

# $C^{1,\alpha}$ ESTIMATES FOR THE FULLY NONLINEAR SIGNORINI PROBLEM

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ABSTRACT. We study the regularity of solutions to the fully nonlinear thin obstacle problem. We establish local  $C^{1,\alpha}$  estimates on each side of the smooth obstacle, for some small  $\alpha > 0$ .

Our results extend those of Milakis-Silvestre [9] in two ways: first, we do not assume solutions nor operators to be symmetric, and second, our estimates are local, in the sense that do not rely on the boundary data.

As a consequence, we prove  $C^{1,\alpha}$  regularity even when the problem is posed in general Lipschitz domains.

## 1. INTRODUCTION

The aim of this work is to study the regularity of the solutions to the Signorini or thin obstacle problem for fully nonlinear operators.

Given a domain  $D \subset \mathbb{R}^n$ , the thin obstacle problem involves a function  $u : D \rightarrow \mathbb{R}$ , an obstacle  $\varphi : S \rightarrow \mathbb{R}$  defined on a  $(n - 1)$ -dimensional manifold  $S$ , a Dirichlet boundary condition given by  $g : \partial D \rightarrow \mathbb{R}$ , and a second order elliptic operator  $L$ ,

$$\begin{cases} Lu = 0 & \text{in } D \setminus \{x \in S : u(x) = \varphi(x)\} \\ Lu \leq 0 & \text{in } D \\ u \geq \varphi & \text{on } S \\ u = g & \text{on } \partial D. \end{cases} \quad (1.1)$$

Intuitively, one can think of it as finding the shape of a membrane with prescribed boundary conditions considering that there is a very thin obstacle forcing the membrane to be above it.

When  $L$  is the Laplacian, the  $C^{1,\alpha}$  regularity of solutions was first proved in 1979 by Caffarelli in [2]. Later, the optimal value of  $\alpha$  was found by Athanasopoulos and Caffarelli in [1], where solutions were proved to be in  $C^{1,\frac{1}{2}}$  on either side of the obstacle. More recently, this has been extended to linear operators with  $x$  dependence  $L = \sum a_{ij}(x)\partial_{ij}u$  in [6, 5, 7].

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Here, we study a nonlinear version of problem (1.1). More precisely, we study (1.1) with  $Lu = F(D^2u)$ , a convex fully nonlinear uniformly elliptic operator. Since all of our estimates are of local character, we consider the problem in  $B_1$ ,

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1 \setminus \{u = \varphi\} \\ F(D^2u) \leq 0 & \text{in } B_1 \\ u \geq \varphi & \text{on } B_1 \cap \{x_n = 0\}. \end{cases} \quad (1.2)$$

Here,  $\varphi : B_1 \cap \{x_n = 0\} \rightarrow \mathbb{R}$  is the obstacle, and we assume that it is  $C^{1,1}$ . We study the regularity of solutions on either side of the obstacle.

We assume that

$$F \text{ is convex, uniformly elliptic} \quad (1.3)$$

with ellipticity constants  $0 < \lambda \leq \Lambda$ , and with  $F(0) = 0$ .

When  $u$  is symmetric, this problem was studied by Milakis and Silvestre in [9], and is equivalent to

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1^+ \\ \max\{u_{x_n}, \varphi - u\} = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{cases} \quad (1.4)$$

Moreover, they also implicitly assume a symmetry condition on the operator  $F$ , in particular, that  $F(A) = F(\tilde{A})$ , where  $\tilde{A}_{in} = \tilde{A}_{ni} = -A_{in} = -A_{ni}$  for  $i < n$  and  $\tilde{A}_{ij} = A_{ij}$  otherwise. Under this assumption, they proved interior  $C^{1,\alpha}$  regularity up to the obstacle on either side by also assuming that  $u \geq \varphi + \varepsilon$  on  $\partial B_1 \cap \{x_n = 0\}$ , for some  $\varepsilon > 0$ . Equivalently, they assume that the coincidence set is contained in some ball  $B_{1-\delta}$  for some  $\delta > 0$ . This assumption is important in [9] to prove semiconvexity of solutions.

Our main result, Theorem 1.1 below, extends the result of [9] in two ways. First, we do not assume anything on the boundary data, so that we give a local estimate. Second, we consider also non-symmetric solutions  $u$  to (1.2) with operators not necessarily satisfying any symmetry assumption, and prove  $C^{1,\alpha}$  regularity for such solutions.

In the linear case, one can symmetrise solutions to (1.2), and then the study of such solutions reduces to problem (1.4). However, in the present nonlinear setting an estimate for (1.4) does not imply one for (1.2).

Our main result is the following, stating that any solution to (1.2) is  $C^{1,\alpha}$  on either side of the obstacle, for some small  $\alpha > 0$ .

**Theorem 1.1.** *Let  $F$  be a nonlinear operator satisfying (1.3) and let  $u$  be any viscosity solution to (1.2) with  $\varphi \in C^{1,1}$ . Then,  $u \in C^{1,\alpha}(\overline{B_{1/2}^+}) \cap C^{1,\alpha}(\overline{B_{1/2}^-})$  and,*

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} + \|u\|_{C^{1,\alpha}(\overline{B_{1/2}^-})} \leq C (\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1 \cap \{x_n = 0\})})$$

for some constants  $\alpha > 0$  and  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

Our proof of the semiconvexity of solutions is completely different from the one done in [9] and follows by means of a Bernstein's technique. On the other hand, to

prove the  $C^{1,\alpha}$  regularity in the non symmetric case we follow [2, 9], but new ideas are needed. We define a symmetrised solution to the problem and follow the steps in [2] and [9] using appropriate inequalities satisfied by the symmetrised solution. This yields the regularity of the symmetrised normal derivative at free boundary points. Then, we show that this implies the  $C^{1,\alpha}$  regularity of the original function  $u$  at free boundary points, by using the ideas from [3]. Finally, we show that the regularity of  $u$  at free boundary points yields the regularity of the symmetrized normal derivative at all points on  $x_n = 0$ , and that this yields the regularity of  $u$  on either side of the obstacle.

As an immediate corollary it follows an estimate when the thin obstacle problem is posed in a bounded Lipschitz domain  $D \subset \mathbb{R}^n$ .

**Corollary 1.2.** *Let  $D \subset \mathbb{R}^n$  be a bounded Lipschitz domain, and let  $K \Subset D$ . Let  $F$  be a nonlinear operator satisfying (1.3). Let  $\varphi : D \cap \{x_n = 0\} \rightarrow \mathbb{R}$  be a  $C^{1,1}$  function, and let  $u$  be the solution to*

$$\begin{cases} F(D^2u) = 0 & \text{in } D \setminus \{u = \varphi\} \\ F(D^2u) \leq 0 & \text{in } D \\ u \geq \varphi & \text{on } D \cap \{x_n = 0\} \\ u = g & \text{on } \partial D, \end{cases} \quad (1.5)$$

for some  $g \in C^0(\partial D)$ . Let  $K^+ := K \cap \{x_n > 0\}$  and  $K^- := K \cap \{x_n < 0\}$ . Then,  $u \in C^{1,\alpha}(\overline{K^+}) \cap C^{1,\alpha}(\overline{K^-})$ , with

$$\|u\|_{C^{1,\alpha}(\overline{K^+})} + \|u\|_{C^{1,\alpha}(\overline{K^-})} \leq C (\|g\|_{L^\infty(\partial D)} + \|\varphi\|_{C^{1,1}(D \cap \{x_n = 0\})})$$

for some constant  $\alpha > 0$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ , and  $C$  depending only on  $n$ ,  $\lambda$ ,  $\Lambda$ ,  $D$ , and  $K$ .

Let us introduce the notation that will be used throughout the work. We denote  $x = (x', x_n) \in \mathbb{R}^n$  and

$$B_1^* := \{x' \in \mathbb{R}^{n-1} : (x', 0) \in B_1\}.$$

The obstacle  $\varphi$  is defined on  $B_1^*$  seen as a subset of  $\mathbb{R}^n$ , and problem (1.2) is written as

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1 \setminus \{(x', 0) : u(x', 0) = \varphi(x')\} \\ F(D^2u) \leq 0 & \text{in } B_1 \\ u(x', 0) \geq \varphi(x') & \text{for } x' \in B_1^*. \end{cases}$$

We also denote

$$B_1^+ := \{(x', x_n) \in B_1 : x_n > 0\}, \quad (\partial B_1)^+ = \partial B_1 \cap \{x_n > 0\},$$

and analogously we define  $B_1^-$  and  $(\partial B_1)^-$ . On the other hand, we call the coincidence set

$$\Delta^* = \{x \in B_1^* : u(x', 0) = \varphi(x')\}, \quad \Delta = \Delta^* \times \{0\},$$

and its complement in  $B_1^*$  is denoted by

$$\Omega^* = B_1^* \setminus \Delta^*, \quad \Omega = \Omega^* \times \{0\}.$$

Our work is organised as follows. In Section 2 we give a Lipschitz bound and prove semiconvexity of solutions. Then, in Section 3 we prove Theorem 1.1.

