OPTIMAL TRANSPORT OF MEASURES VIA AUTONOMOUS VECTOR FIELDS

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Abstract. We study the problem of transporting one probability measure to another via an autonomous velocity field. We rely on tools from the theory of optimal transport. In one space-dimension, we solve a linear homogeneous functional equation to construct a suitable autonomous vector field that realizes the (unique) monotone transport map as the time-1 map of its flow. Generically, this vector field can be chosen to be Lipschitz continuous. We then use Sudakov's disintegration approach to deal with the multi-dimensional case by reducing it to a family of one-dimensional problems.

1. Introduction

We are interested in the problem of transporting one probability measure to another using an autonomous vector field.

This problem can be viewed from two perspectives. The *Lagrangian* one involves pushing the first measure forward to the second via the time-1 map of the flow generated by the vector field:

Problem 1 (Matching measures via the flow generated by an autonomous vector field)**.** *Given two probability measures $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$, with $d \geqslant 1$, construct an autonomous vector field $v : \mathbb{R}^d \to \mathbb{R}^d$ such that the corresponding flow*[1]*, i.e.*

$$\begin{cases} \partial_t \phi(t,x) = v(\phi(t,x)), & t > 0,\ x \in \mathbb{R}^d \\ \phi(0,x) = x, & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

is well-defined and satisfies

$$\phi(1,\cdot)_{\#}\mu_0 \equiv \mu_1. \tag{1.2}$$

We recall that the measure denoted by $\phi(1,\cdot)_{\#}\mu_0$ is defined by

$$\left(\phi(1,\cdot)_{\#}\mu_0\right)(A) := \mu_0\left(\phi(1,\cdot)^{-1}(A)\right), \quad \text{for every measurable set } A \subset \mathbb{R}^d,$$

and is called *image measure* or *push-forward* of μ_0 through $\phi(1,\cdot)$.

Problem 1 amounts to a question about *exact controllability* for ordinary differential equations.

The second perspective is *Eulerian* and involves a question of exact controllability for the continuity (partial differential) equation: steering the solution of the continuity equation from an initial state μ_0 to a target state μ_1 by using an autonomous velocity field v as a control:

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[1] The ODE (1.1) is interpreted in the following sense: $t \mapsto \phi(t,x)$ is absolutely continuous and

$$\phi(t,x) = x + \int_0^t v(\phi(s,x))\,\mathrm{d}s, \quad \text{for all } t \geqslant 0,$$

holds for all $x \in \mathbb{R}^d$.

Problem 2 (Exact controllability of the continuity equation using an autonomous velocity)**.** *Given two probability measures $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$, with $d \geqslant 1$, construct an autonomous vector field $v : \mathbb{R}^d \to \mathbb{R}^d$ such that the solution $\mu : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$ to the Cauchy problem*

$$\begin{cases} \partial_t \mu(t,x) + \operatorname{div}_x(v(x)\,\mu(t,x)) = 0, & t > 0,\ x \in \mathbb{R}^d, \\ \mu(0,x) = \mu_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.3}$$

is well-defined and satisfies

$$\mu(1,\cdot) \equiv \mu_1. \tag{1.4}$$

If v is smooth then, by the *method of characteristics*, the (unique) solution μ of (1.3) can be represented using the (unique) flow ϕ of v, and vice-versa.

More generally, if both μ and ϕ exist and are unique, the equivalence between Problem 1 and Problem 2 is a consequence of the *Lagrangian representation formula* for the solution of (1.3):

$$\mu(t,\cdot) \equiv \phi(t,\cdot)_{\#}\mu_0, \qquad t \geqslant 0. \tag{1.5}$$

This is not generally the case if we drop the uniqueness assumption on μ. For example, even when (1.1) has a unique flow and (1.5) represents a solution (called the *Lagrangian solution*) of (1.3), it does not necessarily encompass all solutions[2]; therefore, solving Problem 1 provides a solution to Problem 2, but the converse does not necessarily hold. We refer to [22, 21] for a discussion on the validity of (1.5).

The construction of an autonomous vector field addressing Problem 1 (or Problem 2) can be performed in a straightforward way if μ_0 and μ_1 are superpositions of Dirac deltas:

$$\mu_0 := \frac{1}{N}\sum_{i=1}^{N} \delta_{\{x=x_i\}}, \quad \mu_1 := \frac{1}{N}\sum_{i=1}^{N} \delta_{\{x=y_i\}},$$
$$\text{with } \{x_i\}_{i\in\{1,\dots,N\}}, \{y_i\}_{i\in\{1,\dots,N\}} \subset \mathbb{R}^d \text{ and } x_i \neq x_j \text{ for } i \neq j.$$

Indeed, in this case, for $d \geqslant 2$, it suffices to build non-intersecting paths (except, maybe, at the end-points) linking x_i to y_i for all $i \in \{1, \dots, N\}$. This is always possible if one uses, for example, the optimal transport map between the points (or small perturbations, if the points are co-linear). In case $d = 1$, the points y_i also need to be distinct, and a further argument is required, but such a construction is not much more difficult (see Lemma 5.1 for details).

Conversely, when the measures are not just superposition of deltas, even for $d = 1$ such a construction becomes more delicate. In this work, we focus our attention on solving Problem 1 and Problem 2 in the case when μ_0 is absolutely continuous with respect to the Lebesgue measure, *i.e.*, $\mu_0 \ll \mathscr{L}^d$, and has a continuous density. Our strategy is based on tools from the theory of optimal transport of measures (see, *e.g.*, [62, 63, 52, 11, 35, 58, 20] for an overview of the topic). In particular, we build an autonomous velocity field from a given Monge's optimal transport map. That is, we turn to the following question.

Problem 3 (Realizing a given optimal transport map as time-1 map of the flow associated with an autonomous vector field)**.** *Given two probability measures $\mu_0, \mu_1 \in \mathcal{P}_{\text{a.c.}}(\mathbb{R}^d)$, with*

[2] In one space-dimension, if the vector field is continuous and autonomous, owing to [21, Proposition 5], uniqueness for the (1.1) implies that every solution of (1.3) is represented by (1.5) and, in particular, uniqueness for (1.3). When we drop the continuity assumption, this is generally false (see [22]).

In any space-dimension, the claim is true for *non-negative measures* by Ambrosio's superposition principle (see [12, Theorem 8.2.1]); however, the superposition principle cannot be extended to signed solutions (see [39]). On the other hand, in the class of signed measures, the claim still holds true, *e.g.*, if the velocity field is either Lipschitz continuous (see [12, Proposition 8.1.7]), or log-Lipschitz continuous (see [15, Théorème 5.1]), or satisfying a quantitative two-sided diagonal Osgood condition (see [10, Theorem 1]).

$d \geqslant 1$, and an optimal transport map T *between them, construct an autonomous vector field $v : \mathbb{R} \to \mathbb{R}$ such that*

$$\phi(1, \cdot) \equiv \mathrm{T},$$

where ϕ solves (1.1).

First, in Section 2, we solve Problem 3 (and, as a byproduct, Problem 1 and Problem 2) in case $d = 1$. In one space-dimension, the results of the theory of optimal transport are very sharp: provided that the source measure μ_0 has no atoms, there exists only one monotone non-decreasing transport map (which is optimal for the cost $\mathrm{c}(x, y) := |x - y|^p$, with $p \geqslant 1$), as recalled in Theorem 2.1. We show that this map can be realized as time-1 map of an autonomous vector field.

Theorem A (Exact controllability, $d = 1$)**.** *Let $\mu_0, \mu_1 \in \mathcal{P}_{\mathrm{a.c.}}(\mathbb{R})$ be two probability measures with convex support*[3]*, and continuous densities positive in their supports. Then, there exists a solution to Problem 1 and Problem 3.*

More precisely, there exists an autonomous velocity field v and a unique solution ϕ to (1.1), up to time $t = 1$, which satisfies $\mathrm{T} \equiv \phi(1, \cdot)$ in $\operatorname{supp} \mu_0$, and thus (1.2), where T *is the unique monotone transport map between μ_0 and μ_1.*

Remark 1.1 (Non-uniqueness of the velocity field). The velocity field is non-unique and, in general, as we will see in its construction in Section 2.1, it is obtained from an arbitrary prescription in an open set. More precisely, given $x_0 \in \operatorname{supp}(\mu_0)$, we *arbitrarily* fix the velocity field in $(x_0, \mathrm{T}(x_0))$ and then extend it *uniquely* to the interval between consecutive fixed points of T containing x_0. Morally, this is enough because the final position of any $y \in (x_0, \mathrm{T}(x_0))$ can be modulated only by the values of the velocity field in $(\mathrm{T}(x_0), \mathrm{T}(y))$.

Remark 1.2 (On the solution to Problem 2). The v constructed also gives a solution to Problem 2 in a suitable sense. That is, the function $\mu(t, \cdot) = \phi(t, \cdot)_{\#}\mu_0$ satisfies (1.3) as follows: There exists a discrete set $\partial\mathcal{S} = \partial\{x = \mathrm{T}(x)\}$ where $v = 0$ such that $\mu(t, \cdot)$ satisfies (1.3) in the distributional sense in $\operatorname{supp}(\mu(t, \cdot))\backslash\partial\mathcal{S}$ and also a no-flow condition through $\partial\mathcal{S}$ (namely, trajectories starting outside of $\partial\mathcal{S}$ never reach $\partial\mathcal{S}$ in finite time).

The need for the previous notion is because, as we will show in Lemma 3.3, the velocity fields constructed do not have to be L^1_{loc} in general at points $\partial\mathcal{S}$, and thus the $\mu(t, \cdot)$ above need not be a distributional solution across $\partial\mathcal{S}$. Somewhat related notions of solutions have been employed in [4, 55, 53] (in different contexts[4]).

Finally, we note that such solutions are the unique distributional solutions on $\operatorname{supp}(\mu(t, \cdot))\backslash\partial\mathcal{S}$. Indeed, this follows from the uniqueness of the flow for continuous and signed velocities in [22, Proposition 5.2], which we can apply in the open intervals between fixed points.

Remark 1.3 (On the positivity assumption). As it will be clear from the proofs, the assumption that the densities of μ_i are positive in $\operatorname{supp}(\mu_i)$ could be weakened to being positive only in $\operatorname{int}(\operatorname{supp}(\mu_i))$ instead. In this case, however, velocities would blow-up or vanish at the endpoints of the supports. On the other hand, removing the positivity assumption in the interior would allow for velocities blowing up or vanishing in the interior as well, interfering with the notions of solution used to make sense of the previous problems.

[3] We recall that, given a (non-negative) measure μ on a measurable space (X, Σ), $\operatorname{supp} \mu := \overline{\{A \in \Sigma : \mu(A) \neq 0\}}$. In particular, the support of a measure is a closed set and, for any compact $K \subset \mathbb{R}$, under our assumptions, we have that the densities are lower bounded by a positive constant in $K \cap \operatorname{supp}(\mu_i)$.

[4] In particular, in the autonomous setting, Aizenman, in [4], proved that a suitable generalized flow avoids a subset $S \subset \mathbb{R}^d$ provided that the vector field is sufficiently regular and S has sufficiently small box-counting dimension. In [55], this result was extended to the non-autonomous setting. More recently, in [53], the authors proved Ambrosio's uniqueness result (see [9]) by allowing the presence of a compact set of singularities $S \subset [0, T] \times \mathbb{R}^d$, such that $b|_{\Omega} \in L^1([0, T]; \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$ for all compact sets $\Omega \subset S^c$.

We refer to Theorem 2.2 for the precise statement of Theorem A. In particular, under suitable assumptions, we obtain further structure and regularity properties for the velocity field. The proof of this result is connected to the theory of linear homogeneous functional equations (see [47, 48, 17, 16]), which, for completeness, we recall in Appendix A.

In the previous construction, the arising vector fields v are not necessarily continuous in general (nor it is expected, even though, in Remark 3.1, we show that they are generically Lipschitz continuous), and they are only piecewise continuous. Even with that, we show the well-posedness (in particular, uniqueness) of the flow. Instead, if we relax the requirement that μ_1 should be achieved *exactly*, it is possible to further improve the (global) regularity of the velocity field v (see Corollary 3.2 for a more precise statement):

Theorem B (Approximate controllability, $d = 1$)**.** *In the setting of Theorem A, if, furthermore, μ_0 has a Lipschitz continuous density, then for any $\varepsilon > 0$, there exists μ_1^ε satisfying the same hypotheses as μ_1 and with*[5] $\operatorname{dist}(\mu_1, \mu_1^\varepsilon) < \varepsilon$ *such that the corresponding vector field v_ε from Theorem A can be taken Lipschitz continuous.*

Remark 1.4. In particular, we also have a unique solution to Problem 2 with vector field v_ε transporting μ_0 into μ_1^ε (*cf.* Remark 1.2).

Building on these results, in Section 4, we deal with the case $d \geqslant 2$. We use *Sudakov's disintegration approach* (see [52, Chapter 18]) to decompose the multi-dimensional optimal transport problem into a family of one-dimensional problems, namely, optimal transport problems on a family of *optimal transport rays* that forms a partition of $\operatorname{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1)$.

Sudakov's optimal transport map can be written as the "gluing" the one-dimensional monotone optimal transport maps built along the transport rays. Correspondingly, we are able to build a vector field in $\mathbb{R}^d$ by putting together the one-dimensional vector fields constructed previously (see Theorem 4.1 for a precise statement).

Theorem C (Exact controllability, $d \geqslant 1$)**.** *Let $\mu_0, \mu_1 \in \mathcal{P}_{\text{a.c.}}(\mathbb{R}^d)$, with $d \geqslant 1$, be two probability measures with convex support and continuous densities positive in their supports. Then there exists a solution to Problem 1 and Problem 3.*

More precisely, there exists an autonomous velocity field v transporting μ_0 into μ_1 in the sense of (1.4) and there exists a solution ϕ to (1.1), up to time $t = 1$, which satisfies $\mathrm{T} \equiv \phi(1, \cdot)$ in $\operatorname{supp}\mu_0$, and thus (1.2), where T is Sudakov's optimal transport map between μ_0 and μ_1.

Finally, in Section 5, we illustrate these results by presenting some one-dimensional examples.

1.1. Some related results.

In the available literature, Problem 1 and Problem 2 have been extensively analyzed in case the requirement on v being autonomous is dropped.

