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Regularity Theory for Thin Obstacle Problems

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presented by

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Abstract

The goal of this PhD thesis is to collect the results of the author in the study of thin obstacle problems. We start by giving an introduction to the Signorini or thin obstacle problem, summarizing some of the most relevant currently known results. The next chapters correspond each to one paper by the author (and collaborators). Thus, we start by studying the regularity of solutions for the fully nonlinear thin obstacle problem, to then move to the study of the free boundary for general fractional obstacle problems with drift, in the critical regime. This is followed by a regularity result for minimizers of the perimeter with lower dimensional obstacles. Finally, the last two chapters focus on the standard thin obstacle problem (and its fractional counter-part) and fine regularity and generic regularity properties for the free boundary.

Sommario

In questa tesi di dottorato si raccolgono i risultati dell'autore nello studio dei problemi di ostacolo sottile. Iniziamo con un'introduzione al problema di Signorini o degli ostacoli sottili, riassumendo alcuni dei risultati più rilevanti attualmente conosciuti. I capitoli successivi corrispondono ciascuno ad un articolo dell'autore e dei collaboratori. Cominciamo con lo studio della regolarità delle soluzioni per il problema degli ostacoli sottili completamente non lineari, per poi passare allo studio della frontiera libera per i problemi generali degli ostacoli frazionari con termine di trasporto, in regime critico. Segue un risultato di regolarità per i minimi del perimetro con ostacoli di dimensioni inferiori. Infine, gli ultimi due capitoli si concentrano sul problema standard dell'ostacolo sottile e la sua controparte frazionaria, e sulle proprietà di regolarità fine e regolarità generica per la frontiera libera.

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Contents

Summary

This thesis revolves around various aspects of the thin (or fractional) obstacle problem (also known as Signorini problem). In the introduction (Chapter 1) we present the problem and the main known results. In the following five chapters (Chapter 2 to Chapter 6) we present the contributions of the author in the field. Each chapter corresponds to a different paper. We summarize here each of the chapters.

- **Chapter 1** is a general introduction to the thin obstacle problem. It is a self-contained survey that aims to cover the main known results regarding the thin (or fractional) obstacle problem. We present the theory with some proofs: from the regularity of the solution to the classification of free boundary points, ending with generic regularity-type results for the free boundary.

- **Chapter 2** corresponds to the paper [Fer16], that is,

  X. Fernández-Real, $C^{1,\alpha}$ *estimates for the fully nonlinear Signorini problem*, Calc. Var. Partial Differential Equations (2016), 55:94.

  In this chapter we study a generalization of the Signorini problem involving more general elliptic operators of second order in place of the Laplacian. We consider general convex fully nonlinear operators, and show the regularity of the solution to the fully nonlinear Signorini problem. This is a generalization of a previous result by Milakis and Silvestre, [MS08], where they showed regularity of solutions under some extra assumptions on the operators and the solution itself.

  Given a fully nonlinear operator defined on the space of $n \times n$ matrices $\mathcal{M}_n$, $F : \mathcal{M}_n \to \mathbb{R}$, satisfying[1]

$$
\begin{aligned}
& F \text{ is convex, uniformly elliptic} \\
& \text{with ellipticity constants } 0 < \lambda \le \Lambda, \text{ and with } F(0) = 0,
\end{aligned}
\tag{0.1}
$$

  we consider the lower dimensional obstacle problem

$$
\left\{
\begin{array}{rcll}
F(D^2u) & = & 0 & \text{in } B_1 \setminus \{x_{n+1} = 0\} \\
\min\{-F(D^2u), u - \varphi\} & = & 0 & \text{on } B_1 \cap \{x_{n+1} = 0\}.
\end{array}
\right.
\tag{0.2}
$$

[1] Notice that, given a function w, we can express the nonlinear operator F as $F(D^2w(x)) = \sup_{\gamma\in\Gamma}\left(L^{ij}_\gamma \partial_{x_i x_j} w(x) + c_\gamma\right)$, for some family of symmetric uniformly elliptic operators with ellipticity constants λ and Λ, $L^{ij}_\gamma \partial_{x_i x_j}$, indexed by $\gamma \in \Gamma$. See [CC95, FR20].

(Notice that the Laplacian corresponds to $F(M) = \mathrm{tr}\, M$.) Then, we show that, if $\varphi \in C^{1,1}$, the solution u is $C^{1,\alpha}$ for some small $\alpha > 0$ in either side of the obstacle (that is, $u \in C^{1,\alpha}(B_{1/2} \cap \{x_{n+1} \geq 0\})$).

- **Chapter 3** corresponds to the paper [FR18],

  X. Fernández-Real, X.Ros-Oton *The obstacle problem for the fractional Laplacian with critical drift*, Math. Ann. 371(3) (2018), 1683-1735.

  Another possible generalization of the thin obstacle problem consists in changing the normal derivative condition with a directional derivative in another (non-tangential) direction. If we denote ∇_n the gradient in the first n variables, we consider the obstacle problem with oblique derivative condition

$$\begin{cases} \Delta u &= 0 \quad \text{in } B_1^+ \\ \min\{-\partial_{x_{n+1}} u + b \cdot \nabla_n u, u - \varphi\} &= 0 \quad \text{on } B_1 \cap \{x_{n+1} = 0\}, \end{cases} \tag{0.3}$$

  for some $b \in \mathbb{R}^n$ fixed (cf. (1.6)). In this case, problem (0.3) can be interpreted as a fractional obstacle problem of the form

$$\min\left\{(-\Delta)^s \bar{u} + b \cdot \nabla \bar{u},\, \bar{u} - \varphi\right\} = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{0.4}$$

  with $s = \frac{1}{2}$. This kind of operators appears as infinitesimal generators of Lévy processes with jumps (see (1.18) below), and in particular, its obstacle-type problem models optimal stopping problems for these processes. Problems of the type (0.4) had been previously studied in [PP15, GPPS17] in the case $s > \frac{1}{2}$, where, as a general intuition, the term involving the gradient can be treated as a lower order term. No regularity results are expected for the case $s < \frac{1}{2}$, but the situation where the gradient has to *compete* with the fractional operator ($s = \frac{1}{2}$) was still open.

  In this chapter we study the free boundary for solutions to (0.4) (or (0.3)) with $s = \frac{1}{2}$, also considering more general nonlocal operators of order 1 (so that no monotonicity formulas are available to be used). Given a solution $\bar{u}$ to (0.4), we establish the $C^{1,\alpha}$ regularity of the free boundary around any regular point $x_\circ$, with an expansion of the form

$$\begin{aligned} \bar{u}(x) - \varphi(x) &= c_\circ\big((x - x_\circ)\cdot e\big)_+^{1+\tilde{\gamma}(x_\circ)} + o\left(|x - x_\circ|^{1+\tilde{\gamma}(x_\circ)+\sigma}\right), \\ \tilde{\gamma}(x_\circ) &= \frac{1}{2} + \frac{1}{\pi}\arctan(b \cdot e), \end{aligned}$$

  where $e \in \mathbb{S}^{n-1}$ is the normal vector to the free boundary, $\sigma > 0$, and $c_\circ > 0$. In particular, we have that the growth of the solution at free boundary points depends on the orientation of the free boundary with respect to the vector b.

- **Chapter 4** corresponds to the paper [FS20],

  X. Fernández-Real, J. Serra, *Regularity of minimal surfaces with lower dimensional obstacles*, J. Reine Angew. Math., to appear.

The Signorini problem can also be interpreted as a linearization of the problem where we want to minimize the area of a surface with prescribed boundary, and constrained to be above a certain lower dimensional obstacle: namely, the Plateau problem, where we restrict the set of admissible solutions to those that contained a fixed lower dimensional smooth manifold (the thin obstacle).

In Chapter 4, we study the regularity of solutions to the minimization problem in $\mathbb{R}^{n+1}$

$$\min\left\{P(E;B_1)\ :\ E\supset\mathcal{O},\ E\setminus B_1 = E_\circ\setminus B_1\right\} \tag{0.5}$$

where $P(E;B_1)$ denotes the (variational) perimeter of E inside B_1, and $\mathcal{O} := \Phi\left(\{x_n = 0, x_{n+1}\le 0\}\right)$ is the thin obstacle (which here is given by the smooth diffeomorphism Φ of a flat thin obstacle).

Perhaps surprisingly, we show that solutions to (0.5) are $C^{1,\frac{1}{2}}$ at free boundary points (in particular, they are a graph). This is opposed to classical smoothness of minimal surfaces, which for dimensions $n\ge 8$ need not be regular. Thus, the thin obstacle is actively acting at contact points and forcing a graphical and regular solution.

The difficulty in studying (0.5) (with respect to the same problem with a thick obstacle) lies on the fact that near a typical point of the contact set the hypersurface ∂E consists of two surfaces that intersect transversally on $\partial\mathcal{O}$. Therefore, ∂E is typically not flat at small scales and thus (0.5) cannot be treated as a perturbation of the Signorini problem.

- **Chapter 5** corresponds to the paper [FJ20],

X. Fernández-Real, Y. Jhaveri, *On the singular set in the thin obstacle problem: higher order blow-ups and the very thin obstacle problem*, Anal. PDE, to appear.

The set of non-regular points of the free boundary can subdivided into the set of *singular points* and the set of *other points*. The set of singular points corresponds to those points where the contact set has zero density (in the thin space) and can be characterized also as those where the blow-up has even homogeneity. It is contained in a countable union of C^1 manifolds. Moreover, under a certain non-degeneracy condition on the obstacle ($\Delta\varphi < 0$), the set of degenerate points consists only of singular points of order 2.

In this chapter we thoroughly investigate the structure of singular points for the Signorini problem (also with weights, $s\in(0,1)$, so to cover the fractional obstacle problem of any order as well). In particular, we adapt the techniques that had been introduced by Figalli and Serra in [FS18] in the context of the classical obstacle problem to our setting. By means of GMT methods we are able to deduce higher regularity properties for the singular set outside of certain exceptional sets with lower dimension, and establish some higher order expansions of the solutions around those points. As a consequence of our study, we encounter a new fractional problem, what we call the *very thin obstacle problem*: an obstacle-type problem with constrains on a co-dimension

2 domain, which only makes sense in the setting $s > \frac{1}{2}$. Thus, we also study the regularity properties of this new problem.

- **Chapter 6** corresponds to the paper [FR19],

  X. Fernández-Real, X. Ros-Oton, *Free boundary regularity for almost every solution to the Signorini problem*, preprint arXiv (2019).

  For general smooth obstacles, without any extra non-degeneracy assumption, the set of non-regular points of the free boundary can be very big, of dimension $n - \varepsilon$ for any $\varepsilon > 0$.

  Thus, while one would expect degenerate (non-regular) points to be always small, we already know it is not true in the context of the Signorini problem. The next natural question is to ask how *frequently* do these degenerate points appear: even if they can exist, we expect them to appear in very particular configurations, or at least, to be large in very particular configurations. This is precisely what we show in this chapter by establishing a first result of this kind in the context of thin obstacle problems.

  In particular, we show that for *almost every* solution to the Signorini problem, the set of degenerate points is $(n-2)$-dimensional (where "*almost every* solution" needs to be understood in the context of the theory of prevalence). That is, if we denote u_0 the solution to

$$\begin{cases} \Delta u = 0 & \text{in } B_1^+ \\ \min\{-\partial_{x_{n+1}} u, u - \varphi\} = 0 & \text{on } B_1 \cap \{x_{n+1} = 0\}, \end{cases} \tag{0.6}$$

  with a certain boundary condition $g \in C^0(\partial B_1)$, and we denote u_λ the solution to (0.6) with boundary data $g_\lambda = g + \lambda$, we show that

$$\dim_{\mathcal{H}} \big(\mathrm{Deg}(u_\lambda)\big) \le n - 2 \quad \text{for a.e.} \quad \lambda \in [0, 1].$$

  In fact, our results are more precise, and are concerned with the Hausdorff dimension of $\Gamma_{\ge\kappa}(u_\lambda)$, the set of points of order greater or equal than κ. We show that if $3 \le \kappa \le n + 1$, then $\Gamma_{\ge\kappa}(u_\lambda)$ has dimension $n - \kappa + 1$, while for $\kappa > n + 1$, then $\Gamma_{\ge\kappa}(u_\lambda)$ is empty for almost every $\lambda \in [0, 1]$. This is the first result, in the context of the Signorini problem, that proves that regular points are *better*, in some sense, than the rest of *degenerate points*.

  We then use similar techniques in the context of the parabolic Signorini problem to show that, for almost all times, the set of non-regular points is lower-dimensional.

Chapter 1

Introduction to the thin obstacle problem

The Signorini problem (also known as the thin or boundary obstacle problem) is a classical free boundary problem that was originally studied by Antonio Signorini in connection with linear elasticity [Sig33, Sig59]. The problem was originally named by Signorini himself *problem with ambiguous boundary conditions*, in the sense that the solution of the problem at each boundary point must satisfy one of two different possible boundary conditions, and it is not known a priori which point satisfies which condition.

Whereas the original problem involved a system of equations, its scalar version gained further attention in the seventies due to its connection to many other areas, which then lead to it being widely studied by the mathematical community. Hence, apart from elasticity, lower dimensional obstacle problems also appear in describing osmosis through semi-permeable membranes as well as boundary heat control (see, e.g., [DL76]). Moreover, they often are local formulations of fractional obstacle problems, another important class of obstacle problems. Fractional obstacle problems can be found in the optimal stopping problem for Lévy processes, and can be used to model American option prices (see [Mer76, CT04]). They also appear in the study of anomalous diffusion, [BG90], the study of quasi-geostrophic flows, [CV10], and in studies of the interaction energy of probability measures under singular potentials, [CDM16]. (We refer to [Ros18] for an extensive bibliography on the applications of obstacle-type problems.)

1.1 A problem from elastostatics

Consider an elastic body $\Omega \subset \mathbb{R}^3$, anisotropic and non-homogeneous, in an equilibrium configuration, that must remain on one side of a frictionless surface. Let us denote $\boldsymbol{u} = (u^1, u^2, u^3) : \Omega \to \mathbb{R}^3$ the displacement vector of the elastic body, Ω, constrained to be on one side of a surface Π (in particular, the elastic body moves from the Ω configuration to $\Omega + \boldsymbol{u}(\Omega)$). We divide the boundary into $\partial\Omega = \Sigma_D \cup \Sigma_S$. The body is free (or clamped, $\boldsymbol{u} \equiv 0$) at Σ_D, whereas Σ_S represents the part of the boundary subject to the constraint, that is, $\Sigma_S = \partial\Omega \cap \Pi$. Alternatively, one can interpret Σ_S itself as the frictionless surface that is constraining the body Ω, understanding that only a subset of Σ_S is actually exerting the constraint on the displacement. This will be more clear below.

Let us assume small displacements, so that we can consider the linearized strain tensor

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u^i_{x_j} + u^j_{x_i}), \quad 1 \le i, j \le 3.$$

Considering an elastic potential energy of the form $W(\varepsilon) = a_{ijkh}(x)\varepsilon_{ij}\varepsilon_{kh}$, for some functions $a_{ijhk}(x) \in C^\infty(\overline{\Omega})$ (where, from now on, we are using the Einstein notation of implicit summation over repeated indices), then the stress tensor has the form

$$\sigma_{ij}(\boldsymbol{u}) = a_{ijhk}(x)\varepsilon_{hk}(\boldsymbol{u}).$$

We also impose that a_{ijhk} are elliptic and with symmetry conditions

$$\begin{aligned} &a_{ijhk}(x)\zeta_{ij}\zeta_{hk} \ge \lambda|\zeta|^2 \quad \text{for all } \zeta \in \mathbb{R}^{n\times n} \text{ such that } \zeta_{ij} = \zeta_{ji},\\ &a_{ijhk}(x) = a_{jihk}(x) = a_{ijkh}(x), \qquad \text{for } x \in \Omega. \end{aligned}$$

Let us also assume that Ω is subject to the body forces $\boldsymbol{f} = (f^1, f^2, f^3)$, so that by the general equilibrium equations we have

$$\frac{\partial \sigma_{ij}(\boldsymbol{u})}{\partial x_j} = f^i, \quad \text{in } \Omega, \quad \text{for } i = 1, 2, 3.$$

From the definitions of $\sigma(\boldsymbol{u})$ and $\varepsilon_{ij}(\boldsymbol{u})$ above, this is a second order system, and from the definition of a_{ijhk}, it is elliptic. Thus, the displacement vector satisfies an elliptic second order linear system inside Ω. We just need to impose boundary conditions on Σ_S (the conditions on Σ_D are given by the problem, we can think of $\boldsymbol{u} \equiv 0$ there).

Let us denote by $\boldsymbol{n}$ the outward unit normal vector to $x \in \partial\Omega$. Notice that, by assumption, the stresses in the normal direction $\boldsymbol{n}$ on Σ_S, $\sigma_{ij}(\boldsymbol{u})\boldsymbol{n}_i$, must be compressive in the normal direction, and zero in the tangential direction (due to the frictionless surface). That is,

$$\begin{aligned} \sigma_{ij}(\boldsymbol{u})\boldsymbol{n}_i\boldsymbol{n}_j &\le 0 \quad \text{on } \Sigma_S, \\ \sigma_{ij}(\boldsymbol{u})\boldsymbol{n}_i\tau_j &= 0 \quad \text{on } \Sigma_S \text{ and for all } \tau \in \mathbb{R}^n \text{ with } \tau\cdot\boldsymbol{n} = 0. \end{aligned} \tag{1.1}$$

On the other hand, we have the kinematical contact condition, encoding the fact that there exists a surface exerting a constraint and the body cannot cross it (under small displacements, or assuming simply that Π is a plane):

$$\boldsymbol{u}\cdot\boldsymbol{n} \le 0, \qquad \text{on } \Sigma_S. \tag{1.2}$$

In fact, conditions (1.1)-(1.2) are complimentary, in the sense that

$$(\boldsymbol{u}\cdot\boldsymbol{n})\cdot(\sigma_{ij}(\boldsymbol{u})\boldsymbol{n}_i\boldsymbol{n}_j) = 0 \qquad \text{on } \Sigma_S, \tag{1.3}$$

and we are dividing Σ_S into two regions: those where the body separates from Π and those where it remains touching Π. That is, if there is an active normal stress at a point $x \in \Sigma_S$, $\sigma_{ij}(\boldsymbol{u}(x))\boldsymbol{n}_i(x)\boldsymbol{n}_j(x) < 0$, then it means that the elastic body is being constrained by Σ_S (or Π) at x, and thus we are in the contact area and there is no normal displacement, $\boldsymbol{u}(x)\cdot\boldsymbol{n}(x)=0$. Alternatively, if there is a normal displacement, $\boldsymbol{u}(x)\cdot\boldsymbol{n}(x)<0$, it means that there is no active obstacle and thus no normal stress, $\sigma_{ij}(\boldsymbol{u}(x))\boldsymbol{n}_i(x)\boldsymbol{n}_j(x)=0$. This is precisely what *ambiguous boundary condition means*:

For each $x\in\Sigma_S$ we have that one of the following two conditions holds

$$\text{either}\quad \begin{cases}\sigma_{ij}(\boldsymbol{u}(x))\boldsymbol{n}_i(x)\boldsymbol{n}_j(x) & \leq & 0\\ \boldsymbol{u}(x)\cdot\boldsymbol{n}(x) & = & 0,\end{cases}\qquad \text{or}\quad \begin{cases}\sigma_{ij}(\boldsymbol{u}(x))\boldsymbol{n}_i(x)\boldsymbol{n}_j(x) & = & 0\\ \boldsymbol{u}(x)\cdot\boldsymbol{n}(x) & < & 0,\end{cases} \tag{1.4}$$

and a priori, we do not know which of the condition is being fulfilled at each point. The Signorini problem is a *free boundary problem* because the set Σ_S can be divided into two different sets according to which of the conditions (1.4) holds, and these sets are, a priori, unknown. The boundary between both sets is what is known as the *free boundary*.

The previous is a strong formulation of the Signorini problem, which assumed a priori that all solutions and data are smooth. In order to prove existence and uniqueness, however, one requires the use of variational inequalities with (convex) constraints in the set of admissible functions.

The first one to approach the existence and uniqueness from a variational point of view was Fichera in [Fic64]. We also refer to the work of Lions and Stampacchia [LS67], where a general theory of variational inequalities was developed, which later led to the scalar version of the Signorini problem, and its interpretation as a minimization problem with admissible functions constrained to be above zero on certain fixed closed sets. Later, in [DL76], Duvaut and Lions studied the problem and its applications to mechanics and physics.

Finally, we refer to [Kin81, KO88] for more details into the strong and weak formulation of the (system) Signorini problem and its properties.

1.2 The thin obstacle problem

In this work we will focus our attention to the scalar version of the Signorini problem from elasticity: our function, u, would correspond to an appropriate limit in the normal components of the displacement vector, $\boldsymbol{u}_n$. Our obstacle, φ, adds generality to the problem, and would correspond to the possible displacement of the frictionless surface $\partial\Omega$ while performing $\boldsymbol{u}$. (We refer the interested reader to [CDV19, Example 1.5] for a deduction of this fact.) As explained above, this problem also appears in biology, physics, and even finance. Thus, from now on, functions are scalar.

Let us denote $x=(x',x_{n+1})\in\mathbb{R}^n\times\mathbb{R}$ and $B_1^+=B_1\cap\{x_{n+1}>0\}$. We say that $u:\overline{B_1^+}\to\mathbb{R}$ is a solution to the Signorini problem or thin obstacle problem with smooth obstacle φ defined on $B_1':=B_1\cap\{x_{n+1}=0\}$, and with smooth boundary data g on $\partial B_1 \cap \{x_{n+1} > 0\}$, if u solves

$$\begin{cases} \Delta u &= 0 \quad \text{in } B_1^+ \\ u &= g \quad \text{on } \partial B_1 \cap \{x_{n+1} > 0\} \\ \partial_{x_{n+1}} u \cdot (u - \varphi) &= 0 \quad \text{on } B_1 \cap \{x_{n+1} = 0\} \\ -\partial_{x_{n+1}} u &\geq 0 \quad \text{on } B_1 \cap \{x_{n+1} = 0\} \\ u - \varphi &\geq 0 \quad \text{on } B_1 \cap \{x_{n+1} = 0\}, \end{cases} \tag{1.5}$$

where we are also assuming that the compatibility condition $g \geq \varphi$ on $\partial B_1 \cap \{x_{n+1} = 0\}$ holds. Notice the analogy with the ambiguous compatibility conditions (1.1)-(1.2)-(1.3) or (1.4): the set with Dirichlet conditions, Σ_D above, is $\partial B_1 \cap \{x_{n+1} > 0\}$, where $u = g$ is imposed; whereas the set with ambiguous boundary conditions, Σ_S above, is now B_1'. That is, at each point $x = (x', 0) \in B_1'$ we have that

$$\text{either} \quad \begin{cases} -\partial_{x_{n+1}} u(x) &\geq 0 \\ u(x) - \varphi(x') &= 0, \end{cases} \quad \text{or} \quad \begin{cases} -\partial_{x_{n+1}} u(x) &= 0 \\ u(x) - \varphi(x') &> 0. \end{cases}$$

An alternative way to write the ambiguous boundary conditions in (1.5) is by imposing a nonlinear condition on B_1' involving u and $\partial_{x_{n+1}} u$ as

$$\begin{cases} \Delta u &= 0 \quad \text{in } B_1^+ \\ \min\{-\partial_{x_{n+1}} u, u - \varphi\} &= 0 \quad \text{on } B_1 \cap \{x_{n+1} = 0\}, \end{cases} \tag{1.6}$$

with $u = g$ on $\partial B_1 \cap \{x_{n+1} > 0\}$. This is the strong formulation of the Signorini problem.

In order to prove existence (and uniqueness) of solutions, we need to study the weak formulation of the problem: a priori, we do not know any regularity for the solution.

Consider a bounded domain $\Omega \subset \mathbb{R}^n$, and a closed set $\mathcal{C} \subset \Omega$. Let, also, $\phi : C(\mathcal{C}) \to \mathbb{R}$ be a continuous function. In [LS67], Lions and Stampacchia prove the existence and uniqueness of a solution to the variational problem

$$\min_{v \in \mathcal{K}} \int_\Omega |\nabla v|^2 \tag{1.7}$$

where $\mathcal{K} = \{v \in H_0^1(\Omega) : v \geq \phi \text{ on } \mathcal{C}\}$. Moreover, they also show that such solution is the smallest supersolution.

If $\mathcal{C} = \overline{\Omega}$, (1.7) is also known as the classical obstacle problem: finding the function with smallest Dirichlet energy among all those which lie above a fixed obstacle ϕ. This problem has been thoroughly studied in the last fifty years (see [LS67, KN77, Caf77, CR77, Wei99, PSU12] and references therein), many times in parallel to the thin obstacle problem, and we will sometimes refer to it also as the *thick* obstacle problem.

Our problem, (1.6), corresponds to the case when $\mathcal{C}$ is lower dimensional, with codimension 1. Notice that simple capacity arguments yield that, if $\mathcal{C}$ has codimension 2 or higher, then the restriction of functions in H_0^1 to $\mathcal{C}$ does not have any effect on the minimization of the Dirichlet energy, and thus we would simply be solving the classical Laplace equation. This means that, in this case, there is in general no minimizer.

Thus, (1.6) are the Euler–Lagrange equations of the following variational problem

$$\min_{v\in\mathcal{K}^*}\int_{B_1^+}|\nabla v|^2, \tag{1.8}$$

where

$$\mathcal{K}^* = \{v\in H^1(B_1^+) : v = g \text{ on } \partial B_1\cap\{x_{n+1}>0\}, v\geq\varphi \text{ on } B_1\cap\{x_{n+1}=0\}\}.$$

Notice that the expressions $v = g$ on $\partial B_1\cap\{x_{n+1}>0\}$ and $v\geq\varphi$ on $B_1\cap\{x_{n+1}=0\}$ must be understood in the trace sense. The existence and uniqueness of a solution, as in [LS67], follows by classical methods: take a minimizing sequence, and by lower semicontinuity of the Dirichlet energy, and the compactness of the trace embeddings into H^1, the limit is also an admissible function. The uniqueness follows by strict convexity of the functional.

In some cases, the thin obstacle problem is posed in the whole ball B_1, and thus we consider

$$\min_{v\in\mathcal{K}^{**}}\int_{B_1}|\nabla v|^2,\quad \mathcal{K}^{**} = \{v\in H^1(B_1) : v = g \text{ on } \partial B_1, v\geq\varphi \text{ on } B_1\cap\{x_{n+1}=0\}\}, \tag{1.9}$$

for some function $g\in C(\partial B_1)$. In this case, the Euler–Lagrange equations are formally

$$\begin{cases} u \geq \varphi & \text{on } B_1\cap\{x_{n+1}=0\}\\ \Delta u = 0 & \text{in } B_1\setminus(\{x_{n+1}=0\}\cap\{u=\varphi\})\\ \Delta u\leq 0 & \text{in } B_1, \end{cases} \tag{1.10}$$

with the added condition that $u = g$ on ∂B_1. Alternatively, making the parallelism with (1.6), one could formally write

$$\begin{cases} \Delta u = 0 & \text{in } B_1\setminus\{x_{n+1}=0\}\\ \min\{-\Delta u, u-\varphi\} = 0 & \text{on } B_1\cap\{x_{n+1}=0\}, \end{cases} \tag{1.11}$$

understanding that Δu is defined only in the distributional sense. Notice that if g is even with respect to x_{n+1}, the solution to (1.10) is even as well, and we recover a problem of the form (1.6). On the other hand, for general g, one can study the symmetrised function $\bar u(x', x_{n+1}) = \frac{1}{2}(u(x',x_{n+1})+u(x',-x_{n+1}))$, which has the same regularity and contact set as u. Thus, in order to study (1.10) one can always assume that u is even in x_{n+1}, and this is enough to study (1.6).

Notice, also, that in (1.10) the condition $\Delta u\leq 0$ needs to be understood in the sense of distributions. In fact, Δu is a (non-positive) measure concentrated on $\{u=0\}$. We can explicitly compute it by taking any test function $\varphi\in C_c^\infty(B_1)$ even in x_{n+1},

$$\begin{aligned} -\langle\Delta u,\varphi\rangle &= 2\int_{B_1^+}\nabla u\cdot\nabla\varphi = 2\lim_{\varepsilon\downarrow 0}\int_{B_1^+\cap\{x_{n+1}\geq\varepsilon\}}\nabla u\cdot\nabla\varphi\\ &= -2\lim_{\varepsilon\downarrow 0}\int_{B_1^+\cap\{x_{n+1}=\varepsilon\}}\partial_{x_{n+1}}u\,\varphi = -2\int_{B_1\cap\{x_{n+1}=0\}}\partial^+_{x_{n+1}}u\,\varphi. \end{aligned}$$

That is,

$$\Delta u = 2\partial^+_{x_{n+1}}u\,\mathcal{H}^n\llcorner(B_1\cap\{x_{n+1}=0\}), \tag{1.12}$$

where $\partial^+_{x_{n+1}}u = \lim_{\varepsilon\downarrow 0}\partial_{x_{n+1}}u(x',\varepsilon)$.

Remark 1.1. In the derivation of (1.12), apart from (1.10), we have also used integrability of ∇u, and that the trace of the normal derivative is well-defined. This follows because, in fact, as we will show later, the solution to the thin obstacle problem is Lipschitz, and is continuously differentiable up to the obstacle.

Remark 1.2. Problem (1.9) can be seen as a first order approximation of the Plateau problem with a lower dimensional obstacle, originally introduced by De Giorgi [DeG73], which has also been studied in the last years [DeA79, FoSp18b, FS20]. Indeed, the Dirichlet functional corresponds to the area functional (up to a constant) for flat graphs. (See Chapter 4 for more discussions on this topic.)

Finally, let us end this section by mentioning other possible constructions of solutions. As mentioned above, the solution to the previous minimization problem can also be recovered as the least supersolution. That is, the minimizer u to (1.9) equals to the pointwise infimum

$$u(x) = \inf \big\{ v(x) : v \in C^2(B_1), -\Delta v \geq 0 \text{ in } B_1, v \geq \varphi \text{ on } B_1 \cap \{x_{n+1} = 0\}, \\ v \geq g \text{ on } \partial B_1 \big\},$$

the least supersolution above the thin obstacle. The fact that such function satisfies (1.10) can be proved by means of Perron's method, analogously to the Laplace equation.

As a final characterization of the construction of the solution, we refer to penalization arguments. In this case there are two ways to penalize:

On the one hand, we can *expand* the obstacle, and work with the classical obstacle problem. That is, we can consider as obstacle $\varphi_\varepsilon(x) = \varphi(x') - \varepsilon^{-1} x_{n+1}^2$ with $\varepsilon > 0$ very small, which is now defined in the whole domain B_1. Then, by taking the solutions to the thick obstacle problem with increasingly thinner obstacles φ_ε (letting $\varepsilon \downarrow 0$), converging to our thin obstacle, we converge to the solution to our problem. Alternatively, we can even avoid the penalization step: the solutions to the thin obstacle problem must coincide with the solution of the thick obstacle problem, with obstacle $\bar\varphi : B_1^+ \to \mathbb{R}$ given by the solution to $\Delta\bar\varphi = 0$ in B_1^+, $\bar\varphi = \varphi$ on $B_1 \cap \{x_{n+1} = 0\}$, $\bar\varphi = g$ on $\partial B_1 \cap \{x_{n+1} > 0\}$. Notice that $\bar\varphi$ itself is not the solution to the thin obstacle problem since, a priori, it is not a supersolution across $\{x_{n+1} = 0\}$.

On the other hand, we can penalize (1.6) by replacing the ambiguous boundary condition on $\{x_{n+1} = 0\}$, by considering solutions u^ε with the Neumann boundary condition $u^\varepsilon_{x_{n+1}} = \varepsilon^{-1} \min\{0, u - \varphi\}$ on $\{x_{n+1} = 0\}$. By letting $\varepsilon \downarrow 0$, u^ε converges to a solution to our problem.

1.3 Relation with the fractional obstacle problem

Let us consider the thin obstacle problem (1.6) posed in the whole $\mathbb{R}^{n+1}$, for some smooth obstacle $\varphi : \mathbb{R}^n \to \mathbb{R}$ with compact support. That is, we denote $\mathbb{R}^{n+1}_+ =$

$\mathbb{R}^{n+1} \cap \{x_{n+1} > 0\}$ and consider a solution to

$$\begin{cases} \Delta u &= 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u(x',0) &\geq \varphi(x') & \text{for } x' \in \mathbb{R}^n \\ \partial_{x_{n+1}} u(x',0) &= 0 & \text{if } u(x',0) > \varphi(x') \\ \partial_{x_{n+1}} u(x',0) &\leq 0 & \text{if } u(x',0) = \varphi(x') \\ u(x) &\to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.13}$$

If we denote by $\bar{u} : \mathbb{R}^n \to \mathbb{R}$ the restriction of u to $\{x_{n+1} = 0\}$, then we can simply reformulate the problem in terms of $\bar{u}$ instead of u, given that u is just the harmonic extension (vanishing at infinity) of $\bar{u}$ to $\mathbb{R}^{n+1}_+$. That is, by means of the Poisson kernel in the half-space,

$$u(x', x_{n+1}) = [P(x_{n+1}, \cdot) * u](x') = c_n \int_{\mathbb{R}^n} \frac{x_{n+1}\bar{u}(y')\,dy'}{(x_{n+1}^2 + |x'-y'|^2)^{\frac{n+1}{2}}}$$

for some dimensional constant c_n. Thus, after a careful computation and taking limits $x_{n+1} \downarrow 0$, one obtains

$$-\partial_{x_{n+1}} u(x',0) = c_n \mathrm{PV} \int_{\mathbb{R}^n} \frac{\bar{u}(x') - \bar{u}(y')}{|x'-y'|^{n+1}}\,dy' =: (-\Delta)^{\frac{1}{2}} \bar{u}(x'),$$

where the integral needs to be understood in the principal value sense. We have introduced here an integro-differential operator, acting on $\bar{u}$, $(-\Delta)^{\frac{1}{2}}$, known as the fractional Laplacian of order 1 (in the sense that $(-\Delta)^{\frac{1}{2}}(\bar{v}(r\cdot)) = r((-\Delta)^{\frac{1}{2}}\bar{v})(r\cdot))$.

Let us very briefly justify the choice of notation $(-\Delta)^{\frac{1}{2}}$ in terms of the discussion above. Given a smooth (say, C^2) function $\bar{u}$, $(-\Delta)^{\frac{1}{2}}\bar{u}$ is the normal derivative of its harmonic extension. If one repeats this procedure, and takes the harmonic extension of $(-\Delta)^{\frac{1}{2}}\bar{u}$, it is simply $\partial_{x_{n+1}} u$. Thus, $(-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{1}{2}}\bar{u} = \partial^2_{x_{n+1}} u = -\Delta_{x'}\bar{u}$, where we are using the fact that $\Delta u = 0$ (up to the boundary), and we denote $\Delta = \Delta'_x + \partial^2_{x_{n+1}}$.

In all, problem (1.13) can be rewritten in terms of $\bar{u}$ as

$$\begin{cases} \bar{u} &\geq \varphi & \text{in } \mathbb{R}^n \\ (-\Delta)^{\frac{1}{2}}\bar{u} &= 0 & \text{if } u > \varphi \\ (-\Delta)^{\frac{1}{2}}\bar{u} &\geq 0 & \text{if } u = \varphi \\ \bar{u}(x') &\to 0 & \text{as } |x'| \to \infty, \end{cases} \tag{1.14}$$

which is the formulation of the classical (or thick) global obstacle problem, with obstacle φ and operator $(-\Delta)^{\frac{1}{2}}$, also referred to as *fractional obstacle problem.* Notice that now, we are considering a function $\bar{u}$ that remains above the obstacle φ in the whole domain (compared to before, where we only needed this condition imposed on a lower dimensional manifold).

Similarly, one can consider the fractional obstacle problem in a bounded domain $\Omega \subset \mathbb{R}^n$ with a (smooth) obstacle $\varphi : \Omega \to \mathbb{R}$ by imposing exterior boundary conditions with sufficient decay, $\bar{g} : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$,

$$\begin{cases} \bar{u} &\geq \varphi & \text{in } \Omega \\ (-\Delta)^{\frac{1}{2}}\bar{u} &= 0 & \text{in } \Omega \cap \{u > \varphi\} \\ (-\Delta)^{\frac{1}{2}}\bar{u} &\geq 0 & \text{in } \Omega \cap \{u = \varphi\} \\ \bar{u} &= \bar{g} & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.15}$$

Thus, in order to study the solution to (1.15), by taking its harmonic extension $\bar{u}$, it is enough to study the solutions to (1.6).

Finally, another characterization of the fractional Laplacian, $(-\Delta)^{\frac{1}{2}}$, is via Fourier transforms. In this way, one can also characterize (up to a constant) general fractional Laplacians of order $2s$, with $0 < s < 1$, as

$$\mathcal{F}((-\Delta)^s\bar{u})(\xi) = |\xi|^{2s}\mathcal{F}(\bar{u})(\xi),$$

where $\mathcal{F}$ denotes the Fourier transform. The operator, which now has order $2s$, can be explicitly written as

$$(-\Delta)^s\bar{u}(x') = c_{n,s}\mathrm{PV}\int_{\mathbb{R}^n}\frac{\bar{u}(x')-\bar{u}(y')}{|x'-y'|^{n+2s}}\,dy'.$$

In this way, one can consider general obstacle problems with nonlocal operator $\mathcal{L} = (-\Delta)^s$

$$\begin{cases} \bar{u} \geq \varphi & \text{in } \Omega \\ \mathcal{L}\bar{u} = 0 & \text{in } \Omega\cap\{u>\varphi\} \\ \mathcal{L}\bar{u} \geq 0 & \text{in } \Omega\cap\{u=\varphi\} \\ \bar{u} = \bar{g} & \text{in } \mathbb{R}^n\setminus\Omega. \end{cases} \tag{1.16}$$

(See, e.g., [Sil07].) As we have seen, the fractional Laplacian $(-\Delta)^{\frac{1}{2}}$ can be recovered as the normal derivative of the harmonic extension towards one extra dimension (cf. (1.15)-(1.6)). Caffarelli and Silvestre showed in [CS07] that the fractional Laplacian of order $(-\Delta)^s$ can also be recovered by extending through suitable operators. Thus, if one considers the operator

$$L_a u := \mathrm{div}(|x_{n+1}|^a\nabla u), \qquad a = 1-2s \in (-1,1),$$

then the even a-harmonic extension of the solution $\bar{u}$ to (1.16) (that is, u with $L_a u = 0$ in $x_{n+1} > 0$ and $u(x', x_{n+1}) = u(x', -x_{n+1})$) solves locally a problem of the type

$$\begin{cases} u \geq \varphi & \text{on } B_1\cap\{x_{n+1}=0\} \\ L_a u = 0 & \text{in } B_1\setminus(\{x_{n+1}=0\}\cap\{u=\varphi\}) \\ L_a u \leq 0 & \text{in } B_1, \end{cases} \tag{1.17}$$

that is, a thin obstacle problem with operator L_a, or a weighted thin obstacle problem (cf. (1.10)) with A_2-Muckenhoupt weight.

It is for this reason that many times one studies the weighted thin obstacle problem (1.17) with $a \in (-1,1)$ (see [CS07, CSS08]). For the sake of simplicity and readability, in this introduction we will always assume $a = 0$, but most of the results mentioned generalize to any $a \in (-1,1)$ accordingly, and therefore, they also apply to solutions to the fractional obstacle problem (1.16).

Fractional obstacle problems such as (1.16), as well as many of its variants (with more general non-local operators, with a drift term, in the parabolic case, etc.), have been a very prolific topic of research in the last years (see [CF13, PP15, GPPS17, DGPT17, CRS17, BFR18b, FR18] and references therein).

1.3.1 The fractional Laplacian and Lévy processes

Integro-differential equations arise naturally in the study of stochastic processes with jumps, namely, Lévy processes. The research in this area is attracting an increasing level of interest, both from an analytical and probabilistic point of view, among others, due to its applications to multiple areas: finance, population dynamics, physical and biological models, etc. (See [DL76, Mer76, CT04, Ros16, Ros18] and references therein.) Infinitesimal generators of Lévy processes are integro-differential operators of the form

$$\mathcal{L}u = b\cdot\nabla u + \mathrm{tr}\,(A\cdot D^2u) + \int_{\mathbb{R}^n}\{u(x+y)-u(x)-y\cdot\nabla u(x)\chi_{B_1}(y)\}\,\nu(dy), \quad (1.18)$$

for some Lévy measure ν such that $\int\min\{1,|y|^2\}\nu(dy)<\infty$. The simplest (nontrivial) example of such infinitesimal generators is the fractional Laplacian introduced above, which arises as infinitesimal generator of a stable and radially symmetric Lévy process.

In particular, obstacle type problems involving general integro-differential operators of the form (1.18) appear when studying the optimal stopping problem for a Lévy process: consider a particle located at X_t at time $t\geq 0$, moving according a Lévy process inside a domain Ω, and let φ be a pay-off function defined in Ω, and $\bar{g}$ an exterior condition defined in $\mathbb{R}^n\setminus\Omega$. At each time we can decide to stop the process and be paid $\varphi(X_t)$ or wait until the particle reaches a region where φ has a higher value. Moreover, if the particle suddenly jumps outside of Ω, we get paid $\bar{g}(X_t)$. The goal is to maximize the expected value of money we are being paid. We refer the interested reader to the aforementioned references as well as [Pha97] and the appendix of [BFR18] for the jump-diffusion optimal stopping problem, as well as [LS09, Eva12, FR20] for the local (Brownian motion) case.

1.4 Regularity of the solution

Once existence and uniqueness is established for solutions to (1.6), the next question that one wants to answer is:

How regular is the solution u to (1.6)?

Of course, its regularity is expected to depend on how smooth is the obstacle φ. We will assume that it is as smooth as needed, so that we do not have to worry about it at this point.

Regularity questions for solutions to the thin obstacle problem were first investigated by Lewy in [Lew68], where he showed, for the case $n=1$, the continuity of the solution of the Signorini problem. He also gave the first proof related to the structure of the free boundary, by showing, also in $n=1$, that if the obstacle φ is concave, the coincidence set $\{u=\varphi\}$ consists of, at most, one connected interval.

The continuity of the solution for any dimension follows from classical arguments. One first shows that the coincidence set $\{u=\varphi\}$ is closed, and then one uses the following fact for harmonic functions: if $\mathcal{C}\subset\Omega$ is closed, and $\Delta v=0$ in $\Omega\setminus\mathcal{C}$ and v is continuous on $\mathcal{C}$, then v is continuous in Ω.