## 2. LIPSCHITZ ESTIMATE AND SEMICONVEXITY

**2.1. Lipschitz estimate.** We begin with a proposition showing that any solution to (1.2) is Lipschitz, as long as the obstacle is  $C^{1,1}$ .

**Proposition 2.1.** *Let  $u$  be any solution to (1.2) with  $F$  satisfying (1.3) and  $\varphi \in C^{1,1}$ . Then  $u$  is Lipschitz in  $B_{1/2}$  with,*

$$\|u\|_{\text{Lip}(B_{1/2})} \leq C (\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}), \quad (2.1)$$

for some  $C$  depending only on  $n$  and the ellipticity constants  $\lambda$  and  $\Lambda$ .

*Proof.* We will extend the obstacle  $\varphi$  to a function  $h$  defined in the whole  $B_1$ , and we treat  $u$  as a solution to a classical ‘‘thick’’ obstacle problem. We define  $h$  separately in  $B_1^+$  and  $B_1^-$ , as the solution to

$$\begin{cases} F(D^2h) = 0 & \text{in } B_1^+ \\ h = -\|u\|_{L^\infty(B_1)} & \text{in } (\partial B_1)^+ \\ h(x', 0) = \varphi(x') & \text{for } x' \in B_1^*, \end{cases} \quad (2.2)$$

and analogously

$$\begin{cases} F(D^2h) = 0 & \text{in } B_1^- \\ h = -\|u\|_{L^\infty(B_1)} & \text{in } (\partial B_1)^- \\ h(x', 0) = \varphi(x') & \text{for } x' \in B_1^*. \end{cases} \quad (2.3)$$

Notice that  $h$  is Lipschitz in  $B_{7/8}$ ; see [8, Proposition 2.2]. By denoting

$$K_0 := \|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)},$$

we have

$$\|h\|_{\text{Lip}(B_{7/8})} \leq CK_0,$$

and by the maximum principle  $u \geq h$ . Moreover,  $u$  is a solution to a classical obstacle problem in  $B_1$  with  $h$  as the obstacle. We show next that this implies  $u$  is Lipschitz, with a quantitative estimate.

To begin with, since  $h$  is Lipschitz, fixed any  $x_0 \in B_{1/2}$  and  $0 < r < 1/4$ , there exists some  $C_0$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$  such that

$$\sup_{B_r(x_0)} |h(x) - h(x_0)| \leq C_0 K_0 r. \quad (2.4)$$

Notice that, by the strong maximum principle, the coincidence set  $\{u = h\}$  is  $\Delta$ , the coincidence set of the thin obstacle problem. Suppose then that  $x_0 \in \Delta$ , i.e.,  $u(x_0) = h(x_0)$ . Since  $u \geq h$ , in particular we have that

$$\inf_{B_r(x_0)} (u(x) - u(x_0)) \geq -C_0 K_0 r. \quad (2.5)$$

because  $h$  is Lipschitz. Now let

$$q(x) = u(x) - u(x_0) + C_0 K_0 r.$$

We already know  $q \geq 0$  in  $B_r(x_0)$ . On the other hand, from (2.4),

$$q(x) \leq 2C_0 K_0 r \quad \text{on} \quad B_r(x_0) \cap \Delta.$$

Moreover,  $q$  is a supersolution,

$$F(D^2 q) = F(D^2 u) \leq 0 \quad \text{in} \quad B_r(x_0).$$

Let  $\bar{q}$  be the viscosity solution to  $F(D^2 \bar{q}) = 0$  in  $B_r(x_0)$  with  $\bar{q} = q$  on  $\partial B_r(x_0)$ . We have  $\bar{q} \leq q$  in  $B_r(x_0)$  and by the non-negativity of  $\bar{q}$  on the boundary,  $\bar{q} \geq 0$  in  $B_r(x_0)$ .

Thus,  $q < \bar{q} + 2C_0 K_0 r$  on  $\partial B_r(x_0)$ , and  $q \leq \bar{q} + 2C_0 K_0 r$  in  $B_r(x_0) \cap \Delta$ . Therefore,

$$q \leq \bar{q} + 2C_0 K_0 r \quad \text{in} \quad B_r(x_0).$$

On the other hand, we know  $0 \leq \bar{q}(x_0) \leq q(x_0) = C_0 K_0 r$ , and by the Harnack inequality,  $\bar{q} \leq C C_0 K_0 r$  in  $B_{r/2}(x_0)$ . Putting all together we obtain that  $u(x) - u(x_0) \leq C C_0 K_0 r$  for some constant  $C > 0$ . Thus, combining this with (2.5),

$$\sup_{B_r(x_0)} |u(x) - u(x_0)| \leq C \left( \|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)} \right) r, \quad (2.6)$$

for some constant  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

We have obtained that the solution is Lipschitz on points of the coincidence set. Let us use interior estimates to deduce Lipschitz regularity inside  $B_{1/2}$ .

Take any points  $x, y \in B_{1/2}$ , and let  $r = |x - y|$ . Define

$$\rho := \min\{\text{dist}(x, \Delta), \text{dist}(y, \Delta)\},$$

and let  $x^*, y^* \in \Delta$ ,  $x^* = (x', 0)$ ,  $y^* = (y', 0)$  for  $x', y' \in \Delta^*$ , be such that  $\text{dist}(x, \Delta) = |x - x^*|$  and  $\text{dist}(y, \Delta) = |y - y^*|$ . We now separate two cases:

- If  $\rho \leq 4r$ , then

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x^*)| + |u(y) - u(y^*)| + |\varphi(x') - \varphi(y')| \\ &\leq C\rho + C(r + \rho) + 2C(r + \rho) \leq Cr \end{aligned}$$

for some constant  $C$ . We are using here that  $\varphi$  is Lipschitz and that if  $|x - x^*| = \rho$ , then  $|y - y^*| \leq r + \rho$  and  $|x^* - y^*| \leq 2(r + \rho)$ .

- If  $\rho > 4r$ , we can use interior estimates. Suppose  $x$  is such that  $\text{dist}(x, \Delta) = \rho$ , and notice  $B_{\rho/2}(x) \subset B_1 \setminus \Delta$ , so that in  $B_{\rho/2}(x)$ ,  $F(D^2 u) = 0$ . We can now use the interior Lipschitz estimates (see, for example, [4, Chapter 5]),

$$[u]_{\text{Lip}(B_{\rho/4})} \leq \frac{C}{\rho} \text{osc}_{B_{\rho/2}(x)} u \leq C$$

for some constant  $C$ . We are using here that the supremum and the infimum of  $u$  in  $B_{\rho/2}(x)$  are controlled respectively by  $C\rho + \varphi(x^*)$  and  $-C\rho + \varphi(x^*)$ .

Thus, we have proved that the solution is Lipschitz in  $B_{1/2}$ , with the estimate (2.1).  $\square$

**2.2. Preliminary consideration.** Before continuing to prove the semiconvexity and semiconcavity result, we introduce a change of variables that will be useful in this section and the next one. Notice that, given a function  $w$ , we can express the nonlinear operator  $F$  as

$$F(D^2w(x)) = \sup_{\gamma \in \Gamma} (L_\gamma^{ij} \partial_{x_i x_j} w(x) + c_\gamma),$$

for some family of symmetric uniformly elliptic operators with ellipticity constants  $\lambda$  and  $\Lambda$ ,  $L_\gamma^{ij} \partial_{x_i x_j}$ , indexed by  $\gamma \in \Gamma$ . Since  $F(0) = 0$ , there is some symmetric uniformly elliptic operator from this family given by a matrix  $\hat{L}$  such that

$$\text{tr}(\hat{L}D^2w(x)) = \hat{L}^{ij} \partial_{x_i x_j} w(x) \leq F(D^2w(x)).$$

We now change coordinates in such a way that the matrix of this operator in the new coordinates, denoted  $\hat{L}_A$ , fulfils  $\hat{L}_A^{in} = \hat{L}_A^{ni} = 0$  for  $i < n$ . More precisely, if we denote  $\hat{L}'$  the matrix in  $\text{Sym}_{n-1}$  given by the  $n-1$  first indices of  $\hat{L}$ , and we denote  $\hat{L}'_n = (\hat{L}^{in})_{1 \leq i \leq n-1}$  the vector of  $\mathbb{R}^{n-1}$ , we change variables as

$$x \mapsto y = Ax,$$

where  $A$  is the matrix given by

$$A := \left( \begin{array}{c|c} \text{Id}_{n-1} & -\bar{a} \\ \hline 0 \dots 0 & 1 \end{array} \right),$$

and  $\bar{a} = (\hat{L}')^{-1} \cdot \hat{L}'_n$  is a vector in  $\mathbb{R}^{n-1}$ . We define the new nonlinear operator  $\tilde{F}$  as

$$\tilde{F}(N) = F(A^T N A), \text{ for all } N \in \text{Sym}_n,$$

so that it is consistent with the change of variables, in the sense that if  $\tilde{w}(y) = w(A^{-1}y)$ , then  $F(D^2w(x)) = \tilde{F}(D^2\tilde{w}(y))$ .

