\bar{z})$ being a real analytic function, we have:

$$h(z,\bar{z}) = \sum_{m,n=0}^{\infty} a_{m,n} z^m \bar{z}^n. \tag{3.9}$$

The equation (3.8), with (3.7), gives then

$$a_{m+2,n} = \frac{1}{(m+2)(m+1)}\cdot\sum_{\nu=0}^{m} b_\nu \cdot a_{m-\nu,n}. \tag{3.10}$$

Since $h = \bar{h}$, we have $a_{m,n} = \bar{a}_{n,m}$. so (3.10) implies that a solution h of (3.8) is completely determined from $a_{0,0}, a_{1,1} \in \mathbf{R}$ and $a_{0,1} = \bar{a}_{1,0} \in \mathbf{C}$. Thus the proposition is proved if one defines the map $Jet^{\infty}(q) \to \mathbf{R}^4$ by $h = \frac{|dz|}{ds} \to (a_{0,0}, a_{1,1}, Re(a_{0,1}), Im(a_{0,1}))$ □

§4 Projective connections with two singularities on the sphere

We are now in position to classify the projective connections with two regular singularities on the sphere :

Proposition 2. *Let η be a projective connection on $S^2 = \mathbf{C}\cup\infty$ with regular singularities at $z = 0$ and $z = \infty$, then we have (in the standard coordinate z) :*

$$\eta(z) = \frac{c}{z^2}\cdot dz^2, \qquad c \in \mathbf{C}. \tag{4.1}$$

In particular, both singularities have the same weight.

As a consequence, the set $\mathcal{PC}(S^2, \beta_1 p_1 + \beta_2 p_2)$ is empty if $\beta_1 \neq \beta_2$.

Proof : If η is a projective connection on $\mathbf{C} \cup \infty$ with regular singularities at $z = 0$ and $z = \infty$, then we have by (3.4)

$$\eta(z) = \left(\frac{c}{z^2} + \frac{d}{z} + \phi_1(z)\right) dz^2,$$

where ϕ_1, being holomorphic in $\mathbf{C}$, is an entire function. Therefore either ϕ_1 is a constant or it has an essential singularity at $z = \infty$.
Setting $w = 1/z$, we have $\{z, w\} = 0$, and so, using (3.2) we have $\eta(w) = \psi(w) dw^2$ with $\psi(w) = \left(\frac{c}{z^2} + \frac{d}{z} + \phi_1(z)\right)\left(\frac{dz}{dw}\right)^2 = \frac{c}{w^2} + \frac{d}{w^3} + \frac{\phi_1(w)}{w^4}$. Since ψ is meromorphic at $w = 0$ with a pole of order 2; we must have $\phi_1 = 0$ and $d = 0$. □

The situation is rather different for surfaces of higher genus :

PROPOSITION 3. *Let $(S, \boldsymbol{\beta})$ be compact Riemann Surface of genus $g \geq 2$ equipped with a divisor $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i$. Then the set $\mathcal{PC}(S, \boldsymbol{\beta})$ of projective connections compatible with $\boldsymbol{\beta}$ is a complex affine space of dimension $3g - 3 + n$.*

This is the first part of theorem 3 in [M], we shortly give the idea of the proof :
As seen in the §3, $\mathcal{PC}(S, \boldsymbol{\beta})$ is either empty or it is an affine space over the space of quadratic differential having at most simple poles at the p_i. The dimension of this space is computed (with the help of the Riemann-Roch theorem) to be $3g - 3 + n$. Hence, we only have to show that there exists at least one projective connection compatible with $\boldsymbol{\beta}$. Using the Riemann-Roch theorem again, we can show that the for each $p \in S$, there exist a (non–unique) quadratic differential ω_p on S such that

$$\omega_p = \left(\frac{1}{z^2} + \varphi(z)\right) dz^2$$

(where z is a uniformizer at p, and φ has at most a simple pole), and ω_p is smooth outside p.
Let $\breve{\eta}$ be a smooth projective connection on S, then

$$\eta = \breve{\eta} - \frac{1}{2} \sum_{i=1}^{n} \beta_i(\beta_i + 2)\omega_{p_i}$$

is the desired connection □

§5 PROOF OF THEOREM II

Let $ds^2 = e^{2u}|dz|^2$ be a conformal metric metric on $\mathbf{C} \cup \infty$ with constant curvature representing the divisor $\boldsymbol{\beta} = \beta_1 0 + \beta_2 \infty$. By the formula (3.6), the metric ds^2 defines a projective connection η on $\mathbf{C} \cup \infty$ compatible with $\boldsymbol{\beta}$, hence Proposition 2 implies that $\beta_1 = \beta_2 (=: \beta)$ and

$$\eta(z) = -\frac{\beta(\beta + 2)}{2} \cdot \frac{dz^2}{z^2}. \tag{5.1}$$

Setting $h = e^{-u}$, we have from (5.1) and (3.8)

$$\frac{\partial^2 h}{\partial z^2} = h \cdot \frac{\beta(\beta+2)}{4} \cdot \frac{1}{z^2}. \tag{5.2}$$