Rather simple arguments also yield that, in fact, the solution is Lipschitz. Indeed, if one considers the solution u to the problem (1.10), and we define $h \in \mathrm{Lip}(B_1)$ as the solution to

$$\begin{cases} \Delta h = 0 & \text{in } B_1 \setminus \{x_{n+1} = 0\} \\ h = -\|u\|_{L^\infty(B_1)} & \text{on } \partial B_1 \\ h = \varphi & \text{on } B_1 \cap \{x_{n+1} = 0\}, \end{cases}$$

then u is a solution to the classical (thick) obstacle problem with h (which is Lipschitz) as the obstacle. In order to close the argument, we just notice that solutions to the thick obstacle problem with Lipschitz obstacles are Lipschitz, so u is Lipschitz as well. This last step is not so immediate, we refer the reader to [AC04, Theorem 1] or [Fer16, Proposition 2.1] for two different ways to conclude this reasoning. These first regularity properties were investigated in the early 1970's (see [Bei69, LS69, Kin71, BC72, GM75]).

In general, we do not expect solutions to (1.10) to be better than Lipschitz. Indeed, across $\{x_{n+1} = 0\}$ on contact points, we have that normal derivatives can change sign, as seen by taking the even extension to (1.6). Nonetheless, we are interested in the regularity of the solution in either side of the obstacle. The fact that normal derivatives jump is *artificial*, in the sense that it does not come from the equations, but from the geometry of the problem. We see that this is not observed in (1.6), where the solution could, a priori, be better than Lipschitz, and it also does not appear when studying the solution restricted to $\{x_{n+1} = 0\}$, as in the situations with the fractional obstacle problem (1.15).

1.4.1 $C^{1,\alpha}$ regularity

The first step to upgrade the regularity of solutions to (1.6) was taken by Frehse in [Fre77] in 1977, where he proved that tangential derivatives of u are continuous up to $\{x_{n+1} = 0\}$, thus showing that the solution is C^1 in B_1^+, up to the boundary.

Later, in 1978 Richardson proved that solutions are $C^{1,1/2}$ for $n = 1$ in [Ric78]; whereas, in parallel, Caffarelli showed in [Caf79] that solutions to the Signorini problem are $C^{1,\alpha}$ for some $0 < \alpha \leq \frac{1}{2}$ up to the boundary on either side (alternatively, tangential derivatives are Hölder continuous). In order to do that, Caffarelli started showing the semiconvexity of the solution in the directions parallel to the thin obstacle. We state this result here for future convenience.

Proposition 1.1 ([Caf79])**.** *Let u be any weak solution to* (1.6)*, and let $\varphi \in C^{1,1}(B_1')$. Let $\boldsymbol{e} \in \mathbb{S}^n$ be parallel to the thin space, $\boldsymbol{e} \cdot \boldsymbol{e}_{n+1} = 0$. Then, u is semiconvex in the $\boldsymbol{e}$ direction. That is,*

$$\inf_{B_{1/2}} \partial^2_{\boldsymbol{ee}} u \geq -C(\|u\|_{L^2(B_1)} + [\nabla \varphi]_{C^{0,1}(B_1')}),$$

for some constant C depending only on n.

As a (not immediate) consequence, Caffarelli deduced the $C^{1,\alpha}$ regularity of solutions.

Theorem 1.2 ([Caf79]). *Let u be any weak solution to* (1.6)*, and let $\varphi \in C^{1,1}(B_1')$. Then, $u \in C^{1,\alpha}(\overline{B_{1/2}^+})$ and*

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \le C\left(\|u\|_{L^2(B_1^+)} + [\nabla\varphi]_{C^{0,1}(B_1')}\right),$$

for some constants $\alpha > 0$ and C depending only on n.

Remark 1.3. In fact, Caffarelli in [Caf79] pointed out how to deal with other smooth operators coming from variational inequalities with smooth coefficients. Thus, in (1.6) one could consider other divergence form operators other than the Laplacian, with smooth and uniformly elliptic coefficients.

Remark 1.4. A posteriori, one can lower the regularity assumptions on the obstacle, the coefficients, and the lower dimensional manifold. We refer to [RuSh17] for a study in this direction, with $C^{1,\alpha}$ obstacles, $C^{0,\alpha}$ coefficients (in divergence form), and with the thin obstacle supported on a $C^{1,\gamma}$ manifold.

The fact that the regularity cannot be better than $C^{1,1/2}$ is due to the simple counter-example,

$$u(x) = \mathrm{Re}\left((x_1 + i|x_{n+1}|)^{3/2}\right) \tag{1.19}$$

which in (x_1, x_{n+1})-polar coordinates can be written as

$$\tilde{u}(r,\theta) = r^{3/2}\cos\left(\tfrac{3}{2}\theta\right).$$

The function u is a solution to the Signorini problem: it is harmonic for $|x_{n+1}| > 0$, the normal derivative $\partial_{x_{n+1}}$ vanishes at $\theta = 0$, and has the right sign at $\theta = \pi$.

It was not until many years later that, in [AC04], Athanasopoulos and Caffarelli showed the optimal $C^{1,1/2}$ regularity of the solution in all dimensions. That is, in the previous theorem $\alpha = \frac{1}{2}$, and by the example above, this is optimal. We leave the discussion of the optimal regularity for the next section, where we deal with the classification of free boundary points.

Historically, the classification of the free boundary was performed *after* having established the optimal regularity. In the next section we show that this was not needed, and in fact one can first study the free boundary, and from that deduce the optimal regularity of the solution.

1.5 Classification of free boundary points

The thin obstacle problem, (1.6) or (1.10), is a *free boundary problem*, i.e., the unknowns of the problem are the solution itself, and the contact set

$$\Lambda(u) := \left\{x' \in \mathbb{R}^n : u(x',0) = \varphi(x')\right\} \times \{0\} \subset \mathbb{R}^{n+1},$$

whose topological boundary in the relative topology of $\mathbb{R}^n$, which we denote $\Gamma(u) = \partial_{\mathbb{R}^n}\Lambda(u) = \partial\{x' \in \mathbb{R}^n : u(x',0) = \varphi(x')\} \times \{0\}$, is known as the *free boundary*.

After studying the regularity of the solution, the next natural step in understanding the thin obstacle problem is the study of the structure and regularity of the free boundary. This is also related to the optimal regularity question presented above, since one expects that the *worst* points in terms of regularity lie on the boundary of the contact set.

Let us suppose, for simplicity, that we have a zero obstacle problem, $\varphi \equiv 0$. Notice that, if the obstacle φ is analytic, we can always reduce to this case by subtracting an even harmonic extension of φ to the solution[1]. This is not possible under lower regularity properties (in particular, this does not include the case where $\varphi \in C^\infty$, see Section 1.9).

Our problem is

$$\begin{cases} u \geq 0 & \text{on } B_1 \cap \{x_{n+1} = 0\} \\ \Delta u = 0 & \text{in } B_1 \setminus (\{x_{n+1} = 0\} \cap \{u = 0\}) \\ \Delta u \leq 0 & \text{in } B_1, \end{cases} \tag{1.20}$$

and the contact set is

$$\Lambda(u) = \{(x', 0) \in \mathbb{R}^{n+1} : u(x', 0) = 0\}.$$

In order to study a free boundary point, $x_\circ \in \Gamma(u)$, one considers *blow-ups* of the solution u around $x_\circ$. That is, one looks at rescalings of the form

$$u_{r,x_\circ}(x) = \frac{u(x_\circ + rx)}{\left(\fint_{\partial B_r(x_\circ)} u^2 \right)^{\frac{1}{2}}}. \tag{1.21}$$

The limit of such rescalings, as $r \downarrow 0$, gives information about the behaviour of the solution around the free boundary point $x_\circ$. Thus, classifying possible blow-up profiles as $r \downarrow 0$ around free boundary points will help us better understand the structure of the free boundary. Notice that, by construction, the blow-up sequence (1.21) is trivially bounded in $L^2(\partial B_1)$. To prove (stronger) convergence results, we need the sequence to be bounded in more restrictive spaces (say, in $W^{1,2}$), by taking advantage of the fact that u solves problem (1.20).

In order to do that, a very powerful tool is *Almgren's frequency function*. If we consider a solution u to the Signorini problem (1.20) and take the odd extension (with respect to x_{n+1}), we end up with a two-valued map that is harmonic on the thin space (and has two branches). Almgren studied in [Alm00] precisely the monotonicity of the frequency function for multi-valued harmonic functions (in fact, Dirichlet energy minimizers), and thus, it is not surprising that such tool is also available in this setting.

Let us define, for a free boundary point $x_\circ \in \Gamma(u)$,

$$N(r, u, x_\circ) := \frac{r \int_{B_r(x_\circ)} |\nabla u|^2}{\int_{\partial B_r(x_\circ)} u^2}.$$

We will often denote $N(r, u)$ whenever we take $x_\circ = 0$. Notice that $N(\rho, u_r) = N(r\rho, u)$, where $u_r := u_{r,0}$ (see (1.21)). Then, we have the following.

[1]If the obstacle φ is analytic, then φ has a harmonic extension to B_1^+, and its even extension in the whole B_1 is harmonic as well. Thus, the function $u - \varphi$ solves a thin obstacle problem with zero obstacle. This is no longer true if φ is not analytic (not even when $\varphi \in C^\infty$), and in such situation one needs to adapt the arguments. However, the ideas are the same.

Lemma 1.3. *Let u be a solution to* (1.20)*, and let us assume $0 \in \Gamma(u)$. Then, Almgren's frequency function*

$$r \mapsto N(r,u) = \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

is nondecreasing. Moreover, $N(r,u)$ is constant if and only if u is homogeneous.

Proof. We very briefly sketch the proof. By scaling ($N(\rho, u_r) = N(r\rho, u)$) it is enough to show that $N'(1,u) \geq 0$. Let us denote

$$D(r,u) = \frac{1}{r^{n+1}}\int_{B_r} |\nabla u|^2 = r^2\int_{B_1} |\nabla u(r\cdot)|^2, \quad H(r,u) = \frac{1}{r^n}\int_{\partial B_r} u^2 = \int_{\partial B_1} u(r\cdot)^2,$$

so that $N(r,u) = \frac{D(r,u)}{H(r,u)}$ and $N'(1,u) = \frac{D(1,u)}{H(1,u)}\left(\frac{D'(1,u)}{D(1,u)} - \frac{H'(1,u)}{H(1,u)}\right)$. Now notice that

$$D'(1,u) = 2\int_{B_1} \nabla u \cdot \nabla(x\cdot\nabla u)\,dx = 2\int_{\partial B_1} u_\nu^2 - 2\int_{B_1} \Delta u(x\cdot\nabla u)\,dx,$$

where u_ν denotes the outward normal derivative to B_1. Since u is a solution to the Signorini problem, either $\Delta u = 0$ or $u = 0$ and $\Delta u > 0$ (in which case, $x \cdot \nabla u = 0$ by C^1 regularity of the solution). Thus, the second term above vanishes. On the other hand, we have that

$$H'(1,u) = 2\int_{\partial B_1} uu_\nu \quad \text{and} \quad D(1,u) = \int_{B_1} |\nabla u|^2 = \int_{\partial B_1} uu_\nu,$$

where in the last equality we have used again that u solves the Signorini problem, $u\Delta u \equiv 0$. Thus,

$$N'(1,u) = 2\frac{D(1,u)}{H(1,u)}\left(\frac{\int_{\partial B_1} u_\nu^2}{\int_{\partial B_1} uu_\nu} - \frac{\int_{\partial B_1} uu_\nu}{\int_{\partial B_1} u^2}\right) \geq 0,$$

by Cauchy-Schwarz inequality. Equality holds if and only if u is proportional to u_ν on ∂B_r for every r (that is, u is homogeneous). □

And from Lemma 1.3 we have the following.

Lemma 1.4. *Let u be a solution to* (1.20)*, and let us assume $0 \in \Gamma(u)$. Let $\lambda := N(0^+, u)$, and let*

$$\varphi(r) := \fint_{\partial B_r} u^2.$$

Then, the function $r \mapsto r^{-2\lambda}\varphi(r)$ is nondecreasing. Moreover, for every $\varepsilon > 0$ there exists some $r_\circ = r_\circ(\varepsilon)$ such that if $r < \rho r \leq r_\circ(\varepsilon) \leq 1$,

$$\varphi(\rho r) \leq \rho^{2(\lambda+\varepsilon)}\varphi(r).$$

Proof. Notice that by Lemma 1.3, λ is well-defined. By differentiating

$$\frac{d}{dr}\left(r^{-2\lambda}\varphi(r)\right) = 2r^{-2\lambda-n-1}\left\{r\int_{B_r}|\nabla u|^2 - \lambda\int_{\partial B_r} u^2\right\} \geq 0,$$

where we are also using the monotonicity of $N(r,u)$ from Lemma 1.3.

On the other hand, choose $r_\circ(\varepsilon)$ such that $N(r_\circ,u) \leq \lambda+\varepsilon$. Then, just noticing that

$$N(r,u) = \frac{r}{2}\frac{d}{dr}\log\varphi(r) \leq \lambda+\varepsilon \tag{1.22}$$

for $r < \rho r \leq r_\circ$ and integrating in $(r,\rho r)$ we get the desired result. □

As a consequence of Almgren's monotonicity formula we get the existence of a (homogeneous) blow-up limit around free boundary points, u_0. Notice that we are not claiming the uniqueness of such blow-up, but its degree of homogeneity is independent of the sequence.

Corollary 1.5. *Let u be a solution to* (1.20)*, and let us assume $0 \in \Gamma(u)$. Let us denote*

$$u_r(x) = \frac{u(rx)}{\left(\fint_{\partial B_r} u^2\right)^{1/2}}.$$

Then, for any sequence $r_k \downarrow 0$ there exists a subsequence $r_{k_j} \downarrow 0$ such that

$$u_{r_{k_j}} \to u_0 \quad \textit{strongly in } L^2_{\rm loc}(\mathbb{R}^{n+1}), \tag{1.23}$$

$$\nabla u_{r_{k_j}} \rightharpoonup \nabla u_0 \quad \textit{weakly in } L^2_{\rm loc}(\mathbb{R}^{n+1}), \tag{1.24}$$

$$u_{r_{k_j}} \to u_0 \quad \textit{strongly in } C^1_{\rm loc}(\overline{\mathbb{R}^{n+1}_+}), \tag{1.25}$$

for some $N(0^+,u)$-homogeneous global solution u_0 to the thin obstacle problem with zero obstacle, (1.20)*, and $\|u_0\|_{L^2(\partial B_1)} = c_n$, for some dimensional constant $c_n > 0$.*

Proof. The proof of the strong convergence in L^2 and weak convergence in $W^{1,2}$ is a consequence of Lemma 1.3, which shows that the sequence u_{r_k} is uniformly bounded in $W^{1,2}(B_1)$. Indeed, take any ball centered at the origin, $B_R \subset \mathbb{R}^n$. Then, using the notation from Lemma 1.4,

$$\int_{B_R}|\nabla u_r|^2 = \frac{r^{1-n}}{\varphi(r)}\int_{B_{rR}}|\nabla u|^2 \leq \frac{R^{n-1}\varphi(Rr)}{\varphi(r)}N(1,u) \leq C(R)N(1,u),$$

where in the last step we are using that r is small enough together with the second part of Lemma 1.4 with $\varepsilon = 1$. Also notice that $\|u_r\|_{L^2(\partial B_1)} = c_n$, so u_r is bounded in $W^{1,2}$ for every compact set (again, by Lemma 1.4).

The homogeneity of u_0 comes from the fact that

$$N(\rho,u_0) = \lim_{r\downarrow 0} N(\rho,u_r) = \lim_{r\downarrow 0} N(r\rho,u) = N(0^+,u),$$

and Lemma 1.3.

Finally, the strong convergence in C^1 follows from the $C^{1,\alpha}$ regularity estimates for the solution, Theorem 1.2. □

Hence, we obtain the following result, describing the structure of blow-ups at free boundary points.

Theorem 1.6. *Let u be a solution to* (1.20)*, and let us assume $0 \in \Gamma(u)$. Let u_0 denote any blow-up at 0. Then, u_0 satisfies*

$$\begin{cases} u_0 \in C^{1,\alpha}_{\rm loc}(\{x_{n+1} \geq 0\}) \\ u_0 \text{ solves the thin obstacle problem } (1.20) \text{ in } \mathbb{R}^{n+1} \\ u_0 \text{ is } \lambda\text{-homogeneous, with } \lambda \in \left\{\frac{3}{2}\right\} \cup [2,\infty). \end{cases} \tag{1.26}$$

Moreover, if $\lambda = \frac{3}{2}$, then u_0 is (after a rotation) of the form (1.19).

Proof. The fact that $u_0 \in C^{1,\alpha}_{\rm loc}(\{x_{n+1} \geq 0\})$ solves the thin obstacle problem (1.20) in $\mathbb{R}^{n+1}$ comes directly from the strong convergence (1.25). Also, from Corollary 1.5, u_0 is a $\lambda := N(0^+, u)$ homogeneous function. We just need to determine the possible values λ can take when $\lambda < 2$.

Thus, from now on, let us assume that $\lambda < 2$. We separate the rest of the proof into two steps.

Step 1: Convexity of u_0. Let us start by showing that u_0 is convex in the directions parallel to the thin space, and thus, in particular, the restriction $u_0|_{\{x_{n+1}=0\}}$ is convex. We do so by means of the semiconvexity estimates from Proposition 1.1 applied to u_0. Indeed, by rescaling Proposition 1.1 to a ball of radius $R \geq 1$ we get

$$R^2 \inf_{B_{R/2}} \partial_{\boldsymbol{ee}} u_0 \geq -CR^{-\frac{n}{2}} \|u_0\|_{L^2(B_R)} = -CR^{\lambda}\|u_0\|_{L^2(B_1)},$$

for some dimensional constant C, and for $\boldsymbol{e} \cdot \boldsymbol{e}_{n+1} = 0$, where in the last equality we are using the λ-homogeneity of u_0. That is, by letting $R \to \infty$,

$$\inf_{B_{R/2}} \partial_{\boldsymbol{ee}} u_0 \geq -CR^{\lambda-2}\|u_0\|_{L^2(B_1)} \to 0, \quad \text{as} \quad R \to \infty.$$

Hence, u_0 is convex in the tangential directions to the thin space.

Step 2: Degree of homogeneity of u_0. From the C^1 convergence of the blow-ups, it is clear that $\lambda > 1$. Let us now consider $\Lambda(u_0) \subset \{x_{n+1} = 0\}$ the contact set for u_0, which is a convex cone, from the convexity and homogeneity of u_0.

If $\Lambda(u_0)$ has empty interior (restricted to the thin space), then $\partial_{x_{n+1}} u_0$ is a harmonic function in $\{x_{n+1} > 0\}$, identically zero on the thin space, and $(\lambda - 1)$-homogeneous. In particular, from the sublinear growth at infinity, $\partial_{x_{n+1}} u_0 \equiv 0$ everywhere, and thus $u_0 \equiv 0$, a contradiction. Hence, $\Lambda(u_0)$ has non-empty interior on the thin space.

Let us denote $\boldsymbol{e} \in \mathbb{S}^{n-1}$ a direction contained in the interior of $\Lambda(u_0)$ (in particular, $\boldsymbol{e} \cdot \boldsymbol{e}_{n+1} = 0$). Let us define, $w_1 := \partial_{-\boldsymbol{e}} u_0$ and $w_2 := -|\partial_{x_{n+1}} u_0|$, which are $(\lambda - 1)$ homogeneous functions, harmonic in $\{x_{n+1} \neq 0\}$.

Notice that $w_1 = 0$ in $\Lambda(u_0)$. In particular, for any $x_\circ \in \{x_{n+1} = 0\}$, $x_\circ + t\boldsymbol{e} \in \Lambda(u_0)$ for $t \in \mathbb{R}$ large enough (since $\Lambda(u_0)$ is a cone with non-empty interior and $\boldsymbol{e}$ is a direction contained in it). Thus, from the convexity of u_0, w_1 has to be monotone along $x_\circ + t\boldsymbol{e}$, and thus $w_1 \geq 0$ on the thin space. Since w_1 is $(\lambda - 1)$-homogeneous (i.e., it has sublinear growth), and is non-negative on the thin space, there is a unique $(\lambda - 1)$-homogeneous harmonic extension that coincides with w_1 (by the Poisson kernel), and it is non-negative as well. Hence, $w_1 \geq 0$ in $\mathbb{R}^{n+1}$.

In addition, $w_2 \geq 0$ on the thin space as well (since u_0 solves the thin obstacle problem), and it has sublinear growth at infinity. That is, its harmonic extension is itself, and thus $w_2 \geq 0$ in $\mathbb{R}^{n+1}$. Moreover, notice that $w_2 = 0$ in $\{x_{n+1} = 0\} \setminus \Lambda(u_0)$ (in particular, $w_1 w_2 \equiv 0$ on $\{x_{n+1} = 0\}$).

On the one hand, we have that the restriction of w_1 to the unit sphere must be the first eigenfunction of the Dirichlet problem for the spherical Laplacian with zero data on $\partial B_1 \cap \Lambda(u_0)$ (since it is non-negative), and it has homogeneity $\lambda - 1$. On the other hand, the restriction of w_2 to the unit sphere must be the first eigenfunction with zero data on $\partial B_1 \cap (\{x_{n+1} = 0\} \setminus \Lambda(u_0))$, and it has the same homogeneity $\lambda - 1$. Since $\Lambda(u_0)$ is a (convex) cone, it is contained in a half-space (of $\{x_{n+1} = 0\}$), and therefore, $\{x_{n+1} = 0\} \setminus \Lambda(u_0)$ contains a half-space. Since the corresponding homogeneities are the same (i.e., $\lambda - 1$), by monotonicity of eigenvalues with respect to the domain we must have that, after a rotation, $\Lambda(u_0)$ and its complement are equal, and hence, they are half-spaces themselves. The homogeneity for the half-space in this situation is $\frac{1}{2}$, so $\lambda = \frac{3}{2}$, and the corresponding eigenfunction is

$$u_0(x) = \mathrm{Re}\left((x_1 + i|x_{n+1}|)^{3/2}\right),$$

as we wanted to see. □

As a consequence of the previous result, we have a dichotomy for free boundary points.

Proposition 1.7 (Classification of free boundary points). *Let u be a solution to* (1.20). *Then, the free boundary can be divided into two sets,*

$$\Gamma(u) = \mathrm{Reg}(u) \cup \mathrm{Deg}(u).$$

The set of regular points,

$$\mathrm{Reg}(u) := \left\{x_\circ \in \Gamma(u) : N(0^+, u, x_\circ) = \tfrac{3}{2}\right\},$$

and the set of degenerate points,

$$\mathrm{Deg}(u) := \left\{x_\circ \in \Gamma(u) : N(0^+, u, x_\circ) \geq 2\right\}.$$

Moreover, $u \in C^{1,1/2}(B_1^+)$ with

$$\|u\|_{C^{1,1/2}(\overline{B_{1/2}^+})} \leq C\|u\|_{L^\infty(B_1)} \tag{1.27}$$

for some C depending only on n, and the set of regular points is open (in the relative topology of the free boundary).

Proof. The classification result is an immediate consequence of Corollary 1.5 and Theorem 1.6.

For the optimal regularity, we observe that by Corollary 1.5, since the sequence u_r is uniformly bounded in r, for $x_\circ \in \Gamma(u)$,

$$\|u\|_{L^\infty(B_r)(x_\circ)} \leq C\left(\fint_{\partial B_r(x_\circ)} u^2\right)^{\frac{1}{2}} \leq C\|u\|_{L^\infty(B_1)} r^{\frac{3}{2}}, \tag{1.28}$$

where in the last inequality we are using Lemma 1.4, together with the fact that, by Theorem 1.6, $N(0^+, u, x_\circ) \geq \frac{3}{2}$. This establishes a uniform growth of the solution around free boundary points. Combined with interior estimates for harmonic functions, this yields that u is $C^{1,1/2}$ on the thin space, and thus $u \in C^{1,1/2}(B_1^+)$ with estimates in $B_{1/2}^+$.

Indeed, take $y \in \{x_{n+1} = 0\} \cap \{u > 0\}$, and let $r = \mathrm{dist}(y, \Gamma(u))$. Then u is harmonic in $B_r(y)$, and by harmonic estimates together with (1.28)

$$\|\nabla_{x'} u\|_{L^\infty(B_{r/2}(y))} \leq Cr^{-1}\|u\|_{L^\infty(B_r(y))} \leq C\|u\|_{L^\infty(B_1)} r^{\frac{1}{2}}.$$

In particular

$$\|\nabla_{x'} u\|_{L^\infty(B_r(x_\circ))} \leq C\|u\|_{L^\infty(B_1)} r^{\frac{1}{2}} \tag{1.29}$$

for $x_\circ \in \Gamma(u)$, since $\nabla_{x'} u \equiv 0$ on the contact set $\{x_{n+1} = 0\} \cap \{u = 0\}$. Take now $y_1, y_2 \in \{x_{n+1} = 0\}$, so that we want to obtain the bound

$$|\nabla_{x'} u(y_1) - \nabla_{x'} u(y_2)| \leq C\|u\|_{L^\infty(B_1)} |y_1 - y_2|^{\frac{1}{2}} \tag{1.30}$$

to get $C^{1,1/2}$ regularity of u on the thin space. Notice that, since $\nabla_{x'} u = 0$ on $\{x_{n+1} = 0\} \cap \{u = 0\}$, we can assume that $y_1, y_2 \in \{x_{n+1} = 0\} \cap \{u > 0\}$.

Let us suppose $r = \mathrm{dist}(y_1, \Gamma(u)) \geq \mathrm{dist}(y_2, \Gamma(u))$. Then, if $\mathrm{dist}(y_1, y_2) \leq \frac{r}{2}$, and since u is harmonic in $B_r(y_1)$, by harmonic estimates we have

$$\frac{|\nabla_{x'} u(y_1) - \nabla_{x'} u(y_2)|}{|y_1 - y_2|^{1/2}} \leq [\nabla_{x'} u]_{C^{1/2}(B_{r/2}(y_1))} \leq Cr^{-1/2}\|\nabla_{x'} u\|_{L^\infty(B_r(y_1))} \leq C\|u\|_{L^\infty(B_1)}$$

where in the last step we have used (1.29). On the other hand, if $\mathrm{dist}(y_1, y_2) \geq \frac{r}{2}$, from (1.29) and $\mathrm{dist}(y_2, \Gamma(u)) \leq r$,

$$\begin{aligned} |\nabla_{x'} u(y_1) - \nabla_{x'} u(y_2)| &\leq |\nabla_{x'} u(y_1)| + |\nabla_{x'} u(y_2)| \\ &\leq C\|u\|_{L^\infty(B_1)} r^{1/2} \leq C\|u\|_{L^\infty(B_1)} |y_1 - y_2|^{1/2}. \end{aligned}$$

In all, (1.30) always holds, and u is $C^{1,1/2}$ on $\{x_{n+1} = 0\}$. By standard harmonic estimates, its harmonic extension to B_1^+ is also $C^{1,1/2}$ with estimates up to the boundary $\{x_{n+1} = 0\}$, which gives (1.27).

Finally, we note that $\Gamma(u) \ni x \mapsto N(r, u, x)$ is continuous for every $r > 0$, and is monotone nondecreasing. Thus, $N(0^+, u, x) = \inf_{r>0} N(r, u, x)$ is the infimum of a family of continuous functions, and therefore, it is upper semi-continuous. In particular, if $\mathrm{Deg}(u) \ni x_k \to x_\circ$, then $N(0^+, u, x_\circ) \geq \limsup_{k\to\infty} N(0^+, u, x_k) \geq 2$, and thus $x_\circ \in \mathrm{Deg}(u)$. The set of degenerate points closed, and the set of regular points is open (in the relative topology of the free boundary). □

1.6 Regular points

We have shown that the free boundary can be divided into two different sets: regular points, and degenerate points, according to the value of the frequency.

As we will show next, the set of regular points received this name because we can show smoothness of the free boundary around them, [ACS08].

Let 0 be a regular free boundary point, and consider the rescalings

$$u_r(x) = \frac{u(rx)}{\left(\fint_{\partial B_r} u^2 \right)^{\frac{1}{2}}}.$$

Since 0 is a regular point, by Theorem 1.6, there exists some sequence $r_j \downarrow 0$ such that, up to a rotation,

$$u_{r_j} \to u_0 := \mathrm{Re}\left((x_1 + i|x_{n+1}|)^{3/2}\right) \quad \text{strongly in } C^1(B^+_{1/2}). \tag{1.31}$$

Notice that, on the thin space, u_0 is a half-space solution of the form $u_0(x', 0) = c(x_1)_+^{3/2}$. In particular, the free boundary is a hyperplane (in $\{x_{n+1} = 0\}$) and thus smooth. We want to show that the smoothness of the free boundary in the limit is inherited by the approximating sequence, u_{r_j}, for j large enough.

Let us start by showing that the free boundary is Lipschitz. In the following proposition, $\mathcal{C}(\boldsymbol{e}_1, \theta)$ denotes a cone with axis $\boldsymbol{e}_1$ an opening $\theta > 0$, in the tangential directions,

$$\mathcal{C}(\boldsymbol{e}_1, \theta) := \left\{\tau \in \mathbb{R}^{n+1} : \tau_{n+1} = 0,\ \tau \cdot \boldsymbol{e}_1 \geq \cos(\theta)\|\tau\|\right\}.$$

Proposition 1.8. *Let u be a solution to* (1.20)*, and let us suppose that the origin is a regular free boundary point,* $0 \in \mathrm{Reg(u)}$*. Suppose, also, that* (1.31) *holds.*

Then, for any fixed $\theta_\circ > 0$, there exists some $\rho > 0$ such that

$$\partial_\tau u \geq 0 \qquad \textit{in } B_\rho, \textit{ for all } \tau \in \mathcal{C}(\boldsymbol{e}_1, \theta_\circ). \tag{1.32}$$

In particular, the free boundary is Lipschitz around regular points. That is, for some neighbourhood of the origin, $\Gamma(u)$ is the graph of a Lipschitz function $x_1 = f(x_2, \dots, x_n)$ in $\{x_{n+1} = 0\}$.

Proof. We use that $\partial_\tau u_{r_j}$ is converging to $\partial_\tau u_0$ uniformly in $B_{1/2}$. Notice that, by assumption, $\partial_\tau u_0 \geq 0$, and in fact, $\partial_\tau u_0 \geq c(\theta_\circ, \sigma) > 0$ in $\{|x_{n+1}| > \sigma\}$.

Thus, from the uniform convergence, for any $\sigma > 0$ there exists some $r_\circ = r_\circ(\theta_\circ, \sigma)$ such that, if $r_j \leq r_\circ$,

$$\begin{array}{ll} \partial_\tau u_{r_j} \geq 0 & \text{in } B_{3/4} \setminus \{|x_{n+1}| \geq \sigma\} \\ \partial_\tau u_{r_j} \geq c(\theta_\circ) > 0 & \text{in } B_{3/4} \setminus \left\{|x_{n+1}| \geq \frac{1}{2}\right\}. \end{array} \tag{1.33}$$

Moreover, from the optimal $C^{1,\frac{1}{2}}$ regularity of solutions,

$$\partial_\tau u_{r_j} \geq -c\sigma^{\frac{1}{2}} \quad \text{in } B_{3/4} \cap \{|x_{n+1} \leq \sigma\}. \tag{1.34}$$

Combining (1.33)-(1.34) with the fact that $\Delta(\partial_\tau u_{r_j}) = 0$ in $B_1 \setminus \Lambda(u_{r_j})$, and $\partial_\tau u_{r_j} = 0$ on $\Lambda(u_{r_j})$, by standard comparison principle arguments (see [ACS08, Lemma 5]) we deduce that there exist some $\sigma_\circ = \sigma_\circ(\theta_\circ)$ such that if $\sigma < \sigma_\circ$, $\partial_\tau u_{r_j} \geq 0$ in $B_{1/2}$. In particular, there exists some ρ (depending only on $\theta_\circ$, but also depending on the regular point) such that $\partial_\tau u_\rho \geq 0$ in B_1. Thus, (1.32) holds.

We finish by showing that (1.32) implies that the free boundary is Lipschitz. We do so by considering the two (half) cones

$$\Sigma_\pm := \pm\mathcal{C}(\boldsymbol{e}_1, \theta_\circ) \cap B_{\rho/2}.$$

Notice that, since $0 \in \Gamma(u)$, $u(0) = 0$, and from $u \geq 0$ on $\{x_{n+1} = 0\}$ together with (1.32) we must have $u \equiv 0$ on Σ_-, so $\Sigma_- \subset \{u = 0\}$.

On the other hand, suppose that $y_\circ \in \Sigma_+$ is such that $u(y_\circ) = 0$. Again, by (1.32) and the non-negativity of u on the thin space, we have $u \equiv 0$ on $y_\circ - \mathcal{C}(\boldsymbol{e}_1, \theta_\circ)$. But notice that, since $y_\circ \in \Sigma_+$, $0 \in y_\circ - \mathcal{C}(\boldsymbol{e}_1, \theta_\circ)$, that is, 0 is not a free boundary point. A contradiction. Therefore, we have that $u(y_\circ) > 0$, so $\Sigma_+ \subset \{u > 0\}$.

Thus, the free boundary at 0 has a cone touching from above and below, and therefore, it is Lipschitz at the origin. We can do the same at the other points around it, so that the free boundary is Lipschitz. □

In fact, the previous proof not only shows that the free boundary is Lipschitz, but letting $\theta_\circ \downarrow 0$ we are showing that it is basically C^1. In order to upgrade the regularity of the free boundary around regular points we use the boundary Harnack principle.

Theorem 1.9 (Boundary Harnack Principle, [ACS08, DS19])**.** *Let $\Omega \subset \{x_{n+1} = 0\} \cap B_1$ be a Lipschitz domain on the thin space, and let $v_1, v_2 \in C(B_1)$ satisfying $\Delta v_1 = \Delta v_2 = 0$ in $B_1 \setminus \Omega$. Assume, moreover, that v_1 and v_2 vanish continuously on Ω, and $v_2 > 0$ in $B_1 \setminus \Omega$. Then, there exists some $\alpha > 0$ such that $\frac{v_1}{v_2}$ is α-Hölder continuous in $B_{1/2} \setminus \Omega$ up to Ω.*

As a consequence, we can show that the Lipschitz part of the free boundary is, in fact, $C^{1,\alpha}$.

Theorem 1.10 ($C^{1,\alpha}$ regularity of the free boundary around regular points)**.** *Let u be a solution to* (1.20)*. Then, the set of regular points,* $\mathrm{Reg}(u)$*, is locally a $C^{1,\alpha}$ $(n-1)$-dimensional manifold.*

Proof. We just need to apply Theorem 1.9 to the right functions. Notice that, by Proposition 1.8 we already know that around regular points, the free boundary is a Lipschitz $(n-1)$-dimensional manifold.

Let us suppose 0 is a regular point. Take $\bar\tau = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \boldsymbol{e}_i)$ with $i \in \{2, \dots, n\}$, and notice that in B_ρ such that (1.32) holds (with $\theta_\circ = \pi/4$) we have that $v_1 := \partial_{\boldsymbol{e}_1} u$ and $v_2 := \partial_{\bar\tau} u$ are positive harmonic functions, vanishing continuously on $\Omega := \Lambda(u) \cap B_\rho$, by Proposition 1.8. Thus, v_1/v_2 is Hölder continuous, which implies that $\partial_{\boldsymbol{e}_i} u/\partial_{\boldsymbol{e}_1} u$ is Hölder continuous, up to $\Lambda(u)$, in B_ρ.

We finish by noticing that, if we take $x \in \{x_{n+1} = 0\}$ such that $u(x) = t$, then $\nu(x)$ denotes the unit normal vector to the level set $\{u = t\}$ on the thin space, where

$$\nu_i(x) := \frac{\partial_{\boldsymbol{e}_i} u}{|(\partial_{\boldsymbol{e}_1} u, \dots, \partial_{\boldsymbol{e}_n} u)|} = \frac{\partial_{\boldsymbol{e}_i} u/\partial_{\boldsymbol{e}_1} u}{\left(1 + \sum_{i=2}^n (\partial_{\boldsymbol{e}_i} u/\partial_{\boldsymbol{e}_1} u)^2\right)^{1/2}}.$$

Thus, $\nu = (\nu_1, \dots, \nu_n)$ is Hölder continuous. In particular, letting $t \downarrow 0$ we obtain that the normal vector to the free boundary is Hölder continuous, and therefore, the free boundary is $C^{1,\alpha}$ in $B_{\rho/2}$. □

It is possible to keep iterating a higher order boundary Harnack principle to obtain higher order free boundary regularity estimates around regular points. Hence, Theorem 1.9 also has a higher order analogy.

Proposition 1.11 (Higher order Boundary Harnack Principle, [DS15])**.** *Let $\Omega \subset \{x_{n+1} = 0\} \cap B_1$ be a $C^{k,\alpha}$ domain on the thin space for $k \geq 1$, and let $v_1, v_2 \in C(B_1)$ satisfying $\Delta v_1 = \Delta v_2 = 0$ in $B_1 \setminus \Omega$. Assume, moreover, that v_1 and v_2 vanish continuously on Ω, and $v_2 > 0$ in $B_1 \setminus \Omega$. Then, $\frac{v_1}{v_2}$ is $C^{k,\alpha}$ in $B_{1/2} \setminus \Omega$ up to Ω.*

Moreover, if $U_0(x') = \sqrt{\mathrm{dist}(x', \Omega)}$, and v_1 is even in x_{n+1}, then $\frac{v_1}{U_0}$ is $C^{k-1,\alpha}$ in $B'_{1/2} \setminus \Omega$ up to Ω.

And from the higher order Boundary Harnack Principle we can deduce higher order regularity of the free boundary (at regular points).

Corollary 1.12 (C^∞ regularity of the free boundary around regular points)**.** *Let u be a solution to (1.20). Then, the set of regular points, $\mathrm{Reg}(u)$, is locally a C^∞ $(n-1)$-dimensional manifold.*

Proof. Follows analogously to the proof of Theorem 1.10 by using Proposition 1.11 instead of Theorem 1.9. □

As a consequence of the previous argumentation we also get an expansion around regular points, proving that, up to lower order terms, the solution behaves like the half-space solution. In particular, this next theorem proves the uniqueness of blow-ups.

Theorem 1.13 (Expansion around regular points)**.** *Let u be a solution to (1.20), and let us assume $0 \in \mathrm{Reg}(u)$. Then, there exists some $c > 0$ and some $\alpha > 0$ (possibly depending on everything) such that*

$$u(x) = cu_0(x) + o\left(|x|^{\frac{3}{2}+\alpha}\right),$$

where u_0 is the blow-up of u at 0 (i.e., $u_0(x) = \mathrm{Re}\left((x_1 + i|x_{n+1}|)^{3/2}\right)$ up to a rotation in the thin space).

Proof. We here use the second part of Proposition 1.11. By taking $\tau \in S^n \cap \{x_{n+1} = 0\}$ and $v_2 = \partial_\tau u$ (a tangential derivative to the thin space), by Proposition 1.11 we have

$$\frac{\partial_\tau u}{U_0} \in C^\alpha$$

in the thin space, for some $\alpha > 0$ (coming from the regularity of the free boundary), outside of the contact set and up to the free boundary. In particular,

$$\left|\frac{\partial_\tau u}{U_0}(x') - c_0\right| \leq C|x'|^\alpha \quad \Longrightarrow \quad |\partial_\tau u(x') - c_0 U_0(x')| \leq CU_0(x')|x'|^\alpha \leq C|x'|^{\frac{1}{2}+\alpha},$$

for some constant $c_0 = \frac{\partial_\tau u}{U_0}(0)$. We recall that $U_0(x') = \sqrt{\mathrm{dist}(x', \Omega)}$. By the $C^{1,\alpha}$ regularity of the free boundary, there exists some c_τ such that $U_0 - c_\tau \partial_\tau u_0 = o\left(|x|^{\frac{1}{2}+\alpha'}\right)$ for some $\alpha' > 0$, where u_0 is the blow-up at 0. Thus, we have that

$$|\partial_{\boldsymbol{e}_i} u(x') - c_i \partial_{\boldsymbol{e}_i} \tilde{u}_0(x')| \leq C|x'|^{\frac{1}{2}+\alpha'}.$$

From the local uniform convergence $\partial_\tau u_r \to \partial_\tau \tilde{u}_0$ we must have $c_i = c \geq 0$ for all $i = 1, \ldots, n$ in the previous expression, where

$$c = \lim_{r\downarrow 0} r^{-\frac{3}{2}} \left(\fint_{\partial B_r} u^2 \right)^{\frac{1}{2}}.$$

Thus,

$$|\nabla_{x'} u(x') - c\nabla_{x'} u_0(x')| \leq C|x'|^{\frac{1}{2}+\alpha'}.$$

Since $\nabla_{x'} u(0) = \nabla_{x'} u_0(0) = 0$, by integrating the previous expression we deduce

$$|u(x') - cu_0(x')| \leq C|x'|^{\frac{3}{2}+\alpha'}.$$

By harmonic estimates, such inequality also holds outside of the thin space. Now, if $c = 0$, it means that the frequency at 0 is at least $\frac{3}{2} + \alpha'$. This contradicts 0 being a regular point, and thus, $c > 0$. This concludes the proof. □

We finish by noticing the uniqueness of blow-ups at regular points.

Corollary 1.14 (Uniqueness of blow-ups at regular points)**.** *Let u be a solution to* (1.20)*, and let us assume* $0 \in \mathrm{Reg}(u)$*. Then, up to a rotation,*

$$\frac{u(r\,\cdot)}{r^{\frac{3}{2}}} \to cu_0 \quad \text{as} \quad r \downarrow 0,$$

locally uniformly, for some $c > 0$. Here, $u_0(x) = \mathrm{Re}\left((x_1 + i|x_{n+1}|)^{3/2}\right)$*.*

Proof. This is a direct consequence of Theorem 1.13. □

1.7 Singular points

In the classical (or thick) obstacle problem, all points of the free boundary have frequency 2, and thus the classification of free boundary points must be performed differently: regular points are those such that the contact set has positive density, whereas singular points are those where the contact set has zero density.

This motivates the definition of singular point. Whereas it is not true that all points of positive density belong to the set $\mathrm{Reg}(u)$ as defined above, one can characterize the points with zero density.

Let us start defining the set of singular points, which was originally studied by Garofalo and Petrosyan in [GP09]. Let u denote a solution to the thin obstacle problem, (1.20), then we define

$$\mathrm{Sing}(u) := \left\{ x \in \Gamma(u) : \liminf_{r\downarrow 0} \frac{\mathcal{H}^n(\Lambda(u) \cap B_r(x))}{\mathcal{H}^n(B_r(x) \cap \{x_{n+1} = 0\})} = 0 \right\}, \tag{1.35}$$

where we recall that $\Lambda(u)$ denotes the contact set, and $\mathcal{H}^n(E)$ denotes the n-dimensional Hausdorff measure of a set E.

The first result in this direction involves the characterization of such points.

Proposition 1.15 (Characterization of singular points, [GP09]). *Let u be a solution to* (1.20)*. Then, the set of singular points* (1.35) *can be equivalently characterized by*

$$\mathrm{Sing}(u) = \left\{x \in \Gamma(u) : N(0^+, u, x) = 2m, \quad m \in \mathbb{N}\right\}.$$

That is, singular points are those with even frequency.