We trivially have that  $\tilde{F}$  is convex and  $\tilde{F}(0) = 0$ . In the new coordinates we still have that  $\hat{L}_A^{ij} \partial_{y_i y_j}$  is a symmetric uniformly elliptic operator, but now the ellipticity constants  $\lambda$  and  $\Lambda$  have changed depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ . The same occurs with all the operators in the family defining  $F$ , so that after changing coordinates,  $F$  is still a convex uniformly elliptic operator with ellipticity constants depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ . Indeed, for any matrices  $N, N_P \in \text{Sym}_{n-1}$  with  $N_P \geq 0$  we have that (using the definition of uniform ellipticity in [4, Chapter 2] and noticing that  $A^T N_P A \geq 0$ ),

$$\|A^{-1}\|^{-2} \|N_P\| \leq \lambda \|A^T N_P A\| \leq \tilde{F}(N + N_P) - \tilde{F}(N) \leq \Lambda \|A^T N_P A\| \leq \Lambda \|A\|^2 \|N_P\|,$$

and it is easy to bound  $\|A^{-1}\|$  and  $\|A\|$  from the definition of  $A$ , depending only on  $n$ ,  $\lambda$  and  $\Lambda$ .

After changing variables, the regularity of the solution remains the same up to multiplicative constants in the bounds depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

As an abuse of notation we will call the new variables  $(x', x_n)$ , the new operator  $F$ , and the new ellipticity constants  $\lambda$  and  $\Lambda$ , understanding that they might depend on the original ellipticity constants and the dimension,  $n$ . This will not be a problem, since in all the statements of the present work  $n$ ,  $\lambda$ , and  $\Lambda$  appear together in the dependence of the constants.

Thus, throughout the paper we will assume that there exists a fixed symmetric uniformly elliptic operator  $\hat{L}$  such that

$$\hat{L}^{ij} \partial_{x_i x_j} w \leq F(D^2 w), \quad \text{and} \quad \hat{L}^{in} = \hat{L}^{ni} = 0 \quad \text{for} \quad i < n. \quad (2.7)$$

This change of variables is useful because, for any function  $w$ ,

$$\hat{L}^{ij} \partial_{x_i x_j} (w(x', -x_n)) = \hat{L}^{ij} (\partial_{x_i x_j} w)(x', -x_n),$$

which will allow us to symmetrise the solution and still have a supersolution for the Pucci extremal operator  $\mathcal{M}^-$ . We also use it to prove a semiconcavity result from semiconvexity in the following proof of Proposition 2.2.

**2.3. Semiconvexity and semiconcavity estimates.** We next prove the semiconvexity of solutions in the directions parallel to the domain of the obstacle. To do it, we use a Bernstein's technique in the spirit of [1].

**Proposition 2.2.** *Let  $u$  be the solution to (1.2). Then*

(a) *(Semiconvexity) If  $\tau = (\tau^*, 0)$ , with  $\tau^*$  a unit vector in  $\mathbb{R}^{n-1}$ ,*

$$\inf_{B_{3/4}} u_{\tau\tau} \geq -C (\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}),$$

*for some constant  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .*

(b) *(Semiconcavity) Similarly, in the direction normal to  $B_1^* \times \{0\}$ ,*

$$\sup_{B_{3/4}} u_{x_n x_n} \leq C (\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}),$$

*for some constant  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .*

*Proof.* The second part, (b), follows from (a) using the definition of uniformly elliptic operator and the fact that we changed variables (in the previous subsection) in order to have matrix  $\hat{L}$  fulfilling (2.7). We denote by  $\hat{L}'$  and  $D_{n-1}^2 u$  the square matrices corresponding to the  $n-1$  first indices of  $\hat{L}$  and  $D^2 u$  respectively. Now, from

$$\hat{L}^{ij} \partial_{x_i x_j} u(x) \leq 0, \quad \hat{L}^{in} = \hat{L}^{ni} = 0 \quad \text{for} \quad i < n,$$

and

$$D_{n-1}^2 u \geq -C (\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}) \text{Id}_{n-1},$$

we directly obtain that

$$\hat{L}^{nn} \partial_{x_n x_n} u \leq - \sum_{i,j=1}^{n-1} \hat{L}^{ij} \partial_{x_i x_j} u \leq C (\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}) \text{tr} \hat{L}'.$$

The desired bound follows because  $\hat{L}^{nn}$  is bounded below by  $\lambda$  and  $\text{tr}(\hat{L}')$  is bounded above by  $(n-1)\Lambda$ .

Let us prove (a). As in the proof of Proposition 2.1, we define  $h$  as the solution to

$$\begin{cases} F(D^2h) = 0 & \text{in } B_1^+ \\ h = -\|u\|_{L^\infty(B_1)} & \text{in } (\partial B_1)^+ \\ h(x', 0) = \varphi(x') & x' \in B_1^* \end{cases} \quad \begin{cases} F(D^2h) = 0 & \text{in } B_1^- \\ h = -\|u\|_{L^\infty(B_1)} & \text{in } (\partial B_1)^- \\ h(x', 0) = \varphi(x') & x' \in B_1^*. \end{cases} \quad (2.8)$$

Recall that  $h$  is Lipschitz and that, by the strong maximum principle,  $u > h$  in  $B_{1/2}^+$  and  $B_{1/2}^-$ .

Define now, for  $\varepsilon > 0$ ,

$$\bar{h}_\varepsilon(x', x_n) := \varphi(x') - \frac{x_n^2}{\varepsilon}$$

and

$$h_\varepsilon(x', x_n) := \max \{h(x', x_n), \bar{h}_\varepsilon(x', x_n)\}.$$

Since,  $h$  is Lipschitz continuous and  $h(x', 0) = \bar{h}_\varepsilon(x', 0)$ , this implies that there exists a constant  $C > 0$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$  such that

$$h(x', x_n) > \bar{h}_\varepsilon(x', x_n) \quad \text{for } |x_n| > CK_0\varepsilon, \quad (2.9)$$

where we define

$$K_0 := \|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}.$$

In particular,  $h_\varepsilon$  is Lipschitz continuous in  $B_{7/8}$ , uniformly on  $\varepsilon$ .

Let  $u_\varepsilon$  be the solution to the ‘‘thick’’ obstacle problem with obstacle  $h_\varepsilon$ ,

$$\begin{cases} F(D^2u_\varepsilon) = 0 & \text{in } B_1 \setminus \{u_\varepsilon = h_\varepsilon\} \\ F(D^2u_\varepsilon) \leq 0 & \text{in } B_1 \\ u_\varepsilon = \max\{u, \bar{h}_\varepsilon\} & \text{on } \partial(B_1^+) \\ u_\varepsilon \geq h_\varepsilon & \text{in } B_1^+, \end{cases} \quad (2.10)$$

and the analogous expression in  $B_1^-$ . By (2.9), the coincidence set satisfies

$$\{u_\varepsilon = h_\varepsilon\} \subset \{\bar{h}_\varepsilon > h\} \subset \{(x', x_n) \in B_1 : |x_n| \leq CK_0\varepsilon\}$$

for some  $C > 0$ . We want to bound  $\partial_{\tau\tau}u_\varepsilon$  from below independently of  $\varepsilon$ .

Notice that  $D^2(u_\varepsilon - h_\varepsilon) \geq 0$  in the coincidence set, and since  $u_\varepsilon \geq h_\varepsilon$ , this also occurs along the free boundary. By the definition of  $\bar{h}_\varepsilon$  and recalling that  $h_\varepsilon = \bar{h}_\varepsilon$  in the coincidence set, this implies  $\partial_{\tau\tau}u_\varepsilon \geq -CK_0$  in  $\{u_\varepsilon = h_\varepsilon\} \cap B_{7/8}$ , for some constant  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ . Thus, it is enough to check that  $\partial_{\tau\tau}u_\varepsilon$  is uniformly bounded from below outside the coincidence set. We proceed by means of a Bernstein’s technique.