For example, the pioneering construction performed by Dacorogna and Moser in [30] provides a *time-dependent* velocity field realizing (1.2):

$$v(t, x) := \frac{\nabla f(x)}{\bar{\mu}_0(1 - t) + \bar{\mu}_1 t},$$

where $\bar{\mu}_i$ denotes the (smooth and positive) density of μ_i (for $i \in \{0, 1\}$), $f \in C^\infty(\mathbb{R}^d)$ is the unique solution of $-\Delta f = \bar{\mu}_1 - \bar{\mu}_0$ with zero mean.

More recently, in [32, 33], Duprez, Morancey, and Rossi constructed a *time-dependent* and localized perturbation of a given velocity field to achieve (1.4) (the localization being in a given non-empty, open, and connected portion of $\mathbb{R}^d$).

Finally, in [57, 6, 56], Problem 1 and Problem 2 were studied with "neural" velocity functions, *i.e.*, under the ansatz $v(t, x) := w(t)\sigma(\langle a(t), x\rangle + b(t))$, with $\sigma(x) := \max\{x, 0\}$

[5] Here $\operatorname{dist}(\mu_1, \mu_1^\varepsilon) < \varepsilon$ can be understood either in the L^1 or in the Wasserstein sense.

(the so-called *activation function of the neural network*) and control parameters $a, w \in L^\infty((0,1);\mathbb{R}^d)$ and $b \in L^\infty((0,1);\mathbb{R})$. The controls a, w, and b were constructed *piecewise-constant in time* (with an explicit bound on the number of jumps).

Problem 3, on the other hand, is somewhat reminiscent of the *dynamic formulation* of the optimal transport problem with quadratic cost introduced by Benamou and Brenier in [18], where the (non-autonomous) vector field is obtained from the minimization of the *kinetic energy*:

$$\min_{(\nu,v)} \int_{[0,1]} \int_{\mathbb{R}^d} |v_t(x)|^2 \, \mathrm{d}\nu_t(x) \, \mathrm{d}t, \tag{1.6}$$

where $(\nu, v) = (\nu_t, v_t)_{t\in[0,1]}$ are *admissible flow plans, i.e.*, $\nu_t \in C^0([0,1];\mathcal{P}(\mathbb{R}^d))$ is a weakly-continuous curve of measures (with $\nu_0 := \mu_0$ and $\nu_1 := \mu_1$), v_t is a *time-dependent* Borel vector field on $\mathbb{R}^d$, and they satisfy the continuity equation

$$\partial_t \nu_t + \mathrm{div}_x (v_t \, \nu_t) = 0, \qquad t \in [0,1], \; x \in \mathbb{R}^d,$$

in the sense of distributions (see [12, Chapter 8] and [24] for further details). We stress, however, that, for Problem 1 and Problem 2, we search for a *time-independent* v and are not concerned with minimizing the kinetic energy (1.6).

Finally, the problem of identifying if a given map can be "embedded in a flow" has a long history in the dynamical system community (see, *e.g.*, [23, 37] and references therein). Moreover, the study of inverse problems for some ODEs (namely, reconstructing a vector field from the time-t_i map of the flow for some $\{t_i\}_{i\in\{1,\dots,N\}}$) has been considered in [5] (and references therein). More recently, the same question was addressed in [49, Sections 3.1 & 3.2] under the additional ansatz that v is of neural type.

2. Construction in the one-dimensional case

If $d = 1$, (1.1) reduces to

$$\begin{cases} \partial_t \phi(t,x) = v(\phi(t,x)), & t > 0, \; x \in \mathbb{R}, \\ \phi(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

We start by remarking that, if the flow is unique (and defined up to time $t = 1$), then the map $\mathbb{R} \ni x \mapsto \phi(1,x)$ is non-decreasing (see [64, Section 8, XI. Theorem, p. 69])[6]. Therefore, if a velocity $v : \mathbb{R} \to \mathbb{R}$ exists such that the corresponding flow ϕ is unique and satisfies $\phi(1,\cdot)_{\#}\mu_0 = \mu_1$, then $\phi(1,\cdot)$ must coincide with the unique *monotone transport map* between μ_0 and μ_1, which is optimal for *Monge's optimal transport problem*

$$\mathrm{M}(\mu_0,\mu_1) := \min \left\{ \int_{\mathbb{R}} \mathrm{c}(\mathrm{T}(x),x) \, \mathrm{d}\mu_0(x) : \; \mathrm{T} : \mathbb{R} \to \mathbb{R} \text{ and } \mu_1 = \mathrm{T}_{\#}\mu_0 \right\},$$

[6] This monotonicity statement is true also for non-autonomous velocities. Let ϕ be the unique solution to

$$\begin{cases} \partial_t \phi = V(t,\phi(t,x)), & t > 0, \\ \phi(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{2.2}$$

where $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. We claim that $x \mapsto \phi(1,x)$ is non-decreasing.

Let us suppose, for the sake of finding a contradiction, that there exists $x_1 \leqslant x_2$ such that $\phi(1,x_2) < \phi(1,x_1)$. Since $t \mapsto \phi(t,\cdot)$ is a continuous function, we can apply the intermediate-value theorem: $x_2 = \phi(0,x_2) > \phi(0,x_1) = x_1$ and $\phi(1,x_2) < \phi(1,x_1)$ imply that $\phi(\bar{t},x_2) = \phi(\bar{t},x_1) =: \bar{\phi}$ for some $\bar{t} \in (0,1)$. This means that $\phi(t,x_1)$ and $\phi(t,x_2)$ solve the Cauchy problem

$$\begin{cases} \partial_t \psi(t) = V(t,\psi(t))), & t > \bar{t}, \\ \psi(\bar{t}) = \bar{\phi}, & x \in \mathbb{R}. \end{cases} \tag{2.3}$$

This yields a contradiction because the solution of (2.2) is unique.

with cost $\mathrm{c}(x,y) := |x-y|^p$ for some $p \geqslant 1$. Owing to the uniqueness of the monotone transport map, Problem 1 and Problem 3 actually coincide in this setting.

For the sake of completeness, we recall some known properties of the one-dimensional optimal transport map in the following theorem (see [8, Theorem 3.1] and [38, Lemma 2.5]).

Theorem 2.1 (One-dimensional Monge's problem). *Let $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R})$ and let us assume that μ_0 is* non-atomic *(i.e., a* diffuse measure*: $\mu_0(\{x\}) = 0$ for any $x \in \mathbb{R}$). Then there exists a unique (modulo countable sets) non-decreasing function $\mathrm{T} : \operatorname{supp}\mu_0 \to \mathbb{R}$ such that $\mathrm{T}_{\#}\mu_0 \equiv \mu_1$, given explicitly by*

$$\mathrm{T}(x) = \sup\left\{z \in \mathbb{R} : \mu_1((-\infty, z]) \leqslant \mu_0((-\infty, x])\right\}, \quad \text{for} \quad x \in \operatorname{supp}\mu_0.$$

Moreover, the function T is an optimal transport map (the unique optimal transport map if $p > 1$) and, provided that $\operatorname{supp}\mu_1$ is connected, it is continuous. Finally, if $\mu_0, \mu_1 \ll \mathscr{L}^1$ and their densities $\bar\mu_0$ and $\bar\mu_1$ are continuous and positive functions in their respective supports, then T is C^1 and its derivative is given by

$$\mathrm{T}'(x) = \frac{\mu_0(x)}{\mu_1(\mathrm{T}(x))}. \tag{2.4}$$

The main result of this section solves, in particular, Problem 3 and, equivalently, Problem 1, for $d = 1$, and is the following:

Theorem 2.2 (Exact controllability, $d = 1$). *Let us consider $\mu_0, \mu_1 \in \mathcal{P}_{\mathrm{a.c.}}(\mathbb{R})$ and suppose that their densities, $\bar\mu_0$ and $\bar\mu_1$, are continuous functions (in their respective supports). Let us suppose that the following conditions hold:*

- ***M-1*** *$\operatorname{supp}\mu_0$ and $\operatorname{supp}\mu_1$ are convex;*
- ***M-2*** *$\bar\mu_0 > 0$ in $\operatorname{supp}(\mu_0)$ and $\bar\mu_1 > 0$ in $\operatorname{supp}(\mu_1)$;*

Then there exists a velocity field $v : \operatorname{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1) \to \mathbb{R}$ such that

$$\begin{aligned} |v| > 0 \quad &\text{in} \quad \operatorname{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1)\setminus\mathcal{S}, \\ v \equiv 0 \quad &\text{in} \quad \mathcal{S}, \end{aligned}$$

and

$$\mathrm{T}(x) = \phi(1, x), \qquad x \in \operatorname{supp}\mu_0,$$

where T is the monotone optimal transport map from Theorem 2.1, $\mathcal{S}$ is the set of fixed points of the map T in $\operatorname{supp}\mu_0$, and ϕ is the unique solution of (2.1) for $x \in \operatorname{supp}\mu_0$.

Moreover, v is continuous except possibly at $\partial\mathcal{S}$. If, additionally, $|\bar\mu_0 - \bar\mu_1| > 0$ in $\partial\mathcal{S}$, then v can be taken to be continuous also at $\partial\mathcal{S}$. If, furthermore, $\bar\mu_0$ and $\bar\mu_1$ are Lipschitz continuous, v can be taken locally Lipschitz continuous up to $\partial\mathcal{S}$.

Remark 2.3 (Continuity equation). In Theorem 2.2, we claim the uniqueness of ϕ. On the other hand, for the continuity equation (1.3), we cannot make any claim, as the velocity field will not, in general, be L^1_{loc} (and therefore, things would depend on the notion of solution chosen in such a case; *cf.* Remark 1.2); see Lemma 3.3 below.

If we are in a situation where v is continuous, we can define weak solutions for (1.3) and their uniqueness follows from the uniqueness for the ODE, (2.1), see [22, Proposition 5.2].

If v is also Lipschitz continuous, then uniqueness for (2.1) follows from the Cauchy–Lipschitz theorem and uniqueness for (1.3) follows from the classical methods of characteristics.

Remark 2.4 (Higher regularity). If, in addition, $\bar\mu_0 \in C^k(\operatorname{supp}\mu_0)$ and $\bar\mu_1 \in C^k(\operatorname{supp}\mu_1)$ for some $k \in \mathbb{N}$, then we can choose $v \in C^k(\operatorname{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1)\setminus\partial S)$. This is a consequence of Corollary 2.7, stated below.

Remark 2.5 (Support of μ_0 with two connected components). If $\operatorname{supp}\mu_0 \equiv C_1 \cup C_2$ with C_1 and C_2 connected sets satisfying $\sup C_1 \leqslant \inf \operatorname{supp}\mu_0 \leqslant \inf C_2$, then we can still solve the problem by splitting it in two. We consider $\mu_0^1 := \mu_0|_{C_1}$, $\mu_0^2 := \mu_0|_{C_2}$ and $\mu_1^1 := \mu_1|_{S_1}$, $\mu_1^2 := \mu_1|_{S_2}$ where S_1 and S_2 are connected sets satisfying $S_1 \cup S_2 = \operatorname{supp}\mu_1$, $\mu_1(S_1) = \mu_0(C_1)$, and $\mu_1(S_2) = \mu_0(C_2)$. Similar strategies can be applied with more interleaved connected components, provided that the corresponding masses allow for it.

The key idea to approach the proof of Theorem 2.2 is as follows. If $v \in C \cap L^\infty$ and $|v| > 0$, by [3, Theorem 1.2.6] (*cf.* also [36]), there exists one and only one[7] solution of (2.1) in the following sense:

$$\begin{aligned} &\phi(\cdot, x) \in C^1((0,+\infty)) \quad \text{for every } x \in \mathbb{R}, \\ &\int_x^{\phi(t,x)} \frac{1}{v(\xi)}\,\mathrm{d}\xi = t, \qquad t > 0. \end{aligned} \tag{2.5}$$

Let us suppose that $\phi(1,\cdot) \equiv \mathrm{T}$. Then, owing to (2.5),

$$\int_x^{\mathrm{T}(x)} \frac{\mathrm{d}\xi}{v(\xi)} = 1. \tag{2.6}$$

That is, a primitive of $1/v$ (*i.e.*, F such that $F' = 1/v$) solves *Abel's functional equation*[8] (introduced in [1]):

$$F(\mathrm{T}(x)) = F(x) + 1, \quad x \in \operatorname{supp}\mu_0. \tag{2.7}$$

Differentiating (2.6) with respect to x yields *Aczél–Jabotinsky–Julia's equation* (introduced in [44, 2, 42][9]; see also [14]):

$$v(\mathrm{T}(x)) = \mathrm{T}'(x)\,v(x), \quad x \in \operatorname{supp}\mu_0. \tag{2.8}$$

Viceversa, a solution $v \in C \cap L^\infty$, with $|v| > 0$, of (2.8) generates a unique flow ϕ that satisfies $\phi(1,\cdot) \equiv \mathrm{T}$ (up to a scaling constant to achieve T at $t = 1$).

Therefore, to prove Theorem 2.2, we will build a suitable solution v to Aczél–Jabotinsky–Julia's equation (2.8), which is more convenient than (2.7) for our purposes (*cf.* Theorem A.1).

2.1. **Measures with bounded supports.** To solve (2.8), we distinguish various cases, according to the number of fixed points, denoted $\mathcal{S}$, of the optimal map T between μ_0 and μ_1, and the boundedness of the supports of the measures μ_0 and μ_1.

In this section, we deal with compactly supported measures, so we will add the following condition:

M-3 $\operatorname{supp}\mu_0$ and $\operatorname{supp}\mu_1$ are compact.

In particular, we will denote $\lambda, \Lambda > 0$ the two constants such that such that

$$0 < \lambda < \bar\mu_0,\ \bar\mu_1 < \Lambda < +\infty \quad \text{in their respective supports} \tag{2.9}$$

(which exist, by compactness of the supports and continuity of the densities).

[7] A (local-in-time) solution ϕ of (2.1) must exist, if v is continuous, by Peano's theorem; moreover, if v is bounded, it can be extended globally-in-time. Let us sketch the proof of uniqueness. The function $G(\psi) := \int_x^\psi \frac{\mathrm{d}\xi}{v(\xi)}$ is of class C^1 and $G' \neq 0$; hence G has a C^1 inverse. We then compute

$$\partial_t G(\phi(t,x)) = \frac{\partial_t \phi(t,x)}{v(\phi(t,x))} = 1,$$

which yields $G(\phi(t,x)) = t$ and thus $\phi(t,x) = G^{-1}(t)$ and uniqueness follows.

[8] The functional equation (2.7) is the *equation of semi-conjugacy of* T *with the standard shift*; see [17, Section 2.2.2, p. 16].