Proof. Let us suppose that $0 \in \mathrm{Sing}(u)$ according to definition (1.35), and take a sequence $r_j \downarrow 0$ such that

$$\frac{\mathcal{H}^n(\Lambda(u) \cap B_{r_j})}{\mathcal{H}^n(B_{r_j} \cap \{x_{n+1} = 0\})} \to 0. \tag{1.36}$$

Consider the sequence u_{r_j}, and after taking a subsequence if necessary, let us assume $u_{r_j} \to u_0$ uniformly in B_1. Notice that Δu_{r_j} is a non-positive measure supported on $\Lambda(u_{r_j})$. By assumption, $\mathcal{H}^n(\Lambda(u_{r_j}) \cap B_1) \to 0$. Thus, since u_{r_j} converges uniformly to u_0, u_0 has Laplacian concentrated on a set with zero harmonic capacity, and thus, it is harmonic.

By Theorem 1.6, u_0 is a global homogeneous solution to the thin obstacle problem, with homogeneity $\kappa := N(0^+, u)$. In particular, being homogeneous and harmonic, it must be a polynomial. Moreover, since u_r is even with respect to $\{x_{n+1} = 0\}$, so is u_0. Thus, u_0 is a non-zero, harmonic polynomial, even with respect to $\{x_{n+1} = 0\}$ and non-negative on the thin space. Its homogeneity must be even, and thus $\kappa = 2m$ for some $m \in \mathbb{N}$.

Suppose now that $0 \in \Gamma(u)$ is such that $N(0^+, u) = 2m$ for some $m \in \mathbb{N}$. Take any blow-up of u at zero, u_0. Then u_0 is a global solution to the thin obstacle problem, with homogeneity $2m$. As a consequence u_0 must be harmonic everywhere, and thus, an homogeneous harmonic polynomial (we refer to [Mon09, Lemma 7.6] or [GP09, Lemma 1.3.4] for a proof of this fact).

Now, since u_0 is non-zero even homogeneous harmonic polynomial, and is non-zero on the thin space (by Cauchy-Kovalevskaya), $\mathcal{H}^n(\{u_0 = 0\} \cap \{x_{n+1} = 0\}) = 0$. Thus, from the uniform convergence $u_{r_j} \to u_0$, we must have that (1.36) holds. □

Thus, the set of singular points consists of those points with even homogeneity. It is then natural to define

$$\Gamma_\lambda(u) := \{x \in \Gamma(u) : N(0^+, u, x) = \lambda\},$$

so that

$$\mathrm{Sing}(u) = \bigcup_{m \in \mathbb{N}} \Gamma_{2m}(u) =: \Gamma_{\mathrm{even}}(u).$$

In fact, singular points present a particularly good structure. At singular points of order $2m$, the solution to the thin obstacle problem is $2m$ times differentiable (in the sense (1.37)) and in particular, the blow-up is unique, and belongs to the set

$$\mathcal{P}_{2m} := \{p : \Delta p = 0, x \cdot \nabla p = 2mp, p(x', 0) \geq 0, p(x', x_{n+1}) = p(x', -x_{n+1})\},$$

$2m$-homogeneous, harmonic polynomials, non-negative on the thin space. That is, the following result from [GP09], which we will not prove, holds.

Theorem 1.16 (Uniqueness of blow-ups at singular points, [GP09])**.** *Let u be a solution to* (1.20)*. Let $x_\circ \in \Gamma_{2m}(u)$ for some $m \in \mathbb{N}$. Then, there exists a non-zero polynomial $p_{x_\circ} \in \mathcal{P}_{2m}$ such that*

$$u(x) = p_{x_\circ}(x - x_\circ) + o(|x - x_\circ|^{2m}). \tag{1.37}$$

In particular, the blow-up at 0 is unique. Moreover, the map $x_\circ \ni \Gamma_{2m}(u) \mapsto p_{x_\circ}$ is continuous.

The proof of the previous theorem is based on a Monneau-type monotonicity formula, saying that if u has a singular points of order $2m$ at the origin, the following function is non-decreasing,

$$r \mapsto M_k(r, u, p_{2m}) = \frac{1}{r^{n+2m}} \int_{\partial B_r} (u - p_{2m})^2,$$

for all $p \in \mathcal{P}_{2m}$ and $0 < r < 1$. From here, in [GP09] they establish first non-degeneracy at singular points, and then the uniqueness of a blow-up. The continuity with respect to the point then follows by a compactness argument.

Theorem 1.16 establishes a connection between singular points and their blow-ups. This also allows to separate between different singular points according to "how big" the contact set is around them. We already know it has zero $\mathcal{H}^n$-density. In fact, the contact set around singular points has the same "size" as the translation invariant set of the blow-up. Thus, we can establish a further stratification within the set of singular points, according to the size of the translation invariant set (which is a subspace) of the blow-up.

Given a solution to the thin obstacle problem, (1.20), u, and given $x \in \Gamma(u)$, let us denote by p_x any blow-up of u at x. In particular, if x is a singular free boundary point, $p_x \in \mathcal{P}_{2m}$ is uniquely determined by the result above.

Let us denote by $L(p)$ the translation invariant set for p, where p is a blow-up,

$$\begin{aligned} L(p) &:= \left\{\xi \in \mathbb{R}^{n+1} : p(x + \xi) = p(x) \text{ for all } x \in \mathbb{R}^{n+1}\right\} \\ &= \left\{\xi \in \mathbb{R}^{n+1} : \xi \cdot \nabla p(x) = 0 \text{ for all } x \in \mathbb{R}^{n+1}\right\}, \end{aligned}$$

where we recall that blow-ups p are homogeneous. Then, if we denote

$$\Gamma_{2m}^\ell := \{x \in \Gamma_{2m} : \dim L(p_x) = \ell\}, \quad \ell \in \{0, \dots, n-1\}, \tag{1.38}$$

we have

$$\mathrm{Sing}(u) = \Gamma_{\mathrm{even}}(u) = \bigcup_{m \in \mathbb{N}} \Gamma_{2m} = \bigcup_{m \in \mathbb{N}} \bigcup_{\ell=0}^{n-1} \Gamma_{2m}^\ell.$$

As a consequence of Theorem 1.16, combined with Whitney's extension theorem and the implicit function theorem, one can prove the following result regarding the structure of the singular set.

Theorem 1.17 ([GP09])**.** *Let u be a solution to* (1.20)*. Then, the set $\Gamma_{2m}^\ell(u)$ (see* (1.38)*) for $\ell \in \{0, \dots, n-1\}$, is contained in a countable union of C^1 ℓ-dimension manifolds.*

Notice that the fact that each stratum of the singular set is contained in countable union of manifolds (rather than a single manifold) is unavoidable: there could be accumulation of lower-order points (say, of order 2) to higher order points (say, of order 4).

On the other hand, the previous result can also be applied to the whole singular set: $\mathrm{Sing}(u)$ can be covered by a countable union of C^1 $(n-1)$-dimensional manifolds. The fact that the manifold id C^1 is due to the expansion of the solution (1.37). In [FJ20], Jhvaeri and the author show higher order expansions at singular points $x_\circ \in \Gamma_{2m}(u)$, analogous to (1.37), as

$$u(x) = p_{x_\circ}(x - x_\circ) + q_{x_\circ}(x - x_\circ) + o(|x - x_\circ|^{2m+1}) \tag{1.39}$$

for some $(2m+1)$-homogeneous, harmonic polynomial $q_{x_\circ}$. Expansion of the form (1.39) hold at almost every singular point, and thus, analogously to the previous case we obtain a structure result, that holds for all singular points up to a lower dimensional set:

Theorem 1.18 ([FJ20])**.** *Let u be a solution to* (1.20)*. Then, there exists a set $E \subset \mathrm{Sing}(u)$ of Hausdorff dimension at most $n-2$ such that $\mathrm{Sing}(u) \setminus E$ is contained in a countable union of C^2 $(n-1)$-dimensional manifolds.*

1.7.1 The non-degenerate case

So far we have been studying the thin obstacle problem with zero obstacle. When solving for an (even) boundary datum

$$g \in C^0(\partial B_1), \quad g(x', x_{n+1}) = g(x', -x_{n+1})$$

the problem looks like

$$\begin{cases} u \geq 0 & \text{on } B_1 \cap \{x_{n+1} = 0\} \\ \Delta u = 0 & \text{in } B_1 \setminus (\{x_{n+1} = 0\} \cap \{u = 0\}) \\ \Delta u \leq 0 & \text{in } B_1 \\ u = g & \text{on } \partial B_1, \end{cases} \tag{1.40}$$

We had reduced to this problem from (1.6) by subtracting the harmonic even extension of the analytic obstacle φ. Alternatively, from (1.40) we can reduce to the case of zero boundary data by subtracting the harmonic extension of g to the unit ball. Thus, we obtain a problem of the form

$$\begin{cases} v \geq \varphi & \text{on } B_1 \cap \{x_{n+1} = 0\} \\ \Delta v = 0 & \text{in } B_1 \setminus (\{x_{n+1} = 0\} \cap \{v = \varphi\}) \\ \Delta v \leq 0 & \text{in } B_1 \\ v = 0 & \text{on } \partial B_1, \end{cases} \tag{1.41}$$

that is, a thin obstacle problem with obstacle φ. Problems (1.40) and (1.41) are the same when

$$\begin{cases} \Delta \varphi = 0 & \text{in } B_1 \\ \varphi = -g & \text{on } \partial B_1. \end{cases} \qquad \text{and} \qquad v = u + \varphi. \tag{1.42}$$

In this setting, we say that problem (1.41) with $\varphi \in C^{3,1}(B_1 \cap \{x_{n+1} = 0\})$ is non-degenerate if

$$\Delta_{x'}\varphi \le -c_\circ < 0 \text{ in } B_1 \cap \{x_{n+1} = 0\} \cap \{\varphi > 0\}, \quad \varnothing \neq \{\varphi > 0\}, \tag{1.43}$$

where $\Delta_{x'}$ denotes the Laplacian in the first n coordinates (Laplacian along the thin space). The last condition above is to avoid having a non-active obstacle. Alternatively, in terms of problem (1.40) we have

$$\text{(1.40) is non-degenerate} \overset{\text{def.}}{\Longleftrightarrow} \varphi_g : \begin{cases} \Delta\varphi_g = 0 & \text{in } B_1 \\ \varphi_g = -g & \text{on } \partial B_1. \end{cases} \text{ satisfies (1.43).} \tag{1.44}$$

In particular, when we deal with concave obstacles, we say that our problem is non-degenerate. In [BFR18], Barrios, Figalli, and Ros-Oton show that, under a non-degeneracy assumption, we have a better characterization of free boundary points.

Theorem 1.19 ([BFR18])**.** *Let u be a solution to* (1.40)*, and suppose that the non-degeneracy condition holds,* (1.43)*. Then, there exists a constant $\bar c$ (depending on $c_\circ$) such that for any $x_\circ \in \Gamma(u) \cap B_{1/2}$,*

$$\sup_{B_r(x_\circ)} u \ge \bar c\, r^2,$$

for all $r \in (0, \frac{1}{4})$. In particular, if (1.43) *holds, then*

$$\Gamma(u) = \mathrm{Reg}(u) \cup \Gamma_2(u),$$

i.e., the free boundary consists only of regular points and singular points of order 2.

Proof. We prove it for v satisfying (1.41) and the proof follows by the transformation (1.42) with $\varphi = \varphi_g$ as in (1.43).

Let us define for $\bar x = (\bar x', 0) \in B_{1/2} \cap \{x_{n+1} = 0\} \cap \{u > \varphi\}$,

$$w_{\bar x}(x', x_{n+1}) = v(x', x_{n+1}) - \varphi(x') - \frac{c_\circ}{2n+2}\left(|x' - \bar x'|^2 + x_{n+1}^2\right),$$

where $c_\circ$ is the constant in (1.43). Notice that, since $\Delta v = 0$ in outside of the contact set $\Lambda(v)$,

$$\Delta w_{\bar x} = -\Delta_{x'}\varphi - c_\circ \ge 0, \quad \text{in} \quad B_r(\bar x) \setminus \Lambda(v).$$

On the other hand, $w_{\bar x}(\bar x', 0) > 0$ and $w < 0$ on $\Lambda(v)$. By maximum principle, we must have $\sup_{\partial B_r(\bar x)} w_{\bar x} > 0$. Letting $\bar x \to x_\circ \in \Gamma(u)$ we deduce

$$\sup_{\partial B_r(x_\circ)} w_{x_\circ} \ge 0,$$

which implies the desired result.

Finally, since the growth at the free boundary is at least quadratic, there cannot be any blow-up at a free boundary point with homogeneity greater than 2. □

In this case, therefore, the non-regular part of the free boundary consists, exclusively, of singular points of order 2. In particular, in Theorem 1.17 we have instead a single C^1 ℓ-dimensional manifold covering the whole of $\Gamma_{2m}^{\ell}(u)$. We can also establish a more refined version of Theorem 1.18,

Theorem 1.20 ([FJ20]). *Let u be a solution to* (1.20)*, and suppose that the non-degenerate condition* (1.43) *holds. Then,*

(i) $\Sigma_2^0(u)$ *is isolated in* $\mathrm{Sing}(u) = \Gamma_2^0(u) \cup \cdots \cup \Gamma_2^{n-1}(u)$.

(ii) There exists an at most countable set $E_1 \subset \Gamma_2^1(u)$ *such that* $\Gamma_2^1(u) \setminus E_1$ *is locally contained in a single one-dimensional* C^2 *manifold.*

(iii) For each $\ell \in \{2, \dots, n-1\}$*, there exists a set* $E_m \subset \Gamma_2^\ell(u)$ *of Hausdorff dimension at most* $\ell - 1$ *such that* $\Gamma_2^\ell(u) \setminus E_\ell$ *is locally contained in a single* ℓ*-dimensional* C^2 *manifold.*

1.8 Other points

The free boundary contains, in general, other points different from *regular* and *singular*. Even in two dimensions ($n = 1$) one can perform the simple (see [FoSp18, Proposition A.1]) task of manually classifying all the possible homogeneities that an homogeneous solution to the thin obstacle problem (with zero obstacle) can present.

Indeed, for $n = 1$ homogeneous solutions to the thin obstacle problem must have homogeneity belonging to the set

$$\left\{2m, 2m - \frac{1}{2}, 2m + 1\right\}_{m \in \mathbb{N}}.$$

Solutions with homogeneity $2m$ are harmonic quadratic polynomials, non-negative on the thin space. On the other hand, homogeneous solutions with homogeneity $2m - \frac{1}{2}$ or $2m + 1$ are of the form

$$\mathrm{Re}\left((x_1 + i|x_2|)^{2m-\frac{1}{2}}\right) \qquad \text{and} \qquad \mathrm{Im}\left((x_1 + i|x_2|)^{2m+1}\right), \quad \text{for } m \in \mathbb{N}.$$

Notice that when the homogeneity is $2m - \frac{1}{2}$ we have *half-space* solutions on the thin space. Indeed, in this case, restricting to $x_2 = 0$, solutions are of the form $u(x_1, 0) = (x_1)_+^{2m-\frac{1}{2}}$. On the other hand, solutions with odd homogeneity are identically zero on the thin space (in particular, this type of homogeneous solution is *not* an example of a free boundary point with odd homogeneity, and in fact, they do not exist in dimension $n = 1$).

Given that no other homogeneities can appear in dimension 2, one can show that, in any dimension, the previous homogeneities comprise all of the free boundary, up to a lower dimensional set. It is for this reason that we separate the possible homogeneities of the free boundary as

$$\Gamma(u) = \Gamma_{3/2}(u) \cup \Gamma_{\mathrm{even}}(u) \cup \Gamma_{\mathrm{odd}}(u) \cup \Gamma_{\mathrm{half}}(u) \cup \Gamma_*(u), \tag{1.45}$$

where $\Gamma_{3/2}(u) = \mathrm{Reg}(u)$ are regular points; $\Gamma_{\mathrm{even}}(u) = \mathrm{Sing}(u)$ are singular points; $\Gamma_{\mathrm{odd}}(u)$ denotes the set of points with odd homogeneity, $2m + 1$ for $m \in \mathbb{N}$; $\Gamma_{\mathrm{half}}(u)$ are the points with homogeneity $2m + \frac{3}{2}$ for $m \in \mathbb{N}$; and $\Gamma_*(u)$ are the rest of possible free boundary points (in particular, $\Gamma_*(u) = \varnothing$ if $n = 1$, $\dim_{\mathcal{H}}(\Gamma_*(u)) \leq n - 2$ in general).

1.8.1 The set $\Gamma_{\text{odd}}(u)$

The free boundary points belonging to $\Gamma_{\text{odd}}(u)$ are those with odd homogeneity, $2m+1$ for $m \in \mathbb{N}$. They are analogous to the singular set, in the sense that in this case, points belonging to $\Gamma_{\text{odd}}(u)$ can also be characterized via the density of the contact set: these points have density 1.

They are not known to exist (no single example has been constructed so far). Notice that the homogeneous solutions presented above are vanishing identically on the thin space, and thus they do not have a free boundary.

In fact, in dimension $n = 1$, if such a point existed its blow-up would be of the form

$$\operatorname{Im}\left((x_1 + i|x_2|)^{2m+1}\right), \quad \text{for } m \in \mathbb{N}. \tag{1.46}$$

(Think, for example, of the x_2-even extension of the harmonic polynomial $x_2^3 - 3x_1^2 x_2$ for $x_2 \geq 0$.) However, solutions of the form (1.46) have non-vanishing normal derivative on the thin space, whereas a free boundary point can be approximated by points with vanishing normal derivative. From the C^1 convergence of blow-ups, we reach a contradiction: free boundary points with odd homogeneity do not exist in dimension $n = 1$.

The set of points belonging to $\Gamma_{\text{odd}}(u)$ has been studied in a recent work by Figalli, Ros-Oton, and Serra [FRS19, Appendix B].

Proposition 1.21 (Characterization of points in $\Gamma_{\text{odd}}(u)$, [FRS19])**.** *Let u be a solution to* (1.20)*. Then, the set of points with odd homogeneity, $\Gamma_{\text{odd}}(u)$, can be equivalently characterized by*

$$\Gamma_{\text{odd}}(u) := \left\{ x \in \Gamma(u) : \limsup_{r \downarrow 0} \frac{\mathcal{H}^n(\Lambda(u) \cap B_r(x))}{\mathcal{H}^n(B_r(x) \cap \{x_{n+1} = 0\})} = 1 \right\}, \tag{1.47}$$

That is, points with odd homegeneity are those where the contact set has density 1.

Proof. Let us suppose that $0 \in \Gamma(u)$ fulfills definition (1.47), that is, we can take a sequence $r_j \downarrow 0$ such that

$$\frac{\mathcal{H}^n(\Lambda(u) \cap B_{r_j})}{\mathcal{H}^n(B_{r_j} \cap \{x_{n+1} = 0\})} \to 1. \tag{1.48}$$

Consider the sequence u_{r_j}, and after taking a subsequence if necessary, let us assume $u_{r_j} \to u_0$ uniformly in B_1. In particular, u_0 vanishes identically on the thin space. Since it is homogeneous, and harmonic on $x_{n+1} > 0$, it must be a polynomial. It cannot have even homogeneity, since by the discussion on singular points it would have zero density. Thus, it is an homogeneous harmonic polynomial with odd homogeneity in $x_{n+1} \geq 0$ (and extend evenly in the whole space). Notice also that it cannot be linear (on each side) because the minimum possible homogeneity is $\frac{3}{2}$.

On the other hand, suppose that $0 \in \Gamma(u)$ is such that $N(0^+, u) = 2m+1$ for some $m \in \mathbb{N}$. Take any blow-up of u at zero, u_0. Then u_0 is a global solution to the thin obstacle problem, with homogeneity $2m+1$. Let us define the global (homogeneous) solution to the thin obstacle problem given by P,

$$P(x) = \sum_{i=1}^{n} \operatorname{Im}\big((x_i + i|x_{n+1}|)\big)^{2m+1},$$

so that $\partial^+_{x_{n+1}} P < 0$ in $\{x_{n+1} = 0\} \setminus \{0\}$. Using (1.12), we obtain that for any test function $\Psi = \Psi(|x|)$ (so that $\nabla\Psi = \Psi'(|x|)\frac{x}{|x|}$),

$$\begin{aligned}2\int_{\{x_{n+1}=0\}} \partial^+_{x_{n+1}} P\,\Psi u_0 &= \int \Delta P \Psi u_0 = -\int (\nabla P\cdot\nabla u_0\Psi + \nabla P\cdot\nabla\Psi u_0)\\ &= \int (P\Delta u_0 \Psi + P\nabla u_0\cdot\nabla\Psi - u_0\nabla P\cdot\nabla\Psi)\\ &= \int \left(P\nabla u_0\cdot x\frac{\Psi'(|x|)}{|x|} - u_0\nabla P\cdot x\frac{\Psi'(|x|)}{|x|}\right) = 0,\end{aligned}$$

where we have used that $P\Delta u_0 \equiv 0$ everywhere, and $\nabla u_0\cdot x = (2m+1)u_0$, $\nabla P\cdot x = (2m+1)P$. Since $u_0 \geq 0$ on the thin space, and $\partial^+_{x_{n+1}} P < 0$ outside of the origin on the thin space, we deduce $u_0 \equiv 0$ on the thin space.

As a consequence u_0 must be harmonic everywhere, vanishing on the thin space. Thus, it is an homogeneous harmonic polynomial with degree $2m+1$. In particular, $\partial_{x_{n+1}} u_0$ is a non-zero $2m$-homogeneous polynomial on $\mathbb{R}^{n+1}_+$. From the C^1 convergence of $u_{r_j} \to u_0$ (that is, the uniform convergence of $\partial_{x_{n+1}} u_{r_j}$ to $\partial_{x_{n+1}} u_0$) we deduce (1.47). □

We also have a result analogous to Theorem 1.16 at odd-frequency points. Let us start by defining for $m \geq 1$

$$\begin{aligned}\mathcal{Q}_{2m+1} := \big\{q : q \text{ solves the thin obstacle problem (1.20) in } \mathbb{R}^{n+1},\\ x\cdot\nabla q = (2m+1)q,\ q(x', x_{n+1}) = q(x', -x_{n+1})\big\},\end{aligned}$$

namely, the set of $(2m+1)$-homogeneous even solutions to the thin obstacle problem (notice that by the proof of Proposition 1.21, in particular, $q(x', 0) \equiv 0$). Then, we have

Theorem 1.22 (Uniqueness of blow-ups at odd-frequency points, [FRS19]). *Let u be a solution to* (1.20). *Let $x_\circ \in \Gamma_{2m+1}(u)$ for some $m \in \mathbb{N}$. Then, there exists a non-zero $q_{x_\circ} \in \mathcal{Q}_{2m+1}$ such that*

$$u(x) = q_{x_\circ}(x - x_\circ) + o(|x - x_\circ|^{2m+1}). \tag{1.49}$$

In particular, the blow-up at 0 is unique. Moreover, the set $\Gamma_{2m+1}(u)$ is $(n-2)$-rectifiable.

1.8.2 The set $\Gamma_{\rm half}(u)$

The free boundary points belonging to $\Gamma_{\rm half}(u)$ are those with homogeneity $2m + \frac{3}{2}$ for $m \in \mathbb{N}$.

They do exist: the homogeneous solutions are themselves examples of solutions to the thin obstacle problem with free boundary points belonging to $\Gamma_{\rm half}(u)$. Whereas they are currently not very well understood, they seem to exhibit a similar behaviour to regular points. However, the fact that they are not an open set (in the free boundary), makes it harder to study regularity properties of the free boundary around them (there could even be, a priori, singular points of order 2 converging to a point of order $\frac{7}{2}$).

The following proposition shows that points in $\Gamma_{\text{half}}(u)$ can present a behaviour similar to that of regular points.

Proposition 1.23 ([FR19]). *Given a C^∞ domain $\Omega \subset B_1 \cap \{x_{n+1} = 0\}$, and $m \in \mathbb{N}$, there exists $\varphi \in C^\infty$, and $g \in C^0(\partial B_1)$, such that the solution u to the thin obstacle problem* (1.10) *with obstacle φ and boundary data g has contact set $\Lambda(u) = \Omega$, and all the points of the free boundary $\Gamma(u)$ have frequency $2m + \frac{3}{2}$.*

The proof of this proposition is an explicit computation based on a previous result by Grubb, [Gru15].

1.8.3 The set $\Gamma_*(u)$

We call $\Gamma_*(u)$ the rest of free boundary points. That is, points with homogeneity not belonging to the set $\{2m, 2m + 1, 2m - \frac{1}{2}\}_{m\in\mathbb{N}}$,

$$\Gamma_*(u) := \left\{ x_\circ \in \Gamma(u) : N(0^+, u, x_\circ) \in (2, \infty) \setminus \bigcup_{m\in\mathbb{N}} \left\{ 2m, 2m + 1, 2m - \frac{1}{2} \right\} \right\}. \tag{1.50}$$

It is currently not known whether such points exist. Nowadays, the only result in this direction is the following by Colombo, Spolaor, and Velichkov, saying that points with order *close* to $2m$ do not exist (except for singular points themselves). Apart from this result, the possible existence (or not) of points with these homogeneities is still an open problem.

Theorem 1.24 ([CSV19]). *Let u be a solution to the thin obstacle problem with zero obstacle,*

$$\begin{cases} u \geq 0 & \text{on } B_1 \cap \{x_{n+1} = 0\} \\ \Delta u = 0 & \text{in } B_1 \setminus (\{x_{n+1} = 0\} \cap \{u = 0\}) \\ \Delta u \leq 0 & \text{in } B_1 \\ u = g & \text{on } \partial B_1, \end{cases} \tag{1.51}$$

Let $\Gamma_\lambda(u)$ denote the points of order $\lambda > 0$. Then,

$$\Gamma_\lambda(u) = \varnothing \qquad \text{for every} \quad \lambda \in \bigcup_{m\in\mathbb{N}} \big((2m - c_m, 2m + c_m) \setminus \{2m\} \big),$$

for some constants c_m depending only on m and n.

The goal of the rest of the subsection is to prove that, if the set $\Gamma_*(u)$ existed, then it would be lower dimensional. That is, we will show the following proposition, stating that points of order $\kappa \in (2, \infty)\setminus\{2m, 2m+1, 2m+\frac{3}{2}\}_{m\in\mathbb{N}}$ are $n-2$ dimensional for general solutions to the thin or fractional obstacle problem. We do that through a dimension reduction argument due to White, [Whi97].

Proposition 1.25. *Let u be a solution to the thin obstacle problem with zero obstacle,* (1.51)*. Let us define $\Gamma_*(u) \subset \Gamma(u)$ by* (1.50)*. Then*

$$\dim_{\mathcal{H}} \Gamma_*(u) \le n-2.$$

Moreover, if $n=2$, $\Gamma_*(u)$ is discrete.

In this proposition, $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension of a set.

In order to prove this result, we will need two lemmas. We will use the notation $u^{x_\circ}(x)$ for $x_\circ \in \Gamma(u)$ to denote translations. That is, we denote

$$u^{x_\circ}(x) = u(x' + x'_\circ, x_{n+1}),$$

so that, in particular, $N(r,u,x_\circ) = N(r,u^{x_\circ})$.

Lemma 1.26. *Let u be a solution to the thin obstacle problem* (1.51)*. Let $\Gamma_*(u)$ as in* (1.50)*.*

Let $y_\circ \in \Gamma_*(u)$. Then, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for every $\rho \in (0,\delta]$, there exists an $(n-2)$-dimensional linear subspace $L_{y_\circ,\rho}$ of $\mathbb{R}^n \times \{0\}$ such that

$$\left\{x \in B_\rho(y_\circ) \cap \{x_{n+1}=0\} : N(0^+,u^x) \ge N(0^+,u^{y_\circ}) - \delta\right\} \subset \{x : \mathrm{dist}(x, y_\circ + L_{y_\circ,\rho}) < \varepsilon\rho\}.$$

Proof. Let us denote $\eta = N(0^+,u^{y_\circ}) \in (2,\infty) \setminus \{2m, 2m+1, 2m+\frac{3}{2}\}_{m\in\mathbb{N}}$. Let us proceed by contradiction. Suppose that there exist $\varepsilon > 0$, and sequences $\delta_k \downarrow 0$ and $\rho_k \downarrow 0$ such that

$$\{x \in B_{\rho_k}(y_\circ) \cap \{x_{n+1}=0\} : N(0^+,u^x) \ge \eta - \delta_k\} \not\subset \{x : \mathrm{dist}(x, y_\circ + L) < \varepsilon\rho_k\} \quad (1.52)$$

for every $(n-2)$-dimensional linear subspace L of $\mathbb{R}^n \times \{0\}$.

In particular, if we denote $u_r^{y_\circ} = u^{y_\circ}(r\,\cdot)$ and $d_r = r^{-n/2}\|u^{x_\circ}\|_{L^2(\partial B_r)}$, then $u_{\rho_k}^{y_\circ}/d_{\rho_k}$ converges, up to subsequences, to some $v_\circ$ a global solution to the thin obstacle problem with zero obstacle, homogeneous of degree η. Let us denote $L(v_\circ)$ the invariant set in $\mathbb{R}^n \times \{0\}$ of $v_\circ$. In particular, it is a subspace of dimension at most $n-2$ (this follows since two dimensional homogeneous solutions to the thin obstacle problem have homogeneity belonging to $\{2m, 2m+1, 2m-\frac{1}{2}\}_{m\in\mathbb{N}}$). As an abuse of notation, let us take as $L(v_\circ)$ any $(n-2)$-dimensional plane containing the invariant set.

Now, by assumption (1.52) and choosing $L = L(v_\circ)$, for every $k \in \mathbb{N}$ there exists some $x_k \in B_{\rho_k}(y_\circ) \cap \{x_{n+1}=0\}$ with $N(0^+,u^{x_k}) \ge \eta - \delta_k$ such that $\mathrm{dist}(x_k, y_\circ + L(v_\circ)) \ge \varepsilon\rho_k$.

Let us denote $z_k = \rho_k^{-1}(x_k - y_\circ) \in B_1(0)$, and notice that $\mathrm{dist}(z_k, L(v_\circ)) \ge \varepsilon$. By scaling, we know that

$$N(0^+,u^{x_k}) = N(0^+,u_{\rho_k}^{y_\circ}(\,\cdot + z_k)).$$

Moreover,

$$d_{\rho_k}^{-1}u_{\rho_k}^{y_\circ} \to v_\circ \quad \text{uniformly in compact sets as } k \to \infty.$$

Thus,

$$\eta - \delta_k = N(0^+,u^{x_k}) = N(0^+,u_{\rho_k}^{y_\circ}(\,\cdot + z_k)) = N(0^+,d_{\rho_k}^{-1}u_{\rho_k}^{y_\circ}(\,\cdot + z_k)),$$

and by the upper semi-continuity of the frequency function (and after taking a subsequence such that $z_k \to z \in B_1(0)$) we get that

$$N(0^+, v_\circ(\,\cdot + z)) \geq \eta,$$

for some $z \in B_1(0)$ such that $\operatorname{dist}(z, L(v_\circ)) \geq \varepsilon$. Since $v_\circ$ is η-homogeneous, $N(0^+, v_\circ(\,\cdot + z)) \geq \eta$ implies that z belongs to the invariant set of $v_\circ$ (see, for instance, [FoSp18, Lemma 5.2]). This contradicts $\operatorname{dist}(z, L(v_\circ)) \geq \varepsilon$, and we are done. □

The following is a very general and standard lemma. We give the proof for completeness. We thank B. Krummel, from whom we learned this proof.

Lemma 1.27. *There exists $\beta : (0,\infty) \to (0,\infty)$ with $\beta(t) \to 0$ as $t \downarrow 0$, such that the following holds true.*

Let $\varepsilon > 0$. Let $A \subseteq \mathbb{R}^n$ such that for each $y \in A$ and $\rho \in (0, \rho_\circ)$ there exists a j-dimensional linear subspace $L_{y,\rho}$ of $\mathbb{R}^n$ for which

$$A \cap B_\rho(y) \subset \{x : \operatorname{dist}(x, y + L_{y,\rho}) < \varepsilon\rho\}.$$

(Note that we do not claim that $L_{y,\rho}$ is unique.) Then $\mathcal{H}^{j+\beta(\varepsilon)}(A) = 0$.

Proof. Let $\beta(t) = n + 1 - j$ for $t \geq 1/8$ and observe that $\mathcal{H}^{n+1}(A) = 0$. Thus it suffices to consider $\varepsilon \in (0, 1/8)$.

By a covering argument, after rescaling and translating, we may assume that $A \subseteq B_1(0)$ and $0 \in A$. By assumption, there exists a subspace $L_{0,1}$ such that

$$A \cap B_1(0) \subset \{x : \operatorname{dist}(x, y + L_{0,1}) < \varepsilon\}.$$

Cover $L_{0,1}$ by a finite collection of balls $\{B_{2\varepsilon}(z_k)\}_{k=1,2,\ldots,N}$ where $z_k \in L_{0,1}$ for each k and $N \leq C(j)\varepsilon^{-j}$. Observe that $\{B_{2\varepsilon}(z_k)\}_{k=1,2,\ldots,N}$ covers $\{x : \operatorname{dist}(x, y + L_{0,1}) < \varepsilon\}$ and thus covers $A \cap B_1(0)$. Throw away the balls $B_{2\varepsilon}(z_k)$ that do not intersect A. For the remaining balls, let $y_k \in A \cap B_{2\varepsilon}(z_k)$. Now $\{B_{4\varepsilon}(y_k)\}_{k=1,2,\ldots,N}$ covers $A \cap B_1(0)$, $y_k \in A$, $N \leq C(j)\varepsilon^{-j}$, and $N(4\varepsilon)^{j+\beta} \leq C(j)\varepsilon^\beta$. Choose $\beta = \beta(\varepsilon)$ so that $C(j)\varepsilon^\beta \leq 1/2$.

Now observe that we can repeat this argument with $B_{4\varepsilon}(y_k)$ in place of $B_1(0)$ to get a new covering $\{B_{(4\varepsilon)^2}(y_{k,l})\}_{l=1,2,\ldots,N_k}$ of $A \cap B_{4\varepsilon}(y_k)$ with $N_k(4\varepsilon)^{j+\beta} < 1/2$. Thus $\{B_{(4\varepsilon)^2}(y_{k,l})\}_{k=1,2,\ldots,N,\, l=1,2,\ldots,N_k}$ covers A with $y_{k,l} \in A$ and $\sum_{k=1}^N N_k(4\varepsilon)^{2\cdot(j+\beta)} < (1/2)^2$. Repeating this argument for a total of p times, we get a finite covering of A by M balls with centers on A, radii $= (4\varepsilon)^p$, and $M(4\varepsilon)^{p(j+\beta)} < (1/2)^p$. Thus $\mathcal{H}^{j+\beta}_{(4\varepsilon)^p}(A) \leq \omega_{j+\beta}(1/2)^p$ for every integer $p = 1, 2, 3, \ldots$. Letting $p \to \infty$, we get $\mathcal{H}^{j+\beta(\varepsilon)}(A) = 0$. □

Thus, we can directly prove Proposition 1.25.

Proof of Proposition 1.25. We want to show that $\Gamma_*(u)$ has Hausdorff dimension at most $n-2$. Let $\varepsilon > 0$ and define, for $i \in \mathbb{N}$, G_i to be the set of all points $x_\circ \in \Gamma_*(u)$ such that the conclusion of Lemma 1.26 holds true with $\delta = 1/i$, so that $\Gamma_*(u) = \bigcup_i G_i$. For each $q \in \mathbb{N}$, define

$$G_{i,q} = \{x_\circ \in G_i : (q-1)/i < N(0^+, u^{x_\circ}) \leq q/i\}.$$

Observe that $\Gamma_*(u) = \bigcup_{i,q} G_{i,q}$, and for every $x_\circ \in G_{i,q}$,

$$G_{i,q} \subset \{y : N(0^+, u^y) > N(0^+, u^{x_\circ}) - 1/i\}$$

so that, by Lemma 1.26, for every $\rho \in (0, 1/i]$ there exists a $(n-2)$-dimensional linear subspace $L_{x_\circ,\rho}$ of $\mathbb{R}^n \times \{0\}$ such that

$$G_{i,q} \cap B_\rho(x_\circ) \subset \{x : \operatorname{dist}(x, x_\circ + L_{x_\circ,\rho}) < \varepsilon\rho\}.$$

Now, thanks to Lemma 1.27 with $A = G_{i,q}$ (taking $\rho_\circ = 1/i$ uniform on $S_{i,q}$), $\mathcal{H}^{n-2+\beta(\varepsilon)}(G_{i,q}) = 0$. Hence $\mathcal{H}^{n-2+\beta(\varepsilon)}(\Gamma_*(u)) = 0$. Since ε is arbitrary, for all $\beta > 0$ we have $\mathcal{H}^{n-2+\beta}(\Gamma_*(u)) = 0$, and thus $\Gamma_*(u)$ has Hausdorff dimension at most $n-2$.

The fact that for $n = 2$, $\Gamma_*(u)$ is discrete, follows by similar arguments in a standard way. □

1.9 C^∞ obstacles

Let us suppose now that the obstacle $\varphi \in C^\infty(B_1')$, and therefore, we cannot reduce the the zero obstacle situation. Our problem is then

$$\begin{cases} u \ \geq \ \varphi & \text{on } B_1 \cap \{x_{n+1} = 0\} \\ \Delta u \ = \ 0 & \text{in } B_1 \setminus (\{x_{n+1} = 0\} \cap \{u = \varphi\}) \\ \Delta u \ \leq \ 0 & \text{in } B_1, \end{cases} \tag{1.53}$$

where, as before, we are assuming that our solution is even in the x_{n+1}-variable.

Let us assume that 0 is a free boundary point, $0 \in \partial_{\mathbb{R}^n}\{u = \varphi\}$. Given $\tau \in \mathbb{N}_{\geq 2}$, let us consider the τ-order expansion of $\varphi(x')$ at 0, given by $Q_\tau(x')$. In particular, $(\varphi - Q_\tau)(x') = O(|x'|^{\tau+1})$. Let $Q_\tau^h(x', x_{n+1})$ be the unique even harmonic extension of Q_τ to B_1. Let us now define

$$\bar{u}(x', x_{n+1}) := u(x', x_{n+1}) - \varphi(x') + Q_\tau(x') - Q_\tau^h(x', x_{n+1}).$$

Then, $\bar{u}$ solves the zero thin obstacle problem with a right-hand side,

$$\begin{cases} \bar{u} \ \geq \ 0 & \text{on } B_1 \cap \{x_{n+1} = 0\} \\ \Delta \bar{u} \ = \ f & \text{in } B_1 \setminus (\{x_{n+1} = 0\} \cap \{u = \varphi\}) \\ -\Delta \bar{u} \ \geq \ f & \text{in } B_1, \end{cases}$$

where

$$f(x) = \Delta_{x'}(Q_\tau(x') - \varphi(x')) = O(|x'|^{\tau-1}).$$

Since $|f| \leq M|x'|^{\tau-1}$ and $\|\nabla u\|_{L^\infty(B_{1/2})} \leq M$ for some constant $M > 0$, we can consider the generalized frequency formula,

$$\Phi_\tau(r, \bar{u}) := (r + C_M r^2)\frac{d}{dr}\log\max\left\{H(r), r^{n+2\tau}\right\}, \quad \text{where} \quad H(r) := \int_{\partial B_r} \bar{u}^2,$$

(cf. (1.22)) and the constant C_M depends only on the dimension and M. Then, there exists some $r_M > 0$ such that $\Phi_\tau(r, \bar{u})$ is non-decreasing for $0 < r < r_M$. In particular, $\Phi_\tau(0^+, \bar{u})$ is well-defined and

$$n + 3 \leq \Phi_\tau(0^+, \bar{u}) \leq n + 2\tau$$

(see [CSS08, GP09]). We say that the origin is a free boundary point of order $\kappa < \tau$ if $\Phi_\tau(0^+, \bar{u}) = n + 2\kappa$ (in particular, as before, $\kappa \geq \frac{3}{2}$). If $\kappa = \tau$, we say that the origin is a free boundary point of order *at least* τ. At this point, all the theory developed above for regular free boundary points and singular points, also applies to the situation where there are non-analytic (i.e., non-zero) obstacles, by using the new generalized frequency formula. Notice that this theory can be developed even if the obstacle φ has lower regularity than C^∞.

Finally, we say that the origin is a free boundary point of infinite order if it is of order *at least* τ for all $\tau \in \mathbb{N}_{\geq 2}$. Notice that this set of free boundary points has not appeared until now, it did not exist in the zero obstacle case.

Intuitively, in the thin obstacle problem (1.53) a point is of order κ when the solution u detaches from the obstacle at order κ on the thin space.

Thus, the free boundary for solutions to the thin obstacle problem with $\varphi \in C^\infty(B_1')$, (1.53), can be split as

$$\Gamma(u) = \Gamma_{3/2}(u) \cup \Gamma_{\text{even}}(u) \cup \Gamma_{\text{odd}}(u) \cup \Gamma_{\text{half}}(u) \cup \Gamma_*(u) \cup \Gamma_\infty(u),$$

(cf. (1.45)), where the new set $\Gamma_\infty(u)$ denotes the set of free boundary points with infinite order.

The set of points in $\Gamma_\infty(u)$ can be very wild. In fact, the following holds.

Proposition 1.28 ([FR19]). *Let $\mathcal{C} \subset B'_{1/2} \subset \mathbb{R}^n$ be any closed set. Then, there exists an an obstacle $\varphi \in C^\infty(B_1')$ and non-trivial solution u to* (1.53) *such that $\Lambda(u) \cap B_{1/2} = \{u = \varphi\} \cap B_{1/2} = \mathcal{C}$.*

Proof. Take any obstacle $\psi \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset\subset B_{1/8}(\frac{3}{4}\boldsymbol{e}_1)$, with $\psi > 0$ somewhere, and take the non-trivial solution to (1.53) with obstacle ψ.

Notice that $u > \psi$ in $B'_{1/2}$ (in particular, $u \in C^\infty(B_{1/2})$). Let $f_\mathcal{C} : B_1' \to \mathbb{R}$ be any C^∞ function such that $0 \leq f_\mathcal{C} \leq 1$ and $\mathcal{C} = \{f_\mathcal{C} = 0\}$.

Now let $\eta \in C_c^\infty(B'_{5/8})$ such that $\eta \geq 0$ and $\eta \equiv 1$ in $B'_{1/2}$. Consider, as new obstacle, $\varphi = \psi + \eta(u - \psi)(1 - f_\mathcal{C}) \in C^\infty(B'_{1/2})$. Notice that $u - \varphi \geq 0$. Notice, also, that for $x' \in B_{1/2}$, $(u - \varphi)(x') = 0$ if and only if $x' \in \mathcal{C}$. Thus, u with obstacle φ gives the desired result. ☐

That is, the contact set can, a priori, be any closed set. In particular, the free boundary can have arbitrary Hausdorff dimension ($n - \varepsilon$ for any $\varepsilon > 0$). It is worth mentioning that the points constructed like this are not really acting as an obstacle (the Laplacian around them vanishes).