Let  $\eta \in C_c^\infty(B_{7/8})$  be a smooth, cutoff function, with  $0 \leq \eta \leq 1$  and  $\eta \equiv 1$  in  $B_{3/4}$ . Define

$$f_\varepsilon(x) = \eta(x)\partial_{\tau\tau}u_\varepsilon(x) - \mu|\nabla u_\varepsilon(x)|^2$$



for some constant  $\mu$  to be determined later. Notice that, since  $h_\varepsilon$  is Lipschitz continuous independently of  $\varepsilon$  in  $B_{7/8}$ , then  $|\nabla u_\varepsilon(x)|$  is bounded independently of  $\varepsilon$  in  $B_{7/8}$ . If the minimum  $x_0$  in  $B_{7/8}$  is attained in the coincidence set, then  $\partial_{\tau\tau}u_\varepsilon(x_0) \geq -CK_0$  and we get that for every  $x \in B_{3/4}$ ,

$$\partial_{\tau\tau}u_\varepsilon(x) \geq -CK_0 - \mu|\nabla u_\varepsilon(x_0)|^2 + \mu|\nabla u_\varepsilon(x)|^2 \geq -CK_0 - \mu\|\nabla u_\varepsilon\|_{L^\infty(B_{7/8})}^2. \quad (2.11)$$

If the minimum  $x_0$  is attained at the boundary,  $\partial B_{7/8}$ , then for every  $x \in B_{3/4}$ ,

$$\partial_{\tau\tau}u_\varepsilon(x) \geq -\mu|\nabla u_\varepsilon(x_0)|^2 + \mu|\nabla u_\varepsilon(x)|^2 \geq -\mu\|\nabla u_\varepsilon\|_{L^\infty(B_{7/8})}^2. \quad (2.12)$$

Let us assume now that the minimum  $x_0$  of  $f_\varepsilon$  in  $B_{7/8}$  is attained at some interior point  $x_0$  outside the coincidence set  $\{u_\varepsilon = h_\varepsilon\}$ .

Let us also assume that the operator  $F$  not only is convex, but also  $F \in C^\infty$ , so that solutions are  $C^4$  outside the coincidence set (see the end of the proof for the general case  $F$  Lipschitz). In this case, the linearised operator of  $F$  at  $x_0$ ,

$$L_0v = a_{ij}v_{ij} := F_{ij}(D^2u_\varepsilon(x_0))v_{ij},$$

is uniformly elliptic with ellipticity constants  $\lambda$  and  $\Lambda$ . Moreover, for any  $\rho \in S^{n-1}$ ,

$$L_0u_\varepsilon(x_0) \geq 0, \quad L_0\partial_\rho u_\varepsilon(x_0) = 0, \quad L_0\partial_{\rho\rho}u_\varepsilon(x_0) \leq 0. \quad (2.13)$$

This is a standard result, which can be found in [4, Lemma 9.2].

For simplicity in the following computations we denote  $w = u_\varepsilon$ . If  $x_0$  is an interior minimum of  $f_\varepsilon$  (which is a  $C^2$  function) in  $B_{7/8}$ , then

$$0 = \nabla f_\varepsilon(x_0) = (\nabla\eta w_{\tau\tau} + \eta\nabla w_{\tau\tau} - 2\mu w_i\nabla w_i)(x_0), \quad (2.14)$$

and by (2.13) and the fact that  $(a_{ij})$  is elliptic,

$$0 \leq a_{ij}f_{\varepsilon,ij}(x_0) \leq (a_{ij}\eta_{ij}w_{\tau\tau} + 2a_{ij}\eta_i w_{\tau\tau,j} - 2\mu a_{ij}w_{kj}w_{ki})(x_0). \quad (2.15)$$

Combining (2.14) and (2.15), we find

$$0 \leq \left( \left( a_{ij}\eta_{ij} - 2\frac{a_{ij}\eta_i\eta_j}{\eta} \right) w_{\tau\tau} - 2\mu a_{ij}w_{kj}w_{ki} + 4\frac{\mu a_{ij}\eta_i w_{kj}w_{ki}}{\eta} \right) (x_0). \quad (2.16)$$

Observe that  $|\nabla\eta|^2 \leq C\eta$  (since  $\sqrt{\eta}$  is Lipschitz). Therefore, for some constants  $C_0$  and  $C_1$  depending only on  $n$  and  $\Lambda$ ,

$$0 \leq (C_0|w_{\tau\tau}| + \mu C_1|D^2w||\nabla w| - 2\mu a_{ij}w_{kj}w_{ki})(x_0).$$

Using  $|w_{\tau\tau}(x_0)| \leq |D^2w(x_0)|$  and the uniform ellipticity of  $(a_{ij})$ ,

$$a_{ij}w_{ki}w_{kj} \geq \lambda C(n)|D^2w|^2,$$

we obtain

$$|D^2w(x_0)| \leq \frac{C_0}{\mu} + C_1|\nabla w(x_0)|,$$

for some constants  $C_0$  and  $C_1$  depending now also on  $\lambda$ . Now, since  $x_0$  is a minimum in  $B_{7/8}$ , for any  $x \in B_{3/4}$ ,

$$\begin{aligned} w_{\tau\tau}(x) &\geq \eta(x_0)w_{\tau\tau}(x_0) - \mu|\nabla u_\varepsilon(x_0)|^2 + \mu|\nabla u_\varepsilon(x)|^2 \\ &\geq -|D^2w(x_0)| - \mu\|\nabla u_\varepsilon\|_{L^\infty(B_{7/8})}^2 \\ &\geq -\frac{C_0}{\mu} - C_1\|\nabla u_\varepsilon\|_{L^\infty(B_{7/8})} - \mu\|\nabla u_\varepsilon\|_{L^\infty(B_{7/8})}^2. \end{aligned} \quad (2.17)$$

We now fix  $\mu = \|\nabla u_\varepsilon\|_{L^\infty(B_{7/8})}^{-1}$ . Notice that, in all three cases (2.11), (2.12), and (2.17), we reach that for some constant  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ ,

$$\inf_{B_{3/4}} \partial_{\tau\tau} u_\varepsilon \geq -C \left( \sup_{B_{7/8}} |\nabla u_\varepsilon| + K_0 \right).$$

We had already seen that  $u_\varepsilon$  is Lipschitz continuous independently of  $\varepsilon > 0$  and controlled by the Lipschitz norm of  $u$ , so that by Proposition (2.1),

$$\inf_{B_{3/4}} \partial_{\tau\tau} u_\varepsilon \geq -C \left( \|u\|_{\text{Lip}(B_{7/8})} + \|\varphi\|_{C^{1,1}(B_1^*)} + K_0 \right) \geq -C \left( \|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)} \right). \quad (2.18)$$

If  $F$  is not smooth, then it can be regularised convoluting with a mollifier in the space of symmetric matrices, so that it can be approximated uniformly in compact sets by a sequence  $\{F_k\}_{k \in \mathbb{N}}$  of convex smooth uniformly elliptic operators with ellipticity constants  $\lambda$  and  $\Lambda$ ; also, by subtracting  $F_k(0)$ , we can assume  $F_k(0) = 0$ . Note that, in  $B_{7/8}$  and for every  $\varepsilon > 0$  we have uniform  $C^{1,\gamma}$  estimates in  $k$  for the solutions to (2.10) with operators  $F_k$ , since the obstacle  $h$  is in  $C^{1,1}$  in a neighbourhood of the free boundary. By Arzelà-Ascoli there exists a subsequence converging uniformly, and therefore, the estimate (2.18) can be extended to solutions of (2.10) with operators not necessarily smooth. Thus, (2.18) follows for any  $F$  not necessarily  $C^\infty$ .

Note that  $u_\varepsilon$  converges uniformly to  $u$ , since for all  $\delta > 0$ , there exists some  $\varepsilon > 0$  small enough such that  $u + \delta > u_\varepsilon \geq u$  in  $B_1$ .

Since the right-hand side of (2.18) is independent of  $\varepsilon$ , and  $u_\varepsilon$  converges uniformly to  $u$  in  $B_{7/8}$  as  $\varepsilon \downarrow 0$ , we finally obtain

$$\inf_{B_{3/4}} u_{\tau\tau} \geq -C \left( \|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)} \right), \quad (2.19)$$

as desired.  $\square$

### 3. $C^{1,\alpha}$ ESTIMATE

**3.1. A symmetrised solution.** By the results in the previous section we know that  $\nabla u$  is bounded in the interior of  $B_1$ . Moreover,  $u_{x_n x_n}$  is bounded from above inside  $B_1$ . In particular, the following limit exists

$$\sigma(x') = \lim_{x_n \downarrow 0^+} u_{x_n}(x', x_n) - \lim_{x_n \uparrow 0^-} u_{x_n}(x', x_n) = \lim_{x_n \downarrow 0^+} (u_{x_n}(x', x_n) - u_{x_n}(x', -x_n)). \quad (3.1)$$

A main step towards Theorem 1.1 consists of proving that  $\sigma \in C^\alpha(B_{1/2}^*)$  for some  $\alpha > 0$ . We will prove this in this section.