[9] In the language of Jabotinsky and Aczél, the function T is the unknown and v is given. On the other hand, in our setting, T is given and v is the unknown.

We start with the case without fixed points.

Lemma 2.6 (Transport map without fixed points)**.** *Let us consider $\mu_0, \mu_1 \in \mathcal{P}_{\text{a.c.}}(\mathbb{R})$ and suppose that their densities, $\bar\mu_0$ and $\bar\mu_1$, are continuous functions (in their respective supports). Let us suppose that the conditions **M-1**–**M-2**–**M-3** hold (with (2.9)) and, moreover, that the map* T *has no fixed points, $\mathcal{S} = \varnothing$.*

Then there exists a continuous velocity field $v : \mathrm{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1) \to \mathbb{R}$ such that

$$|v| > 0 \quad \textit{in} \quad \mathrm{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1), \qquad \mathrm{T}(x) = \phi(1, x), \quad \textit{for} \quad x \in \operatorname{supp}\mu_0,$$

where ϕ is the unique solution of (2.1) for $x \in \operatorname{supp}\mu_0$.

Proof. Let us denote

$$M_0 := \operatorname{supp}\mu_0 = (a_0, b_0), \qquad M_1 := \operatorname{supp}\mu_1 = (a_1, b_1).$$

By previous considerations, such a velocity field v exists if and only if it satisfies (2.8) (up to a multiplicative constant fixing the travel time). Recall that, from Theorem 2.1, we already know that, for any $x \in M_0$,

$$\mathrm{T}'(x) = \frac{\bar\mu_0(x)}{\bar\mu_1(\mathrm{T}(x))}, \quad \text{for} \quad x \in M_0,$$

and in particular, by continuity of μ_0 and μ_1 (and boundedness away from zero for μ_1), T' is continuous and T is C^1. Moreover, by assumption **M-2** (recall (2.9)),

$$\frac{\Lambda}{\lambda} \geqslant \mathrm{T}'(x) \geqslant \frac{\lambda}{\Lambda} > 0, \quad \text{for} \quad x \in M_0.$$

We will split the proof into two cases.

Case 1: $M_0 \cap M_1 = \varnothing$**.** In this case, we can fix $v \equiv 1$ in M_0, so that v in M_1 is given by

$$v(x) := \mathrm{T}'(\mathrm{T}^{-1}(x))v(\mathrm{T}^{-1}(x)) = \mathrm{T}'(\mathrm{T}^{-1}(x)) \in \left[\frac{\lambda}{\Lambda}, \frac{\Lambda}{\lambda}\right], \quad \text{for} \quad x \in M_1.$$

In particular, v can be chosen continuous and with $v(x) \in \left[\frac{\lambda}{\Lambda}, \frac{\Lambda}{\lambda}\right]$ for $x \in \mathrm{Conv}(M_0 \cup M_1)$ to satisfy (2.8). That is, (2.6) holds with a constant non-zero right-hand side. Up to multiplying by a constant to fix the transport time, we get the desired result.

Case 2: $M_0 \cap M_1 \neq \varnothing$**.** Let also assume, without loss of generality, that $a_0 < a_1$ (and therefore, $\mathrm{T}(x) > x$ for $x \in M_0$). Indeed, since T does not have fixed points, we already know that $a_0 \neq a_1$. Furthermore, if we had $a_1 < a_0$, we could swap the roles of μ_0 and μ_1 and consider the vector field $-v$ instead.

Now we define $\alpha_0 := a_0$, $\alpha_1 := a_1 = \mathrm{T}(a_0) = \mathrm{T}(\alpha_0)$, and $\alpha_i := \mathrm{T}(\alpha_{i-1})$ for $i = 1, 2, \ldots$. Then, there exists $N \in \mathbb{N}$ such that $\alpha_N \in (b_0, b_1]$. Indeed, $i \mapsto \alpha_i$ is increasing (owing to the monotonicity of T) and, if $\alpha_i \leqslant b_0$, $\alpha_{i+1} \leqslant b_1$. If the sequence $\{\alpha_i\}_i$ had an accumulation point $\bar\alpha \leqslant b_0$, then $\mathrm{T}(\bar\alpha) = \bar\alpha$ and α is a fixed point for T, which do not exist by assumption. Hence, the sequence must be finite.

Let us now fix $v \in \left[\frac{\lambda}{\Lambda}, \frac{\Lambda}{\lambda}\right]$ to be any smooth function in $[a_0, a_1]$ with

$$v(a_1) = \mathrm{T}'(a_0)v(a_0) = \frac{\mu_0(a_0)}{\mu_1(a_1)}v(a_0). \tag{2.10}$$

We then define, recursively, and denoting $\alpha_{N+1} := b_1$,

$$v(\mathrm{T}(x)) = \mathrm{T}'(x)v(x), \quad \text{for} \quad x \in [\alpha_i, \alpha_{i+1}], \quad i = 1, 2, \ldots, N. \tag{2.11}$$

This defines v in the interval $[a_0, b_1]$ in a continuous way. Indeed, v is continuous in $[\alpha_0, \alpha_1]$ and in $[\alpha_1, \alpha_2]$, and, owing to (2.10)–(2.11), is also continuous at α_1 from both sides, thus being continuous in $[\alpha_0, \alpha_2]$. Then,

$$v(x) = \mathrm{T}'(\mathrm{T}^{-1}(x))v(\mathrm{T}^{-1}(x)), \quad \text{for} \quad x \in [\alpha_1, \alpha_3].$$

Since $\mathrm{T}^{-1}([\alpha_1,\alpha_3]) = [\alpha_0,\alpha_2]$, and v is continuous in $[\alpha_0,\alpha_2]$, T^{-1} is continuous, and T' is continuous, we obtain that v is continuous in $[\alpha_1,\alpha_3]$ as well, and thus, in $[\alpha_0,\alpha_3]$. Proceeding iteratively, it is continuous in the whole interval $[a_0,b_1]$ with a bound $v \leqslant \left[\frac{\Lambda}{\lambda}\right]^{N+1}$. Up to multiplying by a constant to fix the time of transport as before, we get the desired result. □

In the previous proof, higher regularity of $\bar\mu_0$ and $\bar\mu_1$ gives higher regularity for the velocity field v.

Corollary 2.7 (Higher regularity)**.** *In the setting of Lemma 2.6, if, in addition, $\bar\mu_0 \in C^k(\operatorname{supp}\mu_0)$ and $\bar\mu_1 \in C^k(\operatorname{supp}\mu_1)$ for some $k \in \mathbb{N}$, then we can choose $v \in C^k(\mathrm{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1))$.*

Proof. Recalling (2.4), we note that $\mathrm{T}' \in C^k(M_0)$ if $\bar\mu_0 \in C^k(\operatorname{supp}\mu_0)$ and $\bar\mu_1 \in C^k(\operatorname{supp}\mu_1)$. As a consequence, we deduce $\mathrm{T} \in C^{k+1}(M_0)$. Then, to conclude the proof, we just need to choose v in $[a_0,a_1]$ as in the proof of Lemma 2.6, but such that

$$\frac{\mathrm{d}}{\mathrm{d}x^i}\bigg|_{x=a_1^-} v(x) = \frac{\mathrm{d}}{\mathrm{d}x^i}\bigg|_{x=a_0^+} (\mathrm{T}'(x)v(x)),$$

for all $i = 0,\dots,k$. Repeating the reasoning at the end of the proof of Lemma 2.6, we are done. □

We now turn to the other cases, in which there can be fixed points. We start by assuming that T has exactly one fixed point.

Lemma 2.8 (Transport map with exactly one fixed point)**.** *Let us consider $\mu_0,\mu_1 \in \mathcal{P}_{\mathrm{a.c.}}(\mathbb{R})$ and suppose that their densities, $\bar\mu_0$ and $\bar\mu_1$, are continuous functions (in their respective supports). Let us assume that the conditions* ***M-1–M-2–M-3*** *hold (with (2.9)). Suppose, moreover, that the set $\mathcal{S}$ contains a single point, $\bar x$.*

Then there exists a velocity field $v : \operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1 \to \mathbb{R}$ such that

$$\begin{aligned} &|v| > 0 \qquad \text{in} \quad \operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1 \setminus \{\bar x\},\\ &v(\bar x) = 0, \end{aligned}$$

and

$$\mathrm{T}(x) = \phi(1,x), \quad x \in \operatorname{supp}\mu_0,$$

where ϕ is the unique solution of (2.1) for $x \in \operatorname{supp}\mu_0$, and it is continuous except possibly at $\bar x$. If, moreover, $\bar\mu_0(\bar x) \neq \bar\mu_1(\bar x)$, then v can be taken to be continuous also at $\bar x$. If, furthermore, $\bar\mu_0$ and $\bar\mu_1$ are Lipschitz continuous, then v can be taken Lipschitz continuous up to $\bar x$ as well.

Proof. We use the same notation as in the proof of Lemma 2.6. Let $\bar x$ be the unique fixed point for T, and let us assume, without loss of generality, that $\bar x = b_0 = b_1$ (otherwise, we can consider the restrictions of μ_0 and μ_1 to the intervals $(a_0,\bar x)$ and $(b_0,\bar x)$, in which, by assumption, they have the same mass—since $\bar x$ is a fixed point, $\mu_0((-\infty,\bar x)) = \mu_1((-\infty,\mathrm{T}(\bar x) = \bar x))$—; if $\bar x = a_0 = b_0$, instead, we can just flip the x-axis).

We can furthermore assume $a_0 < a_1$ and thus $x < \mathrm{T}(x)$ for $x \in M_0$ (otherwise, we can exchange the roles of μ_0 and μ_1). Then, the sequence α_i constructed in the proof of Lemma 2.6 is no longer finite, and $\alpha_i \to b_0 = b_1$ as $i \to +\infty$. This allows us to recursively define a (continuous) vector field v in (a_0,b_0) by means of (2.11), after fixing it in $(a_0,\mathrm{T}(a_0))$ first. A priori, it could degenerate when approaching $\bar x$, though (cf. Lemma 3.3).

If $\bar\mu_0(\bar x) \neq \bar\mu_1(\bar x)$, then we necessarily have $\mathrm{T}'(x) < 1$ (because we are assuming $x < \mathrm{T}(x)$) and so, in the limit $i \to +\infty$, the sequence of intervals obtained recursively from (2.11) converges to zero:

$$v(\mathrm{T}^i(x)) = v(x)P_i(x) := v(x)\prod_{j=0}^{i-1}\mathrm{T}'(\mathrm{T}^j(x)), \quad \text{for} \quad x \in (a_0, \mathrm{T}(a_0)), \tag{2.12}$$

and, since $\mathrm{T}'(\mathrm{T}^j(x)) < \frac{1}{2}(1+\mathrm{T}'(\bar{x})) < 1$ for j large enough, we get

$$\|v\|_{L^\infty((\alpha_{i+1},\alpha_{i+2}))} \leqslant \|v\|_{L^\infty((\alpha_i,\alpha_{i+1}))} \to 0, \quad \text{as} \quad i \to +\infty,$$

that is, $\lim_{x\to\bar{x}^-} v(x) = 0$, and we can fix $v(\bar{x}) = 0$.

If $\bar{\mu}_0$ and $\bar{\mu}_1$ are Lipschitz continuous, by the same proof as in Corollary 2.7, we immediately get that v is locally Lipschitz continuous in $[a_0, \bar{x})$. Let us check that, in fact, a bound on its derivative holds up to $\bar{x}$.

Differentiating (2.12) and observing that $(\mathrm{T}^i)'(x) = \mathrm{T}'(\mathrm{T}^{i-1}(x))(\mathrm{T}^{i-1})'(x) = \cdots = P_i(x)$, we get

$$v'(\mathrm{T}^i(x)) = v'(x) + v(x)\sum_{j=0}^{i-1}\frac{\mathrm{T}''(\mathrm{T}^j(x))}{\mathrm{T}'(\mathrm{T}^j(x))}P_j(x), \quad \text{for} \quad x \in (a_0, \mathrm{T}(a_0)).$$

In particular, since $\mathrm{T}' \geqslant \frac{\lambda}{\Lambda}$ and T'' is bounded (because $\bar{\mu}_0$ and $\bar{\mu}_1$ are Lipschitz continuous), we can estimate

$$\|v'\|_{L^\infty((\alpha_i,\alpha_{i+1}))} \leqslant \|v'\|_{L^\infty([a_0,\mathrm{T}(a_0)])} + C\|v\|_{L^\infty([a_0,\mathrm{T}(a_0)])}\sum_{j=0}^{i-1}\tilde{P}_j,$$
$$\text{where} \quad \tilde{P}_j := \|P_j(x)\|_{L^\infty([a_0,\mathrm{T}(a_0)])}.$$

We observe that $P_j(x) = \mathrm{T}'(\mathrm{T}^{j-1}(x))P_{j-1}(x)$. In particular, for j large enough, $\mathrm{T}'(\mathrm{T}^{j-1}(x)) \leqslant 1-\varepsilon$ with $\varepsilon = \frac{1-\mathrm{T}'(\bar{x})}{2}$, and we have $P_j(x) \leqslant C(1-\varepsilon/2)^j$ for all $j \in \mathbb{N}$, for some constant C.

In conclusion, the previous sum is bounded and we get

$$\|v'\|_{L^\infty((\alpha_0,\bar{x}))} \leqslant \|v'\|_{L^\infty([a_0,\mathrm{T}(a_0)])} + C\|v\|_{L^\infty([a_0,\mathrm{T}(a_0)])},$$

that is, if v is chosen smooth in $[a_0, \mathrm{T}(a_0)]$, we get a bound on v' up to $\bar{x}$ and obtain that v is Lipschitz continuous.