1.10 Generic regularity

We have seen that, in general, the non-regular (or degenerate) part of the free boundary can be of the same size (or even larger, in the case of C^∞ obstacles) than the regular part. This is not completely satisfactory, since we only know how to prove smoothness of the free boundary around regular points.

It is for this reason that generic regularity results are interesting: even if there exist solutions where degenerate points are larger than regular points, this is not true for a generic solution. That is, for *almost every* solution, the free boundary is smooth up to a lower dimensional set. Let us start by defining what we mean by "almost every" solution.

Let $\varphi \in C^\infty(B_1')$ and let $g \in C^0(\partial B_1)$ even with respect to x_{n+1}. Let $\lambda \in [0,1]$, and let u_λ be the solution to

$$\begin{cases} u_\lambda \geq \varphi & \text{on } B_1 \cap \{x_{n+1} = 0\} \\ \Delta u_\lambda = 0 & \text{in } B_1 \setminus (\{x_{n+1} = 0\} \cap \{u = \varphi\}) \\ \Delta u_\lambda \leq 0 & \text{in } B_1 \\ u_\lambda = g + \lambda & \text{on } \partial B_1. \end{cases} \tag{1.54}$$

That is, we consider the set of solutions $\{u_\lambda\}_{\lambda\in[0,1]}$ with a fixed obstacle φ by raising the boundary datum by λ. Alternatively, we could raise (or lower) the obstacle, or just make small perturbations (monotone) of the boundary value. We say that a property holds for *almost every* solution if it holds for a.e. $\lambda \in [0,1]$ for any such construction of solutions.

Now notice that since points of order κ are detaching from the obstacle with power κ, when raising the boundary datum, the larger κ is, the faster the free boundary is disappearing (and thus, the less common is that type of point). As a consequence, establishing a quantitative characterization of this fact together with a GMT lemma (coming from [FRS19]), one can show the following proposition. We recall that given a solution v to a thin obstacle problem, (1.53), we denote by $\Gamma_{\geq\kappa}(v)$ the set of free boundary points of order greater or equal than κ.

Proposition 1.29 ([FR19]). *Let $\varphi \in C^\infty(B_1')$ and let $g \in C^0(\partial B_1)$ even with respect to x_{n+1}. Let $\{u_\lambda\}_{\lambda\in[0,1]}$ the family of solutions to the thin obstacle problem* (1.54). *Then,*

- *If $3 \leq \kappa \leq n+1$, the set $\Gamma_{\geq\kappa}(u_\lambda)$ has Hausdorff dimension at most $n-\kappa+1$ for almost every $\lambda \in [0,1]$.*
- *If $\kappa > n+1$, the set $\Gamma_{\geq\kappa}(u_\lambda)$ is empty for all $\lambda \in [0,1] \setminus \mathcal{E}_\kappa$, where $\mathcal{E}_\kappa$ has Hausdorff dimension at most $\frac{n}{\kappa-1}$.*
- *The set $\Gamma_\infty(u_\lambda)$ is empty for all $\lambda \in [0,1] \setminus \mathcal{E}$, where $\mathcal{E}$ has Minkowski dimension equal to 0.*

On the other hand, by means of a Monneau-type monotonicity formula one can also show that the set $\bigcup_{\lambda\in[0,1]} \Gamma_2(u_\lambda)$ (union of singular points of order 2 for all $\lambda \in [0,1]$) is contained in a countable union of $(n-1)$-dimensional C^1 manifolds. As a consequence,

Proposition 1.30 ([FR19]). *Let $\varphi \in C^\infty(B_1')$ and let $g \in C^0(\partial B_1)$ even with respect to x_{n+1}. Let $\{u_\lambda\}_{\lambda\in[0,1]}$ the family of solutions to the thin obstacle problem* (1.54). *Then $\Gamma_2(u_\lambda)$ has dimension at most $n-3$ for a.e. $\lambda \in [0,1]$.*

And finally, combining Proposition 1.29, Proposition 1.30, and Proposition 1.25, we get the generic regularity theorem we wanted:

Theorem 1.31 ([FR19]). *Let $\varphi \in C^\infty(B_1')$ and let $g \in C^0(\partial B_1)$ even with respect to x_{n+1}. Let $\{u_\lambda\}_{\lambda\in[0,1]}$ the family of solutions to the thin obstacle problem* (1.54). *Then, the set* $\mathrm{Deg}(u_\lambda)$ *has Hausdorff dimension at most $n-2$ for a.e. $\lambda \in [0,1]$.*

In particular, the free boundary is smooth up to a lower dimensional set, for almost every solution.

The previous theorem also holds true for obstacles with lower regularity. Namely, in the proof of the result, only $C^{3,1}$ regularity of the obstacle is really used.

1.11 Summary

Let us finish with a summary of the known results for the solutions to the thin obstacle problem.

Let $\varphi \in C^\infty(B_1')$ and consider an even solution to the thin obstacle problem, with obstacle φ,

$$\begin{cases} u & \geq & \varphi & \text{on } B_1 \cap \{x_{n+1} = 0\} \\ \Delta u & = & 0 & \text{in } B_1 \setminus (\{x_{n+1} = 0\} \cap \{u = \varphi\}) \\ \Delta u & \leq & 0 & \text{in } B_1. \end{cases} \tag{1.55}$$

Then, the solution u is $C^{1,1/2}$ on either side of the obstacle. That is, there exists a constant C depending only on n such that

$$\|u\|_{C^{1,1/2}(B^+_{1/2})} + \|u\|_{C^{1,1/2}(B^-_{1/2})} \leq C\left(\|\varphi\|_{C^{1,1}(B_1')} + \|u\|_{L^\infty(B_1)}\right).$$

Moreover, if we denote $\Lambda(u) := \{u = \varphi\}$ the contact set, the boundary of $\Lambda(u)$ in the relative topology of $\mathbb{R}^n$, $\partial_{\mathbb{R}^n}\Lambda(u)$, is the free boundary, and can be divided into two sets

$$\Gamma(u) = \mathrm{Reg}(u) \cup \mathrm{Deg}(u),$$

the set of *regular points*,

$$\mathrm{Reg}(u) := \left\{ x = (x', 0) \in \Gamma(u) : 0 < cr^{3/2} \leq \sup_{B_r'(x')} (u - \varphi) \leq Cr^{3/2}, \quad \forall r \in (0, r_\circ) \right\},$$

and the set of non-regular points or *degenerate points*

$$\mathrm{Deg}(u) := \left\{ x = (x', 0) \in \Gamma(u) : 0 \leq \sup_{B_r'(x')} (u - \varphi) \leq Cr^2, \quad \forall r \in (0, r_\circ) \right\},$$

Alternatively, each of the subsets can be defined according to the order of the blow-up (the frequency) at that point. Namely, the set of regular points are those whose blow-up is of order $\frac{3}{2}$, and the set of degenerate points are those whose blow-up is of order κ for some $\kappa \in [2, \infty]$.

The free boundary can be further stratified as

$$\Gamma(u) = \Gamma_{3/2} \cup \Gamma_{\mathrm{even}} \cup \Gamma_{\mathrm{odd}} \cup \Gamma_{\mathrm{half}} \cup \Gamma_* \cup \Gamma_\infty, \tag{1.56}$$

where:

- $\Gamma_{3/2} = \mathrm{Reg}(u)$ is the set of regular points. They are an open $(n-1)$-dimensional subset of $\Gamma(u)$, and it is C^∞ (see [ACS08, KPS15, DS16]).

- $\Gamma_{\mathrm{even}} = \bigcup_{m\geq 1} \Gamma_{2m}(u)$ denotes the set of points whose blow-ups have even homogeneity. Equivalently, they can also be characterised as those points of the free boundary where the contact set has zero density, and they are often called singular points. They are contained in the countable union of C^1 $(n-1)$-dimensional manifolds; see [GP09]. Generically, however, points in $\Gamma_2(u)$ have dimension at most $n-3$, and points in $\Gamma_{2m}(u)$ have dimension at most $n-2m$ for $m \geq 2$; see [FR19].

- $\Gamma_{\mathrm{odd}} = \bigcup_{m\geq 1} \Gamma_{2m+1}(u)$ is, a priori, at most $(n-1)$-dimensional and it is $(n-1)$-rectifiable (see [FoSp18, KW13, FoSp19]), although it is not known whether it exists. Generically, $\Gamma_{2m+1}(u)$ has dimension at most $n-2m$; see [FR19].

- $\Gamma_{\mathrm{half}} = \bigcup_{m\geq 1} \Gamma_{2m+3/2}(u)$ corresponds to those points with blow-ups of order $\frac{7}{2}$, $\frac{11}{2}$, etc. They are much less understood than regular points, although in some situations they have a similar behaviour. The set Γ_{half} is an $(n-1)$-dimensional subset of the free boundary and it is a $(n-1)$-rectifiable set (see [FoSp18, KW13, FoSp19]). Generically, the set $\Gamma_{2m+3/2}(u)$ has dimension at most $n-2m-1/2$.

- Γ_* is the set of all points with homogeneities $\kappa \in (2,\infty)$, with $\kappa \notin \mathbb{N}$ and $\kappa \notin 2\mathbb{N} - \frac{1}{2}$. This set has Hausdorff dimension at most $n-2$, so it is always *small*, see [FoSp18, KW13, FoSp19].

- Γ_∞ is the set of points with infinite order (namely, those points at which $u - \varphi$ vanishes at infinite order). For general C^∞ obstacles it could be a huge set, even a fractal set of infinite perimeter with dimension exceeding $n-1$. When φ is analytic, instead, Γ_∞ is empty. Generically, this set is empty; see [FR19].

Chapter 2

$C^{1,\alpha}$ estimates for the fully nonlinear Signorini problem

We study the regularity of solutions to the fully nonlinear thin obstacle problem. We establish local $C^{1,\alpha}$ estimates on each side of the smooth obstacle, for some small $\alpha > 0$.

Our results extend those of Milakis-Silvestre [MS08] in two ways: first, we do not assume solutions nor operators to be symmetric, and second, our estimates are local, in the sense that do not rely on the boundary data.

As a consequence, we prove $C^{1,\alpha}$ regularity even when the problem is posed in general Lipschitz domains.

2.1 Introduction

The aim of this work is to study the regularity of the solutions to the Signorini or thin obstacle problem for fully nonlinear operators.

Given a domain $D \subset \mathbb{R}^n$, the thin obstacle problem involves a function $u : D \to \mathbb{R}$, an obstacle $\varphi : S \to \mathbb{R}$ defined on a $(n-1)$-dimensional manifold S, a Dirichlet boundary condition given by $g : \partial D \to \mathbb{R}$, and a second order elliptic operator L,

$$\begin{cases} Lu &= 0 & \text{in } D \setminus \{x \in S : u(x) = \varphi(x)\} \\ Lu &\leq 0 & \text{in } D \\ u &\geq \varphi & \text{on } S \\ u &= g & \text{on } \partial D. \end{cases} \tag{2.1}$$

Intuitively, one can think of it as finding the shape of a membrane with prescribed boundary conditions considering that there is a very thin obstacle forcing the membrane to be above it.

When L is the Laplacian, the $C^{1,\alpha}$ regularity of solutions was first proved in 1979 by Caffarelli in [Caf79]. Later, the optimal value of α was found by Athanasopoulos and Caffarelli in [AC04], where solutions were proved to be in $C^{1,\frac{1}{2}}$ on either side of the obstacle. More recently, this has been extended to linear operators with x dependence $L = \sum a_{ij}(x)\partial_{ij}u$ in [Gui09, GS14, KRS16].

Here, we study a nonlinear version of problem (2.1). More precisely, we study (2.1) with $Lu = F(D^2u)$, a convex fully nonlinear uniformly elliptic operator. Since all of our estimates are of local character, we consider the problem in B_1,

$$\begin{cases} F(D^2u) &= 0 \quad \text{in } B_1 \setminus \{u = \varphi\} \\ F(D^2u) &\leq 0 \quad \text{in } B_1 \\ u &\geq \varphi \quad \text{on } B_1 \cap \{x_n = 0\}. \end{cases} \tag{2.2}$$

Here, $\varphi : B_1 \cap \{x_n = 0\} \to \mathbb{R}$ is the obstacle, and we assume that it is $C^{1,1}$. We study the regularity of solutions on either side of the obstacle.

We assume that

$$\begin{aligned} &F \text{ is convex, uniformly elliptic} \\ &\text{with ellipticity constants } 0 < \lambda \leq \Lambda, \text{ and with } F(0) = 0. \end{aligned} \tag{2.3}$$

When u is symmetric, this problem was studied by Milakis and Silvestre in [MS08], and is equivalent to

$$\begin{cases} F(D^2u) &= 0 \quad \text{in } B_1^+ \\ \max\{u_{x_n}, \varphi - u\} &= 0 \quad \text{on } B_1 \cap \{x_n = 0\}. \end{cases} \tag{2.4}$$

Moreover, they also implicitly assume a symmetry condition on the operator F, in particular, that $F(A) = F(\tilde{A})$, where $\tilde{A}_{in} = \tilde{A}_{ni} = -A_{in} = -A_{ni}$ for $i < n$ and $\tilde{A}_{ij} = A_{ij}$ otherwise. Under this assumption, they proved interior $C^{1,\alpha}$ regularity up to the obstacle on either side by also assuming that $u \geq \varphi + \varepsilon$ on $\partial B_1 \cap \{x_n = 0\}$, for some $\varepsilon > 0$. Equivalently, they assume that the coincidence set is contained in some ball $B_{1-\delta}$ for some $\delta > 0$. This assumption is important in [MS08] to prove semiconvexity of solutions.

Our main result, Theorem 2.1 below, extends the result of [MS08] in two ways. First, we do not assume anything on the boundary data, so that we give a local estimate. Second, we consider also non-symmetric solutions u to (2.2) with operators not necessarily satisfying any symmetry assumption, and prove $C^{1,\alpha}$ regularity for such solutions.

In the linear case, one can symmetrise solutions to (2.2), and then the study of such solutions reduces to problem (2.4). However, in the present nonlinear setting an estimate for (2.4) does not imply one for (2.2).

Our main result is the following, stating that any solution to (2.2) is $C^{1,\alpha}$ on either side of the obstacle, for some small $\alpha > 0$.

Theorem 2.1. *Let F be a nonlinear operator satisfying* (2.3) *and let u be any viscosity solution to* (2.2) *with $\varphi \in C^{1,1}$. Then, $u \in C^{1,\alpha}(\overline{B_{1/2}^+}) \cap C^{1,\alpha}(\overline{B_{1/2}^-})$ and,*

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} + \|u\|_{C^{1,\alpha}(\overline{B_{1/2}^-})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1 \cap \{x_n = 0\})}\right)$$

for some constants $\alpha > 0$ and C depending only on n, λ, and Λ.

Our proof of the semiconvexity of solutions is completely different from the one done in [MS08] and follows by means of a Bernstein's technique. On the other hand, to prove the $C^{1,\alpha}$ regularity in the non symmetric case we follow [Caf79, MS08], but new ideas are needed. We define a symmetrised solution to the problem and follow the steps in [Caf79] and [MS08] using appropriate inequalities satisfied by the symmetrised solution. This yields the regularity of the symmetrised normal derivative at free boundary points. Then, we show that this implies the $C^{1,\alpha}$ regularity of the original function u at free boundary points, by using the ideas from [Caf89]. Finally, we show that the regularity of u at free boundary points yields the regularity of the symmetrized normal derivative at all points on $x_n = 0$, and that this yields the regularity of u on either side of the obstacle.

As an immediate corollary it follows an estimate when the thin obstacle problem is posed in a bounded Lipschitz domain $D \subset \mathbb{R}^n$.

Corollary 2.2. *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let $K \Subset D$. Let F be a nonlinear operator satisfying* (2.3). *Let $\varphi : D \cap \{x_n = 0\} \to \mathbb{R}$ be a $C^{1,1}$ function, and let u be the solution to*

$$\begin{cases} F(D^2u) & = & 0 & \text{in } D \setminus \{u = \varphi\} \\ F(D^2u) & \leq & 0 & \text{in } D \\ u & \geq & \varphi & \text{on } D \cap \{x_n = 0\} \\ u & = & g & \text{on } \partial D, \end{cases} \tag{2.5}$$

for some $g \in C^0(\partial D)$. Let $K^+ := K \cap \{x_n > 0\}$ and $K^- := K \cap \{x_n < 0\}$. Then, $u \in C^{1,\alpha}(\overline{K^+}) \cap C^{1,\alpha}(\overline{K^-})$, with

$$\|u\|_{C^{1,\alpha}(\overline{K^+})} + \|u\|_{C^{1,\alpha}(\overline{K^-})} \leq C \left(\|g\|_{L^\infty(\partial D)} + \|\varphi\|_{C^{1,1}(D\cap\{x_n=0\})}\right)$$

for some constant $\alpha > 0$ depending only on n, λ, and Λ, and C depending only on n, λ, Λ, D, and K.

Let us introduce the notation that will be used throughout the work. We denote $x = (x', x_n) \in \mathbb{R}^n$ and

$$B_1^* := \{x' \in \mathbb{R}^{n-1} : (x', 0) \in B_1\}.$$

The obstacle φ is defined on B_1^* seen as a subset of $\mathbb{R}^n$, and problem (2.2) is written as

$$\begin{cases} F(D^2u) & = & 0 & \text{in } B_1 \setminus \{(x', 0) : u(x', 0) = \varphi(x')\} \\ F(D^2u) & \leq & 0 & \text{in } B_1 \\ u(x', 0) & \geq & \varphi(x') & \text{for } x' \in B_1^*. \end{cases}$$

We also denote

$$B_1^+ := \{(x', x_n) \in B_1 : x_n > 0\}, \qquad (\partial B_1)^+ = \partial B_1 \cap \{x_n > 0\},$$

and analogously we define B_1^- and $(\partial B_1)^-$. On the other hand, we call the coincidence set

$$\Delta^* = \{x \in B_1^* : u(x', 0) = \varphi(x')\}, \qquad \Delta = \Delta^* \times \{0\},$$

and its complement in B_1^* is denoted by

$$\Omega^* = B_1^* \setminus \Delta^*, \qquad \Omega = \Omega^* \times \{0\}.$$

Our work is organised as follows. In Section 2.2 we give a Lipschitz bound and prove semiconvexity of solutions. Then, in Section 2.3 we prove Theorem 2.1.

2.2 Lipschitz estimate and semiconvexity

2.2.1 Lipschitz estimate

We begin with a proposition showing that any solution to (2.2) is Lipschitz, as long as the obstacle is $C^{1,1}$.

Proposition 2.3. *Let u be any solution to* (2.2) *with F satisfying* (2.3) *and $\varphi \in C^{1,1}$. Then u is Lipschitz in $B_{1/2}$ with,*

$$\|u\|_{\mathrm{Lip}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}\right), \tag{2.6}$$

for some C depending only on n and the ellipticity constants λ and Λ.

Proof. We will extend the obstacle φ to a function h defined in the whole B_1, and we treat u as a solution to a classical "thick" obstacle problem. We define h separately in B_1^+ and B_1^-, as the solution to

$$\begin{cases} F(D^2h) &= 0 & \text{in } B_1^+ \\ h &= -\|u\|_{L^\infty(B_1)} & \text{in } (\partial B_1)^+ \\ h(x',0) &= \varphi(x') & \text{for } x' \in B_1^*, \end{cases} \tag{2.7}$$

and analogously

$$\begin{cases} F(D^2h) &= 0 & \text{in } B_1^- \\ h &= -\|u\|_{L^\infty(B_1)} & \text{in } (\partial B_1)^- \\ h(x',0) &= \varphi(x') & \text{for } x' \in B_1^*. \end{cases} \tag{2.8}$$

Notice that h is Lipschitz in $B_{7/8}$; see [MS06, Proposition 2.2]. By denoting

$$K_0 := \|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)},$$

we have

$$\|h\|_{\mathrm{Lip}(B_{7/8})} \leq CK_0,$$

and by the maximum principle $u \geq h$. Moreover, u is a solution to a classical obstacle problem in B_1 with h as the obstacle. We show next that this implies u is Lipschitz, with a quantitative estimate.

To begin with, since h is Lipschitz, fixed any $x_0 \in B_{1/2}$ and $0 < r < 1/4$, there exists some C_0 depending only on n, λ, and Λ such that

$$\sup_{B_r(x_0)} |h(x) - h(x_0)| \leq C_0K_0r. \tag{2.9}$$

Notice that, by the strong maximum principle, the coincidence set $\{u = h\}$ is Δ, the coincidence set of the thin obstacle problem. Suppose then that $x_0 \in \Delta$, i.e., $u(x_0) = h(x_0)$. Since $u \geq h$, in particular we have that

$$\inf_{B_r(x_0)} (u(x) - u(x_0)) \geq -C_0K_0r. \tag{2.10}$$

because h is Lipschitz. Now let

$$q(x) = u(x) - u(x_0) + C_0K_0r.$$

We already know $q \geq 0$ in $B_r(x_0)$. On the other hand, from (2.9),

$$q(x) \leq 2C_0K_0r \quad \text{on} \quad B_r(x_0) \cap \Delta.$$

Moreover, q is a supersolution,

$$F(D^2q) = F(D^2u) \leq 0 \quad \text{in} \quad B_r(x_0).$$

Let $\bar{q}$ be the viscosity solution to $F(D^2\bar{q}) = 0$ in $B_r(x_0)$ with $\bar{q} = q$ on $\partial B_r(x_0)$. We have $\bar{q} \leq q$ in $B_r(x_0)$ and by the non-negativity of $\bar{q}$ on the boundary, $\bar{q} \geq 0$ in $B_r(x_0)$.

Thus, $q < \bar{q} + 2C_0K_0r$ on $\partial B_r(x_0)$, and $q \leq \bar{q} + 2C_0K_0r$ in $B_r(x_0) \cap \Delta$. Therefore,

$$q \leq \bar{q} + 2C_0K_0r \quad \text{in} \quad B_r(x_0).$$

On the other hand, we know $0 \leq \bar{q}(x_0) \leq q(x_0) = C_0K_0r$, and by the Harnack inequality, $\bar{q} \leq CC_0K_0r$ in $B_{r/2}(x_0)$. Putting all together we obtain that $u(x) - u(x_0) \leq CC_0K_0r$ for some constant $C > 0$. Thus, combining this with (2.10),

$$\sup_{B_r(x_0)} |u(x) - u(x_0)| \leq C \left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)} \right) r, \tag{2.11}$$

for some constant C depending only on n, λ, and Λ.

We have obtained that the solution is Lipschitz on points of the coincidence set. Let us use interior estimates to deduce Lipschitz regularity inside $B_{1/2}$.

Take any points $x, y \in B_{1/2}$, and let $r = |x - y|$. Define

$$\rho := \min\{\text{dist}(x, \Delta), \text{dist}(y, \Delta)\},$$

and let $x^*, y^* \in \Delta$, $x^* = (x', 0)$, $y^* = (y', 0)$ for $x', y' \in \Delta^*$, be such that $\text{dist}(x, \Delta) = |x - x^*|$ and $\text{dist}(y, \Delta) = |y - y^*|$. We now separate two cases:

• If $\rho \leq 4r$, then

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x^*)| + |u(y) - u(y^*)| + |\varphi(x') - \varphi(y')| \\ &\leq C\rho + C(r + \rho) + 2C(r + \rho) \leq Cr \end{aligned}$$

for some constant C. We are using here that φ is Lipschitz and that if $|x - x^*| = \rho$, then $|y - y^*| \leq r + \rho$ and $|x^* - y^*| \leq 2(r + \rho)$.

• If $\rho > 4r$, we can use interior estimates. Suppose x is such that $\text{dist}(x, \Delta) = \rho$, and notice $B_{\rho/2}(x) \subset B_1 \setminus \Delta$, so that in $B_{\rho/2}(x)$, $F(D^2u) = 0$. We can now use the interior Lipschitz estimates (see, for example, [CC95, Chapter 5]),

$$[u]_{\text{Lip}(B_{\rho/4})} \leq \frac{C}{\rho} \text{osc}_{B_{\rho/2}(x)} u \leq C$$

for some constant C. We are using here that the supremum and the infimum of u in $B_{\rho/2}(x)$ are controlled respectively by $C\rho + \varphi(x^*)$ and $-C\rho + \varphi(x^*)$.

Thus, we have proved that the solution is Lipschitz in $B_{1/2}$, with the estimate (2.6). □

2.2.2 Preliminary consideration

Before continuing to prove the semiconvexity and semiconcavity result, we introduce a change of variables that will be useful in this section and the next one. Notice that, given a function w, we can express the nonlinear operator F as

$$F(D^2w(x)) = \sup_{\gamma\in\Gamma}\left(L_\gamma^{ij}\partial_{x_ix_j}w(x) + c_\gamma\right),$$

for some family of symmetric uniformly elliptic operators with ellipticity constants λ and Λ, $L_\gamma^{ij}\partial_{x_ix_j}$, indexed by $\gamma \in \Gamma$. Since $F(0) = 0$, there is some symmetric uniformly elliptic operator from this family given by a matrix $\hat{L}$ such that

$$\mathrm{tr}(\hat{L}D^2w(x)) = \hat{L}^{ij}\partial_{x_ix_j}w(x) \le F(D^2w(x)).$$

We now change coordinates in such a way that the matrix of this operator in the new coordinates, denoted $\hat{L}_A$, fulfils $\hat{L}_A^{in} = \hat{L}_A^{ni} = 0$ for $i < n$. More precisely, if we denote $\hat{L}'$ the matrix in Sym_{n-1} given by the $n-1$ first indices of $\hat{L}$, and we denote $\hat{L}_n' = (\hat{L}^{in})_{1\le i\le n-1}$ the vector of $\mathbb{R}^{n-1}$, we change variables as

$$x \mapsto y = Ax,$$

where A is the matrix given by

$$A := \left(\begin{array}{c|c} \mathrm{Id}_{n-1} & -\bar{a} \\ \hline 0\dots0 & 1 \end{array}\right),$$

and $\bar{a} = (\hat{L}')^{-1}\cdot\hat{L}_n'$ is a vector in $\mathbb{R}^{n-1}$. We define the new nonlinear operator $\tilde{F}$ as

$$\tilde{F}(N) = F(A^TNA), \text{ for all } N \in \mathrm{Sym}_n,$$

so that it is consistent with the change of variables, in the sense that if $\tilde{w}(y) = w(A^{-1}y)$, then $F(D^2w(x)) = \tilde{F}(D^2\tilde{w}(y))$.

We trivially have that $\tilde{F}$ is convex and $\tilde{F}(0) = 0$. In the new coordinates we still have that $\hat{L}_A^{ij}\partial_{y_iy_j}$ is a symmetric uniformly elliptic operator, but now the ellipticity constants λ and Λ have changed depending only on n, λ, and Λ. The same occurs with all the operators in the family defining F, so that after changing coordinates, F is still a convex uniformly elliptic operator with ellipticity constants depending only on n, λ, and Λ. Indeed, for any matrices $N, N_P \in \mathrm{Sym}_{n-1}$ with $N_P \ge 0$ we have that (using the definition of uniform ellipticity in [CC95, Chapter 2] and noticing that $A^TN_PA \ge 0$),

$$\|A^{-1}\|^{-2}\|N_P\| \le \lambda\|A^TN_PA\| \le \tilde{F}(N+N_P) - \tilde{F}(N) \le \Lambda\|A^TN_PA\| \le \Lambda\|A\|^2\|N_P\|,$$

and it is easy to bound $\|A^{-1}\|$ and $\|A\|$ from the definition of A, depending only on n, λ and Λ.

After changing variables, the regularity of the solution remains the same up to multiplicative constants in the bounds depending only on n, λ, and Λ.

As an abuse of notation we will call the new variables (x', x_n), the new operator F, and the new ellipticity constants λ and Λ, understanding that they might depend on the original ellipticity constants and the dimension, n. This will not be a problem, since in all the statements of the present work n, λ, and Λ appear together in the dependence of the constants.

Thus, throughout the paper we will assume that there exists a fixed symmetric uniformly elliptic operator $\hat{L}$ such that

$$\hat{L}^{ij}\partial_{x_ix_j}w \leq F(D^2w), \quad \text{and} \quad \hat{L}^{in} = \hat{L}^{ni} = 0 \quad \text{for} \quad i < n. \tag{2.12}$$

This change of variables is useful because, for any function w,

$$\hat{L}^{ij}\partial_{x_ix_j}(w(x', -x_n)) = \hat{L}^{ij}(\partial_{x_ix_j}w)(x', -x_n),$$

which will allow us to symmetrise the solution and still have a supersolution for the Pucci extremal operator $\mathcal{M}^-$. We also use it to prove a semiconcavity result from semiconvexity in the following proof of Proposition 2.4.

2.2.3 Semiconvexity and semiconcavity estimates

We next prove the semiconvexity of solutions in the directions parallel to the domain of the obstacle. To do it, we use a Bernstein's technique in the spirit of [AC04].

Proposition 2.4. *Let u be the solution to* (2.2)*. Then*

(a) (Semiconvexity) If $\tau = (\tau^*, 0)$, with τ^* a unit vector in $\mathbb{R}^{n-1}$,

$$\inf_{B_{3/4}} u_{\tau\tau} \geq -C\left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}\right),$$

for some constant C depending only on n, λ, and Λ.

(b) (Semiconcavity) Similarly, in the direction normal to $B_1^* \times \{0\}$,

$$\sup_{B_{3/4}} u_{x_nx_n} \leq C\left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}\right),$$

for some constant C depending only on n, λ, and Λ.

Proof. The second part, (b), follows from (a) using the definition of uniformly elliptic operator and the fact that we changed variables (in the previous subsection) in order to have matrix $\hat{L}$ fulfilling (2.12). We denote by $\hat{L}'$ and $D^2_{n-1}u$ the square matrices corresponding to the $n-1$ first indices of $\hat{L}$ and D^2u respectively. Now, from

$$\hat{L}^{ij}\partial_{x_ix_j}u(x) \leq 0, \quad \hat{L}^{in} = \hat{L}^{ni} = 0 \quad \text{for} \quad i < n,$$

and

$$D^2_{n-1}u \geq -C\left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}\right)\mathrm{Id}_{n-1},$$

we directly obtain that

$$\hat{L}^{nn}\partial_{x_nx_n}u \leq -\sum_{i,j=1}^{n-1}\hat{L}^{ij}\partial_{x_ix_j}u \leq C\left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}\right)\mathrm{tr}\hat{L}'.$$

The desired bound follows because $\hat{L}^{nn}$ is bounded below by λ and $\mathrm{tr}(\hat{L}')$ is bounded above by $(n-1)\Lambda$.

Let us prove (a). As in the proof of Proposition 2.3, we define h as the solution to

$$\begin{cases} F(D^2h) &= 0 & \text{in } B_1^+ \\ h &= -\|u\|_{L^\infty(B_1)} & \text{in } (\partial B_1)^+ \\ h(x',0) &= \varphi(x') & x' \in B_1^* \end{cases} \qquad \begin{cases} F(D^2h) &= 0 & \text{in } B_1^- \\ h &= -\|u\|_{L^\infty(B_1)} & \text{in } (\partial B_1)^- \\ h(x',0) &= \varphi(x') & x' \in B_1^*. \end{cases} \tag{2.13}$$

Recall that h is Lipschitz and that, by the strong maximum principle, $u > h$ in $B_{1/2}^+$ and $B_{1/2}^-$.

Define now, for $\varepsilon > 0$,

$$\bar{h}_\varepsilon(x', x_n) := \varphi(x') - \frac{x_n^2}{\varepsilon}$$

and

$$h_\varepsilon(x', x_n) := \max\left\{h(x', x_n), \bar{h}_\varepsilon(x', x_n)\right\}.$$

Since, h is Lipschitz continuous and $h(x',0) = \bar{h}_\varepsilon(x',0)$, this implies that there exists a constant $C > 0$ depending only on n, λ, and Λ such that

$$h(x', x_n) > \bar{h}_\varepsilon(x', x_n) \quad \text{for} \quad |x_n| > CK_0\varepsilon, \tag{2.14}$$

where we define

$$K_0 := \|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}.$$

In particular, h_ε is Lipschitz continuous in $B_{7/8}$, uniformly on ε.

Let u_ε be the solution to the "thick" obstacle problem with obstacle h_ε,

$$\begin{cases} F(D^2u_\varepsilon) &= 0 & \text{in } B_1 \setminus \{u_\varepsilon = h_\varepsilon\} \\ F(D^2u_\varepsilon) &\le 0 & \text{in } B_1 \\ u_\varepsilon &= \max\{u, \bar{h}_\varepsilon\} & \text{on } \partial(B_1^+) \\ u_\varepsilon &\ge h_\varepsilon & \text{in } B_1^+, \end{cases} \tag{2.15}$$

and the analogous expression in B_1^-. By (2.14), the coincidence set satisfies

$$\{u_\varepsilon = h_\varepsilon\} \subset \{\bar{h}_\varepsilon > h\} \subset \{(x', x_n) \in B_1 : |x_n| \le CK_0\varepsilon\}$$

for some $C > 0$. We want to bound $\partial_{\tau\tau}u_\varepsilon$ from below independently of ε.

Notice that $D^2(u_\varepsilon - h_\varepsilon) \ge 0$ in the coincidence set, and since $u_\varepsilon \ge h_\varepsilon$, this also occurs along the free boundary. By the definition of $\bar{h}_\varepsilon$ and recalling that $h_\varepsilon = \bar{h}_\varepsilon$ in the coincidence set, this implies $\partial_{\tau\tau}u_\varepsilon \ge -CK_0$ in $\{u_\varepsilon = h_\varepsilon\} \cap B_{7/8}$, for some constant C depending only on n, λ, and Λ. Thus, it is enough to check that $\partial_{\tau\tau}u_\varepsilon$ is uniformly bounded from below outside the coincidence set. We proceed by means of a Bernstein's technique.

Let $\eta \in C_c^\infty(B_{7/8})$ be a smooth, cutoff function, with $0 \le \eta \le 1$ and $\eta \equiv 1$ in $B_{3/4}$. Define

$$f_\varepsilon(x) = \eta(x)\partial_{\tau\tau}u_\varepsilon(x) - \mu|\nabla u_\varepsilon(x)|^2$$

for some constant μ to be determined later. Notice that, since h_ε is Lipschitz continuous independently of ε in $B_{7/8}$, then $|\nabla u_\varepsilon(x)|$ is bounded independently of ε in $B_{7/8}$. If the minimum x_0 in $B_{7/8}$ is attained in the coincidence set, then $\partial_{\tau\tau}u_\varepsilon(x_0) \geq -CK_0$ and we get that for every $x \in B_{3/4}$,

$$\partial_{\tau\tau}u_\varepsilon(x) \geq -CK_0 - \mu|\nabla u_\varepsilon(x_0)|^2 + \mu|\nabla u_\varepsilon(x)|^2 \geq -CK_0 - \mu\|\nabla u_\varepsilon\|^2_{L^\infty(B_{7/8})}. \quad (2.16)$$

If the minimum x_0 is attained at the boundary, $\partial B_{7/8}$, then for every $x \in B_{3/4}$,

$$\partial_{\tau\tau}u_\varepsilon(x) \geq -\mu|\nabla u_\varepsilon(x_0)|^2 + \mu|\nabla u_\varepsilon(x)|^2 \geq -\mu\|\nabla u_\varepsilon\|^2_{L^\infty(B_{7/8})}. \quad (2.17)$$

Let us assume now that the minimum x_0 of f_ε in $B_{7/8}$ is attained at some interior point x_0 outside the coincidence set $\{u_\varepsilon = h_\varepsilon\}$.

Let us also assume that the operator F not only is convex, but also $F \in C^\infty$, so that solutions are C^4 outside the coincidence set (see the end of the proof for the general case F Lipschitz). In this case, the linearised operator of F at x_0,

$$L_0 v = a_{ij}v_{ij} := F_{ij}(D^2u_\varepsilon(x_0))v_{ij},$$

is uniformly elliptic with ellipticity constants λ and Λ. Moreover, for any $\rho \in S^{n-1}$,

$$L_0u_\varepsilon(x_0) \geq 0, \quad L_0\partial_\rho u_\varepsilon(x_0) = 0, \quad L_0\partial_{\rho\rho}u_\varepsilon(x_0) \leq 0. \quad (2.18)$$

This is a standard result, which can be found in [CC95, Lemma 9.2].

For simplicity in the following computations we denote $w = u_\varepsilon$. If x_0 is an interior minimum of f_ε (which is a C^2 function) in $B_{7/8}$, then

$$0 = \nabla f_\varepsilon(x_0) = (\nabla\eta w_{\tau\tau} + \eta\nabla w_{\tau\tau} - 2\mu w_i\nabla w_i)(x_0), \quad (2.19)$$

and by (2.18) and the fact that (a_{ij}) is elliptic,

$$0 \leq a_{ij}f_{\varepsilon,ij}(x_0) \leq \left(a_{ij}\eta_{ij}w_{\tau\tau} + 2a_{ij}\eta_i w_{\tau\tau,j} - 2\mu a_{ij}w_{kj}w_{ki}\right)(x_0). \quad (2.20)$$

Combining (2.19) and (2.20), we find

$$0 \leq \left(\left(a_{ij}\eta_{ij} - 2\frac{a_{ij}\eta_i\eta_j}{\eta}\right)w_{\tau\tau} - 2\mu a_{ij}w_{kj}w_{ki} + 4\frac{\mu a_{ij}\eta_i w_{kj}w_k}{\eta}\right)(x_0). \quad (2.21)$$

Observe that $|\nabla\eta|^2 \leq C\eta$ (since $\sqrt{\eta}$ is Lipschitz). Therefore, for some constants C_0 and C_1 depending only on n and Λ,

$$0 \leq \left(C_0|w_{\tau\tau}| + \mu C_1|D^2w||\nabla w| - 2\mu a_{ij}w_{kj}w_{ki}\right)(x_0).$$

Using $|w_{\tau\tau}(x_0)| \leq |D^2w(x_0)|$ and the uniform ellipticity of (a_{ij}),

$$a_{ij}w_{ki}w_{kj} \geq \lambda C(n)|D^2w|^2,$$

we obtain

$$|D^2w(x_0)| \leq \frac{C_0}{\mu} + C_1|\nabla w(x_0)|,$$

for some constants C_0 and C_1 depending now also on λ. Now, since x_0 is a minimum in $B_{7/8}$, for any $x \in B_{3/4}$,

$$\begin{aligned} w_{\tau\tau}(x) &\geq \eta(x_0)w_{\tau\tau}(x_0) - \mu|\nabla u_\varepsilon(x_0)|^2 + \mu|\nabla u_\varepsilon(x)|^2 \\ &\geq -|D^2 w(x_0)| - \mu\|\nabla u_\varepsilon\|^2_{L^\infty(B_{7/8})} \\ &\geq -\frac{C_0}{\mu} - C_1\|\nabla u_\varepsilon\|_{L^\infty(B_{7/8})} - \mu\|\nabla u_\varepsilon\|^2_{L^\infty(B_{7/8})}. \end{aligned} \tag{2.22}$$

We now fix $\mu = \|\nabla u_\varepsilon\|^{-1}_{L^\infty(B_{7/8})}$. Notice that, in all three cases (2.16), (2.17), and (2.22), we reach that for some constant C depending only on n, λ, and Λ,

$$\inf_{B_{3/4}} \partial_{\tau\tau} u_\varepsilon \geq -C\left(\sup_{B_{7/8}} |\nabla u_\varepsilon| + K_0\right).$$

We had already seen that u_ε is Lipschitz continuous independently of $\varepsilon > 0$ and controlled by the Lipschitz norm of u, so that by Proposition (2.3),

$$\inf_{B_{3/4}} \partial_{\tau\tau} u_\varepsilon \geq -C\left(\|u\|_{\mathrm{Lip}(B_{7/8})} + \|\varphi\|_{C^{1,1}(B_1^*)} + K_0\right) \geq -C\left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}\right). \tag{2.23}$$

If F is not smooth, then it can be regularised convoluting with a mollifier in the space of symmetric matrices, so that it can be approximated uniformly in compact sets by a sequence $\{F_k\}_{k\in\mathbb{N}}$ of convex smooth uniformly elliptic operators with ellipticity constants λ and Λ; also, by subtracting $F_k(0)$, we can assume $F_k(0) = 0$. Note that, in $B_{7/8}$ and for every $\varepsilon > 0$ we have uniform $C^{1,\gamma}$ estimates in k for the solutions to (2.15) with operators F_k, since the obstacle h is in $C^{1,1}$ in a neighbourhood of the free boundary. By Arzelà-Ascoli there exists a subsequence converging uniformly, and therefore, the estimate (2.23) can be extended to solutions of (2.15) with operators not necessarily smooth. Thus, (2.23) follows for any F not necessarily C^∞.

Note that u_ε converges uniformly to u, since for all $\delta > 0$, there exists some $\varepsilon > 0$ small enough such that $u + \delta > u_\varepsilon \geq u$ in B_1.

Since the right-hand side of (2.23) is independent of ε, and u_ε converges uniformly to u in $B_{7/8}$ as $\varepsilon \downarrow 0$, we finally obtain

$$\inf_{B_{3/4}} u_{\tau\tau} \geq -C\left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}\right), \tag{2.24}$$

as desired. □

2.3 $C^{1,\alpha}$ estimate

2.3.1 A symmetrised solution

By the results in the previous section we know that ∇u is bounded in the interior of B_1. Moreover, $u_{x_n x_n}$ is bounded from above inside B_1. In particular, the following limit exists

$$\sigma(x') = \lim_{x_n\downarrow 0^+} u_{x_n}(x', x_n) - \lim_{x_n\uparrow 0^-} u_{x_n}(x', x_n) = \lim_{x_n\downarrow 0^+} \big(u_{x_n}(x', x_n) - u_{x_n}(x', -x_n)\big). \tag{2.25}$$

A main step towards Theorem 2.1 consists of proving that $\sigma \in C^\alpha(B^*_{1/2})$ for some $\alpha > 0$. We will prove this in this section.

We begin by noticing that $\sigma(x') = 0$ for $x' \in \Omega^*$ (by the $C^{2,\alpha}$ interior estimates), where we recall that $\Omega^* := \{x' \in B^*_1 : u(x', 0) > \varphi(x')\}$. In general, however, we have the following:

Lemma 2.5. *The function σ defined by* (2.25) *is non-positive, i.e., $\sigma \le 0$ in B^*_1.*

Proof. Suppose it is not true, and there exists some $\bar{x}' \in B^*_1$ such that $\sigma(\bar{x}') > 0$. Let $\delta > 0$ be such that $B^*_\delta(\bar{x}') \subset B^*_1$, so that by the semiconcavity in Proposition 2.4 applied to $B_{\delta/2}((\bar{x}', 0))$, $u_{x_n x_n}(\bar{x}', 0) \le C$ for some constant C, that now depends also on δ. However,

$$\sigma(\bar{x}') = \lim_{x_n \downarrow 0^+} \left(u_{x_n}(\bar{x}', x_n) - u_{x_n}(\bar{x}', -x_n)\right) > 0,$$

which means

$$\frac{u_{x_n}(\bar{x}', x_n) - u_{x_n}(\bar{x}', -x_n)}{2x_n} \to +\infty, \quad \text{as} \quad x_n \downarrow 0^+,$$

a contradiction with the bound in $u_{x_n x_n}$. □

We will now adapt the ideas of [Caf79] to our non-symmetric setting. For this, we use a symmetrised solution, defined as follows

$$v(x', x_n) := \frac{u(x', x_n) + u(x', -x_n)}{2}, \quad \text{for } (x', x_n) \in \overline{B_1}. \tag{2.26}$$

Here u is any solution to (2.2).