We begin by noticing that  $\sigma(x') = 0$  for  $x' \in \Omega^*$  (by the  $C^{2,\alpha}$  interior estimates), where we recall that  $\Omega^* := \{x' \in B_1^* : u(x', 0) > \varphi(x')\}$ . In general, however, we have the following:

**Lemma 3.1.** *The function  $\sigma$  defined by (3.1) is non-positive, i.e.,  $\sigma \leq 0$  in  $B_1^*$ .*

*Proof.* Suppose it is not true, and there exists some  $\bar{x}' \in B_1^*$  such that  $\sigma(\bar{x}') > 0$ . Let  $\delta > 0$  be such that  $B_\delta^*(\bar{x}') \subset B_1^*$ , so that by the semiconcavity in Proposition 2.2 applied to  $B_{\delta/2}((\bar{x}', 0))$ ,  $u_{x_n x_n}(\bar{x}', 0) \leq C$  for some constant  $C$ , that now depends also on  $\delta$ . However,

$$\sigma(\bar{x}') = \lim_{x_n \downarrow 0^+} (u_{x_n}(\bar{x}', x_n) - u_{x_n}(\bar{x}', -x_n)) > 0,$$

which means

$$\frac{u_{x_n}(\bar{x}', x_n) - u_{x_n}(\bar{x}', -x_n)}{2x_n} \rightarrow +\infty, \text{ as } x_n \downarrow 0^+,$$

a contradiction with the bound in  $u_{x_n x_n}$ .  $\square$

We will now adapt the ideas of [2] to our non-symmetric setting. For this, we use a symmetrised solution, defined as follows

$$v(x', x_n) := \frac{u(x', x_n) + u(x', -x_n)}{2}, \text{ for } (x', x_n) \in \overline{B_1}. \quad (3.2)$$

Here  $u$  is any solution to (1.2).

Notice that

$$\sigma(x') = 2 \lim_{x_n \downarrow 0^+} v_{x_n}(x', x_n) \leq 0 \quad (3.3)$$

is well defined, and in particular, we have that

$$\sigma(x') = 2v_{x_n}(x', 0) = 0, \text{ for } x' \in \Omega^*. \quad (3.4)$$

The following result follows from the results in the previous section. We will use the notation  $\mathcal{M}^+$  and  $\mathcal{M}^-$  to refer to the Pucci's extremal operators with the implicit ellipticity constants  $\lambda$  and  $\Lambda$  (see [4, Chapter 2] for the definition and basic properties of such operators).

**Lemma 3.2.** *Let  $u$  be a solution to the nonlinear thin obstacle problem (1.2), and let  $v$  be defined by (3.2). Then  $v$  is Lipschitz in  $\overline{B_{1/2}^+}$  and satisfies*

$$\begin{cases} \mathcal{M}^-(D^2 v) \leq 0 & \text{in } B_1, \\ \max\{v_{x_n}(x', 0), \varphi(x') - v(x', 0)\} = 0 & \text{for } x' \in B_1^*. \end{cases} \quad (3.5)$$

Moreover,

(a) *(Semiconvexity) If  $\tau = (\tau^*, 0)$ , with  $\tau^*$  a unit vector in  $\mathbb{R}^{n-1}$ ,*

$$\inf_{B_{3/4}} v_{\tau\tau} \geq -C (\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}),$$

*for some constant  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .*

(b) (Semiconcavity) In the direction normal to  $B_1^* \times \{0\}$ ,

$$\sup_{B_{3/4}} v_{x_n x_n} \leq C (\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}),$$

for some constant  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

*Proof.* The Lipschitz regularity comes from the Lipschitz regularity in  $u$ , proved in Proposition 2.1.

In (3.5) the first inequality follows thanks to the change of variables introduced in Subsection 2.2. Indeed, there exists some operator given by a matrix  $\hat{L}$  as in (2.7) uniformly elliptic with ellipticity constants  $\lambda$  and  $\Lambda$  such that

$$\hat{L}^{ij} \partial_{x_i x_j} (u(x', -x_n)) = \hat{L}^{ij} (\partial_{x_i x_j} u)(x', -x_n) \leq F((D^2 u)(x', -x_n)) \leq 0,$$

so that

$$\mathcal{M}^-(D^2 v) \leq \hat{L}^{ij} \partial_{x_i x_j} v \leq 0,$$

as we wanted.

The second expression in (3.5) follows from equations (3.3)-(3.4), Lemma 3.1 and the fact that  $v(x', 0) = u(x', 0)$  for  $x' \in B_1^*$ .

Finally, the semiconvexity and semiconcavity follow from Proposition 2.2.  $\square$

**3.2. Regularity for  $\sigma$  on free boundary points.** The next steps are very similar to those in [2] (and [9]), but we adapt them to the symmetrised solution  $v$  instead of  $u$ . For completeness, we provide all the details. We begin with the following lemma, corresponding to [2, Lemma 2] (or [9, Lemma 3.3]).

In the next result, we call  $\varphi$  the extension of the obstacle to  $B_1$ , i.e.  $\varphi(x', x_n) := \varphi(x')$ .

**Lemma 3.3.** *Let  $v$  be the symmetrised solution (3.2). Let  $\kappa$  be a constant such that  $\kappa > \sup |\varphi_{\tau\tau}|$  for any  $\tau$  a unit vector in  $\mathbb{R}^{n-1} \times \{0\}$ . Let  $x_0 \in \Omega$  fixed and  $\psi_{x_0}$  denote the function*

$$\psi_{x_0} = \varphi(x_0) + \nabla \varphi(x_0) \cdot (x - x_0) + \kappa |x - x_0|^2 - \kappa(n-1) \frac{\Lambda}{\lambda} x_n^2.$$

Then, for any open set  $U_{x_0}$  such that  $x_0 \in U_{x_0} \subset B_1$ ,

$$\sup_{\partial U_{x_0} \cap \{x_n > 0\}} (v - \psi_{x_0}) \geq 0.$$

*Proof.* Define  $w = v - \psi_{x_0}$  and notice that by definition of  $\psi_{x_0}$  and the fact that  $v$  is a supersolution for  $\mathcal{M}^-$ , we have  $w(x_0) \geq 0$  and  $\mathcal{M}^-(D^2 w) \leq 0$ . Therefore, we can apply the maximum principle on  $U_{x_0} \setminus \Delta$  (recall  $\Delta$  is the coincidence set) and use the symmetry of  $w$  to obtain that

$$\sup_{\partial(U_{x_0} \setminus \Delta) \cap \{x_n \geq 0\}} (v - \psi_{x_0}) \geq 0.$$

Now notice that on the set  $\{v = u = \varphi\}$  we have that  $\psi_{x_0} > \varphi$ , since  $x_0 \in \Omega$  and  $\kappa > \sup |\varphi_{\tau\tau}|$ . Thus,  $v - \psi_{x_0} < 0$  on this set, so that

$$\sup_{\partial(U_{x_0} \setminus \Delta) \cap \{x_n \geq 0\}} (v - \psi_{x_0}) = \sup_{\partial U_{x_0} \cap \{x_n > 0\}} (v - \psi_{x_0}) \geq 0,$$

and we are done.  $\square$

We now proceed with the following lemma, corresponding to [2, Lemma 2] (or [9, Lemma 3.4]).

**Lemma 3.4.** *Let  $v$  be the symmetrised solution as defined in (3.2), and let  $\sigma$  as defined in (3.1)-(3.3). Let  $x_0 = (x'_0, 0) \in \Omega$  and define  $S_\gamma = \{x' : \sigma(x') > -\gamma\}$ . Then, for suitable positive constants  $C, \bar{C}$ , and  $\gamma_0$  and for all  $\gamma \in (0, \gamma_0)$  there exists a ball  $B_{\bar{C}\gamma}^*(\bar{x}')$  for  $\bar{x}' \in B_1^*$  such that*

$$B_{C\gamma}^*(\bar{x}') \subset B_{\bar{C}\gamma}^*(x'_0) \cap S_\gamma.$$

The constants  $C, \bar{C}$ , and  $\gamma_0$  depend only on  $n, \lambda, \Lambda, \|\varphi\|_{C^{1,1}(B_1^*)}$ , and  $\|u\|_{L^\infty(B_1)}$ .

*Proof.* We apply Lemma 3.3 with  $U_{x_0} = B_{C_1\gamma}(x_0) \times (-C_2\gamma, C_2\gamma)$  for some constants to be chosen  $C_1 \gg C_2$ , and study two cases.