Finally, let us show the uniqueness of the flow. In the previous construction, we fixed $v(\bar{x}) = 0$. Moreover, we have that, for any $\varepsilon > 0$,

$$\int_{A_{\pm,\varepsilon}}\frac{\mathrm{d}x}{|v(x)|} = +\infty \quad \text{whenever} \quad A_{\pm,\varepsilon} \neq \varnothing,$$

where we introduced the notation

$$A_{+,\varepsilon} := (\bar{x}, \bar{x}+\varepsilon) \cap \operatorname{supp}\mu_0, \qquad A_{-,\varepsilon} := (\bar{x}-\varepsilon, \bar{x}) \cap \operatorname{supp}\mu_0.$$

Indeed, let us assume that we are in the same situation as in the construction above, and show an Osgood-type condition at $\bar{x}$:

$$\int_{\bar{x}-\varepsilon}^{\bar{x}}\frac{\mathrm{d}x}{|v(x)|} = +\infty.$$

Fix any $x_0 \in (a_0, \bar{x})$. We know that $\mathrm{T}^i(x_0) \uparrow \bar{x}$ as $i \to +\infty$, but also that

$$\int_{\mathrm{T}^i(x_0)}^{\mathrm{T}^{i+1}(x_0)}\frac{\mathrm{d}x}{|v(x)|} = 1.$$

In particular, taking $j \in \mathbb{N}$ large enough (depending on ε) so that $\mathrm{T}^j(x_0) > \bar{x}-\varepsilon$, we get

$$\int_{\bar{x}-\varepsilon}^{\bar{x}}\frac{\mathrm{d}x}{|v(x)|} \geqslant \sum_{k\geqslant j}\int_{\mathrm{T}^k(x_0)}^{\mathrm{T}^{k+1}(x_0)}\frac{\mathrm{d}x}{|v(x)|} = \sum_{k\geqslant j}1 = +\infty.$$

We now show the uniqueness of the solution to (2.1) up to time 1 at all points $x \in \operatorname{supp}\mu_0$. Since v is continuous and non-zero in $\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1 \backslash \{\bar{x}\}$, we have existence and uniqueness for (1.1) in this set (*cf.* [29]). Indeed, for $x_0 \in \operatorname{supp}\mu_0$ with $x_0 \neq \bar{x}$, the flow can never cross $\bar{x}$ before time 1 because, up until that moment, it would be continuous and, by the previous discussion, it requires infinite time to actually reach $\bar{x}$.

Now, we claim that the ODE

$$\begin{cases} \partial_t \phi(t,\bar{x}) = v(\phi(t,\bar{x})), & t > 0, \\ \phi(0,\bar{x}) = \bar{x}, \end{cases}$$

is also well-posed: it has a unique solution $\phi(\cdot,\bar{x}) \equiv \bar{x}$.

We observe first that $\phi(t,\bar{x})$ must be continuous at $\{t \in [0,1] : \phi(t,\bar{x}) = \bar{x}\}$ and, in particular, since it is continuous outside of $\bar{x}$, it must be continuous at all times $t \in [0,1]$. Indeed, up to extending v smoothly outside of the domain, ϕ cannot instantaneously jump to any other point, since around those we always have local existence and uniqueness of continuous forward and backward flows.

By contradiction, let us assume, for example, that $\phi(t_1, \bar{x}) > \bar{x}$ for some $t_1 > 0$. For every $\delta > 0$, we then have

$$t_1 - \delta \geqslant \int_\delta^{t_1} \frac{\partial_t \phi(t, \bar{x})}{v(\phi(t, \bar{x}))}\, \mathrm{d}t = \int_{\phi(\delta,\bar{x})}^{\phi(t_1,\bar{x})} \frac{1}{v(\xi)}\, \mathrm{d}\xi,$$

which yields a contradiction because the

$$\limsup_{\delta \searrow 0} \int_{\phi(\delta,\bar{x})}^{\phi(t_1,\bar{x})} \frac{1}{v(\xi)}\, \mathrm{d}\xi = \int_{\bar{x}}^{\phi(t_1,\bar{x})} \frac{1}{v(\xi)}\, \mathrm{d}\xi = \infty.$$

We have thus shown that (2.1) has a unique solution up to time 1 at all points $x \in \operatorname{supp}\mu_0$ □

Next, we deal with the case when T has two fixed points.

Lemma 2.9 (Transport map with exactly two fixed points)**.** *Let us consider $\mu_0, \mu_1 \in \mathcal{P}_{\text{a.c.}}(\mathbb{R})$ and suppose that their densities, $\bar{\mu}_0$ and $\bar{\mu}_1$, are continuous functions compactly supported in $[0,1]$. Let us assume that the conditions* ***M-1–M-2–M-3*** *hold (with (2.9)). Suppose, moreover, that the set $\mathcal{S} = \{0,1\}$.*

Then there exists a velocity field $v : [0,1] \to \mathbb{R}$ such that

$$\begin{aligned} &|v| > 0 && \text{in} \quad (0,1), \\ &v(0) = v(1) = 0, \end{aligned}$$

and

$$\mathrm{T}(x) = \phi(1,x), \qquad x \in [0,1],$$

where ϕ solves (2.1), and it is continuous except possibly at $\mathcal{S}$. If, moreover, $|\bar{\mu}_0 - \bar{\mu}_1| > 0$ in $\mathcal{S}$, then v can be taken to be continuous also at $\mathcal{S}$. If, furthermore, $\bar{\mu}_0$ and $\bar{\mu}_1$ are Lipschitz continuous, then v can be taken Lipschitz continuous up to $\mathcal{S}$.

Proof. Up to swapping the roles of μ_0 and μ_1 (and changing the sign of the vector field v), we can assume $\mathrm{T}(x) < x$ for $x \in (0,1)$.

Let ν_0 and ν_1 be the restrictions of μ_0 and μ_1, respectively, in the intervals $(0,1/2)$ and $(0,p)$, where $p := \mathrm{T}(1/2) < 1/2$. By definition of T, ν_0 and ν_1 still have the same mass. Moreover, since $p < \frac{1}{2}$, $\mathrm{T}|_{\operatorname{supp}\nu_0}$ has only one fixed point (because the support of ν_0 is $[0,1/2]$, and $\mathrm{T}(x) < x$ in $(0,1)$), namely, 0. Thus, we can apply Lemma 2.8 and deduce that there exists some v defined in $[0,\frac{1}{2}]$ that is continuous (except, possibly, at 0) and such that ν_0 is transported to ν_1. Such a velocity field is arbitrarily defined in $[p,\frac{1}{2}]$ and then extended to the whole $[0, 1/2]$ by means of (2.6). The compatibility condition that needs to be satisfied to get continuity is given by

$$v(p) = \mathrm{T}'(1/2)\, v(1/2).$$

On the other hand, let $\tilde{\nu}_0$ and $\tilde{\nu}_1$ be, respectively, the restrictions of μ_0 and μ_1 to the intervals $(1/2, 1]$ and $(p, 1]$. Then, we can repeat the proof of Lemma 2.8 with $\mu_0 = \tilde{\nu}_1$ and $\mu_1 = \tilde{\nu}_0$, which have T^{-1} as monotone transport map, to deduce again the existence of a continuous velocity field $\tilde{v}$ in $[p, 1]$ determined from its value on the interval $[p, 1/2]$ through (2.6). The compatibility condition now is

$$\tilde{v}(1/2) = (\mathrm{T}^{-1})'(p)\tilde{v}(p) = \frac{1}{\mathrm{T}'(\mathrm{T}^{-1}(p))}\tilde{v}(p) = \frac{1}{\mathrm{T}'(1/2)}\tilde{v}(p).$$

Namely, we can take $\tilde{v} = -v$ on $[p, 1/2]$. Then, the velocity field v transports ν_0 to ν_1 and is continuous in $[0, 1/2]$, and the velocity field $-\tilde{v}$ transports $\tilde{\nu}_1$ to $\tilde{\nu}_0$ and is continuous in $[p, 1]$. Since $v = -\tilde{v}$ in $[p, 1/2]$, we can continuously extend v by $-\tilde{v}$ to the whole interval $[0, 1]$. In particular, in this construction, we fix $v(0) = v(1) = 0$.

As in the proof of Lemma 2.8, we can show that, for any $\varepsilon > 0$,

$$\int_0^{\varepsilon} \frac{\mathrm{d}x}{|v(x)|} = \int_{1-\varepsilon}^{1} \frac{\mathrm{d}x}{|v(x)|} = +\infty.$$

and that existence and uniqueness of solutions of (2.1) hold also at the fixed points. Indeed, we only need to observe that $\phi(t, 0)$ (resp. $\phi(t, 1)$) is still continuous at $\phi(t, 0) = 0$ (resp. $\phi(t, 1) = 1$), since the only difference with respect to the previous situation would be if ϕ jumped between 0 and 1, which is not possible because in both cases $v(0) = v(1) = 0$. ☐

2.2. **Measures with unbounded supports.** For measures with unbounded supports, we can recover results that are analogous to Lemma 2.6, Corollary 2.7, and Lemma 2.8.

Lemma 2.10 (Transport map without fixed points—unbounded setting)**.** *Let us consider $\mu_0, \mu_1 \in \mathcal{P}_{\mathrm{a.c.}}(\mathbb{R})$ and suppose that their densities, $\bar{\mu}_0$ and $\bar{\mu}_1$, are continuous functions (in their respective supports). Let us assume that the conditions **M-1**–**M-2** hold, that either the support of μ_0 or the support of μ_1 is unbounded, and that the map T has no fixed points, $\mathcal{S} = \varnothing$.*

Then there exists a continuous velocity field $v : \mathrm{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1) \to \mathbb{R}$ such that

$$|v| > 0 \quad \textit{in} \quad \mathrm{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1), \qquad \mathrm{T}(x) = \phi(1, x), \quad \textit{for} \quad x \in \operatorname{supp}\mu_0,$$

where ϕ is the unique solution of (2.1) for $x \in \operatorname{supp}\mu_0$.

Proof. Let us suppose, first, that $\operatorname{supp}(\mu_0) = [a_0, b_0]$ and $\operatorname{supp}(\mu_1) = [a_1, \infty]$, where $a_0, a_1 \in \mathbb{R}$, $b_0 \in \mathbb{R} \cup \{\infty\}$. Up to switching the roles of μ_0 and μ_1 if $b_0 = \infty$, we can assume $a_0 < a_1$ and $\mathrm{T}(x) > x$ for all $x \in \operatorname{supp}(\mu_0)$.

The proof follows as the one in Lemma 2.6 (we use the same notation, as well). In this case, we have that T' is continuous and positive in M_0, but we do not have universal bounds for it.

The case $M_0 \cap M_1 = \varnothing$, follows exactly as Lemma 2.6, without uniform controls on the velocity (which might blow-up or go to zero at infinity).

In the case $M_0 \cap M_1 = \varnothing$ we define α_i again as in Lemma 2.6, and construct v (continuous, and positive) in the same way recursively in the intervals $[\alpha_i, \alpha_{i+1}]$ (basically, v is arbitrarily fixed in $[a_0, a_1]$, and then uniquely continued). Differently from before, however, we lose the control global L^∞ control on the velocity, and in this case, N might even be infinite (when $b_0 = \infty$).

We suppose now $a_0 = -\infty \leqslant a_1$. Let any $c_0 \in (-\infty, b_0)$, $c_1 = \mathrm{T}(c_0) > c_0$, and consider ν_0 and ν_1 the restrictions of μ_0 and μ_1 to the intervals $[c_0, b_0]$ and $[c_1, \infty]$. Then, by the previous argument, we can construct a velocity field in $[c_0, \infty]$ transporting ν_0 into ν_1. On the other hand, if $\bar{\nu}_0$ and $\bar{\nu}_1$ are the restrictions of μ_0 and μ_1 to the intervals $[-\infty, c_0]$ and $[a_1, c_1]$, then the previous velocity field (defined, for these measures, in $[c_0, c_1]$) uniquely extends, by the previous arguments, to the whole interval $[-\infty, c_1]$ as well. □

Corollary 2.11 (Higher regularity—unbounded setting). *In the setting of Lemma 2.10, if, in addition, $\bar{\mu}_0 \in C^k(\operatorname{supp}\mu_0)$ and $\bar{\mu}_1 \in C^k(\operatorname{supp}\mu_1)$ for some $k \in \mathbb{N}$, then we can choose $v \in C^k(\operatorname{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1))$.*

Proof. Cf. proof of Corollary 2.7 by means of Lemma 2.6, using Lemma 2.10 in this case. □

Lemma 2.12 (Transport map with exactly one fixed point—unbounded setting). *Let us consider $\mu_0, \mu_1 \in \mathcal{P}_{\text{a.c.}}(\mathbb{R})$ and suppose that their densities, $\bar{\mu}_0$ and $\bar{\mu}_1$, are continuous functions (in their respective supports). Let us assume that the conditions **M-1**–**M-2** hold.*

Let us suppose, moreover, that $\operatorname{supp}(\mu_0) = [a_0, b_0]$ and $\operatorname{supp}(\mu_1) = [a_0, \infty]$, where $a_0 \in \mathbb{R}$, $b_0 \in \mathbb{R} \cup \{\infty\}$, and that the map T has a single fixed point, $\mathcal{S} = \{a_0\}$.

Then there exists a velocity field $v : \operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1 \to \mathbb{R}$ such that

$$|v| > 0 \qquad \textit{in} \quad \operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1 \setminus \{a_0\},$$
$$v(a_0) = 0,$$

and

$$\mathrm{T}(x) = \phi(1, x), \quad x \in \operatorname{supp}\mu_0,$$

where ϕ is the unique solution of (2.1) for $x \in \operatorname{supp}\mu_0$, and it is continuous except possibly at a_0. If, moreover, $\bar{\mu}_0(a_0) \neq \bar{\mu}_1(a_0)$, then v can be taken to be continuous also at a_0. If, furthermore, $\bar{\mu}_0$ and $\bar{\mu}_1$ are Lipschitz continuous, then v can be taken Lipschitz continuous up to a_0.

Proof. We can assume, without loss of generality, that $\mathrm{T}(x) > x$ (which is directly true if $b_0 < \infty$, since there is only one fixed point).

Let $c_0 = \frac{a_0 + b_0}{2} \in (a_0, b_0)$, $c_1 = \mathrm{T}(c_0) > c_0$. We can now apply Lemma 2.8 to ν_0 and ν_1, the restrictions of μ_0 and μ_1 respectively in the intervals $[a_0, c_0]$ and $[a_0, c_1]$, which will fix a velocity field in $[a_0, c_1]$.

Let $\bar{\nu}_0$ and $\bar{\nu}_1$ be the restrictions of μ_0 and μ_1 in $[c_0, b_0]$ and $[c_1, \infty]$ respectively. Notice that, by assumption, the monotone map from $\bar{\nu}_0$ to $\bar{\nu}_1$ does not have any fixed point. We can now apply the result from Lemma 2.10, where the velocity field has already been fixed in the interval $[c_0, c_1]$, and can be uniquely extended as in the proof of Lemma 2.10 (and Lemma 2.6) to the whole space. □

3. Proofs of the main results

Finally, putting everything together we can deal with the general case, Theorem 2.2.