Notice that

$$\sigma(x') = 2 \lim_{x_n \downarrow 0^+} v_{x_n}(x', x_n) \le 0 \tag{2.27}$$

is well defined, and in particular, we have that

$$\sigma(x') = 2v_{x_n}(x', 0) = 0, \text{ for } x' \in \Omega^*. \tag{2.28}$$

The following result follows from the results in the previous section. We will use the notation $\mathcal{M}^+$ and $\mathcal{M}^-$ to refer to the Pucci's extremal operators with the implicit ellipticity constants λ and Λ (see [CC95, Chapter 2] for the definition and basic properties of such operators).

Lemma 2.6. *Let u be a solution to the nonlinear thin obstacle problem* (2.2)*, and let v be defined by* (2.26)*. Then v is Lipschitz in $\overline{B^+_{1/2}}$ and satisfies*

$$\left\{ \begin{array}{rcll} \mathcal{M}^-(D^2 v) & \le & 0 & \text{in } B_1, \\ \max\{v_{x_n}(x', 0), \varphi(x') - v(x', 0)\} & = & 0 & \text{for } x' \in B^*_1. \end{array} \right. \tag{2.29}$$

Moreover,

(a) *(Semiconvexity) If $\tau = (\tau^*, 0)$, with τ^* a unit vector in $\mathbb{R}^{n-1}$,*

$$\inf_{B_{3/4}} v_{\tau\tau} \ge -C \left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B^*_1)}\right),$$

for some constant C depending only on n, λ, and Λ.

(b) (Semiconcavity) In the direction normal to $B_1^* \times \{0\}$,

$$\sup_{B_{3/4}} v_{x_n x_n} \le C\left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}\right),$$

for some constant C *depending only on* n, λ, *and* Λ.

Proof. The Lipschitz regularity comes from the Lipschitz regularity in u, proved in Proposition 2.3.

In (2.29) the first inequality follows thanks to the change of variables introduced in Subsection 2.2.2. Indeed, there exists some operator given by a matrix $\hat{L}$ as in (2.12) uniformly elliptic with ellipticity constants λ and Λ such that

$$\hat{L}^{ij}\partial_{x_i x_j}(u(x', -x_n)) = \hat{L}^{ij}(\partial_{x_i x_j} u)(x', -x_n) \le F((D^2 u)(x', -x_n)) \le 0,$$

so that

$$\mathcal{M}^-(D^2 v) \le \hat{L}^{ij}\partial_{x_i x_j} v \le 0,$$

as we wanted.

The second expression in (2.29) follows from equations (2.27)-(2.28), Lemma 2.5 and the fact that $v(x', 0) = u(x', 0)$ for $x' \in B_1^*$.

Finally, the semiconvexity and semiconcavity follow from Proposition 2.4. □

2.3.2 Regularity for σ on free boundary points

The next steps are very similar to those in [Caf79] (and [MS08]), but we adapt them to the symmetrised solution v instead of u. For completeness, we provide all the details. We begin with the following lemma, corresponding to [Caf79, Lemma 2] (or [MS08, Lemma 3.3]).

In the next result, we call φ the extension of the obstacle to B_1, i.e. $\varphi(x', x_n) := \varphi(x')$.

Lemma 2.7. *Let* v *be the symmetrised solution* (2.26). *Let* κ *be a constant such that* $\kappa > \sup|\varphi_{\tau\tau}|$ *for any* τ *a unit vector in* $\mathbb{R}^{n-1} \times \{0\}$. *Let* $x_0 \in \Omega$ *fixed and* ψ_{x_0} *denote the function*

$$\psi_{x_0} = \varphi(x_0) + \nabla\varphi(x_0)\cdot(x - x_0) + \kappa|x - x_0|^2 - \kappa(n-1)\frac{\Lambda}{\lambda}x_n^2.$$

Then, for any open set U_{x_0} *such that* $x_0 \in U_{x_0} \subset B_1$,

$$\sup_{\partial U_{x_0} \cap \{x_n > 0\}} (v - \psi_{x_0}) \ge 0.$$

Proof. Define $w = v - \psi_{x_0}$ and notice that by definition of ψ_{x_0} and the fact that v is a supersolution for $\mathcal{M}^-$, we have $w(x_0) \ge 0$ and $\mathcal{M}^-(D^2 w) \le 0$. Therefore, we can apply the maximum principle on $U_{x_0} \setminus \Delta$ (recall Δ is the coincidence set) and use the symmetry of w to obtain that

$$\sup_{\partial(U_{x_0} \setminus \Delta) \cap \{x_n \ge 0\}} (v - \psi_{x_0}) \ge 0.$$

Now notice that on the set $\{v = u = \varphi\}$ we have that $\psi_{x_0} > \varphi$, since $x_0 \in \Omega$ and $\kappa > \sup|\varphi_{\tau\tau}|$. Thus, $v - \psi_{x_0} < 0$ on this set, so that

$$\sup_{\partial(U_{x_0}\setminus\Delta)\cap\{x_n\geq 0\}} (v - \psi_{x_0}) = \sup_{\partial U_{x_0}\cap\{x_n>0\}} (v - \psi_{x_0}) \geq 0,$$

and we are done. □

We now proceed with the following lemma, corresponding to [Caf79, Lemma 2] (or [MS08, Lemma 3.4]).

Lemma 2.8. *Let v be the symmetrised solution as defined in* (2.26)*, and let σ as defined in* (2.25)-(2.27)*. Let $x_0 = (x_0', 0) \in \Omega$ and define $S_\gamma = \{x' : \sigma(x') > -\gamma\}$. Then, for suitable positive constants C, $\overline{C}$, and γ_0 and for all $\gamma \in (0, \gamma_0)$ there exists a ball $B^*_{C\gamma}(\bar{x}')$ for $\bar{x}' \in B^*_1$ such that*

$$B^*_{C\gamma}(\bar{x}') \subset B^*_{\overline{C}\gamma}(x_0') \cap S_\gamma.$$

The constants C, $\bar{C}$, and γ_0 depend only on n, λ, Λ, $\|\varphi\|_{C^{1,1}(B_1^*)}$, and $\|u\|_{L^\infty(B_1)}$.

Proof. We apply Lemma 2.7 with $U_{x_0} = B_{C_1\gamma}(x_0) \times (-C_2\gamma, C_2\gamma)$ for some constants to be chosen $C_1 \gg C_2$, and study two cases.

- Assume $\sup(v - \psi_{x_0})$ is attained at a point (x_1', y_1) (for $x_1' \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$) on the lateral face of the cylinder U_{x_0}, i.e. with $|x_1' - x_0'| = C_1\gamma$ and $0 \leq y_1 \leq C_2\gamma$. Then we have

$$\begin{aligned}\psi_{x_0}(x_1', y_1) - \varphi(x_1') &\geq (\kappa - \sup|\phi_{\tau\tau}|)\,|x_1' - x_0'|^2 - \kappa(n-1)\frac{\Lambda}{\lambda}y_1^2 \\ &\geq (\kappa - \sup|\phi_{\tau\tau}|)\,C_1^2\gamma^2 - \kappa(n-1)\frac{\Lambda}{\lambda}C_2^2\gamma^2 \geq C_3\gamma^2,\end{aligned}$$

provided that $C_1 \gg C_2$. The positive constant C_3 depends only on κ, n, the ellipticity constants, C_1, and C_2. Thus,

$$v(x_1', y_1) \geq \psi_{x_0}(x_1', y_1) \geq \varphi(x_1') + C_3\gamma^2.$$

Now pick a $x_2' \in B^*_{C_4\gamma}(x_1')$ for some positive constant C_4 to be chosen and $(x_2' - x_1') \cdot \nabla_{x'}(v - \varphi)(x_1', y_1) \geq 0$. We are considering here φ in the whole B_1 by simply putting $\varphi(x', y) = \varphi(x')$. Take $\tau = \left(\frac{x_2' - x_1'}{|x_2' - x_1'|}, 0\right)$, and use the semiconvexity from Lemma 2.6 together with the fact that $\varphi \in C^{1,1}$ to get

$$\begin{aligned}&(v - \varphi)(x_2', y_1) = \\ &\quad= (v - \varphi)(x_1', y_1) + (x_2' - x_1') \cdot \nabla_{x'}(v - \varphi)(x_1', y_1) + \iint_{[(x_1', y_1), (x_2', y_1)]} (v - \varphi)_{\tau\tau} \\ &\quad\geq C_3\gamma^2 - C|x_2' - x_1'|^2 \geq (C_3 - CC_4)\gamma^2 > 0,\end{aligned}$$

if C_4 is chosen appropriately, small enough depending only on C_3, $\|\varphi\|_{C^{1,1}}$ and the semiconvexity constant of Lemma 2.6. Here, and in the next steps, $\iint_{[a,b]}$ denotes the double integral over the segment between the points a and b,

$$\iint_{[a,b]} w := \int_0^{|b-a|} \left[\int_0^s w\left(a + \frac{b-a}{|b-a|}t\right) dt\right] ds.$$

To get a contradiction, now suppose that $x_2' \notin S_\gamma$. In particular, this means $v(x_2', 0) = \varphi(x_2')$, and from (2.27) and the semiconcavity in Lemma 2.6 we get

$$\begin{aligned}(v-\varphi)(x_2', y_1) &= (v-\varphi)(x_2', 0) + y_1\frac{\sigma(x_2')}{2} + \iint_{[(x_2',0),(x_2',y_1)]} v_{x_n x_n}\\ &\le -y_1\frac{\gamma}{2} + Cy_1^2 \le y_1\gamma\left(CC_2 - \frac{1}{2}\right) \le 0\end{aligned}$$

if C_2 is small enough depending only on the semiconcavity constant of Lemma 2.6. Thus, we have reached a contradiction.

- Assume now that $\sup(v - \psi_{x_0})$ is attained at a point (x_1', y_1) in the base of the cylinder U_{x_0}, i.e. with $|x_1' - x_0'| \le C_1\gamma$ and $y_1 = C_2\gamma$. Then, from $\kappa > \sup|\varphi_{\tau\tau}|$, we deduce

$$v(x_1', y_1) \ge \psi_{x_0}(x_1', y_1) \ge \varphi(x_1') - \kappa(n-1)\frac{\Lambda}{\lambda}C_2^2\gamma^2.$$

Now choose x_2' such that $|x_2' - x_1'| < C_2\gamma$ and $(x_2' - x_1')\cdot\nabla_{x'}(v-\varphi)(x_1', y_1) \ge 0$. As before,

$$\begin{aligned}&(v-\varphi)(x_2', y_1) =\\ &\quad= (v-\varphi)(x_1', y_1) + (x_2' - x_1')\cdot\nabla_{x'}(v-\varphi)(x_1', y_1) + \iint_{[(x_1',y_1),(x_2',y_1)]} (v-\varphi)_{\tau\tau}\\ &\quad\ge -\kappa(n-1)\frac{\Lambda}{\lambda}C_2^2\gamma^2 - C|x_2' - x_1'|^2 \ge -C_2^2\left(\kappa(n-1)\frac{\Lambda}{\lambda} + C\right)\gamma^2.\end{aligned}$$

Now, if $x_2' \notin S_\gamma$ then $v(x_2', 0) = \varphi(x_2')$,

$$(v-\varphi)(x_2', y_1) \le -C_2\frac{\gamma^2}{2} + \iint_{[(x_2',0),(x_2',y_1)]} v_{x_n x_n} \le \left(\frac{1}{2}CC_2^2 - C_2\right)\gamma^2.$$

The contradiction follows if one chooses C_2 small enough, depending only on κ, n, λ, Λ, and the semiconvexity and semiconcavity constants from Lemma 2.6. □

The following lemma is useful to prove the C^α regularity of σ, and can be found in [MS08, Lemma 3.5]. It follows from an appropriate use of the strong maximum principle for $\mathcal{M}^-$, the Pucci's extremal operator.

Lemma 2.9 ([MS08]). *Let w be a non-negative continuous function in $B_1^* \times (0,1)$ that solves*

$$\mathcal{M}^-(D^2 w) \le 0 \quad \textit{in} \quad B_1^* \times (0,1).$$

Assume

$$\limsup_{x_n \downarrow 0^+} w(x', x_n) \ge 1 \quad \textit{for} \quad x' \in B_\delta^*(\bar{x}'),$$

for some ball $B_\delta^*(\bar{x}') \subset B_1^*$. Then

$$w(x) \ge \varepsilon > 0 \quad \textit{for} \quad x \in B_{1/2}^* \times \left[\frac{1}{4}, \frac{3}{4}\right],$$

for some ε depending only on δ, and the ellipticity constants λ and Λ.

We now show the following lemma, analogous to [Caf79, Lemma 4] (or [MS08, Lemma 3.6]).

Lemma 2.10. *Let* σ *as defined in* (2.25)-(2.27)*, for* u *the solution to the thin obstacle problem* (2.2)*. Let* $x_0' \in \Omega^*$*, then*

$$\sigma(x') \geq -C\left(\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)}\right)|x'-x_0'|^\alpha, \text{ for } x' \in B_1^*$$

for some $\alpha > 0$ *and* C *depending only on* n*,* λ*, and* Λ*.*

Proof. Define

$$K_0 := \|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)},$$

and notice that by taking u/K_0 instead of u if necessary we can assume

$$\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)} \leq 1.$$

Indeed, if $K_0 \geq 1$ then

$$F_{K_0}(D^2u) := \frac{1}{K_0}F(D^2(K_0u)),$$

is a convex elliptic operator with ellipticity constants λ and Λ, and u/K_0 is a solution to the nonlinear thin obstacle problem for the operator F_{K_0} with obstacle φ/K_0. In this case,

$$\|u/K_0\|_{L^\infty(B_1)} + \|\varphi/K_0\|_{C^{1,1}(B_1^*)} = 1,$$

as we wanted to see. Thus, from now on we assume $K_0 \leq 1$.

Using Lemmas 2.5, 2.6, 2.8 and 2.9, now the proof of this lemma is very similar to the proof of [MS08, Lemma 3.6]. We give it here for completeness.

We will show

$$\sigma(x') \geq -C|x'-x_0'|^\alpha, \tag{2.30}$$

with C and $\alpha > 0$ depending only on n, λ, and Λ.

Recall that $\sigma(x') = 2\lim_{x_n \downarrow 0^+} v_{x_n}(x', x_n)$, and that from Lemma 2.6, v_{x_n} is bounded and $v_{x_nx_n} \leq C$. Moreover, σ is non-positive by Lemma 2.5, so that $v_{x_n} \leq Cx_n$ for $x_n > 0$.

In order to reach (2.30) we will prove $v_{x_n}(x) \geq -\theta^k$ for $x \in B^*_{\gamma^k}(x_0') \times (0, \gamma^k)$. Assume this has been already proved for some k with $0 < \gamma \ll \theta < 1$, and consider the function

$$w := \frac{v_{x_n} + \theta^k}{\theta^k - C\mu\gamma^k} \text{ in } B^*_{\mu\gamma^k}(x_0') \times (0, \mu\gamma^k)$$

for μ small enough. Notice that w fulfils the hypotheses of Lemma 2.9, so that using it together with Lemma 2.8 we get

$$v_{x_n}(x) \geq -\theta^k + \varepsilon(\theta^k - C\mu\gamma^k) \geq -\theta^k + \frac{1}{2}\varepsilon\theta^k$$

for $x \in B^*_{\mu\gamma^k/2}(x_0') \times (\mu\gamma^k/4, 3\mu\gamma^k/4)$, since $\gamma \ll \theta$. Now, by means of Lemma 2.6, $v_{x_nx_n} \leq C$, and therefore, for any $y = (y', y_n) \in B^*_{\mu\gamma^k/2}(x_0') \times (0, \mu\gamma^k/4]$,

$$\begin{aligned} v_{x_n}(y) &\geq -\int_{y_n}^{\mu\gamma^k/4} v_{x_nx_n}(y', s)ds + v_{x_nx_n}(y', \mu\gamma^k/4) \\ &\geq -C\left(\frac{\mu\gamma^k}{4} - y_n\right) - \theta^k + \frac{1}{2}\varepsilon\theta^k, \end{aligned}$$

so that we obtain

$$v_{x_n}(x) \geq -\theta^k + \frac{1}{2}\varepsilon\theta^k - \frac{1}{4}\mu C\gamma^k$$

for $x \in B^*_{\mu\gamma^k/2}(x_0') \times (0, 3\mu\gamma^k/4)$. To end the inductive argument we must see

$$\theta^{k+1} \geq \theta^k - \frac{1}{2}\varepsilon\theta^k + \frac{1}{4}\mu C\gamma^k.$$

For this, we pick $\gamma \ll \theta$ so that the right-hand side is smaller than $(1 - \frac{1}{4}\varepsilon)\theta^k$, with θ larger than $1 - \frac{1}{4}\varepsilon$. Then, the inductive argument is completed, and (2.30) follows. □

2.3.3 Proof of Theorem 2.1

Before proving our main result, let us show the following compactness lemma.

Lemma 2.11. *Let F be a nonlinear operator satisfying* (2.3)*, and let w be a continuous function defined on B_1. Suppose that w satisfies the problem*

$$F(D^2 w) = 0 \text{ in } B_1^+ \cup B_1^-, \tag{2.31}$$

and that

$$\|w\|_{L^\infty(B_1)} = 1, \qquad [w]_{\mathrm{Lip}(B_1)} \leq 1.$$

Let ψ be the solution to

$$\begin{cases} F(D^2\psi) &= 0 & \text{in } B_1 \\ \psi &= w & \text{on } \partial B_1, \end{cases} \tag{2.32}$$

and let us define the following operator

$$\tilde{\sigma}(w) := \lim_{h_n \downarrow 0} \left((\partial_{x_n} w)(x', h_n) - (\partial_{x_n} w)(x', -h_n) \right).$$

Then, for every $\varepsilon > 0$ there exists some $\eta = \eta(\varepsilon, n, \lambda, \Lambda) > 0$ such that if

$$\|\tilde{\sigma}(w)\|_{L^\infty(B_1^*)} < \eta$$

then

$$\|\psi - w\|_{L^\infty(B_1)} < \varepsilon,$$

i.e., ψ approximates w as η goes to 0.

Proof. Let us argue by contradiction. Suppose that there exists some fixed $\varepsilon > 0$, a sequence of functions w_k and a sequence of convex nonlinear operators uniformly elliptic with ellipticity constants λ and Λ, F_k, with $F_k(0) = 0$, such that

$$F_k(D^2 w_k) = 0 \text{ in } B_1^+ \cup B_1^- \tag{2.33}$$

and

$$\|w_k\|_{L^\infty(B_1)} = 1, \qquad [w_k]_{\mathrm{Lip}(B_1)} \leq 1,$$

with
$$\|\tilde{\sigma}(w_k)\|_{L^\infty(B_1^*)} < \eta_k \tag{2.34}$$
for some sequence $\eta_k \to 0$, but such that
$$\|\psi_k - w_k\|_{L^\infty(B_1)} \geq \varepsilon, \tag{2.35}$$
for all k, where ψ_k is the solution to
$$\begin{cases} F_k(D^2\psi_k) &= 0 \quad \text{in } B_1 \\ \psi_k &= w_k \quad \text{on } \partial B_1. \end{cases} \tag{2.36}$$

By Arzelà-Ascoli, up to a subsequence, w_k converges to some function $\bar{w}$ uniformly in B_1, with $\|\bar{w}\|_{L^\infty(B_1)} = 1$. On the other hand, since $F_k(0) = 0$ and they are uniformly elliptic and convex, they converge up to subsequences, uniformly over compact sets, to some convex nonlinear operator $\bar{F}$ uniformly elliptic with ellipticity constants λ and Λ such that $\bar{F}(0) = 0$. Notice also that ψ_k converges uniformly to the solution $\bar{\psi}$ to
$$\begin{cases} \bar{F}(D^2\bar{\psi}) &= 0 \quad \text{in } B_1 \\ \bar{\psi} &= \bar{w} \quad \text{on } \partial B_1. \end{cases} \tag{2.37}$$
and in the limit we obtain, from (2.35),
$$\|\bar{\psi} - \bar{w}\|_{L^\infty(B_1)} \geq \varepsilon > 0. \tag{2.38}$$

Now consider the function $w_k + \eta_k|x_n|$ on B_1. From (2.34), $w_k + \eta_k|x_n|$ now has a wedge pointing down in the set $B_1 \cup \{x_n = 0\}$, i.e.,
$$\tilde{\sigma}(w_k + \eta_k|x_n|) \geq \eta_k > 0, \quad \text{in} \quad B_1^*.$$

Therefore, since $F_k(D^2 w_k) = 0$ in $B_1^+ \cup B_1^-$, we have that, in the viscosity sense,
$$F_k(D^2(w_k + \eta_k|x_n|)) \geq 0, \quad \text{in} \quad B_1.$$

Now, passing to the limit, noticing that $w_k + \eta_k|x_n|$ converges uniformly to $\bar{w}$ and using [CC95, Proposition 2.9], we immediately reach that, in the viscosity sense,
$$\bar{F}(D^2\bar{w}) \geq 0, \quad \text{in} \quad B_1.$$
Repeating the same argument for $w_k - \eta_k|x_n|$ we reach $\bar{F}(D^2\bar{w}) \leq 0$ in B_1, to finally obtain
$$\bar{F}(D^2\bar{w}) = 0, \quad \text{in} \quad B_1.$$
This implies $\bar{w} = \bar{\psi}$ in B_1, which is a contradiction with (2.38). □

Using the previous results, we now give the proof of Theorem 2.1.

Proof of Theorem 2.1. We separate the proof into three steps. In the first step we prove that the solution u is $C^{1,\alpha}$ around points in Ω^* by means of Lemmas 2.10 and 2.11. In the second step, we use the result from the first step to deduce that σ is C^α in $B^*_{2/3}$, to finally complete the proof in the third step.

As in the proof of Lemma 2.10 we assume

$$\|u\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1^*)} \le 1,$$

to avoid having this constant on each estimate throughout the proof.

Step 1: Let us suppose that the origin is a free boundary point. Under these circumstances we will prove that there exist some affine function $L = a + b\cdot x$ such that

$$\|u - L\|_{L^\infty(B_r)} \le Cr^{1+\alpha}, \quad \text{for all } r \ge 0, \tag{2.39}$$

for some constants C and $\alpha > 0$ depending only on n, λ, and Λ. To do so, we proceed in the spirit of the proof of [Caf89, Theorem 2].

Notice that from Lemma 2.10 we know that there exists $\eta > 0$ such that

$$|\sigma(x')| \le \eta |x'|^\alpha, \quad \text{for all } x' \in B_1^*. \tag{2.40}$$

Up to replacing from the beginning $u(x)$ by $u(r_0 x)$ with $r_0 \ll 1$, we can make η as small as necessary. The choice of the value of r_0, and consequently the magnitude in which the constant η is made small, will depend only on n, λ, and Λ.

Let us show now that there exists $\rho = \rho(\alpha, n, \lambda, \Lambda) < 1$ and a sequence of affine functions

$$L_k(x) = a_k + b_k \cdot x \tag{2.41}$$

such that

$$\|u - L_k\|_{L^\infty(B_{\rho^k})} \le \rho^{k(1+\alpha)}, \tag{2.42}$$

and

$$|a_k - a_{k-1}| \le C\rho^{k(1+\alpha)}, \qquad |b_k - b_{k-1}| \le C\rho^{k\alpha} \tag{2.43}$$

for some constant C depending only on n, λ, and Λ.

We proceed by induction, taking $L_0 = 0$. Suppose that the k-th step is true, and consider

$$w_k(x) = \frac{(u - L_k)(\rho^k x)}{\rho^{k(1+\alpha)}}, \text{ for } x \in B_1.$$

Begin by noticing that

$$F_k(D^2 w_k) = 0 \text{ in } B_1^+ \cup B_1^-$$

for some operator F_k of the form (2.3). On the other hand, from the induction hypothesis,

$$\|w_k\|_{B_1} \le 1.$$

Moreover, if we define

$$\sigma_k(x') = \lim_{h\downarrow 0} \left(\partial_{x_n} w_k(x', h) - \partial_{x_n} w_k(x', -h)\right), \quad \text{for } x' \in B_1^*,$$

then one can check that, from (2.40),

$$|\sigma_k(x')| \le \eta |x'|^\alpha.$$

We apply now Lemma 2.11. That is, given $\varepsilon > 0$ small, we can choose η small enough such that

$$\|v_k - w_k\|_{L^\infty(B_1)} \le \varepsilon,$$

where v_k is the solution to

$$\begin{cases} F_k(D^2 v_k) &= 0 \quad \text{in } B_1 \\ v_k &= w_k \quad \text{on } \partial B_1. \end{cases} \tag{2.44}$$

Notice that, by interior estimates, v_k is $C^{2,\alpha}$ in $B_{1/2}$ with estimates depending only on n, λ, and Λ. Then, let l_k be the linearisation of v_k around 0, so that up to choosing ρ,

$$\begin{aligned} \|w_k - l_k\|_{L^\infty(B_\rho)} &\le \|w_k - v_k\|_{L^\infty(B_\rho)} + \|v_k - l_k\|_{L^\infty(B_\rho)} \\ &\le \varepsilon + C\rho^2 \le \rho^{1+\alpha}, \end{aligned}$$

where C depends only on n, λ, and Λ, ρ is chosen small enough depending only on α, n, λ, and Λ so that $C\rho^2 \le \frac{1}{2}\rho^{1+\alpha}$, and η is chosen so that $\varepsilon \le \frac{1}{2}\rho^{1+\alpha}$. It is important to remark that the choice of η depends only on n, λ, and Λ.

Now, recalling the definition of w_k, we reach

$$\left\| u - L_k - \rho^{k(1+\alpha)} l_k\left(\frac{\cdot}{\rho^k}\right)\right\|_{L^\infty(B_{\rho^{k+1}})} \le \rho^{(k+1)(1+\alpha)},$$

so that the inductive step is concluded by taking

$$L_{k+1}(x) = L_k(x) + \rho^{k(1+\alpha)} l_k\left(\frac{x}{\rho^k}\right).$$

By noticing that there are bounds on the coefficients of the linearisation of v_k depending only on n, λ, and Λ, the inequalities in (2.43) are obtained.

Once one has (2.41), (2.42), and (2.43), define L as the limit of L_k as $k \to \infty$ (which exists, by (2.43)), and notice that, given any $0 < r = \rho^k$ for some $k \in \mathbb{N}$, then

$$\|u - L\|_{L^\infty(B_r)} \le \|u - L_k\|_{L^\infty(B_r)} + \sum_{j \ge k} \|L_{j+1} - L_j\|_{L^\infty(B_r)} \le C r^{1+\alpha}$$

for some C depending only on n, λ, and Λ; as we wanted.

Step 2: In this step we prove that the function σ defined in (2.25)-(2.27) is $C^\alpha(B^*_{2/3})$ for some $\alpha = \alpha(n, \lambda, \Lambda) > 0$, and

$$\|\sigma\|_{C^\alpha(B^*_{2/3})} \le C, \tag{2.45}$$

for some constant C depending only on n, λ, and Λ.

We already know σ is regular in the interior of Δ^* (by boundary estimates) and Ω^*; respectively the coincidence set and its complement in B_1^*. In particular, from the interior estimates $\sigma \equiv 0$ in Ω^*. From Lemma 2.10 we also obtain C^α regularity at points in $\partial\Delta^*$. Namely, we have that given $(x_0', 0) = x_0 \in \partial\Delta^*$,

$$|\sigma(x')| \le C|x' - x_0'|^\alpha, \ \text{for } x' \in B_1^*, \tag{2.46}$$

for some constant C depending only on n, λ, and Λ.

Therefore, we only need to check that given $x, y \in \Delta$, $x = (x', 0)$, $y = (y', 0)$, then there exists some C depending only on n, λ, and Λ such that, if $|x - y| = r$,

$$|\sigma(x') - \sigma(y')| \leq Cr^{\alpha}.$$

Let $R := \mathrm{dist}(x, \Omega)$ and suppose that $\mathrm{dist}(x, \Omega) \leq \mathrm{dist}(y, \Omega)$. Let $z = (z', 0)$, $z' \in \partial\Delta^*$, be such that $\mathrm{dist}(x, z) = \mathrm{dist}(x, \Omega)$, and assume that $\lim_{x_n \downarrow 0^+} \nabla u(z', x_n) = 0$ and $\nabla_{x'}\varphi(z') = 0$ by subtracting an affine function if necessary. Notice that we can do so because we already know from the first step that u has a $C^{1,\alpha}$ estimate around z'. Let us then separate two cases:

• If $R < 4r$, then using (2.45)

$$\begin{aligned} |\sigma(x') - \sigma(y')| &\leq |\sigma(x') - \sigma(z')| + |\sigma(y') - \sigma(z')| \\ &\leq C\left(R^{\alpha} + (R + r)^{\alpha}\right) \\ &\leq Cr^{\alpha}. \end{aligned}$$

• In the case $R \geq 4r$ we need to use known boundary estimates for this fully nonlinear problem and the previous step of the proof. Notice that $x', y' \in B^*_{R/2}(x') \subset B^*_R(x') \subset \Delta^*$, and u restricted to $B^*_R(x')$ is thus a $C^{1,1}$ function, since $u = \varphi$ there. In particular, we use that under these hypotheses

$$R^{1+\alpha}[u]_{C^{1,\alpha}(\overline{B^+_{R/2}(x)})} \leq C\left(\mathrm{osc}_{B^+_R(x)} u + R^2[\varphi]_{C^{1,1}(B^*_R(x'))}\right);$$

see, for example, [MS06, Proposition 2.2]. Now, remember that the gradient of u at z is 0, so that from the previous step using the bound (2.39) around z,

$$|u(p) - \varphi(z')| \leq C|p - z|^{1+\alpha} \leq CR^{1+\alpha} \text{ for } p \in B^+_R(x). \tag{2.47}$$

In particular, $\mathrm{osc}_{B^+_R(x)} u \leq CR^{1+\alpha}$, and thus, this yields

$$[u]_{C^{1,\alpha}(\overline{B^+_{R/2}(x)})} \leq C,$$

from which (2.45) is proved.

Step 3: Our conclusion now follows by repeating Step 1 around every point on B^*_1. Notice that in the first step we only used that the origin was a free boundary point to be able to apply Lemma 2.10 in (2.40).

Now, given any point $z' \in B^*_{1/2}$, we can consider the function u_z given by

$$u_z(x) := u(x) - \sigma(z')(x_n)^+,$$

where $(x_n)^+$ denotes the positive part of x_n.

Note that this function fulfils the hypotheses of Step 1, in particular,

$$|\sigma_z(x')| := \left|\lim_{h \downarrow 0}\left(\partial_{x_n} u_z(x', h) - \partial_{x_n} u_z(x', -h)\right)\right| \leq C|x' - z'|^{\alpha}, \quad \text{for } x' \in B^*_1,$$

for some constant C depending only on n, λ, and Λ.

By repeating the exact same procedure as in Step 1, we reach that for every point $z \in B_{1/2} \cup \{x_n = 0\}$, and for every $x \in B_1^+$ there exists some L_z^+ affine function such that

$$|u(x) - L_z^+| \leq C|x - z|^{1+\alpha},$$

and the same occurs in B_1^- for a possibly different affine function L_z^-. Therefore, in particular,

$$\|u\|_{C^{1,\alpha}(B^*_{1/2})} \leq C$$

for some C depending only on n, λ, and Λ.

To finish the proof, we could now repeat a procedure like the one done in Step 2, or directly notice that solutions to the nonlinear problem with $C^{1,\alpha}$ boundary data are $C^{1,\alpha}$ up to the boundary (see, for example, [MS06, Proposition 2.2]). □

We finally give the:

Proof of Corollary 2.2. It is an immediate consequence of Theorem 2.1. Indeed, consider balls of radius $R_0 := \text{dist}(K, \partial D)$ around points on $K \cap \{x_n = 0\}$ and apply Theorem 2.1. To cover the rest of K we use interior estimates, and the result follows by noticing that $\|u\|_{L^\infty(D)} \leq \|g\|_{L^\infty(\partial D)} + \|\varphi\|_{L^\infty}$ by the maximum principle. □

Chapter 3

The obstacle problem for the fractional Laplacian with critical drift

We study the obstacle problem for the fractional Laplacian with drift,

$$\min\left\{(-\Delta)^s u + b\cdot\nabla u,\, u-\varphi\right\} = 0 \quad \text{in} \quad \mathbb{R}^n,$$

in the critical regime $s = \frac{1}{2}$.

Our main result establishes the $C^{1,\alpha}$ regularity of the free boundary around any regular point x_0, with an expansion of the form

$$u(x) - \varphi(x) = c_0\big((x-x_0)\cdot e\big)_+^{1+\tilde{\gamma}(x_0)} + o\left(|x-x_0|^{1+\tilde{\gamma}(x_0)+\sigma}\right),$$

$$\tilde{\gamma}(x_0) = \frac{1}{2} + \frac{1}{\pi}\arctan(b\cdot e),$$

where $e \in \mathbb{S}^{n-1}$ is the normal vector to the free boundary, $\sigma > 0$, and $c_0 > 0$.

We also establish an analogous result for more general nonlocal operators of order 1. In this case, the exponent $\tilde{\gamma}(x_0)$ also depends on the operator.

3.1 Introduction

We consider the obstacle problem for the fractional Laplacian with drift,

$$\min\left\{(-\Delta)^s u + b\cdot\nabla u,\, u-\varphi\right\} = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{3.1}$$

where $b \in \mathbb{R}^n$, and $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a smooth obstacle.

Problem (3.1) appears when considering optimal stopping problems for Lévy processes with jumps. In particular, this kind of obstacle problems are used to model prices of (perpetual) American options; see for example [CF13, BFR18] and references therein for more details. See also [Sal12] and [KKP16] for further references and motivation on the fractional obstacle problem.

We study the regularity of solutions and the corresponding free boundaries for problem (3.1). Note that the value of $s \in (0,1)$ plays an essential role. Indeed, if

$s > \frac{1}{2}$, then the gradient term is of lower order with respect to $(-\Delta)^s$, and thus one expects solutions to behave as in the case $b \equiv 0$. When $s < \frac{1}{2}$ the leading term is $b \cdot \nabla u$ and thus one does not expect regularity results for (3.1). Finally, in the borderline case $s = \frac{1}{2}$ there is an interplay between $b \cdot \nabla u$ and $(-\Delta)^{1/2}$, and one may still expect some regularity, but it becomes a delicate issue.

In this work we study this critical regime, $s = \frac{1}{2}$. As explained in detail below, we establish the $C^{1,\alpha}$ regularity of the free boundary near regular points, with a fine description of the solution at such points.

It is important to remark that, when $s = \frac{1}{2}$, problem (3.1) is equivalent to the *thin* obstacle problem in $\mathbb{R}^{n+1}_+$ with an *oblique* derivative condition on $\{x_{n+1} = 0\}$. Thus, our results yield in particular the regularity of the free boundary for such problem, too.

3.1.1 Known results

The regularity of solutions and free boundaries for (3.1) was first studied in [Sil07, CSS08] when $b = 0$. In [CSS08], Caffarelli, Salsa, and Silvestre established the optimal $C^{1,s}$ regularity for the solutions and $C^{1,\alpha}$ regularity of the free boundary around regular points. More precisely, they proved that given any free boundary point $x_0 \in \partial\{u = \varphi\}$, then

(i) either
$$0 < cr^{1+s} \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^{1+s}$$

(ii) or
$$0 \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^2.$$

The set of points satisfying (i) is called the set of *regular points*, and it was proved in [CSS08] that this set is open and $C^{1,\alpha}$.

Later, the singular set — those points at which the contact set has zero density — was studied in [GP09] in the case $s = \frac{1}{2}$. More recently, the regular set was proved to be C^∞ in [JN17, KRS19]; see also [KPS15, DS16]. The complete structure of the free boundary was described in [BFR18] under the assumption $\Delta\varphi \leq 0$. Finally, the results of [CSS08] have been extended to a wide class of nonlocal elliptic operators in [CRS17].

All the previous results are for the case $b = 0$. For the obstacle problem with drift (3.1), Petrosyan and Pop proved in [PP15] the optimal $C^{1,s}$ regularity of solutions in the case $s > \frac{1}{2}$. This result was obtained by means of an Almgren-type monotonicity formula, treating the drift as a lower order term. In [GPPS17], the same authors together with Garofalo and Smit Vega García establish $C^{1,\alpha}$ regularity for the free boundary around regular points, again in the case $s > \frac{1}{2}$. They do so by means of a Weiss-type monotonicity formula and an epiperimetric inequality. The assumption $s > \frac{1}{2}$ is essential in both works in order to treat the gradient as a lower order term.

In the supercritical regime, $s < \frac{1}{2}$, only the linear stationary and evolution problem have been studied. In [Sil12], Silvestre established immediate spatial and temporal Hölder continuity for the solutions to the linear evolution problem; and in [EP16]

Epstein and Pop studied the Sobolev regularity for the linear stationary problem by means of a completely different approach.

3.1.2 Main result

We study the obstacle problem with critical drift

$$\begin{aligned} \min\left\{(-\Delta)^{1/2}u + b\cdot\nabla u,\ u-\varphi\right\} &= 0 \quad \text{in} \quad \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) &= 0. \end{aligned} \tag{3.2}$$

Here b is a fixed vector in $\mathbb{R}^n$, and the obstacle φ is assumed to satisfy

$$\varphi \text{ is bounded},\ \ \varphi\in C^{2,1}(\mathbb{R}^n),\ \text{ and } \{\varphi>0\}\Subset\mathbb{R}^n. \tag{3.3}$$

The solution to (3.2) can be constructed as the smallest supersolution above the obstacle and vanishing at infinity.

Our main result reads as follows.

Theorem 3.1. *Let u be the solution to* (3.2)*, with φ satisfying* (3.3)*, and $b\in\mathbb{R}^n$.*

Let $x_0\in\partial\{u=\varphi\}$ be any free boundary point. Then we have the following dichotomy:

(i) *either*

$$0 < cr^{1+\tilde\gamma(x_0)} \le \sup_{B_r(x_0)} (u-\varphi) \le Cr^{1+\tilde\gamma(x_0)}, \qquad \tilde\gamma(x_0)\in(0,1),$$

for all $r\in(0,1)$,

(ii) *or*

$$0\le \sup_{B_r(x_0)}(u-\varphi)\le C_\varepsilon r^{2-\varepsilon} \qquad \text{for all } \varepsilon>0,\ r\in(0,1).$$

Moreover, the subset of the free boundary satisfying (i) *is relatively open and is locally $C^{1,\alpha}$ for some $\alpha>0$.*

Furthermore, $\tilde\gamma(x_0)$ is given by

$$\tilde\gamma(x_0) = \frac{1}{2} + \frac{1}{\pi}\arctan\big(b\cdot\nu(x_0)\big), \tag{3.4}$$

where $\nu(x_0)$ denotes the unit normal vector to the free boundary at x_0 pointing towards $\{u>\varphi\}$. Finally, for every point x_0 satisfying (i) *we have the expansion*

$$u(x)-\varphi(x) = c_0\Big((x-x_0)\cdot\nu(x_0)\Big)_+^{1+\tilde\gamma(x_0)} + o\left(|x-x_0|^{1+\tilde\gamma(x_0)+\sigma}\right) \tag{3.5}$$

for some $\sigma>0$, and $c_0>0$. The constants σ and α depend only on n and $\|b\|$.

We think it is quite interesting that the growth around free boundary points (and thus, the regularity of the solution) depends on the orientation of the normal vector with respect to the free boundary. To our knowledge, this is the first example of an obstacle-type problem in which this happens.

The previous theorem implies that the solution is C^{1,γ_b} at every free boundary point x_0, with

$$\gamma_b := \frac{1}{2} - \frac{1}{\pi}\arctan(\|b\|). \tag{3.6}$$

Nonetheless, the constants may depend on the point x_0 considered, so that if we want a uniform regularity estimate for u we actually have the following corollary. It establishes almost optimal regularity of solutions.

Corollary 3.2. *Let u be the solution to* (3.2) *for a given obstacle φ of the form* (3.3)*, and a given $b \in \mathbb{R}^n$. Let γ_b given by* (3.6)*. Then, for any $\varepsilon > 0$ we have*

$$\|u\|_{C^{1,\gamma_b-\varepsilon}(\mathbb{R}^n)} \leq C_\varepsilon,$$

where C_ε is a constant depending only on n, $\|b\|$, ε, and $\|\varphi\|_{C^{2,1}(\mathbb{R}^n)}$.

In order to prove Theorem 3.1 we proceed as follows. First, we classify convex global solutions to the obstacle problem by following the ideas in [CRS17]. Then, we show the Lipschitz regularity of the free boundary at regular points, and using the results in [RS19] we find that the free boundary is actually $C^{1,\alpha}$. Finally, to prove (3.5)-(3.4) we need to establish fine regularity estimates up to the boundary in $C^{1,\alpha}$ domains. This is done by constructing appropriate barriers and a blow-up argument in the spirit of [RS16]. Notice that, since we do not have any monotonicity formula for problem (3.2), our proofs are completely different from those in [PP15, GPPS17].

3.1.3 More general nonlocal operators of order 1 with drift

We will show an analogous result for more general nonlocal operators of the form

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x+y)+u(x-y)}{2} - u(x) \right) \frac{\mu(y/|y|)}{|y|^{n+1}} dy, \tag{3.7}$$

with

$$\mu \in L^\infty(\mathbb{S}^{n-1}) \quad \text{satisfying} \quad \mu(\theta) = \mu(-\theta) \;\; \text{and} \;\; 0 < \lambda \leq \mu \leq \Lambda. \tag{3.8}$$

The constants λ and Λ are the ellipticity constants. Notice that the operators L we are considering are of order 1.

The obstacle problem in this case is, then,

$$\begin{array}{rcl} \min\big\{ -Lu + b\cdot\nabla u, u-\varphi \big\} & = & 0 \quad \text{in} \quad \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) & = & 0. \end{array} \tag{3.9}$$

Our main result reads as follows.

Theorem 3.3. *Let L be an operator of the form* (3.7)-(3.8)*. Let u be the solution to* (3.9)*, with φ satisfying* (3.3)*, and $b \in \mathbb{R}^n$.*

Let x_0 be any free boundary point, $x_0 \in \partial\{u = \varphi\}$. Then we have the following dichotomy:

(i) *either*

$$0 < cr^{1+\tilde{\gamma}(x_0)} \leq \sup_{B_r(x_0)} (u-\varphi) \leq Cr^{1+\tilde{\gamma}(x_0)}, \qquad \tilde{\gamma}(x_0) \in (0,1),$$

for all $r \in (0,1)$.