- Assume  $\sup(v - \psi_{x_0})$  is attained at a point  $(x'_1, y_1)$  (for  $x'_1 \in \mathbb{R}^{n-1}, y \in \mathbb{R}$ ) on the lateral face of the cylinder  $U_{x_0}$ , i.e. with  $|x'_1 - x'_0| = C_1\gamma$  and  $0 \leq y_1 \leq C_2\gamma$ . Then we have

$$\begin{aligned} \psi_{x_0}(x'_1, y_1) - \varphi(x'_1) &\geq (\kappa - \sup |\phi_{\tau\tau}|) |x'_1 - x'_0|^2 - \kappa(n-1) \frac{\Lambda}{\lambda} y_1^2 \\ &\geq (\kappa - \sup |\phi_{\tau\tau}|) C_1^2 \gamma^2 - \kappa(n-1) \frac{\Lambda}{\lambda} C_2^2 \gamma^2 \geq C_3 \gamma^2, \end{aligned}$$

provided that  $C_1 \gg C_2$ . The positive constant  $C_3$  depends only on  $\kappa, n$ , the ellipticity constants,  $C_1$ , and  $C_2$ . Thus,

$$v(x'_1, y_1) \geq \psi_{x_0}(x'_1, y_1) \geq \varphi(x'_1) + C_3 \gamma^2.$$

Now pick a  $x'_2 \in B_{C_4\gamma}^*(x'_1)$  for some positive constant  $C_4$  to be chosen and  $(x'_2 - x'_1) \cdot \nabla_{x'}(v - \varphi)(x'_1, y_1) \geq 0$ . We are considering here  $\varphi$  in the whole  $B_1$  by simply putting  $\varphi(x', y) = \varphi(x')$ . Take  $\tau = \left( \frac{x'_2 - x'_1}{|x'_2 - x'_1|}, 0 \right)$ , and use the semiconvexity from Lemma 3.2 together with the fact that  $\varphi \in C^{1,1}$  to get

$$\begin{aligned} (v - \varphi)(x'_2, y_1) &= \\ &= (v - \varphi)(x'_1, y_1) + (x'_2 - x'_1) \cdot \nabla_{x'}(v - \varphi)(x'_1, y_1) + \iint_{[(x'_1, y_1), (x'_2, y_1)]} (v - \varphi)_{\tau\tau} \\ &\geq C_3 \gamma^2 - C|x'_2 - x'_1|^2 \geq (C_3 - CC_4)\gamma^2 > 0, \end{aligned}$$

if  $C_4$  is chosen appropriately, small enough depending only on  $C_3, \|\varphi\|_{C^{1,1}}$  and the semiconvexity constant of Lemma 3.2. Here, and in the next steps,  $\iint_{[a,b]}$  denotes

the double integral over the segment between the points  $a$  and  $b$ ,

$$\iint_{[a,b]} w := \int_0^{|b-a|} \left[ \int_0^s w \left( a + \frac{b-a}{|b-a|} t \right) dt \right] ds.$$

To get a contradiction, now suppose that  $x'_2 \notin S_\gamma$ . In particular, this means  $v(x'_2, 0) = \varphi(x'_2)$ , and from (3.3) and the semiconcavity in Lemma 3.2 we get

$$\begin{aligned} (v - \varphi)(x'_2, y_1) &= (v - \varphi)(x'_2, 0) + y_1 \frac{\sigma(x'_2)}{2} + \iint_{[(x'_2, 0), (x'_2, y_1)]} v_{x_n x_n} \\ &\leq -y_1 \frac{\gamma}{2} + C y_1^2 \leq y_1 \gamma \left( C C_2 - \frac{1}{2} \right) \leq 0 \end{aligned}$$

if  $C_2$  is small enough depending only on the semiconcavity constant of Lemma 3.2. Thus, we have reached a contradiction.

• Assume now that  $\sup(v - \psi_{x_0})$  is attained at a point  $(x'_1, y_1)$  in the base of the cylinder  $U_{x_0}$ , i.e. with  $|x'_1 - x'_0| \leq C_1 \gamma$  and  $y_1 = C_2 \gamma$ . Then, from  $\kappa > \sup |\varphi_{\tau\tau}|$ , we deduce

$$v(x'_1, y_1) \geq \psi_{x_0}(x'_1, y_1) \geq \varphi(x'_1) - \kappa(n-1) \frac{\Lambda}{\lambda} C_2^2 \gamma^2.$$

Now choose  $x'_2$  such that  $|x'_2 - x'_1| < C_2 \gamma$  and  $(x'_2 - x'_1) \cdot \nabla_{x'}(v - \varphi)(x'_1, y_1) \geq 0$ . As before,

$$\begin{aligned} (v - \varphi)(x'_2, y_1) &= \\ &= (v - \varphi)(x'_1, y_1) + (x'_2 - x'_1) \cdot \nabla_{x'}(v - \varphi)(x'_1, y_1) + \iint_{[(x'_1, y_1), (x'_2, y_1)]} (v - \varphi)_{\tau\tau} \\ &\geq -\kappa(n-1) \frac{\Lambda}{\lambda} C_2^2 \gamma^2 - C |x'_2 - x'_1|^2 \geq -C_2^2 \left( \kappa(n-1) \frac{\Lambda}{\lambda} + C \right) \gamma^2. \end{aligned}$$

Now, if  $x'_2 \notin S_\gamma$  then  $v(x'_2, 0) = \varphi(x'_2)$ ,

$$(v - \varphi)(x'_2, y_1) \leq -C_2 \frac{\gamma^2}{2} + \iint_{[(x'_2, 0), (x'_2, y_1)]} v_{x_n x_n} \leq \left( \frac{1}{2} C C_2^2 - C_2 \right) \gamma^2.$$

The contradiction follows if one chooses  $C_2$  small enough, depending only on  $\kappa$ ,  $n$ ,  $\lambda$ ,  $\Lambda$ , and the semiconvexity and semiconcavity constants from Lemma 3.2.  $\square$

The following lemma is useful to prove the  $C^\alpha$  regularity of  $\sigma$ , and can be found in [9, Lemma 3.5]. It follows from an appropriate use of the strong maximum principle for  $\mathcal{M}^-$ , the Pucci's extremal operator.

**Lemma 3.5** ([9]). *Let  $w$  be a non-negative continuous function in  $B_1^* \times (0, 1)$  that solves*

$$\mathcal{M}^-(D^2 w) \leq 0 \quad \text{in } B_1^* \times (0, 1).$$

*Assume*

$$\limsup_{x_n \downarrow 0^+} w(x', x_n) \geq 1 \quad \text{for } x' \in B_\delta^*(\bar{x}'),$$

for some ball  $B_\delta^*(\bar{x}') \subset B_1^*$ . Then

$$w(x) \geq \varepsilon > 0 \quad \text{for } x \in B_{1/2}^* \times \left[ \frac{1}{4}, \frac{3}{4} \right],$$

for some  $\varepsilon$  depending only on  $\delta$ , and the ellipticity constants  $\lambda$  and  $\Lambda$ .

We now show the following lemma, analogous to [2, Lemma 4] (or [9, Lemma 3.6]).

**Lemma 3.6.** *Let  $\sigma$  as defined in (3.1)-(3.3), for  $u$  the solution to the thin obstacle problem (1.2). Let  $x'_0 \in \Omega^*$ , then*

$$\sigma(x') \geq -C (\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}) |x' - x'_0|^\alpha, \quad \text{for } x' \in B_1^*$$

for some  $\alpha > 0$  and  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

*Proof.* Define

$$K_0 := \|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)},$$

and notice that by taking  $u/K_0$  instead of  $u$  if necessary we can assume

$$\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)} \leq 1.$$

Indeed, if  $K_0 \geq 1$  then

$$F_{K_0}(D^2u) := \frac{1}{K_0} F(D^2(K_0u)),$$

is a convex elliptic operator with ellipticity constants  $\lambda$  and  $\Lambda$ , and  $u/K_0$  is a solution to the nonlinear thin obstacle problem for the operator  $F_{K_0}$  with obstacle  $\varphi/K_0$ . In this case,

$$\|u/K_0\|_{L^\infty(B_1)} + \|\varphi/K_0\|_{C^{1,1}(B_1^*)} = 1,$$

as we wanted to see. Thus, from now on we assume  $K_0 \leq 1$ .

Using Lemmas 3.1, 3.2, 3.4 and 3.5, now the proof of this lemma is very similar to the proof of [9, Lemma 3.6]. We give it here for completeness.

We will show

$$\sigma(x') \geq -C|x' - x'_0|^\alpha, \tag{3.6}$$

with  $C$  and  $\alpha > 0$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

Recall that  $\sigma(x') = 2 \lim_{x_n \downarrow 0^+} v_{x_n}(x', x_n)$ , and that from Lemma 3.2,  $v_{x_n}$  is bounded and  $v_{x_n x_n} \leq C$ . Moreover,  $\sigma$  is non-positive by Lemma 3.1, so that  $v_{x_n} \leq Cx_n$  for  $x_n > 0$ .