Proof of Theorem 2.2 (and, therefore, Theorem A). If $\mathcal{S} = \varnothing$, we apply Lemma 2.6 (or Lemma 2.10); if $\mathcal{S} = \{x_1\}$, we apply Lemma 2.8 or Lemma 2.12 (on each side, $(-\infty, x_1)$ and $(x_1, +\infty)$), and if $\mathcal{S} = \{x_1, x_2\}$ with $x_1 < x_2$, we apply Lemma 2.8 (or Lemma 2.12) to the side intervals (namely, $(-\infty, x_1)$ and $(x_2, +\infty)$), and Lemma 2.9 to the middle interval (that is, (x_1, x_2)), after a rescaling and translation if necessary.

Let us, therefore, suppose that $\mathcal{S}$ contains more than two elements. We fix $v \equiv 0$ in $\mathcal{S}$. Since T is continuous, $\mathbb{R} \setminus \mathcal{S}$ can be written as a disjoint countable union of open intervals (being an open set in $\mathbb{R}$), $\mathbb{R} \setminus \mathcal{S} = \bigcup_{i \in \mathbb{N}} I_i$. We can fix $I_0 := (-\infty, x_l)$ if $x_l := \min(\mathcal{S}) > -\infty$ and $I_1 := (x_r, +\infty)$ if $x_r := \max(\mathcal{S}) < \infty$ (them being empty otherwise). We then apply Lemma 2.8 or Lemma 2.12 to I_0 and I_1, and Lemma 2.9 to each I_i with $i \geqslant 2$, to get the desired result.

In order to obtain the conditional continuity and Lipschitz regularity, we notice that, if $|\bar{\mu}_0 - \bar{\mu}_1| > 0$ on $\partial\mathcal{S}$, then we cannot have an accumulation point of $\partial\mathcal{S}$ (alternatively, in any compact set, there are finitely many intervals I_i). Indeed, otherwise there would be a sequence of points $x_k \in \partial I_i \subset \mathcal{S}$ with $x_k \to \bar{x} \in \mathcal{S}$ as $k \to +\infty$. In particular, from $\mathrm{T}(x_k) = x_k$, we get

$$\mathrm{T}'(\bar{x}) = \frac{\mathrm{T}(x_k) - \mathrm{T}(\bar{x})}{x_k - \bar{x}} = 1;$$

in turn, by (2.1), this implies $\bar{\mu}_0(\bar{x}) = \bar{\mu}_1(\bar{x})$, contradicting our assumption. Hence, we can apply Lemma 2.9 finitely many times in any given compact set, and use the fact that the concatenation of finitely many continuous or Lipschitz continuous functions remains continuous or Lipschitz continuous, to obtain that v can be taken locally Lipschitz continuous in this case.

Finally, on the uniqueness of the flow, as in the proof of Lemma 2.8, we can show that, for any $\bar{x} \in \partial\mathcal{S}$ and $\varepsilon > 0$,

$$\int_{A_{\pm,\varepsilon}} \frac{\mathrm{d}x}{|v(x)|} = +\infty \quad \text{whenever} \quad A_{\pm,\varepsilon} \neq \varnothing,$$

where we use again the notation

$$A_{+,\varepsilon} := (\bar{x}, \bar{x} + \varepsilon) \cap \operatorname{supp} \mu_0, \qquad A_{-,\varepsilon} := (\bar{x} - \varepsilon, \bar{x}) \cap \operatorname{supp} \mu_0.$$

Indeed, if $A_{\pm,\varepsilon} \cap \mathring{\mathcal{S}} \neq \varnothing$, since $v \equiv 0$ in $\mathcal{S}$, the result follows. There are now two cases.

Case 1. If $A_{\pm,\varepsilon} \cap \partial\mathcal{S} = \varnothing$ for $\varepsilon > 0$ small enough, we are in the same situation as Lemma 2.8, so the result follows.

Case 2. If $A_{\pm,\varepsilon} \cap \partial\mathcal{S} \neq \varnothing$ for all $\varepsilon > 0$ small, then there is a monotone sequence of fixed points $\partial\mathcal{S} \ni \bar{x}_k \to \bar{x}$ as $k \to \infty$, with $x_k \in A_{\pm,\varepsilon}$. Let us assume, without loss of generality, that x_k is an increasing sequence and we are looking at $A_{-,\varepsilon}$. Notice that, if $y_k \in (x_k, x_{k+1})$ is not a fixed point (which we can always find, otherwise (x_k, x_{k+1}) is an open interval contained in $A_{\pm,\varepsilon}$), then $\mathrm{T}(y_k) \in (x_k, x_{k+1})$ as well, with $\mathrm{T}(y_k) \neq y_k$. Then, we have

$$\int_{A_{-,\varepsilon}} \frac{\mathrm{d}x}{|v(x)|} \geqslant \sum_{k\in\mathbb{N}} \left| \int_{y_k}^{\mathrm{T}(y_k)} \frac{\mathrm{d}x}{v(x)} \right| = \sum_{k\in\mathbb{N}} 1 = +\infty.$$

In both cases, we get the desired result.

The existence and uniqueness of solutions to (2.1) now follow arguing as in Lemma 2.8. □

Remark 3.1 (On the assumption "$|\bar{\mu}_0 - \bar{\mu}_1| > 0$ in $\mathcal{S}$"). While the condition $|\bar{\mu}_0 - \bar{\mu}_1| > 0$ in $\mathcal{S}$ can be violated for a suitable choice of measures μ_0 and μ_1, such a situation is *uncommon*. Namely, given μ_0, by Theorem 2.2 we have that, for "almost every" μ_1, we can construct a Lipschitz continuous field v. This can be formalized as follows.

Given a measure μ_0 and a monotone map T, we define $\mu_1 := \mathrm{T}_{\#}\mu_0$. Let us show that, given any measure μ_0 satisfying **M-1**–**M-2**, the set of (monotone and smooth) maps T for which μ_0 and $\mu_1 := \mathrm{T}_{\#}\mu_0$ satisfy $|\bar{\mu}_0 - \bar{\mu}_1| > 0$ in $\mathcal{S} = \mathcal{S}(\mathrm{T})$ is *prevalent*[10] among (monotone and smooth) maps. Namely, *for almost every* map T, we have $|\bar{\mu}_0 - \bar{\mu}_1| > 0$ in $\mathcal{S}$. To show that, given any μ_0 and T, we construct the following *1-probe* (*cf.* [54, Definition 3.5]):

$$\mathrm{T}_\lambda(x) := \mathrm{T}(x) - \lambda, \quad \text{for} \quad x \in \mathbb{R}, \ \lambda \in \mathbb{R}.$$

We now have to prove that, for almost every $\lambda \in \mathbb{R}$,

$$|\bar{\mu}_0 - \bar{\mu}_\lambda| > 0 \quad \text{in} \quad \mathcal{S}_\lambda := \mathcal{S}(\mathrm{T}_\lambda) = \{x : x = \mathrm{T}_\lambda(x)\},$$

where $\mu_\lambda = (\mathrm{T}_\lambda)_{\#}\mu_0$ and $\bar{\mu}_\lambda$ is its density as an absolutely continuous measure. Equivalently, by Theorem 2.1, it is enough to show that

$$\mathrm{T}'_\lambda(x) \neq 1 \quad \text{in} \quad \mathcal{S}(\mathrm{T}_\lambda) \quad \Leftrightarrow \quad \widetilde{\mathrm{T}}' \neq 0 \quad \text{in} \quad \{\widetilde{\mathrm{T}} = \lambda\}, \qquad \text{for a.e. } \lambda \in \mathbb{R},$$

[10] We use the language of *prevalence* from [40, 41, 54].

where we have denoted $\widetilde{\mathrm{T}} := \mathrm{T} - \mathrm{Id}$. And this immediately holds, because the set $\{\mathrm{T}' = 0\}$ is a closed set (and thus the countable union of closed intervals) inside each of which $\widetilde{\mathrm{T}}$ is constant. Hence, only for countably many λ will we have $\widetilde{\mathrm{T}}'(x) = 0$ for some x with $\widetilde{\mathrm{T}}(x) = \lambda$.

From Remark 3.1, we deduce the following corollary of Theorem 2.2.

Corollary 3.2 (Approximate controllability, $d = 1$). *Let us consider $\mu_0, \mu_1 \in \mathcal{P}_{\mathrm{a.c.}}(\mathbb{R})$ and suppose that their densities, $\bar{\mu}_0$ and $\bar{\mu}_1$, are continuous functions (in their respective supports). Let us assume that* ***M-1***–***M-2*** *hold. For every $\varepsilon > 0$, there exists $\mu_1^\varepsilon \in \mathcal{P}_{\mathrm{a.c.}}(\mathbb{R})$ such that $\operatorname{dist}(\mu_1, \mu_1^\varepsilon) < \varepsilon$ (in the sense of the L^1 or of the Wasserstein distance) and there exists a continuous velocity field $v^\varepsilon : \operatorname{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1^\varepsilon) \to \mathbb{R}$ such that*

$$\begin{aligned} |v^\varepsilon| > 0 \quad & \textit{in} \quad \operatorname{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1) \setminus \mathcal{S}, \\ v^\varepsilon \equiv 0 \quad & \textit{in} \quad \mathcal{S}, \end{aligned}$$

where T is the monotone optimal transport map from Theorem 2.1, $\mathcal{S}$ of fixed points of the map T in $\operatorname{supp}\mu_0$, and ϕ is the unique solution of (2.1). If, furthermore, $\bar{\mu}_0$ is Lipschitz continuous, then v^ε can be taken Lipschitz continuous.

Corollary 3.2 implies, in particular, Theorem B as well.

Proof of Corollary 3.2 (and, therefore, Theorem B). Up to a standard smoothing argument, we can assume that $\bar{\mu}_1$ is Lipschitz. We consider the monotone map T between μ_0 and μ_1 and define $\mu_1^\varepsilon := (\mathrm{T}(\cdot) - \lambda)_{\#}\mu_0$ for some arbitrarily small λ such that we are in the context of Remark 3.1. Then the result follows from Theorem 2.2. □

3.1. **On the assumptions of Theorem 2.2.** We conclude this section by showing that, in Lemma 2.8 (and, consequently, in Theorem 2.2), if $\bar{\mu}_0$ and $\bar{\mu}_1$ have the same value at the fixed point $\bar{x}$, continuity of v may indeed fail. Moreover, in general, v does not need to belong to L^1_{loc} around $\bar{x}$.

Lemma 3.3. *In the context of Lemma 2.8, there exist measures μ_0 and μ_1, satisfying the hypotheses, such that $\bar{\mu}_0(\bar{x}) = \bar{\mu}_1(\bar{x})$ and either v cannot be taken bounded, or there is no uniqueness of the flow (2.1) (more precisely, it is not true $|v| > 0$ outside of $\mathcal{S}$). Moreover, if v is continuous outside of $\bar{x}$, then it does not belong to L^1_{loc} around $\bar{x}$.*

Before proving Lemma 3.3, we need the following preliminary result.

Lemma 3.4. *There exists a map $\mathrm{T} : [0,1] \to [0,1]$ such that*

$$\mathrm{T} \in C^1([0,1]), \qquad \frac{1}{2} \leqslant \mathrm{T}' \leqslant \frac{3}{2} \quad \textit{in} \quad [0,1], \qquad \mathrm{T}(x) < x, \quad \textit{for} \quad x \in (0,1],$$

and

$$\prod_{i \geqslant 0} \mathrm{T}'(\mathrm{T}^i(1/2)) = +\infty.$$

Proof. Let us first construct a map $\mathrm{S} \in C^1([0,1])$ such that $\mathrm{S}(0) = 0$, $\mathrm{S} > 0$ in $(0,1)$, $-\frac{1}{2} \leqslant \mathrm{S}' \leqslant \frac{1}{2}$, and $\mathrm{S}'(\mathrm{T}^i(1/2)) = \gamma_i$. By taking

$$\mathrm{T}(x) := x - \mathrm{S}(x),$$

we get our result.

We fix first the points $\mathrm{T}^i(1/2) := \alpha_i$, taking $\alpha_1 < \frac{1}{2}$ and $i \mapsto \alpha_i$ strictly decreasing and converging to zero. At each of these points, we then fix a value $\beta_i = \mathrm{S}(\alpha_i)$ such that the following compatibility condition holds:

$$\alpha_{i+1} = \alpha_i - \beta_i, \qquad \beta_{i+1} < \beta_i, \qquad \beta_i \downarrow 0 \quad \text{as} \quad i \to +\infty. \tag{3.1}$$

Now, the map S is defined at each point α_i. Let us fix it between in the interval $[\alpha_{i+1}, \alpha_i]$ for each $i \in \mathbb{N}$. To do that, let, for $0 < \bar\gamma < \frac{1}{2}$,

$$\begin{aligned}&\varphi_{\bar\gamma} \in C^\infty([0,1]), \quad \text{such that } -\frac{1}{4} \leqslant \varphi_{\bar\gamma} \leqslant \frac{5}{4},\\ &\text{with } \varphi'_{\bar\gamma}(0) = -\bar\gamma, \quad \varphi'_{\bar\gamma}(1) = -\frac{1}{4}, \quad \varphi_{\bar\gamma}(0) = 0, \quad \varphi_{\bar\gamma}(1) = 1, \quad -\frac{1}{2} \leqslant \varphi'_{\bar\gamma} \leqslant \frac{3}{2}.\end{aligned}$$

We then define, for some $0 < \bar\gamma_i < \frac{1}{2}$ to be chosen,

$$\mathrm{S}(x) := \beta_{i+1} + (\beta_i - \beta_{i+1})\varphi_{\bar\gamma_i}\left(\frac{x - \alpha_{i+1}}{\alpha_i - \alpha_{i+1}}\right), \quad \text{for} \quad x \in (\alpha_{i+1}, \alpha_i).$$

To ensure $\mathrm{S} \in C^1((0, 1/2])$, we impose $\frac{\beta_i - \beta_{i+1}}{4(\alpha_i - \alpha_{i+1})} = \bar\gamma_i \frac{\beta_{i-1} - \beta_i}{(\alpha_{i-1} - \alpha_i)}$, that is, $\bar\gamma_i = \frac{\beta_{i-1}(\beta_i - \beta_{i+1})}{4\beta_i(\beta_{i-1} - \beta_i)}$ from (3.1), and have $\mathrm{S}'(\alpha_i) = -\frac{\beta_i - \beta_{i+1}}{4\beta_i}$ for all $i \geqslant 0$. Moreover, if we want to bound T', we need

$$-\frac{1}{2} \leqslant -\frac{1}{2}\frac{\beta_i - \beta_{i+1}}{\beta_i} \leqslant \mathrm{S}'(x) \leqslant \frac{3}{2}\frac{\beta_i - \beta_{i+1}}{\alpha_i - \alpha_{i+1}} = \frac{3}{2}\frac{\beta_i - \beta_{i+1}}{\beta_i} \leqslant \frac{1}{2} \quad \text{in} \quad [\alpha_{i+1}, \alpha_i];$$

if we also want T' to be continuous at 0, we necessarily need $\mathrm{S}'(x) \to 0$ as $x \to 0$, or,

$$\frac{\beta_i - \beta_{i+1}}{\beta_i} \to 0, \quad \text{as} \quad i \to +\infty.$$

Finally, we notice that, to have $\mathrm{S} > 0$ in $(0, 1/2]$, we need $\beta_{i+1} - \frac{1}{4}(\beta_i - \beta_{i+1}) > 0$, that is, $\frac{\beta_{i+1}}{\beta_i} > \frac{1}{5}$.