(ii) *or*

$$0 \leq \sup_{B_r(x_0)} (u-\varphi) \leq C_\varepsilon r^{2-\varepsilon} \qquad \text{for all } \varepsilon > 0,\ r \in (0,1).$$

Moreover, the subset of the free boundary satisfying (i) *is relatively open and is locally* $C^{1,\alpha}$ *for some* $\alpha > 0$.

Furthermore, the value of $\tilde{\gamma}(x_0)$ *is given by*

$$\tilde{\gamma}(x_0) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{b\cdot\nu(x_0)}{\chi(\nu(x_0))}\right), \tag{3.10}$$

where $\nu(x_0)$ *denotes the unit normal vector to the free boundary at* x_0 *pointing towards* $\{u > \varphi\}$, *and*

$$\chi(e) = \frac{\pi}{2}\int_{\mathbb{S}^{n-1}} |\theta\cdot e|\mu(\theta)d\theta \qquad \text{for} \quad e \in \mathbb{S}^{n-1}. \tag{3.11}$$

Finally, for any point x_0 *satisfying* (i) *we have the expansion*

$$u(x)-\varphi(x) = c_0\Big((x-x_0)\cdot\nu(x_0)\Big)_+^{1+\tilde{\gamma}(x_0)} + o\left(|x-x_0|^{1+\tilde{\gamma}(x_0)+\sigma}\right)$$

for some $\sigma > 0$, *and* $c_0 > 0$. *The constants* σ *and* α *depend only on* n, *the ellipticity constants, and* $\|b\|$.

This result extends Theorem 3.1, and the dependence on the operator L is reflected in (3.10). For the fractional Laplacian we have $\chi \equiv 1$, and thus (3.10) becomes (3.4).

We will also prove an analogous result to Corollary 3.2 regarding the almost optimal regularity of solutions; see Corollary 3.29.

3.1.4 Structure of the work

We will focus on the proof of Theorem 3.3, from which in particular will follow Theorem 3.1. The paper is organised as follows.

In Section 3.2 we introduce the notation and give some preliminary results regarding nonlocal elliptic problems with drift. In Section 3.3 we establish $C^{1,\tau}$ estimates for solutions to the obstacle problem with critical drift. In Section 3.4 we classify convex global solutions to the problem. In Section 3.5 we introduce the notion of regular points and we prove that blow-ups of solutions around such points converge to convex global solutions. In Section 3.6 we prove $C^{1,\alpha}$ regularity of the free boundary around regular points. In Section 3.7 we establish estimates up to the boundary for the Dirichlet problem with drift in $C^{1,\alpha}$ domains, in particular, finding an expansion of solutions around points of the boundary. In Section 3.8 we combine the results from Sections 3.6 and 3.7 to prove Theorems 3.1 and 3.3. Finally, in Section 3.9, we establish a non-degeneracy property at all points of the free boundary when the obstacle is concave near the coincidence set.

3.2 Notation and preliminaries

We begin our work with a section of notation and preliminaries. Here, we recall some known results regarding nonlocal operators with drift, and we also find a 1-dimensional solution.

Throughout the work we will use the following function in order to avoid a heavy reading, $\gamma : \mathbb{R} \to (0,1)$, given by

$$\gamma(t) := \frac{1}{2} + \frac{1}{\pi} \arctan(t) . \tag{3.12}$$

We next introduce some known results regarding the elliptic problem with drift that will be used. The first one is the following interior estimate.

Proposition 3.4. *Let L be an operator of the form (3.7)-(3.8), and let $b \in \mathbb{R}^n$. Let u solve*

$$(-L + b\cdot\nabla)u = f, \quad in \quad B_1,$$

for some f. Then, if $f \in L^\infty(B_1)$, and for any $\varepsilon > 0$,

$$[u]_{C^{1-\varepsilon}(B_{1/2})} \le C\left(\|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} + \int_{\mathbb{R}^n} \frac{|u(y)|}{1+|y|^{n+1}}dy\right),$$

where C depends only on n, ε, the ellipticity constants, and $\|b\|$.

The proof of Proposition (3.4) is given in [Ser15] in case $b = 0$ (in the much more general context of fully nonlinear equations). The proof of [Ser15] uses the main result in [CD14]. The proof of Proposition 3.4 follows simply by replacing the use of the result [CD14] in [Ser15] by [SS16, Theorem 7.2] or [CD16, Corollary 7.1].

We also need the following boundary Harnack inequality from [RS19].

Theorem 3.5 ([RS19])**.** *Let $U \subset \mathbb{R}^n$ be an open set, let L be an operator of the form (3.7)-(3.8), and let $b \in \mathbb{R}^n$.*

Let $u_1, u_2 \in C(B_1)$ be viscosity solutions to

$$\begin{cases} (-L + b\cdot\nabla)u_i &= 0 \quad in \quad U \cap B_1 \\ u_i &= 0 \quad in \quad B_1 \setminus U, \end{cases} \quad i = 1, 2,$$

and such that

$$u_i \ge 0 \quad in \quad \mathbb{R}^n, \qquad \int_{\mathbb{R}^n} \frac{u_i(y)}{1+|y|^{n+1}}dy = 1, \quad i = 1, 2.$$

Then,

$$0 < cu_2 \le u_1 \le Cu_2 \quad in \quad U \cap B_{1/2},$$

for some constants c and C depending only on n, $\|b\|$, U, and the ellipticity constants.

We will also need the following result.

Theorem 3.6 ([RS19]). *Let $U \subset \mathbb{R}^n$ be a Lipschitz set, let L be an operator of the form (3.7)-(3.8), and let $b \in \mathbb{R}^n$.*

Let $u_1, u_2 \in C(B_1)$ be viscosity solutions to

$$\left\{ \begin{array}{rcll} (-L + b\cdot\nabla)u_i & = & g_i & \text{in} \quad U \cap B_1 \\ u_i & = & 0 & \text{in} \quad B_1 \setminus U, \end{array} \right. , \qquad i = 1, 2,$$

for some functions $g_i \in L^\infty(U \cap B_1)$, $i = 1, 2$. Assume also that

$$u_i \geq 0 \quad \text{in} \quad \mathbb{R}^n, \qquad \int_{\mathbb{R}^n} \frac{u_i(y)}{1 + |y|^{n+1}} dy = 1, \quad i = 1, 2.$$

Then, there exists $\delta > 0$ depending only on n, U, the ellipticity constants, and $\|b\|$ such that, if

$$\|g_i\|_{L^\infty(U\cap B_1)} \leq \delta \quad \text{in} \quad U \cap B_1, \qquad\qquad i = 1, 2,$$

then

$$\left\| \frac{u_1}{u_2} \right\|_{C^\sigma(U \cap B_{1/2})} \leq C,$$

for some constants σ and C depending only on n, U, the ellipticity constants, and $\|b\|$.

Finally, to conclude this section we study how 1-dimensional powers behave with respect to the operator, and in particular, we find a 1-dimensional solution to the problem. This solution is the same as the one that appears as a travelling wave solution in the parabolic fractional obstacle problem for $s = \frac{1}{2}$; see [CF13, Remark 3.7].

Proposition 3.7. *Let $b \in \mathbb{R}$, and let $u \in C(\mathbb{R})$ be defined by*

$$u(x) := (x_+)^\beta,$$

for $\beta \in (0, 1)$. Then u satisfies

$$\begin{aligned} (-\Delta)^{1/2}u + bu' = \beta\big(b\sin(\beta\pi) + \cos(\beta\pi)\big)(x_+)^{\beta-1} \quad &\text{in} \quad \mathbb{R}_+, \\ u \equiv 0 \quad &\text{in} \quad \mathbb{R}_-. \end{aligned}$$

In particular, let us define

$$u_0(x) := C(x_+)^{\gamma(b)},$$

where

$$\gamma(t) := \frac{1}{2} + \frac{1}{\pi}\arctan(t) \in (0, 1).$$

Then, u_0 satisfies

$$\begin{aligned} (-\Delta)^{1/2}u_0 + bu_0' = 0 \quad &\text{in} \quad \mathbb{R}_+, \\ u_0 \equiv 0 \quad &\text{in} \quad \mathbb{R}_-, \end{aligned}$$

i.e., u_0 is a solution to the 1-dimensional non-local elliptic problem with critical drift and with zero Dirichlet conditions in $\mathbb{R}_-$.

Proof. Define the harmonic extension to $\mathbb{R}^2_+$, $\bar{u} = \bar{u}(x, y)$, via the Poisson kernel, so that $\bar{u}(x, 0) = u(x)$, and $-\partial_y \bar{u}(x, 0) = (-\Delta)^{1/2} u(x)$. We have that $\bar{u}$ solves,

$$\begin{cases} \Delta \bar{u} = 0 & \text{in} \quad \mathbb{R}^2 \cap \{y > 0\} \\ \bar{u} = 0 & \text{in} \quad \{x \le 0\} \cap \{y = 0\}. \end{cases} \tag{3.13}$$

For simplicity, define the reflected function $w(x, y) = \bar{u}(-x, y)$, and let us consider that, by separation of variables in polar coordinates, $w(r, \theta) = g(r)h(\theta)$, for $r \ge 0$, $\theta \in [0, \pi]$ (we use the standard variables, $x = r\cos\theta$, $y = r\sin\theta$). Notice that we are considering homogeneous solutions, so that $g(r) = r^\beta$. Then, from (3.13) we get

$$\begin{cases} g''h + r^{-1}g'h + r^{-2}gh'' = 0 & \text{in} \quad \{r > 0\} \cap \{\theta \in (0, \pi)\} \\ h(0) = 0 & \end{cases} \tag{3.14}$$

from which arise that w can be expressed as

$$w(r, \theta) = r^\beta \sin(\beta\theta).$$

Now notice that, for $r > 0$,

$$((-\Delta)^{1/2} u + bu')(r) = (r^{-1}\partial_\theta + b\partial_r) w(r, \theta)\big|_{\theta = \pi} = \beta \left(b \sin(\beta\pi) + \cos(\beta\pi) \right) r^{\beta - 1}.$$

Solving for β we obtain that it is a solution for $\beta = \gamma(b)$. Moreover, notice that for $\beta < \gamma(b)$ it is a supersolution, and for $\beta > \gamma(b)$ a subsolution. □

3.3 $C^{1,\tau}$ regularity of solutions

In this section we prove $C^{1,\tau}$ regularity of solutions to the obstacle problem with critical drift. For this, we use the method in [CRS17, Section 2].

Throughout this section we can consider the wider class of nonlocal operators

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x + y) + u(x - y)}{2} - u(x) \right) \frac{a(y)}{|y|^{n+1}} dy, \tag{3.15}$$

with

$$a \in L^\infty(\mathbb{R}^n) \quad \text{satisfying} \quad a(y) = a(-y) \quad \text{and} \quad \lambda \le a \le \Lambda, \tag{3.16}$$

so that we are dropping the homogeneity condition of the kernel.

Lemma 3.8. *Let L be an operator of the form* (3.15)-(3.16) *and let $b \in \mathbb{R}^n$. Let φ be any obstacle satisfying* (3.3), *and let u be a solution to* (3.9). *Then,*

(a) u is semiconvex, with

$$\partial_{ee} u \ge -\|\varphi\|_{C^{1,1}(\mathbb{R}^n)} \quad \textit{for all} \quad e \in \mathbb{S}^{n-1}.$$

(b) u is bounded, with

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le \|\varphi\|_{L^\infty(\mathbb{R}^n)}.$$

(c) u is Lipschitz, with

$$\|u\|_{\mathrm{Lip}(\mathbb{R}^n)} \le \|\varphi\|_{\mathrm{Lip}(\mathbb{R}^n)}.$$

Proof. The proof is exactly the same as in [CRS17, Lemma 2.1], since the operator $-L + b\cdot\nabla$ still has maximum principle and is translation invariant. □

We next prove the lemma that will yield the $C^{1,\tau}$ regularity of solutions.

Lemma 3.9. *There exist constants $\tau > 0$ and $\delta > 0$ such that the following statement holds true.*

Let L be and operator of the form (3.15)-(3.16), let $b \in \mathbb{R}^n$, and let $u \in \mathrm{Lip}(\mathbb{R}^n)$ be a solution to

$$\begin{array}{rcll} u & \ge & 0 & \text{in } \mathbb{R}^n \\ \partial_{ee} u & \ge & -\delta & \text{in } B_2 \quad \text{for all } e \in \mathbb{S}^{n-1} \\ (-L + b\cdot\nabla)(u - u(\cdot - h)) & \le & \delta|h| & \text{in } \{u > 0\} \cap B_2 \quad \text{for all } h \in \mathbb{R}^n, \\ & & & \text{in the viscosity sense.} \end{array}$$

satisfying the growth condition

$$\sup_{B_R} |\nabla u| \le R^\tau \quad \text{for} \quad R \ge 1.$$

Assume that $u(0) = 0$. Then,

$$|\nabla u(x)| \le 2|x|^\tau.$$

The constants τ and δ depend only on n, the ellipticity constants and $\|b\|$.

Proof. The proof is very similar to that of [CRS17, Lemma 2.3].

Define

$$\theta(r) := \sup_{\bar r \ge r} \left\{ (\bar r)^{-\tau} \sup_{B_{\bar r}} |\nabla u| \right\}$$

Note that, by the growth control on the gradient, $\theta(r) \le 1$ for $r \ge 1$. Note also that θ is nonincreasing by definition.

To get the desired result, it is enough to prove $\theta(r) \le 2$ for all $r \in (0,1)$. Assume by contradiction that $\theta(r) > 2$ for some $r \in (0,1)$, so that from the definition of θ, there will be some $\bar r \in (r, 1)$ such that

$$(\bar r)^{-\tau} \sup_{B_{\bar r}} |\nabla u| \ge (1-\varepsilon)\theta(r) \ge (1-\varepsilon)\theta(\bar r) \ge \frac{3}{2},$$

for some small $\varepsilon > 0$ to be chosen later.

We now define

$$\bar u(x) := \frac{u(\bar r x)}{\theta(\bar r)(\bar r)^{1+\tau}},$$

and

$$L_{\bar r} w(x) := \int_{\mathbb{R}^n} \left(\frac{w(x+y) + w(x-y)}{2} - w(x) \right) \frac{a(\bar r y)}{|y|^{n+1}} dy$$

Notice that $L_{\bar r}$ is still of the form (3.15)-(3.16).

The rescaled function satisfies

$$\begin{array}{rcll} \bar u & \geq & 0 & \text{in } \mathbb{R}^n \\ D^2\bar u & \geq & -(\bar r)^{2-1-\tau}\delta \mathrm{Id} \geq -\delta\mathrm{Id} & \text{in } B_{2/\bar r} \supset B_2 \\ (-L_{\bar r} + b\cdot\nabla)(\bar u - \bar u(\cdot - \bar h)) & \leq & (\bar r)^{-\tau}\delta|\bar r \bar h| \leq \delta |\bar h| & \text{in } \{\bar u > 0\}\cap B_2 \\ & & & \text{for all } h \in \mathbb{R}^n, \end{array}$$

Moreover, by definition of θ and $\bar r$, the rescaled function $\bar u$ also satisfies

$$1-\varepsilon \leq \sup_{|\bar h|\leq 1/4}\sup_{B_1} \frac{\bar u - \bar u(\cdot - \bar h)}{|\bar h|} \quad \text{and} \quad \sup_{|\bar h|\leq 1/4}\sup_{B_R} \frac{\bar u - \bar u(\cdot - \bar h)}{|\bar h|} \leq (R+1/4)^\tau \tag{3.17}$$

for all $R \geq 1$.

Let $\eta \in C^2_c(B_{3/2})$ with $\eta \equiv 1$ in B_1, $\eta \leq 1$ in $B_{3/2}$. Then,

$$\sup_{|\bar h|\leq 1/4}\sup_{B_{3/2}} \left(\frac{\bar u - \bar u(\cdot - \bar h)}{|\bar h|} + 3\varepsilon\eta\right) \geq 1 + 2\varepsilon.$$

Fix $h_0 \in B_{1/4}$ such that

$$t_0 := \max_{\overline{B_{3/2}}} \left(\frac{\bar u - \bar u(\cdot - h_0)}{|h_0|} + 3\varepsilon\eta\right) \geq 1 + \varepsilon.$$

and let $x_0 \in \overline{B_{3/2}}$ be such that

$$\frac{\bar u(x_0) - \bar u(x_0 - h_0)}{|h_0|} + 3\varepsilon\eta(x_0) = t_0. \tag{3.18}$$

Let us denote

$$v(x) := \frac{\bar u(x) - \bar u(x - h_0)}{|h_0|}.$$

Then, we have

$$v + 3\varepsilon\eta \leq v(x_0) + 3\varepsilon\eta(x_0) = t_0 \quad \text{in} \quad \overline{B_{3/2}}.$$

Moreover, if τ is taken small enough then

$$\sup_{B_4} v \leq (4 + 1/4)^\tau < 1 + \varepsilon \leq t_0,$$

so that in particular x_0 is in the interior of $B_{3/2}$, and

$$v + 3\varepsilon\eta \leq t_0 \quad \text{in } \overline{B_3}. \tag{3.19}$$

Note also that $x_0 \in \{\bar u > 0\}$ since otherwise $\bar u(x_0) - \bar u(x_0 - h_0)$ would be a nonpositive number.

We now evaluate the equation for v at x_0 to obtain a contradiction. To do so, recall that $D^2\bar u \geq -\delta\mathrm{Id}$ in B_2, $\bar u \geq 0$ in $\mathbb{R}^n$, and $\bar u(0) = 0$. It follows that, for $z \in B_2$ and $t' \in (0,1)$,

$$\bar u(t'z) \leq t'\bar u(z) + (1-t')\bar u(0) + \frac{\delta|z|^2}{2}t'(1-t') \leq \bar u(z) + \frac{\delta|z|^2}{2}t'(1-t')$$

and thus, for $t \in (0,1)$, setting $z = x(1+t/|x|)$ and $t' = 1/(1+t/|x|)$ we obtain, for $x \in B_1$,

$$\bar u(x) - \bar u\left(x + t\frac{x}{|x|}\right) \le \frac{\delta}{2}(|x|+t)^2 \frac{t/|x|}{(1+t/|x|)^2} = \frac{\delta |x| t}{2} \le \delta t.$$

Therefore, denoting $e = h_0/|h_0|$, $t = |h_0| \le 1$ and using that by (3.17), if τ small enough,

$$\|\bar u\|_{\mathrm{Lip(B_1)}} \le \frac{4}{3},$$

we obtain

$$\begin{aligned} v(x) = \frac{\bar u(x) - \bar u(x-te)}{t} &\le \frac{\bar u(x) - \bar u(x-te)}{t} + \frac{\bar u\left(x + t\frac{x}{|x|}\right) - \bar u(x)}{t} + \delta \\ &\le \frac{\bar u\left(x + t\frac{x}{|x|}\right) - \bar u(x-te)}{t} + \delta \\ &\le \frac{4}{3}\left|e + \frac{x}{|x|}\right| + \delta \le \frac{1}{4} \end{aligned} \tag{3.20}$$

in $\mathcal{C}_e \cap B_1$ provided δ is taken smaller than $1/12$; where $\mathcal{C}_e$ is the cone,

$$\mathcal{C}_e := \left\{ x \,:\, \left| e + \frac{x}{|x|}\right| \le \frac{1}{8}\right\}.$$

On the other hand, we know that

$$v(x_0+y) - v(x_0) \le 3\varepsilon\big(\eta(x_0) - \eta(x_0+y)\big) \quad \text{in} \quad B_3. \tag{3.21}$$

This allows us to define

$$\phi(x_0+y) = \begin{cases} v(x_0) + 3\varepsilon\big(\eta(x_0) - \eta(x_0+y)\big) & \text{in } B_{1/8} \\ v(x_0+y) & \text{otherwise.} \end{cases}$$

Notice that ϕ is regular around x_0 and that $\phi \ge v$ everywhere, and recall that $(-L_{\bar r} + b\cdot\nabla)v(x_0) \le \delta$ in the viscosity sense. Therefore, we have

$$-L_{\bar r}\phi(x_0) - C\|b\|\varepsilon \le (-L_{\bar r} + b\cdot\nabla)\phi(x_0) \le \delta. \tag{3.22}$$

Now, using

$$1 - 2\varepsilon \le v(x_0) \le 1 + \varepsilon,$$

and defining

$$\delta\phi(x,y) := \frac{\phi(x+y) + \phi(x-y)}{2} - \phi(x),$$

we can bound $\delta\phi(x_0, y)$ as

$$\delta\phi(x_0,y) \le \begin{cases} C\varepsilon|y|^2 & \text{in} \quad B_2 \\ (|y|+2)^\tau - 1 + 2\varepsilon & \text{in} \quad \mathbb{R}^n \setminus B_1 \\ -3/8 + C\varepsilon & \text{in} \quad (-x_0 + \mathcal{C}_e \cap B_1) \setminus B_{1/4}. \end{cases}$$

The first inequality follows because around x_0 and from (3.21) we have the bound $\delta\phi(x_0,y) \le \frac{3}{2}\varepsilon\left(2\eta(x_0) - \eta(x_0+y) - \eta(x_0-y)\right)$ and η is a C^2 function. The second inequality follows from (3.17), and using that $\frac{1}{2}\left(|x_0+y|+\frac{1}{4}\right)^\tau + \frac{1}{2}\left(|x_0-y|+\frac{1}{4}\right)^\tau \le (|y|+2)^\tau$. For the third inequality, notice that

$$\begin{aligned}\delta\phi(x_0,y) &= \frac{v(x_0+y)-v(x_0)}{2} + \frac{v(x_0-y)-v(x_0)}{2}\\ &\le \frac{1}{8} - \frac{1}{2} + \epsilon + C\varepsilon \le -\frac{3}{8} + C\varepsilon \quad \text{ in } \quad (-x_0 + C_e \cap B_1)\setminus B_{1/4},\end{aligned}$$

where we have used (3.20) to bound the first term and (3.21) to bound the second one. The constant C depends only on the η, so it is independent of everything else.

We then find

$$\begin{aligned}L_{\bar r}\phi(x_0) &\le \Lambda\int_{B_1} C\varepsilon|y|^2|y|^{-n-1}dy + \Lambda\int_{\mathbb{R}^n\setminus B_1}\left\{(|y|+2)^\tau - 1 + 2\varepsilon\right\}|y|^{-n-1}dy\\ &\quad + \lambda\int_{(-x_0+\mathcal{C}_e\cap B_1)\setminus B_{1/4}}\left(-\frac{3}{8}+C\varepsilon\right)|y|^{-n-1}dy\\ &\le C\varepsilon + C\int_{\mathbb{R}^n\setminus B_{1/2}}\left\{(|y|+2)^\tau - 1\right\}|y|^{-n-1}dy - c,\end{aligned}$$

with $c > 0$ independent of δ and τ (for ε small).

Thus, combining with (3.22) we get

$$c - C\left((\|b\|+1)\varepsilon + \int_{\mathbb{R}^n\setminus B_{1/2}}\frac{(|y|+2)^\tau - 1}{|y|^{n+1}}dy\right) \le -C\|b\|\varepsilon - \tilde{L}_{\bar r}\phi(x_0) \le \delta. \quad (3.23)$$

If ε and τ are taken small enough so that the left-hand side in (3.23) is greater than $c/2$, we get a contradiction for $\delta \le c/4$. ☐

The following proposition implies that the solution to the obstacle problem (3.9) is $C^{1,\tau}$ for some $\tau > 0$.

Proposition 3.10. *Let L be any operator of the form (3.15)-(3.16), let $b \in \mathbb{R}^n$, and let $u \in \mathrm{Lip}(\mathbb{R}^n)$ with $u(0) = 0$ be any function satisfying, for all $h \in \mathbb{R}^n$ and $e \in \mathbb{S}^{n-1}$, and for some $\varepsilon > 0$,*

$$\begin{array}{rcll} u & \ge & 0 & \text{in } \mathbb{R}^n\\ \partial_{ee}u & \ge & -K & \text{in } B_2\\ (-L + b\cdot\nabla)(u - u(\cdot - h)) & \le & K|h| & \text{in } \{u>0\}\cap B_2\\ |\nabla u| & \le & K(1+|x|^{1-\varepsilon}) & \text{in } \mathbb{R}^n. \end{array}$$

Then, there exists a small constant $\tau > 0$ such that

$$\|u\|_{C^{1,\tau}(B_{1/2})} \le CK.$$

The constants τ and C depend only on n, $\|b\|$, ε, and the ellipticity constants.

Proof. The proof is standard and it is exactly the same as the proof of [CRS17, Proposition 2.4] by means of Lemma 3.9. ☐

3.4 Classification of convex global solutions

In this section we prove the following theorem, that classifies all convex global solutions to the obstacle problem with critical drift.

Theorem 3.11. *Let L be an operator of the form* (3.7)-(3.8)*. Let $\Omega \subset \mathbb{R}^n$ be a closed convex set, with $0 \in \Omega$. Let $u \in C^1(\mathbb{R}^n)$ a function satisfying, for all $h \in \mathbb{R}^n$,*

$$\left\{\begin{array}{rcll} (-L + b\cdot\nabla)(\nabla u) & = & 0 & \text{in} \quad \mathbb{R}^n \setminus \Omega \\ (-L + b\cdot\nabla)(u - u(\cdot - h)) & \leq & 0 & \text{in} \quad \mathbb{R}^n \setminus \Omega \\ D^2 u & \geq & 0 & \text{in} \quad \mathbb{R}^n \\ u & = & 0 & \text{in} \quad \Omega \\ u & \geq & 0 & \text{in} \quad \mathbb{R}^n. \end{array}\right. \tag{3.24}$$

Assume also the following growth control satisfied by u,

$$\|\nabla u\|_{L^\infty(B_R)} \leq R^{1-\varepsilon} \quad \text{for all} \quad R \geq 1, \tag{3.25}$$

for some $\varepsilon > 0$. Then, either $u \equiv 0$, or

$$\Omega = \{e \cdot x \leq 0\} \quad \text{and} \quad u(x) = C(e\cdot x)_+^{1+\gamma(b\cdot e/\chi(e))}, \tag{3.26}$$

for some $e \in \mathbb{S}^{n-1}$ and $C > 0$. The value of $\chi(e)$ is given by (3.11) *with the kernel μ of L, and γ is given by* (3.12)*.*

We start by proving the following proposition.

Proposition 3.12. *Let Σ be a non-empty closed convex cone, and let L be an operator of the form* (3.7)-(3.8)*. Let u_1 and u_2 be two non-negative continuous functions satisfying*

$$\int_{\mathbb{R}^n} \frac{u_i(y)}{1+|y|^{n+1}}dy < \infty, \quad i = 1, 2.$$

Assume, also, that they are viscosity solutions to

$$\left\{\begin{array}{rcll} (-L + b\cdot\nabla)u_i & = & 0 & \text{in} \quad \mathbb{R}^n \setminus \Sigma \\ u_i & = & 0 & \text{in} \quad \Sigma \\ u_i & > & 0 & \text{in} \quad \mathbb{R}^n \setminus \Sigma. \end{array}\right.$$

Then,

$$u_1 \equiv K u_2 \quad \text{in} \quad \mathbb{R}^n,$$

for some constant K.

Proof. The proof is the same as the proof of [CRS17, Theorem 3.1], using the boundary Harnack inequality in Theorem 3.5.

Suppose, without loss of generality, that $\Sigma \subsetneq \mathbb{R}^n$. Take P a point with $|P| = 1$ and $B_r(P) \subset \mathbb{R}^n \setminus \Sigma$ for some $r > 0$, and assume that $u_i(P) = 1$. We want to prove $u_1 \equiv u_2$.

Define, given $R \geq 1$,

$$\bar{u}_i(x) = \frac{u_i(Rx)}{C_i},$$

with C_i such that $\int_{\mathbb{R}^n} \bar u_i(y)(1+|y|)^{-n-1}dy = 1$. Thus, by Theorem 3.5 there exists some $c > 0$ such that

$$\bar u_1 \geq c\bar u_2 \quad \text{and} \quad \bar u_2 \geq c\bar u_1 \quad \text{in} \quad B_{1/2}. \tag{3.27}$$

In particular, $\bar u_1(P/R)$ and $\bar u_2(P/R)$ are comparable, so that C_1 and C_2 are comparable. Thus, from (3.27),

$$u_1 \geq cu_2 \quad \text{and} \quad u_2 \geq cu_1 \quad \text{in} \quad B_{R/2},$$

for any $R \geq 1$, so that the previous inequalities are true in $\mathbb{R}^n$.

Now take

$$\bar c := \sup\{c > 0 : u_1 \geq cu_2 \quad \text{in} \quad \mathbb{R}^n\} < \infty.$$

Define

$$v = u_1 - \bar c u_2 \geq 0.$$

Either $v \equiv 0$ in $\mathbb{R}^n$ or $v > 0$ in $\mathbb{R}^n \setminus \Sigma$ by the strong maximum principle. If $v \equiv 0$ we are done, because in this case $\bar c = 1$ due to the fact that $u_1(P) = u_2(P) = 1$.

Let us assume then that $v > 0$ in $\mathbb{R}^n \setminus \Sigma$. Apply the first part of the proof to $v/v(P)$ and u_2 to deduce that, for some $\delta > 0$, $v > \delta u_2$. This contradicts the definition of $\bar c$, so $v \equiv 0$ as we wanted. □

We can now prove the classification of convex global solutions in Theorem 3.11

Proof of Theorem 3.11. First, by the same blow-down argument in [CRS17, Theorem 4.1], we can restrict ourselves to the case in which $\Omega = \Sigma$ for Σ a closed convex cone in $\mathbb{R}^n$ with vertex at 0.

We now split the proof into two cases:

Case 1: When Σ has non empty interior there are n linearly independent unitary vectors e_i such that $-e_i \in \Sigma$. Define

$$v_i := \partial_{e_i} u,$$

and note that, since $D^2u \geq 0$ and $-e_i \in \Sigma = \{u = 0\}$, we have

$$\begin{cases} (-L + b\cdot\nabla)v_i &= 0 & \text{in} & \mathbb{R}^n \setminus \Sigma \\ v_i &= 0 & \text{in} & \Sigma \\ v_i &\geq 0 & \text{in} & \mathbb{R}^n. \end{cases} \tag{3.28}$$

From Proposition 3.12, we must have $v_i = a_i v_k$ for some $1 \leq k \leq n$, $a_i \in \mathbb{R}$, and for all $i = 1, \dots, n$, so that $\partial_{e_i - a_i e_k} u \equiv 0$ in $\mathbb{R}^n$ for all $i \neq k$. Thus, there exists a non-negative function $\phi : \mathbb{R} \to \mathbb{R}$, $\phi \in C^1$, such that $u = \phi(e \cdot x)$ for some $e \in \mathbb{S}^{n-1}$; so that, since $0 \in \partial\Sigma$, $\Sigma = \{e \cdot x \leq 0\}$.

Notice that $\phi' \geq 0$ solves $(-L + (b\cdot e)\partial)(\phi') = 0$ in $\mathbb{R}_+$ and $\phi' \equiv 0$ in $\mathbb{R}_-$, with the growth $\phi'(t) \leq C(1 + t^{1-\varepsilon})$. From [RS14, Lemma 2.1], we have

$$(\chi(e)(-\Delta)^{1/2} + (b\cdot e)\partial)(\phi') = 0 \quad \text{in} \quad \mathbb{R}_+,$$

where $\chi(e)$ is given by (3.11). Now, a non-negative solution to the previous equation is given by Proposition 3.7. Such solution is unique up to a multiplicative constant thanks to Proposition 3.12. Indeed, notice that the hypotheses of the lemma are fulfilled due to the growth control of ϕ' and the fact that $\phi' \geq 0$. Thus, we obtain

$$\phi(t) = (t_+)^{1+\gamma(b\cdot e)/\chi(e)} \quad \text{for} \quad t \in \mathbb{R},$$

where γ and χ are given by (3.12) and (3.11) respectively.

Case 2: If Σ has empty interior then by convexity it must be contained in some hyperplane $H = \{x \cdot e = 0\}$. From Proposition 3.10, rescaling,

$$[\nabla u]_{C^\tau(B_R)} \leq C(R),$$

for some constant $C(R)$ depending on R; and for any $R \geq 1$. In particular, for any $h \in \mathbb{R}^n$, if we define

$$v(x) = u(x) - u(x-h) \quad \text{for} \quad x \in \mathbb{R}^n,$$

then $v \in C^{1,\tau}_{\text{loc}}(\mathbb{R}^n)$. This implies that $(-L + b\cdot\nabla)v \in C^\tau_{\text{loc}}(\mathbb{R}^n)$, but we already knew that $(-L + b\cdot\nabla)v = 0$ in $\mathbb{R}^n \setminus H$, so we must have

$$(-L + b\cdot\nabla)v = 0 \quad \text{in} \quad \mathbb{R}^n.$$

Now, from the interior estimates in Proposition 3.4 rescaled on balls B_R we have

$$R^{1-\varepsilon/2}[v]_{C^{1-\varepsilon/2}(B_{R/2})} \leq C\left(\|v\|_{L^\infty(B_R)} + \int_{\mathbb{R}^n} \frac{|v(Ry)|}{1+|y|^{n+1}}dy\right).$$

On the other hand, from the growth control on the gradient, we have

$$\|v\|_{L^\infty(B_R)} \leq |h| R^{1-\varepsilon}.$$

Putting the last two expressions together we reach

$$[v]_{C^{1-\varepsilon/2}(B_{R/2})} \leq \frac{C|h|}{R^{\varepsilon/2}}.$$

Now let $R \to \infty$ to obtain that v must be constant for all h. That means that u is affine, but $u(0) = 0$ and $u \geq 0$ in $\mathbb{R}^n$, so $u \equiv 0$. $\square$

3.5 Blow-ups at regular points

By subtracting the obstacle if necessary and dividing by $C\|\varphi\|_{C^{2,1}(\mathbb{R}^n)}$, we can assume that we are dealing with the following problem,

$$\left\{\begin{array}{rcll} u & \geq & 0 & \text{in} \quad \mathbb{R}^n \\ (-L + b\cdot\nabla)u & \leq & f & \text{in} \quad \mathbb{R}^n \\ (-L + b\cdot\nabla)u & = & f & \text{in} \quad \{u > 0\} \\ D^2 u & \geq & -\text{Id} & \text{in} \quad \mathbb{R}^n. \end{array}\right. \tag{3.29}$$

Moreover, dividing by a bigger constant if necessary, we can also assume that

$$\|f\|_{C^1(\mathbb{R}^n)} \leq 1, \tag{3.30}$$

and that

$$\|u\|_{C^{1,\tau}(\mathbb{R}^n)} \leq 1. \tag{3.31}$$

The validity of the last expression and the constant τ come from Proposition 3.10 and Lemma 3.8.

Let us now introduce the notion of *regular* free boundary point.

Definition 3.1. We say that $x_0 \in \partial\{u > 0\}$ is a *regular* free boundary point with exponent ε if

$$\limsup_{r \downarrow 0} \frac{\|u\|_{L^\infty(B_r(x_0))}}{r^{2-\varepsilon}} = \infty$$

for some $\varepsilon > 0$.

The following proposition states that an appropriate blow up sequence of the solution around a regular free boundary point converges in C^1 norm to a convex global solution.

Proposition 3.13. *Let L be an operator of the form* (3.7)-(3.8)*, and let $b \in \mathbb{R}^n$. Let u be a solution to* (3.29)-(3.30)-(3.31)*. Assume that* 0 *is a regular free boundary point with exponent ε.*

Then, given $\delta > 0$, $R_0 \geq 1$, there exists $r > 0$ such that the rescaled function

$$v(x) := \frac{u(rx)}{r\|\nabla u\|_{L^\infty(B_r)}}$$

satisfies

$$\|\nabla v\|_{L^\infty(B_R)} \leq 2R^{1-\varepsilon} \quad \textit{for all} \quad R \geq 1,$$

$$\left|(-L + b \cdot \nabla)(\nabla v)\right| \leq \delta \quad \textit{in} \quad \{v > 0\},$$

and

$$|v - u_0| + |\nabla v - \nabla u_0| \leq \delta \quad \textit{in} \quad B_{R_0},$$

for some u_0 of the form (3.26) *and with $\|\nabla u_0\|_{L^\infty(B_1)} = 1$.*

Before proving the previous proposition, let us prove the following lemma.

Lemma 3.14. *Assume $u \in C^1(B_1)$ satisfies $\|\nabla u\|_{L^\infty(\mathbb{R}^n)} = 1$, $u(0) = 0$, and*

$$\sup_{\rho \leq r} \frac{\|u\|_{L^\infty(B_r)}}{r^{2-\varepsilon}} \to \infty \quad \textit{as} \quad \rho \downarrow 0.$$

Then, there exists a sequence $r_k \downarrow 0$ such that $\|\nabla u\|_{L^\infty(B_{r_k})} \geq \frac{1}{2} r_k^{1-\varepsilon}$, and for which the rescaled functions

$$u_k(x) = \frac{u(r_k x)}{r_k \|\nabla u\|_{L^\infty(B_{r_k})}}$$

satisfy

$$|\nabla u_k(x)| \leq 2(1 + |x|^{1-\varepsilon}) \quad \textit{in} \quad \mathbb{R}^n.$$

Proof. Define

$$\theta(\rho) := \sup_{r\geq\rho} \frac{\|\nabla u\|_{L^\infty(B_r)}}{r^{1-\varepsilon}}.$$

Notice that, since $u(0) = 0$, we have

$$\frac{\|u\|_{L^\infty(B_r)}}{r^{2-\varepsilon}} \leq \frac{\|\nabla u\|_{L^\infty(B_r)}}{r^{1-\varepsilon}}.$$

Therefore, $\theta(\rho) \to \infty$ as $\rho \downarrow 0$, and notice also that θ is non-increasing.

Now, for every $k \in \mathbb{N}$, there is some $r_k \geq \frac{1}{k}$ such that

$$r_k^{\varepsilon-1}\|\nabla u\|_{L^\infty(B_{r_k})} \geq \frac{1}{2}\theta\left(1/k\right) \geq \frac{1}{2}\theta(r_k). \tag{3.32}$$

Since $\|\nabla u\|_{L^\infty(\mathbb{R}^n)} = 1$, then

$$r_k^{\varepsilon-1} \geq \frac{1}{2}\theta(1/k) \to \infty \quad \text{as} \quad k \to \infty,$$

so that $r_k \to 0$ as $k \to \infty$. We also have $\theta(r_k) \geq 1$, and therefore $\|\nabla u\|_{L^\infty(B_{r_k})} \geq \frac{1}{2}r_k^{1-\varepsilon}$.

Finally, from the definition of θ and (3.32), and for any $R \geq 1$, we have

$$\|\nabla u_k\|_{L^\infty(B_R)} = \frac{\|\nabla u\|_{L^\infty(B_{r_k R})}}{\|\nabla u\|_{L^\infty(B_{r_k})}} \leq \frac{\theta(r_k R)(r_k R)^{1-\varepsilon}}{\frac{1}{2}(r_k)^{1-\varepsilon}\theta(r_k)} \leq 2R^{1-\varepsilon},$$

which follows from the monotonicity of θ. □

We can now prove Proposition 3.13, which follows taking the sequence of rescalings given by Lemma 3.14 together with a compactness argument.

Proof of Proposition 3.13. Let $r_k \downarrow 0$ be the sequence given by Lemma 3.14. Therefore, the functions

$$v_k(x) = \frac{u(r_k x)}{r_k\|\nabla u\|_{L^\infty(B_{r_k})}}$$

satisfy

$$\|\nabla v_k\|_{L^\infty(B_R)} \leq 2R^{1-\varepsilon} \quad \text{for all} \quad R \geq 1,$$

and

$$\|\nabla v_k\|_{L^\infty(B_1)} = 1, \quad v_k(0) = 0.$$

Moreover,

$$D^2 v_k = \frac{r_k}{\|\nabla u\|_{L^\infty(B_{r_k})}} D^2 u \geq -\frac{r_k}{\|\nabla u\|_{L^\infty(B_{r_k})}}\mathrm{Id},$$

and, in $\{v_k > 0\}$,

$$\begin{aligned}\left|(-L + b\cdot\nabla)(\nabla v_k)\right| &= \frac{r_k}{\|\nabla u\|_{L^\infty(B_{r_k})}}\left|(-L + b\cdot\nabla)(\nabla u)\right| \\ &\leq \frac{r_k}{\|\nabla u\|_{L^\infty(B_{r_k})}}\|\nabla f\|_{L^\infty} \leq \frac{r_k}{\|\nabla u\|_{L^\infty(B_{r_k})}}.\end{aligned}$$

Notice that, from (3.32) and with the notation from the proof of Lemma 3.14,

$$\frac{1}{\eta_k} := \frac{\|\nabla u\|_{L^\infty(B_{r_k})}}{r_k} \geq \frac{\theta(r_k)}{2r_k^\varepsilon} \to \infty, \quad \text{as} \quad r_k \downarrow 0.$$

Thus, in all we have a sequence v_k such that $v_k \in C^1$, $v_k(0) = 0$, and

$$\|\nabla v_k\|_{L^\infty(B_R)} \leq 2R^{1-\varepsilon} \quad \text{for all} \quad R \geq 1,$$

$$\left|(-L + b\cdot\nabla)(\nabla v_k)\right| \leq \eta_k \quad \text{in} \quad \{v_k > 0\},$$

$$D^2 v_k \geq -\eta_k \mathrm{Id},$$

with $\eta_k \downarrow 0$. From the estimates in Proposition 3.10,

$$\|\nabla v_k\|_{C^\tau(B_R)} \leq C(R) \quad \text{for all} \quad R \geq 1,$$

for some constant depending on R, $C(R)$. Thus, up to taking a subsequence, v_k converges in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ to some v_∞ which by stability of viscosity solutions is a convex global solution to the obstacle problem (3.24) fulfilling (3.25).

By the classification theorem, Theorem 3.11, v_∞ must be of the form (3.26). Taking limits

$$\|\nabla v_\infty\|_{L^\infty(B_1)} = 1$$

and $v_\infty(0) = 0$. Now the result follows because $\eta_k \downarrow 0$ and v_k converge in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ to v_∞. □

3.6 $C^{1,\alpha}$ regularity of the free boundary around regular points

In this section we prove $C^{1,\alpha}$ regularity of the free boundary around regular points.

We begin by proving the Lipschitz regularity of the free boundary, as stated in the following proposition.