All solutions of (5.2) are of the form

$$h(z,\bar{z}) = f(\bar{z}) z^{-\beta/2} + g(\bar{z}) z^{1+\beta/2}. \tag{5.3}$$

Since h is real ($h = \bar{h}$) and analytic, we must have

$$h(z,\bar{z}) = a(z\bar{z})^{-\beta/2} + b z^{1+\beta/2}\bar{z}^{-\beta/2} + \bar{b}\bar{z}^{1+\beta/2} z^{-\beta/2} + c(z\bar{z})^{1+\beta/2} \tag{5.4}$$

with $a, c \in \mathbf{R}, b \in \mathbf{C}$. It follows from (1.2) that $a \neq 0$, so one may define $\mu := b/a$ and $\nu = (ac - b\bar{b})/a^2$. In this way, we get from (5.4) :

$$h(z,\bar{z}) = a \cdot \left(\frac{|1 + \mu z^{\beta+1}|^2 + \nu |z|^{2\beta+2}}{|z|^\beta} \right). \tag{5.4'}$$

The condition that h is a uniform function is equivallent to :

$$\mu = 0 \quad \underline{or} \quad \beta \in \mathbf{N}, \tag{5.5}$$

and the condition that $h = e^{-u}$ is a globally defined positive function on $\mathbf{C} \setminus \{0\}$ is equivalent to :

$$a > 0 \quad \underline{and} \quad \nu > 0. \tag{5.6}$$

So far, we have proved that a conformal metric on $\mathbf{C} \cup \infty$ with constant curvature representing the divisor $\boldsymbol{\beta} = \beta 0 + \beta\infty$ is given by

$$ds^2 = \frac{|dz|^2}{h^2} = \frac{1}{a^2} \cdot \frac{|z|^{2\beta}|dz|^2}{(|1 + \mu z^{\beta+1}|^2 + \nu |z|^{2\beta+2})^2}, \tag{5.7}$$

with a, μ, ν subjected to the conditions (5.5) and (5.6).
Setting $z = pz'$ ($p \in \mathbf{C}$), (5.7) becomes

$$ds^2 = \frac{1}{a'^2} \cdot \frac{|z'|^{2\beta}|dz'|^2}{(|1 + \mu' z'^{\beta+1}|^2 + \nu' |z'|^{2\beta+2})^2}, \tag{5.8}$$

with $a' = a|p|^{-(\beta+1)}, \mu' = \mu p^{\beta+1}, \nu' = \nu |p|^{2\beta+2}$. We may, by appropriately choosing p, obtain

$$a' = \frac{1}{2\beta + 2} \quad , \quad \mu' \in [0, \infty[; \tag{5.9}$$

(indeed choose

$$p = \begin{cases} (a(2\beta+2)\frac{|\mu|}{\mu})^{1/(\beta+1)} & if \mu \neq 0; \\ (a(2\beta+2))^{1/(\beta+1)} & if \mu = 0). \end{cases}$$

We thus have with this choice of p :

$$ds^2 = (2+2\beta)^2 \frac{|z'|^{2\beta}|dz'|^2}{(|1+\mu' z'^{\beta+1}|^2 + \nu'|z'|^{2\beta+2})^2}, \tag{5.10}$$

with $\nu' > 0, \mu' \in [o,\infty[, \beta \in]-1,\infty[$ and $\mu' = 0$ if $\beta \notin \mathbf{N}$. □

APPENDIX : ABOUT THE SCHWARZIAN DERIVATIVE.

Let $f(t)$ be a non constant meromorphic function, its Schwarzian derivative is defined by

$$\{f,t\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = \frac{2f'f''' - 3f''^2}{2f'^2},$$

for instance $\{e^t,t\} = -\frac{1}{2}$, $\{logt,t\} = \frac{1}{2t^2}$, $\{t^\alpha,t\} = \frac{1-\alpha^2}{2t^2}(\alpha \neq 0)$, $\{\frac{at+b}{ct+d},t\} = 0\,(if\,ad - bc \neq 0)$.

The Schwarzian enjoys the following properties :

(1) $\{y,t\} = \{x,t\}$ iff. $x = \frac{ay+b}{cy+d}$,

(2) $\{x,y\} = -\{y,x\}\left(\frac{dx}{dy}\right)^2$,

(3) $\{y,x\} = \left(\frac{dt}{dx}\right)^2(\{y,t\} - \{x,t\})$,

(4) $\{x^\alpha,t\} = \frac{1-\alpha^2}{2}\left(\frac{x'}{x}\right)^2 + \{x,t\}$,

Furthermore, if u and v are linearly independant solutions of the equation $u'' = -\frac{1}{2}\phi(t)\cdot u$, then $\{\frac{u}{v},t\} = \phi$.

Property (1) states that the Schwarzian derivative is invariant under the action of the projective group $SL_2\mathbf{C}$, it is therefore a usefull tool in studying the projective geometry of a Riemann surface.

The properties (2) and (3) above tell us that $\sigma(z,w) = \{z,w\}dw^2$ is a one cocycle in the sheaf of germ of holomorphic quadratic differential. Thus, by definition, a projective connection is a meromorphic 0–cochain η (in the same sheaf) such that $\delta\eta = \sigma$. In particular, there always exists a smooth projective connection a compact Riemann surface, since $H^1(S.\kappa^2) = 0$ (where κ is the canonical bundle on S).

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