In conclusion, we just need to construct a decreasing sequence β_i such that, from (3.1) and the considerations above,

$$\sum_{i\geqslant 0} \beta_i = \alpha_0 = \frac{1}{2}, \quad \frac{\beta_{i+1}}{\beta_i} \geqslant \frac{2}{3} \quad \text{for} \quad i \geqslant 0, \quad \frac{\beta_{i+1}}{\beta_i} \to 1, \quad \text{as} \quad i \to +\infty,$$

and

$$\bar\gamma_i = \frac{\beta_{i-1}(\beta_i - \beta_{i+1})}{4\beta_i(\beta_{i-1} - \beta_i)} < \frac{1}{2}.$$

We can take, for example, $\beta_i := \frac{\gamma}{(i+10)^2}$, where $\gamma := \frac{1}{2}(\sum_{j\geqslant 10} j^{-2})^{-1}$. Then, S is a map with $\mathrm{S} \in C^1([0, 1/2])$, such that $\mathrm{S}(0) = 0$, $\mathrm{S} > 0$ in $(0, 1)$, $-\frac{1}{2} \leqslant \mathrm{S}' \leqslant \frac{1}{2}$, $\mathrm{S}'(0) = 0$, and $\mathrm{S}'(\mathrm{T}^i(1/2)) = -\frac{\beta_i - \beta_{i+1}}{4\beta_i}$.

In this situation, we have that $\mathrm{T}(x) = x - \mathrm{S}(x)$ satisfies

$$\mathrm{T}(\alpha_i) = \alpha_i - \mathrm{S}(\alpha_i) = \alpha_i - \beta_i = \alpha_{i+1},$$

$\mathrm{T} \in C^1([0, 1/2])$, with $\mathrm{T}'(x) = 1 - \mathrm{S}'(x) \in [1/2, 3/2]$, and $\mathrm{T}'(\alpha_i) = 1 - \mathrm{S}'(\alpha_i) = 1 + \frac{\beta_i - \beta_{i+1}}{4\beta_i}$; therefore,

$$\prod_{i\geqslant 0} \mathrm{T}'(\alpha_i) \geqslant \frac{1}{4}\sum_{i\geqslant 0}\left(1 - \frac{\beta_{i+1}}{\beta_i}\right) = +\infty,$$

since $1 - \frac{\beta_{i+1}}{\beta_i} = 1 - \frac{(i+10)^2}{(i+11)^2} = \frac{2i+21}{(i+11)^2}$. By extending the map to $[1/2, 1]$ as needed, we get the desired result. □

Thanks to the previous construction, we can prove that v is not regular or even integrable in general.

Proof of Lemma 3.3. Let μ_0 be the uniform measure in $[0, 1/2]$ and let T be the map constructed in Lemma 3.4. Let $\mu_1 := \mathrm{T}_\#\mu_0$. Then μ_0 and μ_1 satisfy the hypotheses of Lemma 2.8.

Let us suppose now that there exists a vector field v defined in $[0, 1/2]$ inducing a unique flow, with $v \leqslant 0$ in $(0, 1/2]$, transporting μ_0 to μ_1. By assumption, $v(0) = 0$ (since 0 must be transported to 0), and $v(1/2) < 0$ (since $\mathrm{T}(1/2) < 1/2$ and there is uniqueness of the flow). Then, from (2.12), we deduce

$$v(\mathrm{T}^i(1/2)) = v(1/2)\prod_{j=0}^{i-1}\mathrm{T}'(\mathrm{T}^j(1/2)) = -\infty,$$

so v is not bounded around 0. This proves the first part of the result.

For the second part, we continue from the construction in Lemma 3.4, using the same notation. Let us assume, in such a construction, moreover, that $\varphi'(t) = -\frac{1}{4}$ for all $t \in [\frac{9}{10}, 1]$. We have that, if $\bar{\alpha} \in (\alpha_{i+1}, \alpha_i)$ is such that $\alpha_i - \bar{\alpha} < \frac{1}{10}\beta_i$, then

$$\mathrm{T}^{-1}(\alpha_i) - \mathrm{T}^{-1}(\bar{\alpha}) = \frac{\alpha_i - \bar{\alpha}}{1 + \frac{\beta_i - \beta_{i+1}}{4\beta_i}} < \frac{1}{10}\frac{\beta_i}{1 + \frac{\beta_i - \beta_{i+1}}{4\beta_i}} < \frac{1}{10}\beta_{i-1}$$

whenever $(\beta_i)_i$ is decreasing.

Hence, we have that, for any $\eta \in (0, \frac{1}{10}]$,

$$\begin{aligned} v(x) &\geqslant \prod_{j=0}^{i}\mathrm{T}'(\alpha_i)\inf_{\xi\in(1/2-\eta\beta_0,1/2]} v(\xi) \\ &\geqslant \frac{1}{4}\sum_{j=0}^{i}\frac{\beta_i - \beta_{i+1}}{\beta_i}\inf_{\xi\in(1/2-\eta\beta_0,1/2]} v(\xi), \quad \text{for all} \quad x \in (\alpha_i - \eta\beta_i, \alpha_i]. \end{aligned}$$

Let us take now $\beta_i = f(i) := \frac{\gamma}{i\log^2 i}$ for a suitable universal constant γ. Then we have, on the one hand, that

$$\alpha_i \asymp \int_i^\infty f(s)\,\mathrm{d}s \asymp \frac{1}{\log i}, \quad \text{for} \quad i \geqslant 2,$$

where $\asymp$ denotes comparable quantities by universal constants. On the other hand,

$$\begin{aligned} C\sum_{j=0}^{i}\frac{\beta_i - \beta_{i+1}}{\beta_i} &\geqslant -\sum_{j=0}^{i}\frac{f'(i)}{f(i)} = \sum_{j=0}^{i}|\log(f)'(i)| \\ &\asymp |\log(f(i))| = |\log(\gamma e^{-\frac{1}{\alpha_i}}\alpha_i^2)| = \frac{1}{\alpha_i} - C - 2\log(\alpha_i). \end{aligned}$$

We always fix $\inf_{\xi\in(1/2-\eta\beta_0,1/2]} v(\xi) > c_0 > 0$ by continuity of v (this is the only place where continuity is used, to say $\inf_{\xi\in(1/2-\eta\beta_0,1/2]} v(\xi) > 0$ for some $\eta > 0$). Then, from the computations above, we deduce

$$v(x) \geqslant c_0\left(\frac{1}{\alpha_i} - C - 2\log(\alpha_i)\right), \quad \text{for all} \quad x \in (\alpha_i - \eta\beta_i, \alpha_i].$$

In particular, since $\beta_i/\beta_{i+1} \approx 1$ for i large, v does not belong to L^1. □

Remark 3.5. In fact, by choosing $\beta_i := \frac{\gamma}{i\log i(\log\log i)^2}$, we can take the previous v not in L^ε for any $\varepsilon > 0$.

4. Construction in the multi-dimensional case

To solve Problem 1 and Problem 2 in multiple space-dimensions, we rely on the approach to Monge's optimal transport problem proposed by Sudakov in [59]. It consists of writing (through a disintegration) μ_0 and μ_1 as the superposition of measures concentrated on lower-dimensional sets (typically, 1D segments); solving the lower-dimensional transport problems; and, finally, "gluing" all the partial transport maps into a single transport map. After a technical gap was found in Sudakov's paper, his program was still carried out successfully: by Ambrosio and Pratelli in [8, 13] and by Trudinger and Wang in [61] for the Euclidean distance; by Ambrosio, Kirchheim, and Pratelli in [7] for crystalline norms;

by Caffarelli, Feldman, and McCann in [25] for distances induced by norms that satisfy certain smoothness and convexity assumptions; by Caravenna for general strictly convex norms [26]; and by Bianchini and Daneri [19] for general convex norms on finite-dimensional normed spaces.

We consider Sudakov's optimal transport map (associated, *e.g.*, with a strictly convex norm cost) and its decomposition along one-dimensional transport rays[11]. We can then directly apply Theorem 2.2 to realize these one-dimensional monotone transport maps as the time-1 map of the flows associated with suitable (one-dimensional) vector fields. The last step is to piece them together to define a (unique) flow in $\mathbb{R}^d$.

Theorem 4.1 (Exact controllability, $d \geqslant 1$)**.** *Let us consider $\mu_0, \mu_1 \in \mathcal{P}_{\text{a.c.}}(\mathbb{R}^d)$ (with densities $\bar\mu_0$ and $\bar\mu_1$, respectively) satisfying conditions **M-1**–**M-2**. Then, there exists a vector field $v : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$\mathrm{T}(x) = \phi(1, x), \qquad x \in \mathbb{R}^d,$$

where T is Sudakov's transport map (see Theorem 4.2 below) and ϕ is the unique solution of (1.1) for $x \in \operatorname{supp} \mu_0$.

The vector field v constructed in Theorem 4.1 is defined on the Borel partition of $\mathbb{R}^d$ into optimal transport rays $\{I^1_\alpha\}_\alpha$ (*cf.* Theorem 4.2): *i.e.*, $v(x) := v_\alpha(x)$ for $\mu_0|_{I_\alpha}$-a.e. $x \in I^1_\alpha$, and v_α is the one-dimensional velocity field obtained in Theorem 2.2 corresponding to the one-dimensional monotone optimal transport map T_α on the oriented line associated with I^1_α.

We observe that v does not need to be Lipschitz continuous (or even continuous) or satisfy the assumptions of the theory developed by DiPerna–Lions–Ambrosio (see [9, 31]).

4.1. **Preliminaries on Sudakov's theorem.** For completeness, following the notation of [52, Chapter 18], let us outline Sudakov's result. We consider Monge's problem with Euclidean norm cost:

$$\mathrm{M}^d(\mu_0, \mu_1) := \min\left\{ \int_{\mathbb{R}^d} \|\mathrm{T}(x) - x\| \, \mathrm{d}\mu_0(x) : \mathrm{T} : \mathbb{R}^d \to \mathbb{R}^d \text{ and } \mu_1 = \mathrm{T}_{\#}\mu_0 \right\}. \tag{4.1}$$

The first step in Sudakov's approach consists in finding a suitable partition of $\mathbb{R}^d$ on which the transport occurs (namely such that Kantorovich's optimal plans move the initial mass inside the elements of the partition). Given a *Kantorovich potential*[12] f from μ_0 to μ_1 (see [52, Theorem 3.17]), we define

$$G(f) := \{(a, b) \in \mathbb{R}^d \times \mathbb{R}^d : f(b) - f(a) = \|a - b\|\},$$

and consider open oriented segments $I^1_\alpha :=]a_\alpha, b_\alpha[\subset \mathbb{R}^d$ (where $\alpha \in \mathfrak{A}$ is a continuous parameter) whose extreme points belong to $G(f)$ and which are maximal with respect to set-inclusion are called *optimal rays*. By definition of Kantorovich potential, all transport has to occur on these rays. We will use the notation $\mathbb{R}^1_\alpha$ for the oriented line corresponding to I^1_α. We call *transport set* (relative to G), the union of all transport rays: $\bigcup_\alpha I^1_\alpha$. The optimal rays $\{I^1_\alpha\}_\alpha$ form a Borel partition of $\operatorname{Conv}(\operatorname{supp} \mu_0 \cup \operatorname{supp} \mu_1)$ into one-dimensional open segments, up to the sets of their initial points and final points (which are $\mathscr{L}^d$-negligible and then also μ_0-negligible).

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[11] If the cost is given by a norm that is not strictly convex, then these transport rays need not be one-dimensional.

[12] Let us recall the duality formula Monge's problem. We have the equality

$$\mathrm{M}(\mu_0, \mu_1) \equiv \sup\left\{ \int_{\mathbb{R}} f(y) \, \mathrm{d}\mu_1(y) - \int_{\mathbb{R}} f(x) \, \mathrm{d}\mu_0(x) : f \in \mathrm{Lip}_1\left(\mathbb{R}^d\right) \right\},$$

where $\mathrm{Lip}_1(\mathbb{R}^d)$ denotes the set of 1-Lipschitz functions $\mathbb{R}^d \to \mathbb{R}$. The optimal functions f are called *Kantorovich potentials.*

The second step is decomposing the transport problem and reducing it to a family of independent one-dimensional transport problems. If $\mu_0^\alpha := \mu_0|_{I^1_\alpha}$ has no atoms, then the unique transference plan concentrated on a monotone graph in $I^1_\alpha \times I^1_\alpha$ is actually concentrated on an optimal transport map T_α (*cf.* Theorem 2.1). Then the transport map T for the multi-dimensional problem is obtained by assembling the family $\{\mathrm{T}_\alpha\}_\alpha$ of one-dimensional maps[13]. The main technical difficulty in Sudakov's approach (and the flawed point in Sudakov's original contribution [59], which was subsequently amended in the references mentioned above) is proving that the disintegration of $\mathscr{L}^d$ on the optimal rays has indeed non-atomic conditional measures.

We recall the final statement below, following [52, Theorems 18.1 and 18.7].