Proposition 3.15. *Let L be an operator of the form* (3.7)-(3.8)*, and let $b \in \mathbb{R}^n$. Let u be a solution to* (3.29)-(3.30)-(3.31)*. Assume that* 0 *is a regular free boundary point.*

Then, there exists a vector $e \in \mathbb{S}^{n-1}$ such that for any $\ell > 0$, there exists an $r > 0$ and a Lipschitz function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\{u > 0\} \cap B_r = \big\{y_n > g(y_1, \dots, y_{n-1})\big\} \cap B_r,$$

where $y = Rx$ is a change of coordinates given by a rotation R with $Re = e_n$, and g fulfils

$$\|g\|_{\mathrm{Lip}(B_r)} \leq \ell.$$

Moreover, $\partial_{e'} u \geq 0$ in B_r for all $e' \cdot e \geq \frac{\ell}{\sqrt{1+\ell^2}}$.

The following lemma will be needed in the proof, and it is analogous to [CRS17, Lemma 6.2].

Lemma 3.16. *There exists* $\eta = \eta(n, \Lambda, \lambda, \|b\|)$ *such that the following statement holds.*

Let L *be an operator of the form* (3.7)-(3.8), *and let* $b \in \mathbb{R}^n$. *Let* $E \subset B_1$ *be relatively closed, and assume that, in the viscosity sense,* $w \in C(B_1)$ *satisfies*

$$\left\{ \begin{array}{rcll} (-L + b \cdot \nabla) w & \geq & -\eta & \textit{in} \quad B_1 \setminus E \\ w & = & 0 & \textit{in} \quad E \cup (\mathbb{R}^n \setminus B_2) \\ w & \geq & -\eta & \textit{in} \quad B_2 \setminus E, \end{array} \right. \tag{3.33}$$

and

$$\int_{B_1} w_+ \geq 1.$$

Then, w *is non-negative in* $B_{1/2}$, *i.e.,*

$$w \geq 0 \quad \textit{in} \quad B_{1/2}.$$

Proof. Let us argue by contradiction, and suppose that the statement does not hold for any $\eta > 0$. Define $\psi \in C^2_c(B_{3/4})$ be a radial function with $\psi \geq 0$, $\psi \equiv 1$ in $B_{1/2}$ and with $|\nabla \psi| \leq C(n)$. Let

$$\psi_t(x) := -\eta - t + \eta\psi(x).$$

If w attains negative values on $B_{1/2}$, then there exists some $t_0 > 0$ and $z \in B_{3/4}$ such that ψ_{t_0} touches w from below at z, i.e. $\psi_{t_0} \leq w$ everywhere and $\psi_{t_0}(z) = w(z) < 0$. Let $\delta > 0$ be such that $w < 0$ in $B_\delta(z)$ (recall w continuous). Let us now define

$$\bar{w}(x) := \left\{ \begin{array}{ll} w(x) & \text{if} \quad x \in \mathbb{R}^n \setminus B_\delta(z) \\ \psi_{t_0}(x) & \text{if} \quad x \in B_\delta(z). \end{array} \right. \tag{3.34}$$

Notice that $\bar{w}$ is C^2 around z, and is such that $\bar{w} \leq w$. By definition of viscosity supersolution, we have

$$(-L + b \cdot \nabla)\bar{w}(z) \geq -\eta.$$

On the one hand, this implies

$$(-L + b \cdot \nabla)(\bar{w} - \psi_{t_0})(z) \geq -C\eta,$$

for some C depending on n, the ellipticity constants, and $\|b\|$. On the other hand, we can evaluate $\bar{w} - \psi_{t_0}$ classically at z,

$$\begin{aligned} (-L+b \cdot \nabla)(\bar{w} - \psi_{t_0})(z) &= -L(\bar{w} - \psi_{t_0})(z) \\ &\leq -\lambda \int_{\mathbb{R}^n} (\bar{w} - \psi_{t_0})(z+y)|y|^{-n-1} dy \leq -c(n)\lambda \int_{B_1 \setminus B_\delta(z)} (\bar{w} - \psi_{t_0}) dy \\ &\leq -c(n)\lambda \int_{B_1} w^+ dy \leq -c(n)\lambda. \end{aligned}$$

We used here that $(\bar{w} - \psi_{t_0})\chi_{B_1 \setminus B_\delta(z)} \geq w^+$ in B_1.

In all, for η small enough depending only on n, the ellipticity constants, and $\|b\|$, we reach a contradiction. □

With the previous lemma and the results from the previous section, we can now prove Proposition 3.15.

Proof of Proposition 3.15. Let $\delta > 0$ and R_0 to be chosen, and consider the rescaled function from Proposition 3.13,

$$v(x) = \frac{u(rx)}{r\|\nabla u\|_{L^\infty(B_r)}}.$$

Thanks to Proposition 3.13, there exists some $e \in \mathbb{S}^{n-1}$ such that

$$\left|\nabla v - (x\cdot e)_+^{\gamma(b\cdot e/\chi(e))} e\right| \le \delta \quad \text{in} \quad B_{R_0}.$$

Recall γ and χ are given by (3.12)-(3.11).

Now let $e' \in \mathbb{S}^{n-1}$ be such that (assuming $\ell \le 1$)

$$e'\cdot e \ge \frac{\ell}{\sqrt{1+\ell^2}} \ge \frac{\ell}{2}.$$

Notice that

$$\nabla v\cdot e' \ge \frac{\ell}{2}(x\cdot e)_+^{\gamma(b\cdot e/\chi(e))} - \delta \quad \text{in} \quad B_{R_0},$$

and

$$\left|(-L + b\cdot\nabla)(\nabla v\cdot e')\right| \le \delta \quad \text{in} \quad \{v > 0\}.$$

Define

$$w = \frac{C_1}{\ell}(\nabla v\cdot e')\chi_{B_2},$$

for some C_1 such that

$$\int_{B_1} w^+ \ge 1.$$

Notice that, if δ is small enough, then C_1 depends only on n, ℓ, $\|b\|$, and the ellipticity constants.

Let us call $E = \{v = 0\}$. If R_0 is large enough, depending only on n, ℓ, ε, $\|b\|$, δ, and the ellipticity constants, then w satisfies

$$\begin{cases} (-L + b\cdot\nabla)w \ge -\frac{CC_1}{\ell}\delta \ge -\eta & \text{in} \quad B_1\setminus E \\ w = 0 & \text{in} \quad E\cup(\mathbb{R}^n\setminus B_2) \\ w \ge -\frac{C_1}{\ell}\delta \ge -\eta & \text{in} \quad B_2\setminus E. \end{cases} \tag{3.35}$$

We are using here that, for $x \in B_1\setminus E$,

$$\begin{aligned} (-L + b\cdot\nabla)w(x) &\ge -\frac{C_1}{\ell}\delta - (-L+b\cdot\nabla)\left(\frac{C_1}{\ell}(\nabla v\cdot e')\chi_{B_2^c}\right)(x) \\ &\ge -\frac{C_1}{\ell}\delta + \frac{C_1}{\ell}L(\nabla v\cdot e')\chi_{B_2^c}(x) \\ &\ge -\frac{C_1}{\ell}\delta + \lambda\frac{C_1}{\ell}\int_{B_{R_0-1}} \frac{(\nabla v\cdot e')\chi_{B_2^c}(x+y) + (\nabla v\cdot e')\chi_{B_2^c}(x-y)}{2|y|^{n+1}} \\ &\quad + \lambda\frac{C_1}{\ell}\int_{B_{R_0-1}^c} \frac{(\nabla v\cdot e')\chi_{B_2^c}(x+y) + (\nabla v\cdot e')\chi_{B_2^c}(x-y)}{2|y|^{n+1}} \\ &\ge -\frac{C_1}{\ell}\delta - \lambda\frac{C_1}{\ell}\hat{C}\delta - \hat{c} \ge -\frac{CC_1}{\ell}\delta, \end{aligned}$$

where R_0 is chosen large enough so that $\hat{c}$ can be comparable to the other terms (which can be done, thanks to the fact that ∇v grows as $R^{1-\varepsilon}$). Notice that C depends only on λ and n.

In all, we can choose δ small enough so that

$$\frac{CC_1}{\ell}\delta \le \eta$$

for the constant η given in Lemma 3.16.

Therefore, applying Lemma 3.16 to the function w we get that

$$w \ge 0 \quad \text{in} \quad B_{1/2},$$

or equivalently,

$$\partial_{e'} u \ge 0 \quad \text{in} \quad B_{r/2},$$

for all $e' \in \mathbb{S}^{n-1}$ such that $e' \cdot e \ge \frac{\ell}{\sqrt{1+\ell^2}}$. This implies that $\partial\{u > 0\}$ is Lipschitz in B_r, with Lipschitz constant smaller than ℓ. □

Finally, combining Proposition 3.15 with the boundary regularity result in Theorem 3.6 we show that the free boundary is $C^{1,\alpha}$ around regular points.

Proposition 3.17. *Let L be an operator of the form* (3.7)-(3.8)*, and let $b \in \mathbb{R}^n$. Let u be a solution to* (3.29)-(3.30)-(3.31)*. Assume that x_0 is a regular free boundary point.*

Then, there exists $r > 0$ such that the free boundary is $C^{1,\alpha}$ in $B_r(x_0)$ for some $\alpha > 0$ depending only on n, $\|b\|$, and the ellipticity constants.

Proof. Without loss of generality assume $x_0 = 0$ and that $\nu(0) = e_n$, where $\nu(0)$ denotes the normal vector to the free boundary at 0 pointing towards $\{u > 0\}$.

By Proposition 3.15, we already know the free boundary is Lipschitz around 0, with Lipschitz constant 1 in a ball B_ρ. Let $v_1 = \frac{1}{\sqrt{2}}(\partial_i u + \partial_n u)$ for any fixed $i \in \{1, \dots, n-1\}$, and let $v_2 = \partial_n u$. We first show that for some $r > 0$ and $\alpha > 0$,

$$\left\|\frac{v_1}{v_2}\right\|_{C^\alpha(\{u>0\}\cap B_r)} = \frac{1}{\sqrt{2}}\left\|1 + \frac{\partial_i u}{\partial_n u}\right\|_{C^\alpha(\{u>0\}\cap B_r)} \le C. \tag{3.36}$$

Define w as in the proof of Proposition 3.15, i.e., $w = C_1(\nabla v \cdot e')\chi_{B_2}$, where v is the rescaling given by Proposition 3.13, and e' is such that $e' \cdot e \ge \frac{\ell}{2}$ (choose $\ell = 1$ for example).

From the proof of Proposition 3.15 we know that $w \ge 0$ in $B_{1/2}$ (if, using the same notation, R_0 is large enough and δ is small enough; i.e., the rescaling defining v is appropriately chosen). Now define

$$\tilde{w} = C_1(\nabla v \cdot e')_+$$

and notice that

$$\left|(-L + b\cdot\nabla)\tilde{w}\right| \le \eta \quad \text{in} \quad B_{1/4} \setminus \{v = 0\}$$

for some $\eta > 0$ that can be made arbitrarily small by choosing the appropriate (small) $\delta > 0$ and (large) R_0 in the rescaling given by Proposition 3.13. The previous inequality follows from the fact that $(\nabla v\cdot e')_- \le \delta$ in B_{R_0}, $(\nabla v\cdot e')_- \le 2\,(1+|x|^{1-\varepsilon})$ in $B_{R_0^c}$, and $(\nabla v\cdot e')_- \equiv 0$ in $B_{1/2}$.

Let $e_{in} := \frac{1}{\sqrt{2}}(e_i+e_n)$, and define $w_1 = C\,(\nabla v\cdot e_{in})_+$ and $w_2 = C(\nabla v\cdot e_n)_+$ (taking $e' = e_{in}$ and $e' = e_n$). Now notice that w_1 and w_2 fulfil the hypotheses of the boundary regularity result in Theorem 3.6, and $w_1 = C(\nabla v\cdot e_{in})$ and $w_2 = C(\nabla v\cdot e_n)$ in $B_{1/2}$. Thus, applying Theorem 3.6 to w_1 and w_2 we obtain that there exists some $\alpha > 0$ such that

$$\left\|\frac{w_1}{w_2}\right\|_{C^\alpha(\{v>0\}\cap B_{1/8})} \le C.$$

Going back to the rescalings defining $\tilde{w}$ we reach that for some $r>0$, (3.36) holds.

Once we have (3.36) the procedure is standard. Notice that the components of the normal vector to the level sets $\{u=t\}$ for $t>0$ can be written as

$$\nu^i(x) = \frac{\partial_i u}{|\nabla u|}(x) = \frac{\partial_i u/\partial_n u}{\left(\sum_{j=1}^{n-1}(\partial_j u/\partial_n u)^2+1\right)^{1/2}},$$

$$\nu^n(x) = \frac{\partial_n u}{|\nabla u|}(x) = \frac{1}{\left(\sum_{j=1}^{n-1}(\partial_j u/\partial_n u)^2+1\right)^{1/2}},$$

for $u(x) = t > 0$. In particular, from the regularity of $\partial_i u/\partial_n u$ given by (3.36), we obtain ν is C^α on these level sets; that is, $|\nu(x)-\nu(y)| \le C|x-y|^\alpha$ whenever $x, y \in \{u=t\}\cap B_r$. Now let $t \downarrow 0$ and we are done. □

3.7 Estimates in $C^{1,\alpha}$ domains

Once we know that the free boundary is $C^{1,\alpha}$ around regular points, we need to find the expansion of the solution (3.5) around such points. To do so, we establish fine boundary regularity estimates for solutions to elliptic problem with critical drift in arbitrary $C^{1,\alpha}$ domains. That is the aim of this section.

The main result of this section is the following, for the Dirichlet problem with the operator $-L + b\cdot\nabla$ in $C^{1,\alpha}$ domains. We will use it on the derivatives of the solution to the obstacle problem.

Theorem 3.18. *Let L be an operator of the form* (3.7)-(3.8)*, let $b\in\mathbb{R}^n$ and let Ω be a $C^{1,\alpha}$ domain.*

Let $f \in L^\infty(\Omega\cap B_1)$, and suppose $u\in L^\infty(\mathbb{R}^n)$ satisfies

$$\begin{cases} (-L+b\cdot\nabla)u &= f \quad \text{in } \Omega\cap B_1 \\ \qquad\qquad u &= 0 \quad \text{in } B_1\setminus\Omega. \end{cases} \tag{3.37}$$

Then, for each boundary point $x_0 \in B_{1/2}\cap\partial\Omega$, there exists a constant Q with $|Q| \le C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega\cap B_1)}\right)$ such that for all $x\in B_1$

$$\left|u(x) - Q\big((x-x_0)\cdot\nu(x_0)\big)_+^{\tilde\gamma(x_0)}\right| \le C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega\cap B_1)}\right)|x-x_0|^{\tilde\gamma(x_0)+\sigma},$$

where $\sigma>0$ and $\nu(x_0)$ is the normal unit vector to $\partial\Omega$ at x_0 pointing towards the interior of Ω, and $\tilde\gamma(x_0)$ is defined in (3.10)*. The constant C depends only on n, α,*

Ω, the ellipticity constants, and $\|b\|$; and the constant σ depends only on n, α, the ellipticity constants, and $\|b\|$.

To prove Theorem 3.18 we will need several ingredients.

3.7.1 A supersolution and a subsolution

In this section we denote

$$d(x) := \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega).$$

We will also use the following.

Definition 3.2. Given a $C^{1,\alpha}$ domain Ω, we consider ϱ a regularised distance function to $C^{1,\alpha}$; i.e., a function that satisfies

$$\tilde{K}^{-1} d \le \varrho \le \tilde{K} d,$$

$$\|\varrho\|_{C^{1,\alpha}(\Omega)} \le \tilde{K} \quad \text{and} \quad |D^2\varrho| \le \tilde{K} d^{\alpha-1},$$

where the constant $\tilde{K}$ depends only on α and the domain Ω.

The existence of such regularised distance was discussed, for example, in [RS15, Remark 2.2].

We next construct a supersolution, needed in our proof of Theorem 3.18.

Proposition 3.19 (Supersolution)**.** *Let L be an operator of the form* (3.7)-(3.8), *and let $b \in \mathbb{R}^n$. Let Ω be a $C^{1,\alpha}$ domain for some $\alpha > 0$, and suppose $0 \in \partial\Omega$.*

Let $\nu : \partial\Omega \to \mathbb{S}^{n-1}$ be the outer normal vector at the points of the boundary of Ω, let γ be defined by (3.12)*, and χ by* (3.11)*. Let us also define*

$$\gamma_0 := \gamma\left(\frac{b\cdot\nu(0)}{\chi(\nu(0))}\right),$$

and

$$\eta_\nu := \inf\left\{\eta \ge 0 : \gamma\left(\frac{b\cdot\nu(x)}{\chi(\nu(x))}\right) \ge \gamma_0 - \eta \quad \forall x \in \partial\Omega \cap B_1\right\}. \tag{3.38}$$

Let $\phi := \varrho^\kappa$ for a fixed $0 < \kappa < \gamma_0 - 2\eta_\nu$, and where ϱ is the regularised distance given by Definition 3.2. Then, there exist $\delta > 0$ and $\hat{C} > 0$ such that

$$\begin{cases} \hat{C}(-L + b\cdot\nabla)\phi \ \ge \ 1 & \text{in } B_{1/2} \cap \{x : 0 < d(x) \le \delta\} \\ \hat{C}\phi \ \ge \ 1 & \text{in } B_{1/2} \cap \{x : d(x) \ge \delta\}. \end{cases} \tag{3.39}$$

The constants δ and $\hat{C}$ depend only on n, Ω, κ, the ellipticity constants, and $\|b\|$.

Proof. Pick any $x_0 \in B_{1/2} \cap \{x : d(x) \le \delta\}$, and define

$$l_0(x) = \big(\varrho(x_0) + \nabla\varrho(x_0)\cdot(x - x_0)\big)_+.$$

Notice that, whenever $l_0 > 0$, if we define $\hat{\varrho}_0 := \frac{\nabla\varrho(x_0)}{|\nabla\varrho(x_0)|}$ and $z = \hat{\varrho}_0 \cdot x$ then

$$\begin{aligned} (-L + b\cdot\nabla)l_0^\kappa(x) &= \big(\chi(\hat{\varrho}_0)(-\Delta)^{1/2} + (b\cdot\hat{\varrho}_0)\,\partial\big)\,\big(|\nabla\varrho(x_0)|z + c_0\big)_+^\kappa \\ &= |\varrho(x_0)|\chi(\hat{\varrho}_0)c\big(\kappa, b\cdot\hat{\varrho}_0/\chi(\hat{\varrho}_0)\big)\big(|\nabla\varrho(x_0)|z + c_0\big)_+^{\kappa-1}, \end{aligned}$$

where $c_0 = \varrho(x_0) - \nabla\varrho(x_0)\cdot x_0$, and $c(\kappa, b\cdot\hat{\varrho}_0/\chi(\hat{\varrho}_0))$ is the constant arising from Proposition 3.7. We want to check that this constant is positive, which is equivalent to saying (again, from Proposition 3.7) that

$$\kappa < \gamma\left(\frac{b\cdot\hat{\varrho}_0}{\chi(\hat{\varrho}_0)}\right).$$

To see this, it is enough to check that

$$\gamma_0 - 2\eta_\nu \leq \gamma\left(\frac{b\cdot\hat{\varrho}_0}{\chi(\hat{\varrho}_0)}\right),$$

which will be true for some small $\delta > 0$ and for any $x_0 \in B_{1/2}\cap\{x : d(x)\leq\delta\}$ if

$$\lim_{\delta\downarrow 0}\inf_{\substack{y\in B_{1/2}\\ 0<d(y)\leq\delta}}\sup_{x\in\partial\Omega\cap B_{3/4}}\frac{\nabla\varrho(y)}{|\nabla\varrho(y)|}\cdot\nu(x) = 1,$$

i.e., $\nabla\varrho$ normalised is close to some unit normal vector to the boundary as δ goes to zero (notice that γ and χ are continuous). But this is true since ϱ is a $C^{1,\alpha}$ function, so in particular, its gradient is continuous, and the boundary is a level set of ϱ; i.e., $\nabla\varrho(y) = |\nabla\varrho(y)|\nu(y)$ for any y on the boundary. It is important to remark that the modulus of continuity of $\nabla\varrho$ depends only on Ω.

Now notice that

$$l_0(x_0) = \varrho(x_0) \qquad \nabla l_0(x_0) = \nabla\varrho(x_0). \tag{3.40}$$

Let $\tilde{\varrho}$ be a $C^{1,\alpha}(\mathbb{R}^n)$ extension of ϱ to the whole $\mathbb{R}^n$ with $\varrho\leq 0$ in $\mathbb{R}^n\setminus\Omega$. Then we have

$$\big|\varrho(x_0) + \nabla\varrho(x_0)\cdot y - \tilde{\varrho}(x_0+y)\big| \leq C|y|^{1+\alpha}.$$

By using that $|a_+ - b_+| \leq |a-b|$ we find

$$\big|l_0(x_0+y) - \varrho(x_0+y)\big| \leq C|y|^{1+\alpha}.$$

Now, also using that $|a^t - b^t| \leq |a-b|(a^{t-1}+b^{t-1})$ for $a, b\geq 0$, $|a^t-b^t|\leq C|a-b|^t$, and saying $d_0 = d(x_0)$ we get

$$|\phi - l_0^\kappa|(x_0+y) \leq \begin{cases} Cd_0^{\kappa-1}|y|^{1+\alpha} & \text{for} \quad y\in B_{d_0/(\tilde{K}+1)} \\ C|y|^{(1+\alpha)\,\kappa} & \text{for} \quad y\in B_1\setminus B_{d_0/(\tilde{K}+1)} \\ C|y|^\kappa & \text{for} \quad y\in\mathbb{R}^n\setminus B_1. \end{cases} \tag{3.41}$$

We have used here that, in $B_{d_0/(\tilde{K}+1)}$, $l_0^{\kappa-1}\leq Cd_0^{\kappa-1}$ and $\varrho^{\kappa-1}\leq Cd_0^{\kappa-1}$. Here, $\tilde{K}$ denotes the constant given in Definition 3.2. Putting all together

$$\begin{aligned}
(-L+&b\cdot\nabla)\phi(x_0) = \\
&= (-L+b\cdot\nabla)(\phi - l_0^\kappa)(x_0) + (-L+b\cdot\nabla)l_0^\kappa(x_0)\\
&\geq L(l_0^\kappa - \phi)(x_0) + c(\kappa)d_0^{\kappa-1}\\
&= \int_{\mathbb{S}^{n-1}}\int_0^\infty \left((l_0^\kappa-\phi)(x_0+r\theta) + (l_0^\kappa-\phi)(x_0-r\theta)\right)\frac{dr}{r^2}d\mu(\theta) + c(\kappa)d_0^{\kappa-1}\\
&\geq -C\left(\int_0^{d_0/(\tilde{K}+1)}\frac{d_0^{\kappa-1}r^{1+\alpha}}{r^2}dr + \int_{d_0/(\tilde{K}+1)}^1\frac{r^{(1+\alpha)\,\kappa}}{r^2}dr + \int_1^\infty\frac{r^\kappa}{r^2}dr\right) + c(\kappa)\rho^{\kappa-1}\\
&\geq -Cd_0^{\kappa-1+\alpha} - Cd_0^{(1+\alpha)\,\kappa-1} + c(\kappa)d_0^{\kappa-1}.
\end{aligned}$$

Notice that the right-hand side tends to $+\infty$ as $\delta \downarrow 0$ independently of the x_0 chosen. Thus, we can choose δ small enough so that the right-hand side is greater than 1. Then, by choosing $\hat{C} \geq 1$ such that $\hat{C}\phi \geq 1$ in $B_{1/2} \cap \{x : d(x) > \delta\}$ we are done. $\square$

We can similarly find a subsolution for the problem. It will be used in the next section.

Lemma 3.20 (Subsolution). *Let L be an operator of the form* (3.7)-(3.8), *and let $b \in \mathbb{R}^n$. Let Ω be a $C^{1,\alpha}$ domain for some $\alpha > 0$, and suppose $0 \in \partial\Omega$.*

Let $\nu : \partial\Omega \to \mathbb{S}^{n-1}$ be the outer normal vector at the points of the boundary of Ω, let γ be defined by (3.12), *and χ by* (3.11). *Let us also define*

$$\gamma_0 := \gamma\left(\frac{b \cdot \nu(0)}{\chi(\nu(0))}\right),$$

and

$$\eta_\nu^{(2)} := \inf\left\{\eta \geq 0 : \gamma\left(\frac{b \cdot \nu(x)}{\chi(\nu(x))}\right) \leq \gamma_0 + \eta \quad \forall x \in \partial\Omega \cap B_1\right\}. \tag{3.42}$$

Let $\phi := \varrho^{\kappa_2}$ for any fixed $1 > \kappa_2 > \gamma_0 + 2\eta_\nu^{(2)}$. Then, there exist $\delta > 0$ and $\hat{C} > 0$ such that

$$\begin{cases} (-L + b \cdot \nabla)\phi \leq -1 & \text{in } B_{1/2} \cap \{x : 0 < d(x) \leq \delta\} \\ \phi \leq \hat{C} & \text{in } B_{1/2} \cap \{x : d(x) > \delta\}. \end{cases} \tag{3.43}$$

The constants δ and $\hat{C}$ depend only on n, Ω, κ_2, the ellipticity constants, and $\|b\|$.

Proof. The proof follows by the same steps as the proof of Proposition 3.19. Using the same notation, one just needs to notice that when evaluating

$$(-L + b \cdot \nabla)l_0^{\kappa_2}(x) = c\big(\kappa_2, b \cdot \hat{\varrho}_0/\chi(\hat{\varrho}_0)\big)\big(|\nabla\varrho(x_0)|z + c_0\big)_+^{\kappa_2-1},$$

now the constant $c(\kappa_2)$ is negative (independently of the κ_2 chosen, as before). Thus,

$$(-L + b \cdot \nabla)\phi(x_0) \leq Cd_0^{\kappa_2-1+\alpha} + Cd_0^{(1+\alpha)\,\kappa_2-1} + c(\kappa)d_0^{\kappa_2-1},$$

for negative $c(\kappa_2)$, so that if d_0 is small enough we obtain the desired result. $\square$

3.7.2 Hölder continuity up to the boundary in $C^{1,\alpha}$ domains

The aim of this subsection is to prove Proposition 3.21 below. Before doing that, let us introduce a definition.

Definition 3.3. We say that $\Gamma \subset \mathbb{R}^n$ is a $C^{1,\alpha}$ graph splitting B_1 into U^+ and U^- if there exists some $f_\Gamma \in C^{1,\alpha}(\mathbb{R}^{n-1})$ such that

- $\Gamma := \{(x', f_\gamma(x')) \cap B_1 \text{ for } x' \in \mathbb{R}^{n-1}\}$;
- $U^+ := \{(x', x_n) \in B_1 : x_n > f_\Gamma(x')\}$;

- $U^- := \{(x', x_n) \in B_1 : x_n < f_\Gamma(x')\}$.

Under these circumstances, we refer to the $C^{1,\alpha}$ norm of Γ as $\|f_\Gamma\|_{C^{1,\alpha}(D')}$, where $D' := \{x' \in \mathbb{R}^n : (x', f_\Gamma(x')) \in B_1\}$.

Proposition 3.21. *Let L be an operator of the form (3.7)-(3.8), and let $b \in \mathbb{R}^n$. Let Γ be a $C^{1,\alpha}$ graph splitting B_1 into U^+ and U^-, according to Definition 3.3, and suppose $0 \in \Gamma$.*

Let $f \in L^\infty(U^+)$, let $g \in C^\beta(\overline{U_-})$, and suppose $u \in C(\overline{B_1})$ satisfying the growth condition $|u(x)| \le M(1+|x|)^\Upsilon$ in $\mathbb{R}^n$ for some $\Upsilon < 1$. Assume also that u satisfies in the viscosity sense

$$\begin{cases} (-L + b\cdot\nabla)u &= f \quad \text{in } U^+ \\ u &= g \quad \text{in } U^-. \end{cases} \tag{3.44}$$

Then there exists some $\sigma > 0$ such that $u \in C^\sigma(\overline{B_{1/2}})$ with

$$\|u\|_{C^\sigma(B_{1/2})} \le C\big(\|u\|_{L^\infty(B_1)} + \|g\|_{C^\beta(U^-)} + \|f\|_{L^\infty(U^+)} + M\big).$$

The constants C and σ depend only on n, α, the $C^{1,\alpha}$ norm of Γ, Υ, the ellipticity constants, and $\|b\|$.

Proof. Let $\tilde u = u\chi_{B_1}$ so that $(-L + b\cdot\nabla)\tilde u = f + L(u\chi_{B_1^c}) =: \tilde f$ in $U^+ \cap B_{3/4}$, and $\tilde u = g$ in U^-. Note that $\|\tilde f\|_{L^\infty(U^+\cap B_{3/4})} \le C(\|f\|_{L^\infty(U^+)} + M) =: C_0$ for some constant C depending only on n, Υ, and the ellipticity constants.

We begin by proving that for some small $\epsilon > 0$, and for some C, we have

$$\|\tilde u - g(z)\|_{L^\infty(B_r(z))} \le Cr^\epsilon \quad \text{for all} \quad r \in (0,1), \quad \text{and for all} \quad z \in \Gamma \cap B_{1/2}, \tag{3.45}$$

where $\epsilon > 0$ and C depend only on n, C_0, $\|u\|_{L^\infty(B_1)}$, $\|g\|_{C^\beta(U^-)}$, the ellipticity constants, and $\|b\|$.

Let us define a $C^{1,\alpha}$ domain that will be used in this proof, analogous to a fixed ball if the surface Γ was $C^{1,1}$.

Thus, we define P as a fixed $C^{1,\alpha}$ bounded convex domain with diameter 1 that coincides with $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n \ge |(x_1, \dots, x_{n-1})|^{1+\alpha}\}$ in $B_{1/2}$. Let y_P be a fixed point inside the domain, which will be treated as the *center*. Let us call P_R the rescaled version of such domain with diameter R and *center* y_{P_R}, and let us define

$$P_R^{(\delta)} := \{x \in \mathbb{R}^n : \operatorname{dist}(x, P_R) \le \delta\}.$$

As an abuse of notation we will also call P_R any rotated and translated version that will be given by the context.

Note that, since Γ is $C^{1,\alpha}$, there exists some $\rho_0 \in (0,1)$ depending on the $C^{1,\alpha}$ norm of Γ such that any point $z \in \Gamma \cap B_{1/2}$ can be touched by some P_{ρ_0} rotated and translated correspondingly and contained completely in U^-.

Let us now consider the supersolution given by Proposition 3.19 with respect to the domain $\mathbb{R}^n \setminus P$.

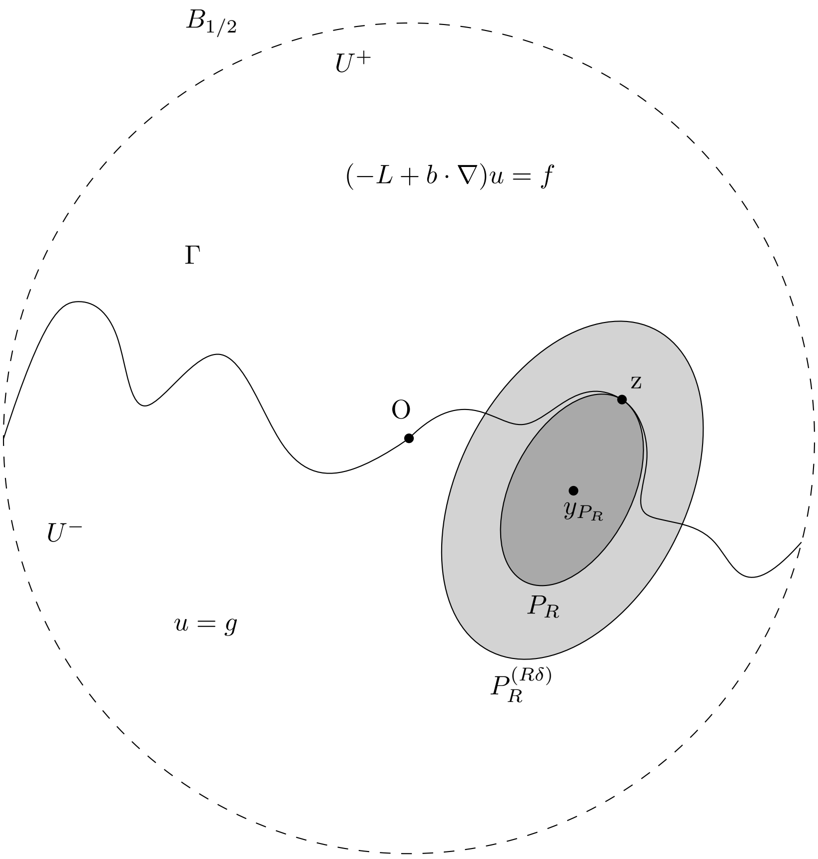

Figure 3.1: Sketch of the ball $B_{1/2}$ split into U^+ and U^-, and a domain P_R tangentially touching the boundary Γ.

That is, there is some function ϕ_P such that, for some constants $\delta > 0$ and C fixed,

$$\begin{cases} (-L+b\cdot\nabla)\phi_P & \geq & 1 & \text{in} & P^{(\delta)}\setminus P \\ \phi_P & \geq & 1 & \text{in} & \mathbb{R}^n\setminus P^{(\delta)} \\ \phi_P & = & 0 & \text{in} & P \\ \phi_P & \leq & Cd^\kappa & \text{in} & \mathbb{R}^n, \end{cases} \tag{3.46}$$

where $d = \mathrm{dist}(x, P)$ and $0 < \kappa < \min\left\{\gamma\left(\frac{b'\cdot e}{\chi(e)}\right) : \|b'\| = \|b\|, e \in \mathbb{S}^{n-1}\right\}$ can also be fixed — recall that γ and χ are given by (3.12)-(3.11).

Let P' be a rotated version of P, and let $\phi_{P'}$ be the corresponding rotated supersolution. Notice that we can assume that $\phi_{P'}$ also fulfils (3.46) (with P' instead of P), since while the operator $(-L+b\cdot\nabla)$ is not rotation invariant, only an extra positive constant arises depending on the ellipticity constants and $\|b\|$.

Given a rotated, scaled and translated version of the domain P, P_R, we will denote the corresponding supersolution (the rotated, scaled and translated version of ϕ_P) by ϕ_{P_R}.

Let now $z \in \Gamma\cap B_{1/2}$. For any $R \in (0, \rho_0)$ there exists some rescaled, rotated and translated domain $P_R \subset U^-$ touching Γ at z. Recall that y_{P_R} is the *center* of the domain P_R, so that in particular $|z - y_{P_R}| = C_P R$ for some constant C_P that only depends on the domain P chosen ($C_P \in (0, 1)$ because the domain P_R has diameter

R). See Figure 3.1 for a representation of this situation.

Recall that ϕ_{P_R} is the supersolution corresponding to the domain P_R, with the δ given by Proposition 3.19 (now, when rescaling, δ becomes $R\delta$). Define the function

$$\psi(x) = g(y_{P_R}) + \|g\|_{C^\beta(U^-)}\big((1+\delta)R\big)^\beta + \big(C_0 + \|u\|_{L^\infty(B_1)}\big)\phi_{P_R}.$$

Note that ψ is above $\tilde{u}$ in $U^- \cap P_R^{(R\delta)}$, since $\tilde{u} = g$ there and the distance from y_{P_R} to any other point in $P_R^{(R\delta)}$ is at most $(1+\delta)R$.

On the other hand, in $P_R^{(R\delta)} \setminus P_R$ we have $(-L + b\cdot\nabla)\psi \geq (C_0 + \|u\|_{L^\infty(B_1)})R^{-1} \geq C_0 \geq (-L + b\cdot\nabla)\tilde{u}$ since $R \leq \rho_0 < 1$; and outside $P_R^{(R\delta)}$ we have $\tilde{u} \leq \psi$. In all, $\tilde{u} \leq \psi$ everywhere by the maximum principle, and thus for any $R \in (0, \rho_0)$

$$\tilde{u}(x) - g(z) \leq C\big(R^\beta + (r/R)^\kappa\big) \quad \text{for all} \quad x \in B_r(z) \quad \text{and for all} \quad r \in (0, R\delta),$$

for some constant C that depends only on n, C_0, $\|u\|_{L^\infty(B_1)}$, $\|g\|_{C^\beta(U^-)}$, the ellipticity constants, and $\|b\|$. If R is small enough we can take $r = R^2$, and repeat this reasoning upside down to get that

$$\|\tilde{u} - g(z)\|_{L^\infty(B_r(z))} \leq C\left(r^{\beta/2} + r^{\kappa/2}\right) \leq Cr^\epsilon \quad \text{for all} \quad r \in (0, \delta^2),$$

for $\epsilon = \min\left\{\frac{\beta}{2}, \frac{\kappa}{2}\right\}$. This yields the result (3.45) by taking a larger C if necessary.

Now let $x, y \in B_{1/2}$, and let $r = |x - y|$. We will show

$$|u(x) - u(y)| \leq Cr^\sigma,$$

for some $\sigma > 0$. If $x, y \in U^-$ we are done by the regularity of g. If $x \in U^+$, $y \in U^-$, we can take z in the segment between x and y, on the boundary Γ, and compare x and y to z, so that it is enough to consider $x, y \in U^+$.

Let $R = \mathrm{dist}(x, \Gamma) \geq \mathrm{dist}(y, \Gamma)$, and suppose $x_0, y_0 \in \Gamma$ are such that $\mathrm{dist}(x, \Gamma) = \mathrm{dist}(x, x_0)$ and $\mathrm{dist}(y, \Gamma) = \mathrm{dist}(y, y_0)$. By interior estimates for the problem (see Proposition 3.4),

$$[u]_{C^\epsilon(B_{R/2}(x))} \leq CR^{-\epsilon}. \tag{3.47}$$

Let $r < 1$, and let us separate two different cases

- Suppose $r \geq R^2/2$. Then, using (3.45) and the regularity of g we obtain

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\ &\leq CR^\epsilon + C(2R + r)^\beta \\ &\leq C(r^{\epsilon/2} + r^{\beta/2}) \leq Cr^{\epsilon/2}. \end{aligned}$$

- Assume $r \leq R^2/2$, so that $y \in B_{R/2}(x)$. Thus, using (3.47),

$$|u(x) - u(y)| \leq CR^{-\epsilon}r^\epsilon \leq Cr^{\epsilon/2}.$$

In all, we have found $u \in C^\sigma(B_{1/2})$ for $\sigma = \epsilon/2$. □

Remark 3.1. When U is C^∞, the above Hölder estimate follows from the results in [S94], [CD01]. We thank G. Grubb for pointing this out to us.

3.7.3 A Liouville theorem

We next prove a Liouville-type theorem in the half-space for non-local operators with critical drift, that will be used to prove Theorem 3.18.

Theorem 3.22. *Let L be an operator of the form* (3.7)-(3.8)*, and let $b \in \mathbb{R}^n$. Let u be any weak solution to*

$$\begin{cases} (-L + b\cdot\nabla)u = 0 & \text{in } \mathbb{R}^n_+ \\ u = 0 & \text{in } \mathbb{R}^n_-. \end{cases} \tag{3.48}$$

Assume also that for some $\varepsilon > 0$ and some constant C, u satisfies

$$\|u\|_{L^\infty(B_R)} \le CR^{1-\varepsilon} \quad \text{for all} \quad R \ge 1.$$

Then,

$$u(x) = C(x_n)_+^{\gamma(b_n/\chi)}, \tag{3.49}$$

for some $C > 0$, and where b_n is the n-th component of b. The constant χ is defined by $\chi = \chi(e_n)$ where $\chi(e)$ is given by (3.11)*, and γ is given by* (3.12)*.*

Before proving the Liouville theorem, let us prove it in the 1-dimensional case.

Notice that from Proposition 3.12 it already follows that any non-negative solution must be either $u \equiv 0$ or the one found in Proposition 3.7. Here, however, we need the same result for solutions that may change sign.

Proposition 3.23. *Let $b \in \mathbb{R}$, and let $u \in C(\mathbb{R})$ be a function satisfying*

$$(-\Delta)^{1/2}u + bu' = 0 \quad \text{in} \quad \mathbb{R}_+, \qquad u \equiv 0 \quad \text{in} \quad \mathbb{R}_-,$$

and $|u(x)| \le C(1 + |x|^{1-\varepsilon})$ for some $\varepsilon > 0$. Then,

$$u(x) = C_0(x_+)^{\gamma(b)},$$

where γ is given by (3.12)*.*

Proof. We first claim that

$$\left\|u/(x_+)^{\gamma(b)}\right\|_{C^\sigma([0,1])} \le C \tag{3.50}$$

for some $\sigma > 0$.

Indeed, let

$$w = \chi_{[0,2]}u + \kappa\chi_{[3/2,2]},$$

and recall that, for some $\hat{C}$,

$$\|u\|_{L^\infty([0,R])} \le \hat{C}R^{1-\varepsilon}.$$

Notice that $w(0) = 0$, and that $w \le C_0(x)_+^{\gamma(b)}$ for $x \ge 1$, if C_0 is big enough depending only on κ and $\hat{C}$. Choose κ so that $(-\Delta)^{1/2}w \le 0$ in $[0,1]$ so that by the maximum principle $u = w \le C_0(x)_+^{\gamma(b)}$ in $[0,1]$. Doing the same for $-u$ we reach that

$$|u| \le C_0(x)_+^{\gamma(b)} \quad \text{for} \quad x \in [0,1].$$

Define now $\tilde{u} = u\chi_{(0,m)} + M(x_+)^{\gamma(b)}$, where $M = M(m)$ is such that $\tilde{u} \geq 0$ in $(0, m)$. Notice that $\tilde{u}$ solves an equation of the form $(-\Delta)^{1/2}\tilde{u} + b\tilde{u}' = f_m(x)$ in $(0, 1)$ for some bounded f_m with $\|f_m\|_{L^\infty(0,1)} \downarrow 0$ as $m \to \infty$. We can now apply Theorem 3.6 with $\tilde{u}$ and $(x_+)^{\gamma(b)}$ to get that for some large enough m,

$$\left\|\tilde{u}/(x_+)^{\gamma(b)}\right\|_{C^\sigma([0,1])} \leq C,$$

for some $\sigma > 0$. Thus, we get (3.50).