In order to reach (3.6) we will prove  $v_{x_n}(x) \geq -\theta^k$  for  $x \in B_{\gamma^k}^*(x'_0) \times (0, \gamma^k)$ . Assume this has been already proved for some  $k$  with  $0 < \gamma \ll \theta < 1$ , and consider the function

$$w := \frac{v_{x_n} + \theta^k}{\theta^k - C\mu\gamma^k} \quad \text{in } B_{\mu\gamma^k}^*(x'_0) \times (0, \mu\gamma^k)$$

for  $\mu$  small enough. Notice that  $w$  fulfils the hypotheses of Lemma 3.5, so that using it together with Lemma 3.4 we get

$$v_{x_n}(x) \geq -\theta^k + \varepsilon(\theta^k - C\mu\gamma^k) \geq -\theta^k + \frac{1}{2}\varepsilon\theta^k$$

for  $x \in B_{\mu\gamma^k/2}^*(x'_0) \times (\mu\gamma^k/4, 3\mu\gamma^k/4)$ , since  $\gamma \ll \theta$ . Now, by means of Lemma 3.2,  $v_{x_n x_n} \leq C$ , and therefore, for any  $y = (y', y_n) \in B_{\mu\gamma^k/2}^*(x'_0) \times (0, \mu\gamma^k/4]$ ,

$$\begin{aligned} v_{x_n}(y) &\geq - \int_{y_n}^{\mu\gamma^k/4} v_{x_n x_n}(y', s) ds + v_{x_n x_n}(y', \mu\gamma^k/4) \\ &\geq -C \left( \frac{\mu\gamma^k}{4} - y_n \right) - \theta^k + \frac{1}{2}\varepsilon\theta^k, \end{aligned}$$

so that we obtain

$$v_{x_n}(x) \geq -\theta^k + \frac{1}{2}\varepsilon\theta^k - \frac{1}{4}\mu C\gamma^k$$

for  $x \in B_{\mu\gamma^k/2}^*(x'_0) \times (0, 3\mu\gamma^k/4)$ . To end the inductive argument we must see

$$\theta^{k+1} \geq \theta^k - \frac{1}{2}\varepsilon\theta^k + \frac{1}{4}\mu C\gamma^k.$$

For this, we pick  $\gamma \ll \theta$  so that the right-hand side is smaller than  $(1 - \frac{1}{4}\varepsilon)\theta^k$ , with  $\theta$  larger than  $1 - \frac{1}{4}\varepsilon$ . Then, the inductive argument is completed, and (3.6) follows.  $\square$

**3.3. Proof of Theorem 1.1.** Before proving our main result, let us show the following compactness lemma.

**Lemma 3.7.** *Let  $F$  be a nonlinear operator satisfying (1.3), and let  $w$  be a continuous function defined on  $B_1$ . Suppose that  $w$  satisfies the problem*

$$F(D^2w) = 0 \text{ in } B_1^+ \cup B_1^-, \quad (3.7)$$

and that

$$\|w\|_{L^\infty(B_1)} = 1, \quad [w]_{\text{Lip}(B_1)} \leq 1.$$

Let  $\psi$  be the solution to

$$\begin{cases} F(D^2\psi) = 0 & \text{in } B_1 \\ \psi = w & \text{on } \partial B_1, \end{cases} \quad (3.8)$$

and let us define the following operator

$$\tilde{\sigma}(w) := \lim_{h_n \downarrow 0} ((\partial_{x_n} w)(x', h_n) - (\partial_{x_n} w)(x', -h_n)).$$

Then, for every  $\varepsilon > 0$  there exists some  $\eta = \eta(\varepsilon, n, \lambda, \Lambda) > 0$  such that if

$$\|\tilde{\sigma}(w)\|_{L^\infty(B_1^*)} < \eta$$

then

$$\|\psi - w\|_{L^\infty(B_1)} < \varepsilon,$$



i.e.,  $\psi$  approximates  $w$  as  $\eta$  goes to 0.

*Proof.* Let us argue by contradiction. Suppose that there exists some fixed  $\varepsilon > 0$ , a sequence of functions  $w_k$  and a sequence of convex nonlinear operators uniformly elliptic with ellipticity constants  $\lambda$  and  $\Lambda$ ,  $F_k$ , with  $F_k(0) = 0$ , such that

$$F_k(D^2w_k) = 0 \text{ in } B_1^+ \cup B_1^- \quad (3.9)$$

and

$$\|w_k\|_{L^\infty(B_1)} = 1, \quad [w_k]_{\text{Lip}(B_1)} \leq 1,$$

with

$$\|\tilde{\sigma}(w_k)\|_{L^\infty(B_1^*)} < \eta_k \quad (3.10)$$

for some sequence  $\eta_k \rightarrow 0$ , but such that

$$\|\psi_k - w_k\|_{L^\infty(B_1)} \geq \varepsilon, \quad (3.11)$$

for all  $k$ , where  $\psi_k$  is the solution to

$$\begin{cases} F_k(D^2\psi_k) = 0 & \text{in } B_1 \\ \psi_k = w_k & \text{on } \partial B_1. \end{cases} \quad (3.12)$$

By Arzelà-Ascoli, up to a subsequence,  $w_k$  converges to some function  $\bar{w}$  uniformly in  $B_1$ , with  $\|\bar{w}\|_{L^\infty(B_1)} = 1$ . On the other hand, since  $F_k(0) = 0$  and they are uniformly elliptic and convex, they converge up to subsequences, uniformly over compact sets, to some convex nonlinear operator  $\bar{F}$  uniformly elliptic with ellipticity constants  $\lambda$  and  $\Lambda$  such that  $\bar{F}(0) = 0$ . Notice also that  $\psi_k$  converges uniformly to the solution  $\bar{\psi}$  to

$$\begin{cases} \bar{F}(D^2\bar{\psi}) = 0 & \text{in } B_1 \\ \bar{\psi} = \bar{w} & \text{on } \partial B_1. \end{cases} \quad (3.13)$$

and in the limit we obtain, from (3.11),

$$\|\bar{\psi} - \bar{w}\|_{L^\infty(B_1)} \geq \varepsilon > 0. \quad (3.14)$$

Now consider the function  $w_k + \eta_k|x_n|$  on  $B_1$ . From (3.10),  $w_k + \eta_k|x_n|$  now has a wedge pointing down in the set  $B_1 \cup \{x_n = 0\}$ , i.e.,

$$\tilde{\sigma}(w_k + \eta_k|x_n|) \geq \eta_k > 0, \quad \text{in } B_1^*.$$

Therefore, since  $F_k(D^2w_k) = 0$  in  $B_1^+ \cup B_1^-$ , we have that, in the viscosity sense,

$$F_k(D^2(w_k + \eta_k|x_n|)) \geq 0, \quad \text{in } B_1.$$

Now, passing to the limit, noticing that  $w_k + \eta_k|x_n|$  converges uniformly to  $\bar{w}$  and using [4, Proposition 2.9], we immediately reach that, in the viscosity sense,

$$\bar{F}(D^2\bar{w}) \geq 0, \quad \text{in } B_1.$$

Repeating the same argument for  $w_k - \eta_k|x_n|$  we reach  $\bar{F}(D^2\bar{w}) \leq 0$  in  $B_1$ , to finally obtain

$$\bar{F}(D^2\bar{w}) = 0, \quad \text{in } B_1.$$

This implies  $\bar{w} = \bar{\psi}$  in  $B_1$ , which is a contradiction with (3.14).  $\square$

Using the previous results, we now give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* We separate the proof into three steps. In the first step we prove that the solution  $u$  is  $C^{1,\alpha}$  around points in  $\Omega^*$  by means of Lemmas 3.6 and 3.7. In the second step, we use the result from the first step to deduce that  $\sigma$  is  $C^\alpha$  in  $B_{2/3}^*$ , to finally complete the proof in the third step.

As in the proof of Lemma 3.6 we assume

$$\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)} \leq 1,$$

to avoid having this constant on each estimate throughout the proof.

**Step 1:** Let us suppose that the origin is a free boundary point. Under these circumstances we will prove that there exist some affine function  $L = a + b \cdot x$  such that

$$\|u - L\|_{L^\infty(B_r)} \leq Cr^{1+\alpha}, \quad \text{for all } r \geq 0, \quad (3.15)$$

for some constants  $C$  and  $\alpha > 0$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ . To do so, we proceed in the spirit of the proof of [3, Theorem 2].

Notice that from Lemma 3.6 we know that there exists  $\eta > 0$  such that

$$|\sigma(x')| \leq \eta|x'|^\alpha, \quad \text{for all } x' \in B_1^*. \quad (3.16)$$

Up to replacing from the beginning  $u(x)$  by  $u(r_0x)$  with  $r_0 \ll 1$ , we can make  $\eta$  as small as necessary. The choice of the value of  $r_0$ , and consequently the magnitude in which the constant  $\eta$  is made small, will depend only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

Let us show now that there exists  $\rho = \rho(\alpha, n, \lambda, \Lambda) < 1$  and a sequence of affine functions

$$L_k(x) = a_k + b_k \cdot x \quad (3.17)$$

such that

$$\|u - L_k\|_{L^\infty(B_{\rho^k})} \leq \rho^{k(1+\alpha)}, \quad (3.18)$$

and

$$|a_k - a_{k-1}| \leq C\rho^{k(1+\alpha)}, \quad |b_k - b_{k-1}| \leq C\rho^{k\alpha} \quad (3.19)$$

for some constant  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

We proceed by induction, taking  $L_0 = 0$ . Suppose that the  $k$ -th step is true, and consider

$$w_k(x) = \frac{(u - L_k)(\rho^k x)}{\rho^{k(1+\alpha)}}, \quad \text{for } x \in B_1.$$