Theorem 4.2 (Sudakov's optimal transport map)**.** *If $\mu_0, \mu_1 \in \mathcal{P}\left(\mathbb{R}^d\right)$ and $\mu_0 \ll \mathscr{L}^d$ is absolutely continuous, then there exists a (Borel measurable) transport map T from μ_0 to μ_1 satisfying (4.1). Furthermore, T is obtained as follows: $\mathrm{T}(x) := \mathrm{T}_\alpha(x)$ if $x \in I^1_\alpha$, where $\mathrm{T}_\alpha : I^1_\alpha \to I^1_\alpha$ is the monotone transport map on the ray I^1_α. Moreover, if we also have $\mu_1 \ll \mathscr{L}^d$, then we can find an optimal transport map T for $\mathrm{M}^d(\mu_0, \mu_1)$ and an optimal transport map S for $\mathrm{M}^d(\mu_1, \mu_0)$ such that $\mathrm{S} \circ \mathrm{T} = \mathrm{Id}$, μ_0-a.e. on $\mathbb{R}^d$ and $\mathrm{T} \circ \mathrm{S} = \mathrm{Id}$, μ_1-a.e. on $\mathbb{R}^d$.*

Such a *ray-monotone* map is unique: that is, there exists a unique transport map T between μ_0 and μ_1 such that, for each maximal transport ray I^1_α, T is non-decreasing from the segment $I^1_\alpha \cap \operatorname{supp} \mu_0$ to the segment $I^1_\alpha \cap \operatorname{supp} \mu_1$.

4.2. Proof of Theorem 4.1 (and Theorem C).

In this section, we prove Theorem 4.1. If we are able to show that $\mu_0|_{I^1_\alpha}$ and $\mu_1|_{I^1_\alpha}$ satisfy the assumptions of Theorem 2.2, we can then conclude by invoking Sudakov's decomposition.

Proof of Theorem 4.1 (and, therefore, Theorem C). By Theorem 4.2, there exists a (Borel measurable) transport map T from μ_0 to μ_1, satisfying (4.1), such that $\mathrm{T}(x) = \mathrm{T}_\alpha(x)$ for μ_0^α-a.e. $x \in I^1_\alpha \cap \operatorname{supp} \mu_0$, where $\mathrm{T}_\alpha : I^1_\alpha \cap \operatorname{supp} \mu_0 \to I^1_\alpha \cap \operatorname{supp} \mu_1$ is the monotone transport map on the line $\mathbb{R}^1_\alpha$.

If the measures μ_0^α and μ_1^α satisfy assumptions of Theorem 2.2, then T_α is induced by a vector field $v_\alpha : I^1_\alpha \cap \operatorname{supp} \mu_0 \to \mathbb{R}$ and we conclude the proof.

In other words, it suffices to show that the following conditions hold:

M-0$_\alpha$ $\mu_0^\alpha, \mu_1^\alpha \in \mathcal{P}_{\text{a.c.}}(I^1_\alpha)$ and their densities, $\bar\mu_0^\alpha$ and $\bar\mu_1^\alpha$, are continuous functions (in their respective supports);

M-1$_\alpha$ $\operatorname{supp} \mu_0^\alpha$ and $\operatorname{supp} \mu_1^\alpha$ are convex;

M-2$_\alpha$ $\bar\mu_0^\alpha > 0$ in $\operatorname{supp} \mu_0^\alpha$ and $\bar\mu_1^\alpha > 0$ in $\operatorname{supp} \mu_1^\alpha$.

The validity of the first part of **M-0**$_\alpha$ is a key contribution in Sudakov's disintegration result; in our setting, though, assuming additionally that the densities $\bar\mu_0$ and $\bar\mu_1$ are continuous functions, it is straightforward, and so is the second part of **M-0**$_\alpha$; **M-1**$_\alpha$ follows from the fact that $\operatorname{supp} \mu_0$ and $\operatorname{supp} \mu_1$ are convex, which yields that their intersection with each transport ray is an interval; and **M-2**$_\alpha$ holds by definition of restriction.

[13] The regularity of T is a delicate issue (see, *e.g.*, the discussion in [28, 51, 34]). For $d = 2$ and using the cost of [13], the continuity of the Sudakov's ray-monotone optimal transport map T on the interior of $\operatorname{supp} \mu_0$ was established in [38] assuming the following conditions hold:

R-1 $\mu_0, \mu_1 \in \mathcal{P}_{\text{a.c.}}(\mathbb{R}^2)$ with densities $\bar\mu_0$ and $\bar\mu_1$;

R-2 $\operatorname{supp} \mu_0$ and $\operatorname{supp} \mu_1$ are compact, convex, and disjoint subsets of $\mathbb{R}^2$;

R-3 $\bar\mu_0$ and $\bar\mu_1$ are continuous functions on their respective supports;

R-4 $\bar\mu_0$ and $\bar\mu_1$ are strictly positive in the interior of their respective supports.

A more recent refinement is contained in [50], removing the strict separation assumption, and using a different geometric set of hypotheses.

The uniqueness for (1.1), for a.e. x, follows from the one-dimensional uniqueness result of Theorem 2.2, because the trajectories are contained in the transport rays. □

5. Examples

We conclude by presenting a few examples about the construction of a (one-dimensional) velocity $v : \mathbb{R} \to \mathbb{R}$ addressing Problem 1, Problem 2, and Problem 3.

First, we study the case in which μ_0 and μ_1 are given by a superposition of Dirac deltas, mentioned in Section 1.

Lemma 5.1 (Superposition of Dirac deltas, $d = 1$)**.** *Let $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R})$ be of the form*

$$\mu_0 := \frac{1}{N}\sum_{i=1}^{N} \delta_{\{x=x_i\}}, \quad \mu_1 := \frac{1}{N}\sum_{i=1}^{N} \delta_{\{x=y_i\}}, \tag{5.1}$$
$$\text{with } \{x_i\}_{i\in\{1,\dots,N\}}, \{y_i\}_{i\in\{1,\dots,N\}} \subset \mathbb{R} \text{ and } x_i \neq x_j,\ y_i \neq y_j, \text{ for } i \neq j,$$

Then, there exists $v \in C^\infty(\mathbb{R})$ that solves Problem 1 (and Problem 2).

Proof. Let us assume that $x_1 < x_2 < \cdots < x_N$ and $y_1 < \cdots < y_N$. We now construct inductively a smooth velocity field v that sends x_i to y_i for all $1 \leqslant i \leqslant N$.

For $N = 1$, we can immediately take $v(x) := y_1 - x_1$ for x in the segment with endpoints x_1 and y_1, $x \in \{tx_1 + (1-t)y_1 : t \in [0,1]\}$.

Assuming that the statement holds for $N = k$, let us prove it for $N = k+1$. We have a smooth vector field, defined in the interval $I_k := [\min\{x_1, y_1\}, \max\{x_k, y_k\}]$, that sends each x_i to y_i for $1 \leqslant i \leqslant k$. We can assume, without loss of generality, that $x_{k+1} \leqslant y_{k+1}$: otherwise, we exchange the roles of μ_0 and μ_1 and consider as our vector field $-v$ instead.

We now consider two separate cases.

Case 1. If $x_{k+1} \notin I_k$, then $x_{k+1} > \max\{x_k, y_k\}$ and $y_{k+1} \geqslant x_{k+1}$ by assumption, so we can put as before $v \equiv y_{k+1} - x_{k+1}$ in the interval $[x_{k+1}, y_{k+1}]$, which is disjoint from I_k.

Case 2. If $x_{k+1} \in I_k$, then we necessarily have $I_k := [\min\{x_1, y_1\}, y_k]$. Since $x_k < x_{k+1} \leqslant y_k$ and x_k is sent to y_k, we have $v > 0$ in (x_k, y_k) and

$$\int_{x_{k+1}}^{y_k} \frac{\mathrm{d}x}{v(x)} = \int_{x_k}^{y_k} \frac{\mathrm{d}x}{v(x)} - \int_{x_k}^{x_{k+1}} \frac{\mathrm{d}x}{v(x)} < 1.$$

We can then define v in the interval (y_k, y_{k+1}) such that v is C^∞ in $I_k \cup (y_k, y_{k+1})$ and

$$\int_{y_k}^{y_{k+1}} \frac{\mathrm{d}x}{v(x)} = 1 - \int_{x_{k+1}}^{y_k} \frac{\mathrm{d}x}{v(x)} > 0,$$

and we have extended v outside of I_k so that the statement holds for $N = k+1$.

This completes the construction. □

Now, we turn to presenting some solutions of Problem 3 for simple transport maps.

Example 5.2 ($\mu_0 \equiv \mu_1$). If $\mu_0 \equiv \mu_1$, then $\mathrm{T}(x) = x$; then $v \equiv 0$, which yields $\phi(t,x) = x$, solves Problem 3.

Example 5.3 (Translated measures). Let $\mu_0 := \chi_{[0,1]}\mathscr{L}^1$ and $\mu_1 := \chi_{[1,2]}\mathscr{L}^1$. The monotone transport map between μ_0 and μ_1 is $\mathrm{T}(x) = x+1$. We fix a velocity $v \equiv 1$, which generates a flow $\phi(t,x) = t + x$ that brings μ_0 into μ_1 in time $t = 1$, thus solving Problem 3.

Example 5.4 (Transport map with no fixed points). Let $\mu_0 := \chi_{[0,1]}\mathscr{L}^1$ and $\mu_1 := \frac{1}{2}\chi_{[2,4]}\mathscr{L}^1$. The monotone transport map between μ_0 and μ_1 is $\mathrm{T}(x) = 2x + 2$. Using Theorem 2.2 (in particular, Lemma 2.6), we can build a suitable velocity field v. In particular, a solution to Abel's and Julia's equations can be given explicitly as follows:

$$F(x) = c + \frac{\log(|x+2|)}{\log(2)}, \qquad x \in \mathbb{R}, \text{ for any } c \in \mathbb{R},$$
$$v(x) = x\log(2) + \log(4), \qquad x \in \mathbb{R}.$$

This yields $\phi(t,x) = 2^t(2+x) - 2$, so that $\phi(1,x) = 2x+2$ solves Problem 3.

The map T has a fixed point, $\bar{x} = -2$, but it does not belong to the intervals where μ_0 and μ_1 are supported (and F is not defined there).

Example 5.5 (Transport map with one "good" fixed point). Let $\mu_0 := \chi_{[1,2]}\mathscr{L}^1$ and $\mu_1 := \frac{1}{3}\chi_{[0,3]}\mathscr{L}^1$. The monotone transport map between μ_0 and μ_1 is $\mathrm{T}(x) = 3x - 3$. A solution to Abel's and Julia's equations can be given explicitly as follows:

$$F(x) = c + \frac{\log\left(\left|x - \frac{3}{2}\right|\right)}{\log(3)}, \qquad x \in \mathbb{R}, \text{ for any } c \in \mathbb{R},$$
$$v(x) = x\log(3) - \frac{3}{2}\log(3), \qquad x \in \mathbb{R}.$$

This yields $\phi(t,x) = -3/2(-1+3^t) + 3^t x$, so that $\phi(1,x) = 3x-3$ solves Problem 3.

We observe that the map T has a fixed point, $\bar{x} = 3/2$ and $v(3/2) = 0$, while F is not defined there.

Example 5.6 (Gaussian measures). Let us consider $\mu_0 := \mathcal{N}\left(m_0, \sigma_0^2\right)$ and $\mu_1 := \mathcal{N}\left(m_1, \sigma_1^2\right)$ be two Gaussian measures[14] in $\mathbb{R}$. The monotone transport map between μ_0 to μ_1 is given by

$$\mathrm{T}(x) = \frac{\sigma_1}{\sigma_0}x - \frac{\sigma_1}{\sigma_0}m_0 + m_1$$

(here, we take σ_0, $\sigma_1 > 0$). T coincides with the identity map if $m_0 = m_1$ and $\sigma_0 = \sigma_1$; has no fixed points if $\sigma_0 = \sigma_1$ and $m_0 \neq m_1$; and has one fixed point at $\bar{x} = \sigma_0\frac{m_0 - m_1}{\sigma_0 - \sigma_1}$ if $\sigma_0 \neq \sigma_1$. At $\bar{x}$, the densities of the two measures measures do not coincide. The first case is trivial (as we can take $v \equiv 0$); in the other two, using Theorem 2.2 (in particular, Lemma 2.10 or Lemma 2.12), we can build a suitable velocity field v. In particular, we note that a solution to Abel's and Julia's equations can be given explicitly as follows:

$$F(x) = c + \frac{\log\left(\left|x - \frac{\sigma_1 m_0 - \sigma_0 m_1}{\sigma_1 - \sigma_0}\right|\right)}{\log\left(\frac{\sigma_1}{\sigma_0}\right)}, \qquad x \in \mathbb{R}, \text{ for any } c \in \mathbb{R},$$
$$v(x) = x\log\left(\frac{\sigma_1}{\sigma_0}\right) - \log\left(\frac{\sigma_1}{\sigma_0}\right)\frac{\sigma_1 m_0 - \sigma_0 m_1}{\sigma_1 - \sigma_0}, \qquad x \in \mathbb{R}.$$

See Figure 1 for an illustration.

Example 5.7 (Affine transport maps). The previous examples are particular cases of equivalent measures under affine transformations. Namely, if, in general,

$$\mu_0(\mathrm{d}x) := f(x)\mathscr{L}^1(\mathrm{d}x) \quad \text{and} \quad \mu_1(\mathrm{d}x) := \alpha f(\alpha(x-\beta))\mathscr{L}^1(\mathrm{d}x) \quad \text{for some} \quad \alpha > 0,\ \beta \in \mathbb{R},$$

where the density f is positive and continuous in its (convex) support, then the monotone map transporting μ_0 into μ_1 is

$$\mathrm{T}(x) = \frac{x}{\alpha} + \beta$$

which has a single fixed point at

$$x_{\alpha\beta} := \frac{\alpha\beta}{\alpha - 1}.$$

[14] We recall that, by definition, $\mathcal{N}(m,\sigma^2)$ has density $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$.

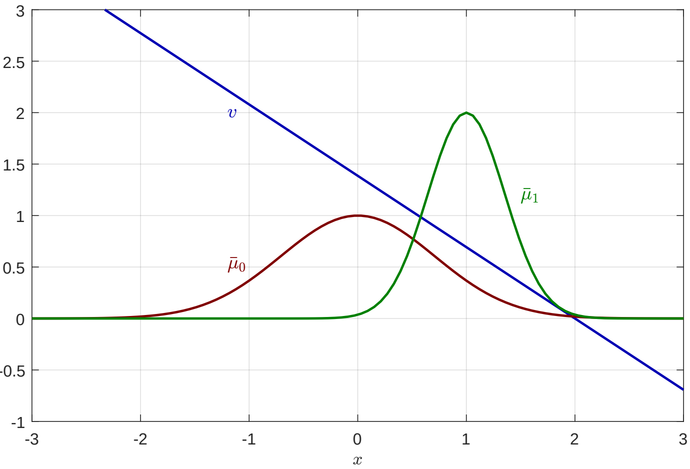

FIGURE 1. The vector field transporting a Gaussian $\bar{\mu}_0(x) = e^{-x^2}$ into a translated and rescaled Gaussian $\bar{\mu}_1(x) = 2e^{-4(x-1)^2}$ is given by the linear function v here depicted (as explained in Example 5.6 and Example 5.7). In particular, since the supports are unbounded, even if we are in a setting where the velocity field is smooth, it does not need to be globally bounded. Plot created with MATLAB [60].