Begin by noticing that

$$F_k(D^2 w_k) = 0 \text{ in } B_1^+ \cup B_1^-$$

for some operator  $F_k$  of the form (1.3). On the other hand, from the induction hypothesis,

$$\|w_k\|_{B_1} \leq 1.$$

Moreover, if we define

$$\sigma_k(x') = \lim_{h \downarrow 0} (\partial_{x_n} w_k(x', h) - \partial_{x_n} w_k(x', -h)), \quad \text{for } x' \in B_1^*,$$

then one can check that, from (3.16),

$$|\sigma_k(x')| \leq \eta |x'|^\alpha.$$

We apply now Lemma 3.7. That is, given  $\varepsilon > 0$  small, we can choose  $\eta$  small enough such that

$$\|v_k - w_k\|_{L^\infty(B_1)} \leq \varepsilon,$$

where  $v_k$  is the solution to

$$\begin{cases} F_k(D^2 v_k) = 0 & \text{in } B_1 \\ v_k = w_k & \text{on } \partial B_1. \end{cases} \quad (3.20)$$

Notice that, by interior estimates,  $v_k$  is  $C^{2,\alpha}$  in  $B_{1/2}$  with estimates depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ . Then, let  $l_k$  be the linearisation of  $v_k$  around 0, so that up to choosing  $\rho$ ,

$$\begin{aligned} \|w_k - l_k\|_{L^\infty(B_\rho)} &\leq \|w_k - v_k\|_{L^\infty(B_\rho)} + \|v_k - l_k\|_{L^\infty(B_\rho)} \\ &\leq \varepsilon + C\rho^2 \leq \rho^{1+\alpha}, \end{aligned}$$

where  $C$  depends only on  $n$ ,  $\lambda$ , and  $\Lambda$ ,  $\rho$  is chosen small enough depending only on  $\alpha$ ,  $n$ ,  $\lambda$ , and  $\Lambda$  so that  $C\rho^2 \leq \frac{1}{2}\rho^{1+\alpha}$ , and  $\eta$  is chosen so that  $\varepsilon \leq \frac{1}{2}\rho^{1+\alpha}$ . It is important to remark that the choice of  $\eta$  depends only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

Now, recalling the definition of  $w_k$ , we reach

$$\left\| u - L_k - \rho^{k(1+\alpha)} l_k \left( \frac{\cdot}{\rho^k} \right) \right\|_{L^\infty(B_{\rho^{k+1}})} \leq \rho^{(k+1)(1+\alpha)},$$

so that the inductive step is concluded by taking

$$L_{k+1}(x) = L_k(x) + \rho^{k(1+\alpha)} l_k \left( \frac{x}{\rho^k} \right).$$

By noticing that there are bounds on the coefficients of the linearisation of  $v_k$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ , the inequalities in (3.19) are obtained.

Once one has (3.17), (3.18), and (3.19), define  $L$  as the limit of  $L_k$  as  $k \rightarrow \infty$  (which exists, by (3.19)), and notice that, given any  $0 < r = \rho^k$  for some  $k \in \mathbb{N}$ , then

$$\|u - L\|_{L^\infty(B_r)} \leq \|u - L_k\|_{L^\infty(B_r)} + \sum_{j \geq k} \|L_{j+1} - L_j\|_{L^\infty(B_r)} \leq Cr^{1+\alpha}$$

for some  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ ; as we wanted.

**Step 2:** In this step we prove that the function  $\sigma$  defined in (3.1)-(3.3) is  $C^\alpha(B_{2/3}^*)$  for some  $\alpha = \alpha(n, \lambda, \Lambda) > 0$ , and

$$\|\sigma\|_{C^\alpha(B_{2/3}^*)} \leq C, \quad (3.21)$$

for some constant  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

We already know  $\sigma$  is regular in the interior of  $\Delta^*$  (by boundary estimates) and  $\Omega^*$ ; respectively the coincidence set and its complement in  $B_1^*$ . In particular, from

the interior estimates  $\sigma \equiv 0$  in  $\Omega^*$ . From Lemma 3.6 we also obtain  $C^\alpha$  regularity at points in  $\partial\Delta^*$ . Namely, we have that given  $(x'_0, 0) = x_0 \in \partial\Delta^*$ ,

$$|\sigma(x')| \leq C|x' - x'_0|^\alpha, \text{ for } x' \in B_1^*, \quad (3.22)$$

for some constant  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

Therefore, we only need to check that given  $x, y \in \Delta$ ,  $x = (x', 0)$ ,  $y = (y', 0)$ , then there exists some  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$  such that, if  $|x - y| = r$ ,

$$|\sigma(x') - \sigma(y')| \leq Cr^\alpha.$$

Let  $R := \text{dist}(x, \Omega)$  and suppose that  $\text{dist}(x, \Omega) \leq \text{dist}(y, \Omega)$ . Let  $z = (z', 0)$ ,  $z' \in \partial\Delta^*$ , be such that  $\text{dist}(x, z) = \text{dist}(x, \Omega)$ , and assume that  $\lim_{x_n \downarrow 0^+} \nabla u(z', x_n) = 0$  and  $\nabla_{x'} \varphi(z') = 0$  by subtracting an affine function if necessary. Notice that we can do so because we already know from the first step that  $u$  has a  $C^{1,\alpha}$  estimate around  $z'$ . Let us then separate two cases:

- If  $R < 4r$ , then using (3.21)

$$\begin{aligned} |\sigma(x') - \sigma(y')| &\leq |\sigma(x') - \sigma(z')| + |\sigma(y') - \sigma(z')| \\ &\leq C(R^\alpha + (R+r)^\alpha) \\ &\leq Cr^\alpha. \end{aligned}$$

- In the case  $R \geq 4r$  we need to use known boundary estimates for this fully nonlinear problem and the previous step of the proof. Notice that  $x', y' \in B_{R/2}^*(x') \subset B_R^*(x') \subset \Delta^*$ , and  $u$  restricted to  $B_R^*(x')$  is thus a  $C^{1,1}$  function, since  $u = \varphi$  there. In particular, we use that under these hypotheses

$$R^{1+\alpha}[u]_{C^{1,\alpha}(\overline{B_{R/2}^+(x)})} \leq C \left( \text{osc}_{B_R^+(x)} u + R^2[\varphi]_{C^{1,1}(B_R^*(x'))} \right);$$

see, for example, [8, Proposition 2.2]. Now, remember that the gradient of  $u$  at  $z$  is 0, so that from the previous step using the bound (3.15) around  $z$ ,

$$|u(p) - \varphi(z')| \leq C|p - z|^{1+\alpha} \leq CR^{1+\alpha} \text{ for } p \in B_R^+(x). \quad (3.23)$$

In particular,  $\text{osc}_{B_R^+(x)} u \leq CR^{1+\alpha}$ , and thus, this yields

$$[u]_{C^{1,\alpha}(\overline{B_{R/2}^+(x)})} \leq C,$$

from which (3.21) is proved.

**Step 3:** Our conclusion now follows by repeating Step 1 around every point on  $B_1^*$ . Notice that in the first step we only used that the origin was a free boundary point to be able to apply Lemma 3.6 in (3.16).

Now, given any point  $z' \in B_{1/2}^*$ , we can consider the function  $u_z$  given by

$$u_z(x) := u(x) - \sigma(z')(x_n)^+,$$

where  $(x_n)^+$  denotes the positive part of  $x_n$ .

Note that this function fulfils the hypotheses of Step 1, in particular,

$$|\sigma_z(x')| := \left| \lim_{h \downarrow 0} (\partial_{x_n} u_z(x', h) - \partial_{x_n} u_z(x', -h)) \right| \leq C|x' - z'|^\alpha, \quad \text{for } x' \in B_1^*,$$

for some constant  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

By repeating the exact same procedure as in Step 1, we reach that for every point  $z \in B_{1/2} \cup \{x_n = 0\}$ , and for every  $x \in B_1^+$  there exists some  $L_z^+$  affine function such that

$$|u(x) - L_z^+| \leq C|x - z|^{1+\alpha},$$

and the same occurs in  $B_1^-$  for a possibly different affine function  $L_z^-$ . Therefore, in particular,

$$\|u\|_{C^{1,\alpha}(B_{1/2}^*)} \leq C$$

for some  $C$  depending only on  $n$ ,  $\lambda$ , and  $\Lambda$ .

To finish the proof, we could now repeat a procedure like the one done in Step 2, or directly notice that solutions to the nonlinear problem with  $C^{1,\alpha}$  boundary data are  $C^{1,\alpha}$  up to the boundary (see, for example, [8, Proposition 2.2]).  $\square$

We finally give the:

*Proof of Corollary 1.2.* It is an immediate consequence of Theorem 1.1. Indeed, consider balls of radius  $R_0 := \text{dist}(K, \partial D)$  around points on  $K \cap \{x_n = 0\}$  and apply Theorem 1.1. To cover the rest of  $K$  we use interior estimates, and the result follows by noticing that  $\|u\|_{L^\infty(D)} \leq \|g\|_{L^\infty(\partial D)} + \|\varphi\|_{L^\infty}$  by the maximum principle.  $\square$

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