If $\alpha = 1$, this was just a translation and we can fix $v \equiv c$ constant in the whole space. Otherwise, we can take

$$v(x) = \begin{cases} x - x_{\alpha\beta} & \text{if } \quad \alpha \in (0,1), \\ x_{\alpha\beta} - x & \text{if } \quad \alpha > 1, \end{cases}$$

and then adjust a multiplicative constant on v so that

$$\left| \int_0^\beta \frac{\mathrm{d}x}{x - x_{\alpha\beta}} \right| = 1.$$

See Figure 1 for a sketch in the case of transporting Gaussian measures one into another.

Example 5.8 (Transport map with two "good" fixed points). Let $\mu_0 := (1-x)\chi_{[-1/2,1/2]}\mathscr{L}^1$ and $\mu_1 := (1+x)\chi_{[-1/2,1/2]}\mathscr{L}^1$. The monotone transport map between μ_0 and μ_1 is $\mathrm{T} = \frac{1}{2}(-2 + \sqrt{2(3+4x-2x^2)})$, which has two fixed points, $\mathcal{S} = \{-1/2, 1/2\}$. Moreover, $\bar{\mu}_0 \neq \bar{\mu}_1$ on $\mathcal{S}$. We let $v(-1/2) = v(1/2) = 0$ and, using Theorem 2.2 (in particular, Lemma 2.9), we can construct a Lipschitz continuous velocity field in $[-1/2, 1/2]$ solving Problem 3.

Example 5.9 (Transport map with one "bad" fixed point). Let $\mu_0 := \frac{1}{2}\chi_{[0,2]}\mathscr{L}^1$ and $\mu_1(\mathrm{d}x) := \left(\frac{1}{2} - \frac{1}{9}x\right)\chi_{[0,3]}(x)\mathscr{L}^1(\mathrm{d}x)$. The monotone transport map that brings μ_1 to μ_0 is

$$\mathrm{T}^{-1}(x) = x - \frac{1}{9}x^2$$

It has a single fixed point at $\bar{x} = 0$, where the densities of both measures coincide. Thanks to Theorem 2.2, we can construct a velocity field in $[0,3]$, which follows for any arbitrary v fixed in $[2,3]$. See Figure 2 for one such example.

Example 5.10 (Transport map with a sequence of "good" fixed points). Let $\mu_0 := \chi_{[0,1]}\mathscr{L}^1$ and $\mathrm{T}(x) := x + \frac{1}{5}x^3 \sin(\pi/x) \in C^1([0,+\infty))$, which has fixed points

$$\mathcal{S} = \{0\} \cup \left\{ \frac{1}{n} : \ n \in \mathbb{Z} \backslash \{0\} \right\}.$$

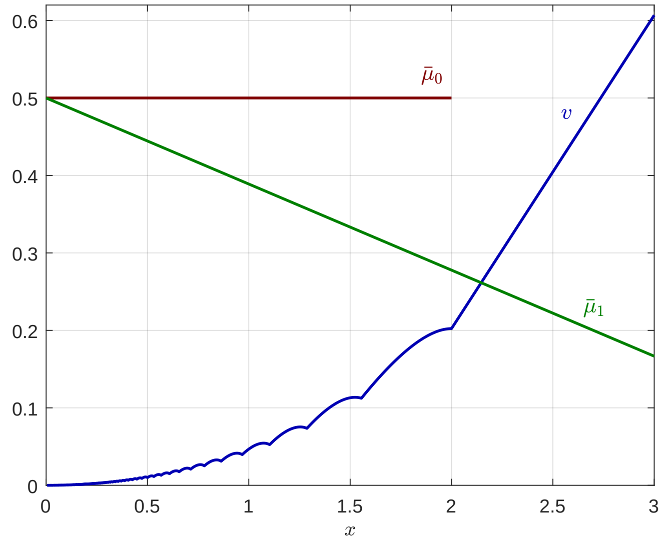

FIGURE 2. The densities $\bar\mu_0$ (in red) and $\bar\mu_1$ (in green) from Example 5.9. The velocity field v (in blue) can be constructed arbitrarily in the interval $[2, 3]$, and this fixes the values uniquely in $[0, 2]$ as well. In this case, we have chosen a linear construction (with the integral of the reciprocal equal to 1, to make time-1 maps), that matches the end-points in $[2, 3]$ following the rule in (2.10). As proven in Theorem 2.2, this extends to a continuous map, but since we are not trying to match higher derivatives (as in Corollary 2.7), such a v is not C^1. Plot created with MATLAB [60].

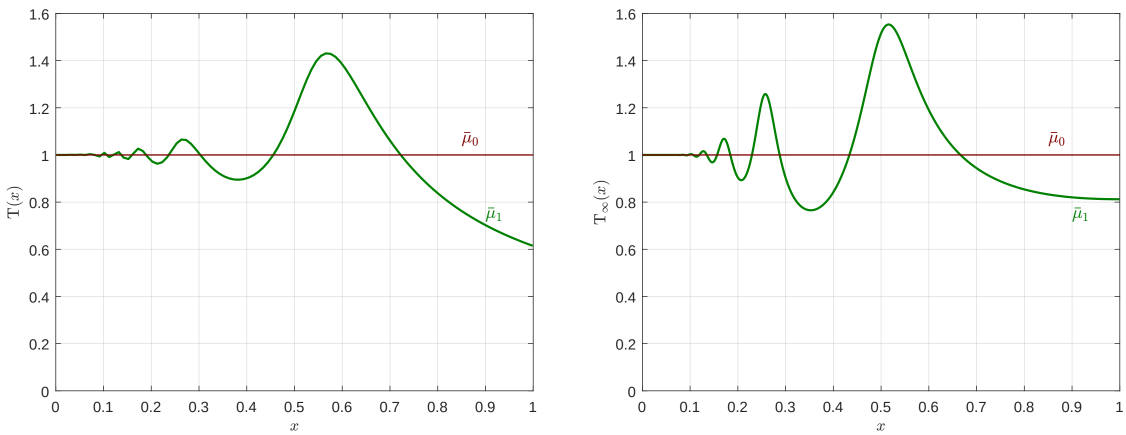

FIGURE 3. The densities $\bar\mu_0$ (in red) and $\bar\mu_1$ (in green) from Example 5.10 for the C^1 map T (left) and for the C^∞ map T_∞ (right). Plot created with MATLAB [60].

In $\mathcal{S}$, 0 is an accumulation point. We define $\mu_1 := \mathrm{T}_\# \mu_0$ (so we have $\mu_1 = \bar\mu_1 \mathscr{L}^1$, with $\bar\mu_1 = (\mathrm{T}^{-1})' \chi_{[0,1]} \in C([0, 1]) \cap C^\infty((0, 1))$; see Figure 3 for an illustration). Moreover, $\bar\mu_0 \neq \bar\mu_1$ in $\mathcal{S}\backslash\{0\}$. Then μ_0 and μ_1 satisfy the hypotheses of Theorem 2.2, and we can construct a Lipschitz continuous velocity field solving Problem 3 in $(0, 1]$.

In fact, we can even make the previous transport map, and thus μ_1, to be smooth up to the endpoints (*i.e.*, $C^\infty([0, 1])$), by taking, for example, $\mathrm{T}_\infty(x) := x + \frac{1}{5} e^{-\frac{1}{x}} \sin(\pi/x) \in C^\infty([0, +\infty))$.

Appendix A. The theory of linear homogeneous functional equations

In order to provide some context to our results, in this appendix, we recall some general statements on the solvability of (2.8) and (2.7).

By [17, Theorem 2.3, p. 16], the following theorem holds.

Theorem A.1 (Solvability of the cohomological equation)**.** *The* cohomological equation*,*

$$F(\mathrm{T}(x)) = F(x) + \gamma(x), \quad x \in \mathbb{R}, \tag{A.1}$$

where $\mathrm{T} : \mathbb{R} \to \mathbb{R}$ *and* $\gamma : \mathbb{R} \to \mathbb{R}$ *are continuous functions*[15]*, has a continuous solution* $F : \mathbb{R} \to \mathbb{R}$ *for every continuous function* $\gamma : \mathbb{R} \to \mathbb{R}$ *if and only if the following conditions are satisfied:*

A-1 T *has no fixed points;*

A-2 *there exists* $c \in \mathbb{R}$ *such that* T *is strictly increasing either on* $[c, +\infty)$ *if* $\mathrm{T}(x) > x$*, or on* $(-\infty, c]$ *if* $\mathrm{T}(x) < x$*.*

Let us now turn to studying the solvability of (2.8). If T does not have fixed points, the problem is well-understood (*cf.* [47, Theorem 2.1, p. 46])). On the other hand, the situation is much more delicate if T admits fixed points. To simplify the notation, let us focus our analysis on an interval $I := [a, b)$ and suppose that a is the only fixed point of T. Crucially, the solvability of (2.8) (in the space of continuous functions on I) depends on whether the value of $\mathrm{T}'(a)$ equals 1 or not, which is in line with the discussion in Section 3.1.

In particular, the following result holds (see [45, 27]; *cf.* [47, Theorems 2.2, 2.3, and 2.4] and [48, Sections 3.1.A–3.1.B, pp. 97–101] and also [66]).

Theorem A.2 (Solvability of Julia's equation (case of one fixed point))**.** *Let us suppose that the following conditions hold:*

H-1 $\mathrm{T} : I \to I$ *is* C^1*,* $a < \mathrm{T}(x) < x$ *for* $x \in (a, b)$*, and* $\mathrm{T}(a) = a$*;*

H-2 T' *is continuous and* $\mathrm{T}' > 0$*.*

Let us define the function

$$G_n := \prod_{i=0}^{n-1} \mathrm{T}' \circ \mathrm{T}^i, \quad n \in \mathbb{N};$$

there are three possible cases:

C-1 *there exists a continuous function* $G : I \to I$ *such that* $G := \lim_{n\to+\infty} G_n$ *and* $G(x) \neq 0$ *for every* $x \in I$*;*

C-2 $\lim_{n\to+\infty} G_n = 0$ *uniformly on a subinterval of* I*;*

C-3 *neither* ***C-1*** *nor* ***C-2*** *occurs.*

If $|\mathrm{T}'(a)| < 1$*, then case* ***C-2*** *occurs; and if* $|\mathrm{T}'(a)| = 1$ *(called* indeterminate case*), either* ***C-1****,* ***C-2****, or* ***C-3*** *may occur.*

If ***C-1*** *holds, then equation* (2.8) *has in* I *a unique one-parameter family of continuous solutions:* $v(x) = c/G(x)$*, where* $c \in \mathbb{R}$ *is an arbitrary real constant;*

If ***C-2*** *holds, then equation* (2.8) *has in* I *a continuous solution depending on an arbitrary function—which means that, for any* $\tilde{x} \in I$ *and an arbitrary continuous function* $\tilde{v} : [\min\{\tilde{x}, \mathrm{T}(\tilde{x})\}, \max\{\tilde{x}, \mathrm{T}(\tilde{x})\}) \to [\min\{\tilde{x}, \mathrm{T}(\tilde{x})\}, \max\{\tilde{x}, \mathrm{T}(\tilde{x})\})$ *fulfilling the condition*

$$\tilde{v}(\mathrm{T}(\tilde{x})) = \mathrm{T}'(\tilde{x})\, \tilde{v}(\tilde{x}),$$

there exists exactly one solution v *of equation* (2.8) *in* I *such that*

$$v(x) = \tilde{v}(x), \qquad x \in [\min\{\tilde{x}, \mathrm{T}(\tilde{x})\}, \max\{\tilde{x}, \mathrm{T}(\tilde{x})\}),$$

—and, moreover, $v(a) = 0$*;*

If ***C-3*** *holds, then* $v \equiv 0$ *is the unique continuous solution of equation* (2.8) *in* I*.*

[15] The cohomological equation (A.1) reduces to Abel's equation (2.7) if we take $\gamma \equiv 1$.

Remark A.3 (The "indeterminate case"). Following [46, p. 43], let us discuss the three cases presented in Theorem A.2. Dealing with (2.8) amounts to solving a fixed point problem in a suitable space of functions. If, for instance, we are dealing with continuous functions, as $x \to a^+$, we can approximate $v(\mathrm{T}(x)) \approx v(x)$ and $\mathrm{T}'(x) \approx \mathrm{T}'(a)$ and, heuristically, reduce (2.8) to finding a fixed point of the operator

$$H[v](x) \approx \frac{1}{\mathrm{T}'(a)} v(x).$$

For two functions v_1, v_2 we obtain

$$H[v_1](x) - H[v_2](x) \approx \frac{1}{\mathrm{T}'(a)}(v_1(x) - v_2(x)).$$

If $|\mathrm{T}'(a)| < 1$, then there exists a continuous solution depending on an arbitrary function; the "indeterminate case", $|\mathrm{T}'(a)| = 1$, is more delicate. If $|\mathrm{T}'(a)| = 1$, then it was shown in [43] that for almost all equations of the form (2.8) (in the sense of Baire's category theorem), case **C-3** holds. However, under some additional assumptions, **C-2** holds. For example, **C-2** holds assuming one of the following conditions:

- **E-1** T convex or concave (see [48, Theorem 3.5.2, p. 124]);
- **E-2** there exists c, k, $\lambda > 0$ such that $\mathrm{T}(x) = x - cx^{k+1} + \mathcal{O}\left(x^{k+1+\lambda}\right)$, for $x \in [a, b)$ (see [65, Theorems 1 and 3]).

Remark A.4 (Analytic transport maps). In Theorem 2.2, if $\bar{\mu}_0$ and $\bar{\mu}_1$ are also analytic, then T is analytic. Then we are in the setting of **E-2** and we may argue that v can be taken analytic in $\mathrm{Conv}(\operatorname{supp} \mu_0 \cup \operatorname{supp} \mu_1)$.

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