Define $v = u - k(x_+)^{\gamma(b)}$, where $k = \lim_{x\downarrow 0} \frac{u(x)}{(x_+)^{\gamma(b)}}$. Then we have

$$|v(x)| \leq C|x|^{1-\varepsilon} \quad \text{for} \quad x \geq 1, \tag{3.51}$$

$$|v(x)| \leq C|x|^{\gamma(b)+\sigma} \quad \text{for} \quad x \in [0, 2], \tag{3.52}$$

and we can assume, without loss of generality, that $1 - \varepsilon > \gamma(b) + \sigma$. Combining this with the interior estimates from Proposition 3.4 we obtain $v \in C^{\gamma(b)+\sigma}([0, 1])$. Indeed, take $x, y \in [0, 1]$, $x < y$. Let $r = y - x$ and $R = |y|$. Now separate two cases

- If $2r \geq R$, by (3.52)

$$\begin{aligned} |v(x) - v(y)| &\leq |v(x)| + |v(y)| \leq C(|x|^{\gamma(b)+\sigma} + |y|^{\gamma(b)+\sigma}) \\ &\leq C\big((R - r)^{\gamma(b)+\sigma} + R^{\gamma(b)+\sigma}\big) \leq Cr^{\gamma(b)+\sigma}. \end{aligned}$$

- If $2r < R$, then $x, y \in (y - R/2, y + R/2)$. By rescaling the estimates from Proposition 3.4 and using (3.51)

$$R^{\gamma(b)+\sigma}[v]_{C^{\gamma(b)+\sigma}\left(y-\frac{R}{2},y+\frac{R}{2}\right)} \leq C\left(\|v\|_{L^\infty(y-R,y+R)} + R^{1-\varepsilon}\right).$$

  Now, from (3.52)

$$\|v\|_{L^\infty(y-R,y+R)} \leq CR^{\gamma(b)+\sigma},$$

  so that

$$[v]_{C^{\gamma(b)+\sigma}\left(y-\frac{R}{2},y+\frac{R}{2}\right)} \leq C.$$

This implies

$$\|v\|_{C^{\gamma(b)+\sigma}([0,1])} \leq C,$$

as desired.

Now, we claim that using the interior estimates from Proposition 3.4 we obtain

$$|v'(x)| \leq C|x|^{-\varepsilon} \quad \text{for} \quad x \geq 1, \tag{3.53}$$

and

$$|v'(x)| \leq C|x|^{\gamma(b)+\sigma-1} \quad \text{for} \quad x \in [0, 1]. \tag{3.54}$$

Let us show that these last inequalities hold. The first one, (3.53), follows using that $|v(x)| \leq C(1+|x|^{1-\varepsilon})$, and that (3.51)-(3.52) combined with the rescaled interior estimates in Proposition 3.4 yield

$$[v]_{C^{\gamma(b)+\sigma}(R,2R)} \leq CR^{1-\varepsilon-\gamma(b)-\sigma} \quad \text{for} \quad R \geq 1. \tag{3.55}$$

Indeed, take $0 < \alpha < \gamma(b) + \sigma$, and any $h \in \mathbb{R}$ with $|h| \le R/2$. Then by interior estimates applied to the incremental quotients,

$$\left[\frac{v(x+h) - v(x)}{|h|^{\gamma(b)+\sigma}}\right]_{C^{1-\alpha}(R,2R)} \le C R^{\alpha-\varepsilon-\gamma(b)-\sigma} \quad \text{for} \quad R \ge 1,$$

with C independent of the h chosen. In particular, this yields

$$[v']_{C^{\gamma(b)+\sigma-\alpha}(R,2R)} \le C R^{\alpha-\varepsilon-\gamma(b)-\sigma} \quad \text{for} \quad R \ge 1.$$

The inequality in (3.53) now follows comparing the value of $v'(2^k)$ for any $k \in \mathbb{N}$ with $v'(1)$ dyadically.

For the second inequality, (3.54), we proceed similarly. Take $0 < \alpha < \gamma(b) + \sigma$, and for any $R > 0$ fixed take $|h| \le R/2$ and notice that

$$\left[\frac{v(x+h) - v(x)}{|h|^{\gamma(b)+\sigma}}\right]_{C^{1-\alpha}(R,2R)} \le C R^{\alpha-1} \quad \text{for} \quad 0 < R < 1, \tag{3.56}$$

with C independent of h. This follows from the interior estimates in Proposition 3.4 and the growth of $\frac{v(x+h)-v(x)}{|h|^{\gamma(b)+\sigma}}$ given by (3.55). As before, this implies

$$[v']_{C^{\gamma(b)+\sigma-\alpha}(R,2R)} \le C R^{\alpha-1} \quad \text{for} \quad 0 < R < 1.$$

Finally, the inequality (3.54) follows comparing the value of $v'(2^{-k})$ with $v'(1)$ dyadically. Thus, (3.53) and (3.54) are proved.

Define now the function

$$\psi_A(x) = A\left((x_+)^{\gamma(b)} + (x_+)^{\gamma(b)-1}\right),$$

and notice that ψ_A and v' solve

$$(-\Delta)^{1/2}\psi_A + b\psi_A' = 0 \quad \text{in} \quad x > 0, \tag{3.57}$$

$$(-\Delta)^{1/2}v' + b(v')' = 0 \quad \text{in} \quad x > 0. \tag{3.58}$$

We have that $\psi_A > v'$ in $\{x > 0\}$ for some large enough A, thanks to the growth of v' in (3.53)-(3.54). Choose the smallest nonnegative A such that $\psi_A \ge v'$. Then, by the growth at zero and infinity of both v' and ψ_A they touch at some point in $(0, \infty)$. Moreover, if $A > 0$, then we must have $\psi_A \not\equiv v'$.

Let $x_0 > 0$ be a point where $\psi_A(x_0) = v'(x_0)$. Notice that $\psi_A - v'$ is a non-negative (and non-zero) function with a minimum at x_0. Thus,

$$\left((-\Delta)^{1/2}(\psi_A - v') + b(\psi_A - v')'\right)(x_0) = (-\Delta)^{1/2}(\psi_A - v')(x_0) < 0,$$

which contradicts the fact that both ψ_A and v' are solutions to the problem, (3.57)-(3.58). Thus, there is no positive A such that ψ_A and v' touch at at least one point, so we must have $v' \le 0$. Doing the same from below we reach $v' \ge 0$, and therefore $v' \equiv 0$. Hence, since $u(0) = 0$ we find $v \equiv 0$. In particular, this means that

$$u = k(x_+)^{\gamma(b)},$$

as desired. □

We can now prove the Liouville theorem.

Proof of Theorem 3.22. Let us first see that the solution is 1-dimensional in the direction e_n.

Given $\rho \geq 1$, define

$$v_\rho(x) = \rho^{-\varepsilon+1} u(\rho x).$$

Notice that

$$\|v_\rho\|_{L^\infty(B_R)} = \rho^{-\varepsilon+1} \|u(\rho\cdot)\|_{L^\infty(B_R)} = \rho^{-\varepsilon+1} \|u\|_{L^\infty(B_{\rho R})} \leq CR^{1-\varepsilon}.$$

Moreover, by the homogeneity of $(-L + b\cdot\nabla)$,

$$\begin{cases} (-L + b\cdot\nabla)v_\rho &= 0 \quad \text{in} \quad \mathbb{R}^n_+ \\ v_\rho &= 0 \quad \text{in} \quad \mathbb{R}^n_-. \end{cases} \tag{3.59}$$

Define now $\tilde{v}_\rho = v_\rho \chi_{B_2}$, so that $\tilde{v}_\rho \in L^\infty(\mathbb{R}^n)$. We now have

$$\begin{cases} (-L + b\cdot\nabla)\tilde{v}_\rho &= g_\rho \quad \text{in} \quad B_1^+ \\ \tilde{v}_\rho &= 0 \quad \text{in} \quad B_1^-, \end{cases} \tag{3.60}$$

for some g_ρ with $\|g_\rho\|_{L^\infty(B_1^+)} \leq C_0$ with C_0 independent of ρ. Indeed,

$$(-L + b\cdot\nabla)\tilde{v}_\rho = (-L + b\cdot\nabla)(v_\rho - v_\rho\chi_{B_2^c}) = L(v_\rho\chi_{B_2^c}) \leq C_0 \quad \text{in} \quad B_1^+,$$

where the last inequality follows thanks to the uniform growth control on v_ρ.

Now, by Proposition 3.21,

$$\|v_\rho\|_{C^\sigma(B_{1/2})} = \|\tilde{v}_\rho\|_{C^\sigma(B_{1/2})} \leq C,$$

from which

$$[u]_{C^\sigma(B_{\rho/2})} = \rho^{-\sigma}[u(\rho\cdot)]_{C^\sigma(B_{1/2})} = \rho^{-\sigma+1-\varepsilon}[v_\rho]_{C^\sigma(B_{1/2})} \leq C\rho^{-\sigma+1-\varepsilon}. \tag{3.61}$$

Now, given $e \in \mathbb{S}^{n-1}$ with $e_n = 0$, and for any $h > 0$, define

$$w(x) = \frac{u(x+eh) - u(x)}{h^\sigma}.$$

By (3.61),

$$\|w\|_{L^\infty(B_R)} \leq CR^{-\sigma+1-\varepsilon} \quad \text{for all} \quad R \geq 1.$$

We also have

$$\begin{cases} (-L + b\cdot\nabla)w &= 0 \quad \text{in} \quad \mathbb{R}^n_+ \\ w &= 0 \quad \text{in} \quad \mathbb{R}^n_-, \end{cases} \tag{3.62}$$

thanks to the fact that e does not have component in the n-th direction, $e_n = 0$.

Repeat the previous argument applied to w instead of u, to get

$$[w]_{C^\sigma(B_R)} \leq CR^{-2\sigma+1-\varepsilon} \quad \text{for all} \quad R \geq 1.$$

Repeating iteratively we get that, for $m = \lfloor \frac{1-\varepsilon}{\sigma} + 1 \rfloor$, then

$$[w_m]_{C^\sigma(B_R)} \leq CR^{-m\sigma+1-\varepsilon} \quad \text{for all} \quad R \geq 1,$$

where w_m is an incremental quotient of order m of u. Letting $R \to \infty$ we observe that $w_m \equiv 0$.

Since w_m is any incremental quotient of order m, this means that for any fixed x, $q_x(y') := u(x + (y', 0))$ for $y' \in \mathbb{R}^{n-1}$ is a polynomial of order $m-1$ in the y' variables. However, from the growth condition on u the polynomial must grow less than linearly at infinity, and therefore it is constant. This means that for any x, $u(x + eh) = u(x)$ for all $h \in \mathbb{R}$ and for all $e \in \mathbb{S}^{n-1}$ with $e_n = 0$; i.e., $u(x) = u(x_n)$, as we wanted to see.

Now we can proceed as in the proof of the classification theorem, Theorem 3.11, and use the classification of 1-dimensional solutions from Proposition 3.23. □

3.7.4 Proof of Theorem 3.18

We now prove the following result, which will directly yield Theorem 3.18. For this, we combine the ideas in [RS16] with Propositions 3.21 and 3.23.

Proposition 3.24. *Let L be an operator of the form (3.7)-(3.8), and let $b \in \mathbb{R}^n$. Let Γ be a $C^{1,\alpha}$ graph splitting B_1 into U^+ and U^- (see Definition 3.3), and suppose $0 \in \Gamma$ and that $\nu(0) = e_n$, where $\nu(0)$ is the normal vector to Γ at 0 pointing towards U^+.*

Let $f \in L^\infty(U^+)$, and suppose $u \in L^\infty(\mathbb{R}^n)$ satisfies

$$\begin{cases} (-L + b\cdot\nabla)u = f & \text{in } U^+ \\ u = 0 & \text{in } U^-. \end{cases} \tag{3.63}$$

Let us denote $\gamma := \gamma\left(\frac{b\cdot\nu(0)}{\chi(\nu(0))}\right) = \gamma(b_n/\chi(e_n))$ and $\chi = \chi(e_n)$ as defined in (3.12)-(3.11), and suppose that $\gamma \in \left[\gamma_0, \gamma_0\left(1+\frac{\alpha}{8}\right)\right]$ for some $\gamma_0 \in (0,1)$ such that $\gamma_0\left(1+\frac{\alpha}{4}\right) < 1$. Suppose also that η_ν as defined in (3.38) satisfies $\eta_\nu \le \frac{\alpha\gamma_0}{64}$, and let $\Upsilon = \gamma_0\left(1+\frac{\alpha}{4}\right)$.

Then, there exists Q with $|Q| \le C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(U^+)}\right)$ such that

$$\left|u(x) - Q(x_n)_+^\gamma\right| \le C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(U^+)}\right)|x|^\Upsilon \quad \text{for all} \quad x \in B_1,$$

where the constant C depends only on n, α, the $C^{1,\alpha}$ norm of Γ, γ_0, the ellipticity constants, and $\|b\|$.

Before proving the previous result let us state a useful lemma. It can be found in [RS16, Lemma 5.3].

Lemma 3.25 ([RS16])**.** *Let $1 > \Upsilon > \beta_0 \ge \beta$ and $\nu \in \mathbb{S}^{n-1}$ some unit vector. Let $u \in C(B_1)$ and define*

$$\phi_r(x) := Q_*(r)(x\cdot\nu)_+^\beta,$$

where

$$Q_*(r) := \arg\min_{Q\in\mathbb{R}} \int_{B_r} \left(u(x) - Q(x\cdot\nu)_+^\beta\right)^2 dx = \frac{\int_{B_r} u(x)(x\cdot\nu)_+^\beta dx}{\int_{B_r}(x\cdot\nu)_+^{2\beta} dx}.$$

Assume that for all $r \in (0,1)$ *we have*

$$\|u - \phi_r\|_{L^\infty(B_r)} \le C_0 r^\Upsilon.$$

Then, there is $Q \in \mathbb{R}$ *with* $|Q| \le C(C_0 + \|u\|_{L^\infty(B_1)})$ *such that*

$$\|u - Q(x \cdot \nu)_+^\beta\|_{L^\infty(B_r)} \le CC_0 r^\Upsilon$$

for some constant C *depending only on* Υ *and* β_0.

We can now prove Proposition 3.24.

Proof of Proposition 3.24. Let us argue by contradiction. Suppose that there are sequences Γ_i, U_i^+, U_i^-, L_i, b_i, u_i, and f_i that satisfy the assumptions

- Γ_i is a $C^{1,\alpha}$ graph with bounded $C^{1,\alpha}$ norm independently of i, splitting B_1 into U_i^+ and U_i^- with $0 \in \Gamma_i$ and with e_n being the normal vector at 0 pointing towards U_i^+.
- L_i are of the form (3.7)-(3.8), and $\|b_i\| = \|b\|$;
- For each Γ_i, the corresponding η_ν as defined in (3.38) fulfils $\eta_\nu \le (\alpha\gamma_0)/64$;
- $\|u_i\|_{L^\infty(\mathbb{R}^n)} + \|f_i\|_{L^\infty(U^+)} = 1$;
- u_i solves $(-L_i + b_i \cdot \nabla)u_i = f_i$ in U_i^+, $u_i = 0$ in U_i^-;
- If we define $\gamma_i := \gamma(b_i \cdot e_n/\chi_i)$ with γ as in (3.12) and $\chi_i = \chi_i(e_n)$ as in (3.11) with the operator L_i, then $\gamma_i \in [\gamma_0, \gamma_0(1+\alpha/8)]$;

but they are such that for all $C > 0$ there exists some i such that there is no constant Q satisfying

$$\left|u_i(x) - Q(x_n)_+^{\gamma_i}\right| \le C|x|^\Upsilon \quad \text{for all} \quad x \in B_1.$$

Step 1: Construction and properties of the blow up sequence.

Let us denote

$$\phi_{i,r} := Q_i(r)(x_n)_+^{\gamma_i},$$

where

$$Q_i(r) := \arg\min_{Q\in\mathbb{R}} \int_{B_r} (u_i(x) - Q(x_n)_+^{\gamma_i})^2 dx = \frac{\int_{B_r} u_i(x)(x_n)_+^{\gamma_i} dx}{\int_{B_r} (x_n)_+^{2\gamma_i} dx}.$$

From Lemma 3.25 with $\beta = \gamma_i$ and $\beta_0 = \gamma_0(1+\alpha/8)$ we have that

$$\sup_i \sup_{r>0} \left\{ r^{-\Upsilon} \|u_i - \phi_{i,r}\|_{L^\infty(B_r)} \right\} = \infty.$$

Define the monotone function

$$\theta(r) := \sup_i \sup_{r'>r} \left\{ (r')^{-\Upsilon} \|u_i - \phi_{i,r'}\|_{L^\infty(B_{r'})} \right\}.$$

Note that for $r > 0$, $\theta(r) < \infty$, and $\theta(r) \to \infty$ as $r \downarrow 0$. Now take a sequences $r_m \downarrow 0$ and i_m such that

$$(r_m)^{-\Upsilon} \|u_{i_m} - \phi_{i_m,r_m}\|_{L^\infty(B_{r_m})} \geq \frac{\theta(r_m)}{2},$$

and denote $\phi_m = \phi_{i_m,r_m}$.

Consider now

$$v_m(x) = \frac{u_{i_m}(r_m x) - \phi_m(r_m x)}{r_m^\Upsilon \theta(r_m)}.$$

By definition of ϕ_m we have the orthogonality condition for all $m \geq 1$,

$$\int_{B_1} v_m(x)(x_n)_+^{\gamma_i} dx = 0. \tag{3.64}$$

Note that also from the choice of r_m we have a nondegeneracy condition for v_m,

$$\|v_m\|_{L^\infty(B_1)} \geq \frac{1}{2}. \tag{3.65}$$

From the definition of $\phi_{i,r}$, $\phi_{i,2r} - \phi_{i,r} = \big(Q_i(2r) - Q_i(r)\big)(x_n)_+^{\gamma_i}$ so that

$$\begin{aligned} |Q_i(2r) - Q_i(r)| r^{\gamma_i} &= \|\phi_{i,2r} - \phi_{i,r}\|_{L^\infty(B_r)} \\ &\leq \|\phi_{i,2r} - u\|_{L^\infty(B_{2r})} + \|\phi_{i,r} - u\|_{L^\infty(B_r)} \leq C r^\Upsilon \theta(r). \end{aligned}$$

Proceeding inductively, if $R = 2^N$, then

$$\begin{aligned} \frac{r^{\gamma_i - \Upsilon}|Q_i(Rr) - Q_i(r)|}{\theta(r)} &\leq \sum_{j=0}^{N-1} 2^{j(\Upsilon - \gamma_i)} \frac{(2^j r)^{\gamma_i - \Upsilon}|Q_i(2^{j+1}r) - Q_i(2^j r)|}{\theta(r)} \\ &\leq C \sum_{j=0}^{N-1} 2^{j(\Upsilon - \gamma_i)} \frac{\theta(2^j r)}{\theta(r)} \leq C 2^{N(\Upsilon - \gamma_i)} = C R^{\Upsilon - \gamma_i}. \end{aligned} \tag{3.66}$$

Thus, we obtain a bound on the growth control of v_m given by

$$\|v_m\|_{L^\infty(B_R)} \leq C R^\Upsilon \quad \text{for all} \quad R \geq 1. \tag{3.67}$$

Indeed,

$$\begin{aligned} \|v_m\|_{L^\infty(B_R)} &= \frac{1}{\theta(r_m) r_m^\Upsilon} \|u_i - Q_{i_m}(r_m)(x_n)_+^{\gamma_i}\|_{L^\infty(Rr_m)} \\ &\leq \frac{1}{\theta(r_m) r_m^\Upsilon} \|u_i - Q_{i_m}(Rr_m)(x_n)_+^{\gamma_i}\|_{L^\infty(Rr_m)} + \\ &\qquad + \frac{1}{\theta(r_m) r_m^\Upsilon} |Q_{i_m}(Rr_m) - Q_{i_m}(r_m)|(Rr_m)^{\gamma_i} \\ &\leq \frac{R^\Upsilon \theta(Rr_m)}{\theta(r_m)} + C R^\Upsilon, \end{aligned}$$

and the result follows from the monotonicity of θ.

Notice also that the previous computation in (3.66) also gives a bound for $Q_i(r)$ given by

$$|Q_i(r)| \leq C\theta(r), \tag{3.68}$$

which follows by putting $R = r^{-1}$.

Step 2: Convergence of the blow up sequence.

In this second step we show that v_m converges locally uniformly in $\mathbb{R}^n$ to some function v satisfying

$$\begin{cases} (-\tilde{L} + \tilde{b} \cdot \nabla)v &= 0 \quad \text{in} \quad \mathbb{R}^n_+ \\ v &= 0 \quad \text{in} \quad \mathbb{R}^n_-, \end{cases} \tag{3.69}$$

for some operator $\tilde{L}$ of the form (3.7)-(3.8), $\|\tilde{b}\| = \|b\|$.

To do so, define

$$U^+_{R,m} := B_R \cap \left(r_m^{-1} U^+_{i_m}\right) \cap \{x_n > 0\},$$

and suppose that it is well defined by assuming m is large enough so that $Rr_m < 1/2$.

Notice that in $U^+_{R,m}$, v_m satisfies an elliptic equation with drift,

$$(-L_{i_m} + b_{i_m} \cdot \nabla)v_m(x) = \frac{r_m}{r_m^{\Upsilon}\theta(r_m)} f_{i_m}(r_m x) \quad \text{in} \quad U^+_{R,m},$$

since we know that $(-L_i + b_i \cdot \nabla)\phi_m = 0$ in $\{x_n > 0\}$. In particular, since $\Upsilon < 1$, the right-hand side converges uniformly to 0 as $r_m \downarrow 0$.

We will now show that

$$\|u_{i_m} - \phi_m\|_{L^\infty(B_r \cap (U^-_{i_m} \cup \mathbb{R}^n_-))} \leq C\theta(r_m) r^{(1+\alpha)\kappa} \quad \text{for all} \quad r < 1/4, \tag{3.70}$$

and where the constant C is independent of m, and $\kappa := \gamma_0 \left(1 - \frac{\alpha}{16}\right)$. Notice that $\kappa < \gamma_0 - 2\eta_\nu$, so that we can use the supersolution from Proposition 3.19 to get

$$|u_{i_m}| \leq C \left(\operatorname{dist}(x, U^-)\right)^\kappa,$$

with C depending only on n, the $C^{1,\alpha}$ norm of Γ, α, the ellipticity constants, and $\|b\|$. On the other hand, by definition of ϕ_m,

$$|\phi_m(x)| \leq CQ_{i_m}(r_m) \left(\operatorname{dist}(x, \mathbb{R}^n_-)\right)^{\gamma_i} \leq C\theta(r_m) \left(\operatorname{dist}(x, \mathbb{R}^n_-)\right)^\kappa \quad \text{for all} \quad x \in B_1,$$

where we used (3.68). Finally, since the domain is $C^{1,\alpha}$, we have that

$$\operatorname{dist}(x, U^-_{i_m}) \leq Cr^{1+\alpha}, \quad \operatorname{dist}(x, \mathbb{R}^n_-) \leq Cr^{1+\alpha} \quad \text{in} \quad B_r \cap (U^-_{i_m} \cup \mathbb{R}^n_-),$$

where the constant C depends only on the $C^{1,\alpha}$ norm of the domain $U^+_{i_m}$, and therefore, it is independent of m. Thus, combining the last two expressions we get (3.70).

Now, from Proposition 3.21 we have

$$\|u_{i_m}\|_{C^\sigma(B_{1/8})} \leq C,$$

uniformly in m, for some $\sigma \in (0, \gamma_0)$.

From the regularity of ϕ_m this yields, in particular,

$$\|u_{i_m} - \phi_m\|_{C^\sigma\left(B_r\cap(U^-\cup\mathbb{R}^n_-)\right)} \le C\theta(r_m), \tag{3.71}$$

where we have used again the bound (3.68).

Thus, interpolating (3.70) and (3.71) there exists some $\sigma_0 < \sigma$ (depending on σ, γ_0, and α) such that

$$\|u_{i_m} - \phi_m\|_{C^{\sigma_0}(B_r\cap(U^-_{i_m}\cup\mathbb{R}^n_-))} \le C\theta(r_m)r^{\Upsilon}.$$

Notice that we can do so because $\Upsilon < \kappa(1+\alpha)$. Scaling the previous expression we obtain

$$\|v_m\|_{C^{\sigma_0}(B_R\setminus U^+_{R,m})} \le C(R) \quad \text{for all } m \text{ with } Rr_m < 1/4, \tag{3.72}$$

for some constant $C(R)$ that depends on R, but is independent of m.

We now want to apply Proposition 3.21 to v_m, rescaled to balls B_R. Recall that

$$(-L_{i_m} + b_{i_m}\cdot\nabla)v_m(x) = \frac{r_m}{r_m^{\Upsilon}\theta(r_m)} f_{i_m}(r_m x) \quad \text{in} \quad U^+_{R,m},$$

and v_m is C^{σ_0} outside $U^+_{R,m}$ by (3.72). Notice also that the boundary $\partial U^+_{R,m}$ has $C^{1,\alpha}$ norm smaller than the $C^{1,\alpha}$ norm of Γ thanks to the fact that we are rescaling with smaller r_m and $Rr_m < 1/4$. Thus, Proposition 3.21 can be applied and we obtain that there exists some $\sigma' > 0$ small such that

$$\|v_m\|_{C^{\sigma'}(B_{R/2})} \le C(R) \quad \text{for } m \text{ with} \quad Rr_m < 1/4.$$

we have again that the constant $C(R)$ depends on R, but is independent of m; i.e, we have reached a uniform $C^{\sigma'}$ bound on v_m over compact subsets.

Thus, up to taking a subsequence, v_m converge locally uniformly to some v.

Step 3: Contradiction. Up to taking a subsequence if necessary, L_{i_m} converges weakly to some operator $\tilde{L}$ of the form (3.7)-(3.8), and b_{i_m} converges to some $\tilde{b}$ with $\|\tilde{b}\| = \|b\|$. Notice that, in particular, this means that γ_i converges to some $\gamma_* \in [\gamma_0, \gamma_0(1+\alpha/8)]$, and $\gamma_* = \gamma(\tilde{b}\cdot e_n/\tilde{\chi})$, where $\tilde{\chi} = \tilde{\chi}(e_n)$ is the associated constant defined as in (3.11) with the operator $\tilde{L}$.

On the other hand, the domains $U^+_{i_m}$ converge uniformly to $\mathbb{R}^n_+$ over compact subsets by construction. Thus, passing all this to the limit, we reach that v satisfies (3.69).

Now, passing the growth control (3.67) to the limit, we reach

$$\|v\|_{L^\infty(B_R)} \le CR^{\Upsilon} \quad \text{for all} \quad R \ge 1,$$

so that we can apply the Liouville theorem in the half space, Theorem 3.22, to get

$$v(x) = C(x_n)_+^{\gamma_*}.$$

Passing to the limit (3.64) and using this last expression, we obtain $v \equiv 0$. However, by passing (3.65) to the limit we get

$$\|v\|_{L^\infty(B_1)} \ge \frac{1}{2},$$

a contradiction. □

Proof of Theorem 3.18. The result follows from Proposition 3.24 applied to small enough balls so that the condition on η_ν is fulfilled. Notice that the constant σ cannot go to 0, because $\tilde{\gamma}(x_0)$ cannot be made arbitrarily small for a given L and b. □

3.8 Proof of Theorems 3.1 and 3.3

In this section, we will prove Theorems 3.1 and 3.3. We already know that if x_0 is a regular free boundary point, then the free boundary is $C^{1,\alpha}$ in a neighbourhood. Next, using the results of the previous section, we show that the regular set is open, and that at any regular free boundary point we have (3.73) below.

Proposition 3.26. *Let L be an operator of the form* (3.7)-(3.8)*, and let $b \in \mathbb{R}^n$. Let u be a solution to* (3.29)-(3.30)-(3.31)*.*

Then the set of regular free boundary points is relatively open. Moreover, around each regular point x_0

$$0 < cr^{1+\tilde{\gamma}(x_0)} \leq \sup_{B_r(x_0)} u \leq Cr^{1+\tilde{\gamma}(x_0)} \quad \textit{for all} \quad r \leq 1, \tag{3.73}$$

for some positive constants c and C depending only on n, $\|b\|$, and the ellipticity constants. Here, $\tilde{\gamma}(x_0)$ is given by (3.10) *with $\nu(x_0)$ being the normal vector to the free boundary at x_0 pointing towards $\{u > 0\}$.*

Proof. Suppose without loss of generality that $x_0 = 0$ and $\nu(x_0) = e_n$. The free boundary, Γ, is $C^{1,\alpha}$ in B_{r_0} for some $\alpha, r_0 > 0$ by Proposition 3.17. Apply now Theorem 3.18 to the partial derivative $\partial_n u$ around points $z \in B_{r_0/2} \cap \Gamma$. We obtain

$$\left|\partial_n u(x) - Q(z)\big((x-z)\cdot\nu(z)\big)_+^{\tilde{\gamma}(z)}\right| \leq C|x-z|^{\tilde{\gamma}(z)+\sigma}, \tag{3.74}$$

for some $\sigma > 0$, and some constant C independent of z.

Step 1: Q is continuous and positive at the origin. Let us first check that Q is a continuous function on the free boundary at 0. Indeed, suppose it is not continuous, so that there exists a sequence $z_k \to 0$ on the free boundary such that $\lim_{k\to\infty} Q(z_k) = \bar{Q} \neq Q(0)$. Then, we have

$$\left|\partial_n u(x) - Q(z_k)\big((x-z_k)\cdot\nu(z_k)\big)_+^{\tilde{\gamma}(z_k)}\right| \leq C|x-z_k|^{\tilde{\gamma}(z_k)+\sigma}.$$

Thus, taking limits as $k \to \infty$, for any fixed x, we obtain

$$\left|\partial_n u(x) - \bar{Q}(x_n)_+^{\tilde{\gamma}(0)}\right| \leq C|x|^{\tilde{\gamma}(0)+\sigma}.$$

We have used here that ν and $\tilde{\gamma}$ are continuous. On the other hand, we had

$$\left|\partial_n u(x) - Q(0)(x_n)_+^{\tilde{\gamma}(0)}\right| \leq C|x|^{\tilde{\gamma}(0)+\sigma},$$

so that

$$|\bar{Q} - Q(0)|(x_n)_+^{\tilde{\gamma}(0)} \leq C|x|^{\tilde{\gamma}(0)+\sigma}.$$

Now take $x = (0, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ for $t \in \mathbb{R}^+$ and let $t \to 0$. It follows $\bar{Q} = Q(0)$, a contradiction; i.e., Q is continuous at 0.

We now prove that $Q(0) > 0$ (notice that we already know that $Q(0) \geq 0$ because $u \geq 0$). To do so, we proceed by creating an appropriate subsolution using Lemma 3.20.

First of all, consider a fixed bounded strictly convex $C^{1,\alpha}$ domain $P \subset \{u > 0\}$ touching the free boundary at 0, similar to the domains considered in the proof of Proposition 3.21. Suppose that P has diameter less than 1, and take an $h > 0$ such that, if we denote $\nu_P(z)$ the normal vector to ∂P pointing towards the interior of P at $z \in \partial P$, then

$$\tilde{\gamma}_h := \max\left\{\gamma\left(\frac{b \cdot \nu_P(z)}{\chi(\nu_P(z))}\right) \quad \text{for} \quad z \in \partial P \cap \{x_n < h\}\right\} \leq \tilde{\gamma}(0) + \frac{\sigma}{4},$$

where σ is the small constant following from Theorem 3.18 that appears in (3.74). Let us call

$$\eta_\nu^{(h)} := \tilde{\gamma}_h - \tilde{\gamma}(0) \geq 0$$

Such $h > 0$ exists because P is $C^{1,\alpha}$, and γ and χ are continuous. Take now $\kappa = \tilde{\gamma}(0) + 3\eta_\nu^{(h)}$, and let ϱ be a regularised distance to $\mathbb{R}^n \setminus P$ as in Definition 3.2. In particular, $\varrho \equiv 0$ in $\mathbb{R}^n \setminus P$. We will see that $\phi := \varrho^\kappa \leq C\partial_n u$ for an appropriate C.

By Lemma 3.20 used in B_h we get that for some constant $\delta_0 < h/2$,

$$(-L + b \cdot \nabla)\phi \leq -1 \quad \text{in} \quad B_{h/2} \cap \{x : 0 < d(x, \mathbb{R}^n \setminus P) \leq \delta_0\}.$$

Now, since P is strictly convex, we have that there exists some δ_P with $0 < \delta_P \leq \delta_0$ such that

$$(-L + b \cdot \nabla)\phi \leq -1 \quad \text{in} \quad \{0 < x_n < \delta_P\} \cap P.$$

Now consider v_r as the one defined in Proposition 3.13 (there it is called v),

$$v_r(x) = \frac{u(rx)}{r\|\nabla u\|_{L^\infty(B_r)}}.$$

By the same reasoning as in the proof of Proposition 3.15 rescaling to a larger ball we have that

$$\tilde{w}_r = C_1(\partial_n v_r)\chi_{B_2} \geq 0$$

for r small enough.

From Proposition 3.13 we can choose r small enough so that for some positive constant c,

$$\tilde{w}_r > c > 0 \quad \text{in} \quad P \cap \{x_n \geq \delta_P\}.$$

Moreover, also proceeding as in the proof of Proposition 3.15, $(-L+b\cdot\nabla)\tilde{w}_r > -\eta$ in $B_1 \cap \{v_r > 0\}$ for some arbitrarily small constant η, making r even smaller if necessary. Thus, we can assume

$$(-L + b \cdot \nabla)\tilde{w}_r > -\frac{\tilde{c}}{2} \quad \text{in} \quad B_1 \cap \{v_r > 0\},$$

for some $0 < \tilde{c} < c$ to be chosen later.

Now compare the functions ϕ and $\tilde{c}^{-1}\tilde{w}_r$. Notice that in $\mathbb{R}^n \setminus P$, $\tilde{w}_r \geq \phi \equiv 0$. In $P \cap \{x_n \geq \delta_P\}$, $\tilde{c}$ can be chosen small enough depending on δ_P and P so that $\tilde{c}^{-1}\tilde{w}_r \geq \phi$ there, because $\tilde{w}_r > c > 0$ in $P \cap \{x_n \geq \delta_P\}$. Finally,

$$(-L + b \cdot \nabla)\phi \leq (-L + b \cdot \nabla)\tilde{w}_r \quad \text{in} \quad \{0 < x_n < \delta_P\} \cap P.$$

Thus, by the maximum principle, for this particular r fixed we have that $\tilde{w}_r \geq \tilde{c}\phi$. Going back to the definition of $\tilde{w}_r$, this means that for some ρ and c positive constants

$$\partial_n u(te_n) \geq c\varrho(te_n) \quad \text{for} \quad 0 < t < \rho.$$

For ρ small enough, ϱ is comparable to $(x_n)_+^\kappa$ along the segment te_n, so that we actually have

$$\partial_n u(te_n) \geq ct^\kappa \quad \text{for} \quad 0 < t < \rho. \tag{3.75}$$

Now, if $Q(0) = 0$ then

$$|\partial_n u(x)| \leq C|x|^{\tilde{\gamma}(0)+\sigma}.$$

Since $\kappa < \tilde{\gamma}(0) + \sigma$ we get a contradiction with (3.75). Thus, $Q(0) > 0$.

Step 2: Conclusion of the proof. For $z \in \Gamma \cap B_r$ for r small enough we have that $Q(z) > 0$, because Q is continuous and $Q(0) > 0$. In particular,

$$\left|\partial_n u(x) - Q(z)\big((x - z) \cdot \nu(z)\big)_+^{\tilde{\gamma}(z)}\right| \leq C|x - z|^{\tilde{\gamma}(z)+\sigma}.$$

By taking $x = z + te_n$ for $t > 0$ we get

$$\left|\partial_n u(z + te_n) - Q(z)\big(\nu_n(z)t\big)_+^{\tilde{\gamma}(z)}\right| \leq Ct^{\tilde{\gamma}(z)+\sigma}.$$

Integrating with respect to t from 0 to $t' < 1$, using that $\partial_n u(z) = 0$ and $\nu_n(z) > 1/2$ for r small enough and recalling that $Q(z) > 0$, we get

$$u(z + t'e_n) \geq ct'^{1+\tilde{\gamma}(z)} > 0,$$

so that in particular, z is a regular point; i.e., the set of regular points is relatively open. Doing the same for $z = 0$ we get one of the inequalities from (3.73),

$$\sup_{B_r} u \geq cr^{1+\tilde{\gamma}(0)} > 0 \quad \text{for all} \quad r \leq 1. \tag{3.76}$$

On the other hand, we can also find the expansion at 0 for $\partial_i u$ for any $i \in \{1, \ldots, n\}$,

$$\left|\partial_i u(x) - Q_i(x_n)_+^{\tilde{\gamma}(0)}\right| \leq C|x|^{\tilde{\gamma}(0)+\sigma}.$$

Therefore,

$$|\nabla u(x)| \leq C\left(|x|^{\tilde{\gamma}(0)} + |x|^{\tilde{\gamma}(0)+\sigma}\right).$$

Integrating, and using $\nabla u(0) = 0$

$$u(x) \leq C\left(|x|^{1+\tilde{\gamma}(0)} + |x|^{1+\tilde{\gamma}(0)+\sigma}\right),$$

i.e.,

$$\sup_{B_r} u \leq Cr^{1+\tilde{\gamma}(0)} \quad \text{for all} \quad r \leq 1.$$

Thus, combined with (3.76), this proves (3.73). □

Proposition 3.27. *Let L be an operator of the form* (3.7)-(3.8)*, and let $b \in \mathbb{R}^n$. Let u be a solution to* (3.29)-(3.30)-(3.31) *and let x_0 be a free boundary regular point. Then*

$$u(x) = c_0\big((x - x_0) \cdot \nu(x_0)\big)_+^{1+\tilde{\gamma}(x_0)} + o\left(|x - x_0|^{1+\tilde{\gamma}(x_0)+\sigma}\right) \tag{3.77}$$

with $c_0 > 0$ and for some $\sigma > 0$. Here $\tilde{\gamma}(x_0)$ is given by (3.10)*, with $\nu(x_0)$ being the normal vector to the free boundary at* 0 *pointing towards $\{u > 0\}$; and σ depends only on n, the ellipticity constants, and $\|b\|$.*

Proof. Assume that $x_0 = 0$ and $\nu(x_0) = e_n$. From the expansions in the proof of Proposition 3.26 we have

$$\partial_i u(x) = Q_i(x_n)_+^{\tilde{\gamma}(0)} + o\left(|x|^{\tilde{\gamma}(0)+\sigma}\right), \tag{3.78}$$

for some Q_i, with $Q_n > 0$, and $\sigma > 0$. Now, let $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Integrating the expression (3.78) in the segment with endpoints 0 and $(x', 0)$ we get

$$u(x', 0) = o\left(|x|^{1+\tilde{\gamma}(0)+\sigma}\right).$$

Then, integrating in the segment with endpoints $(x', 0)$ and (x', x_n) we find

$$u(x', x_n) = \frac{Q_n}{1 + \tilde{\gamma}(0)}(x_n)_+^{1+\tilde{\gamma}(0)} + o\left(|x|^{1+\tilde{\gamma}(0)+\sigma}\right).$$

Thus, (3.76) is proved. □

We finally can put all elements together to prove our main results, Theorems 3.1 and 3.3.

Proof of Theorem 3.3. After subtracting the obstacle and dividing by a constant, we can assume u is a solution to (3.29)-(3.30)-(3.31). Then the result we want is a combination of Propositions 3.17, 3.26, and 3.27. □

Proof of Theorem 3.1. It is a particular case of Theorem 3.3; we only need to check that $\chi \equiv 1$. For this, notice that the kernel is constant and given by $\mu(\theta) = c_{n,1/2}$, where the constant $c_{n,s}$ is the one appearing in the definition of fractional Laplacian,

$$c_{n,s} := \left(\int_{\mathbb{R}^n} \frac{1 - \cos(x_1)}{|x|^{n+2s}} dx\right)^{-1};$$

see for example [DPV12]. Thus, the value of χ for $(-\Delta)^{1/2}$ is

$$\chi(e) = \frac{\pi c_{n,1/2}}{2} \int_{\mathbb{S}^{n-1}} |\theta \cdot e| d\theta.$$

Notice that, by changing variables to polar coordinates,

$$c_{n,1/2}^{-1} = \int_{\mathbb{R}^n} \frac{1 - \cos(x_1)}{|x|^{n+1}} dx = \int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{1 - \cos(r\theta_1)}{r^2} dr d\theta = \frac{\pi}{2} \int_{\mathbb{S}^{n-1}} |\theta_1| d\theta,$$

where we have used that $\int_0^\infty (1 - \cos(t))t^{-2} dt = \pi/2$. This immediately yields that $\chi \equiv 1$ for $(-\Delta)^{1/2}$, as desired. □

We next prove the almost optimal regularity of solutions. Given an operator L of the form (3.7)-(3.8), the associated χ defined as in (3.11), and $b \in \mathbb{R}^n$, we define

$$\gamma_{L,b}^{-} := \inf_{e\in\mathbb{S}^{n-1}} \gamma\left(\frac{b\cdot e}{\chi(e)}\right), \tag{3.79}$$

where γ is given by (3.12). Notice that $\gamma_{L,b}^{-} \in (0, 1/2]$.

Proposition 3.28. *Let L be an operator of the form* (3.7)-(3.8)*, and let $b \in \mathbb{R}^n$. Let u be a solution to* (3.29)-(3.30)-(3.31)*. Then, for any $\varepsilon > 0$,*

$$\|u\|_{C^{1,\gamma_{L,b}^{-}-\varepsilon}(\mathbb{R}^n)} \le C_\varepsilon,$$

where the constant C_ε depends only on n, L, b, and ε. The constant $\gamma_{L,b}^{-}$ is given by (3.79)*.*

Proof. In order to prove the bound we first check the growth of the solution at the free boundary, and then we combine it with interior estimates.

For simplicity, we will denote $\gamma_\varepsilon = \gamma_{L,b}^{-} - \varepsilon$.

Step 1: Growth at the free boundary. We first prove that, if 0 is a free boundary point, then

$$\sup_{r>0} \frac{\|\nabla u\|_{L^\infty(B_r)}}{r^{\gamma_\varepsilon}} \le C, \tag{3.80}$$

for some constant C depending only on n, L, b, and ε.

We proceed by contradiction, using a compactness argument. Suppose that it is not true, so that there exists a sequence of functions u_k, f_k, with $\|u_k\|_{C^{1,\tau}} \le 1$ for some $\tau > 0$ fixed and $\|f_k\|_{C^1(\mathbb{R}^n)} \le 1$, such that

$$\left\{\begin{array}{rcll} u_k & \ge & 0 & \text{in } \mathbb{R}^n \\ (-L + b\cdot\nabla)u_k & \le & f_k & \text{in } \mathbb{R}^n \\ (-L + b\cdot\nabla)u_k & = & f_k & \text{in } \{u_k > 0\} \\ D^2 u_k & \ge & -1 & \text{in } \mathbb{R}^n, \end{array}\right. \tag{3.81}$$

but u_k are such that

$$\theta(r) := \sup_i \sup_{r'>r} (r')^{-\gamma_\varepsilon} \|\nabla u_k\|_{L^\infty(B_{r'})} \to \infty \quad \text{as} \quad r \downarrow 0.$$

Notice that for $r > 0$, $\theta(r) < \infty$ and that θ is a monotone function, with $\theta(r) \to \infty$ as $r \downarrow 0$. Now take sequences $r_m \downarrow 0$ and i_m such that

$$r_m^{-\gamma_\varepsilon} \|\nabla u_{i_m}\| \ge \frac{\theta(r_m)}{2},$$

and define the functions

$$v_m(x) := \frac{u_{i_m}(r_m x)}{r_m^{1+\gamma_\varepsilon}\theta(r_m)}.$$

Notice that

$$\|\nabla v_m\|_{L^\infty(B_1)} \ge \frac{1}{2}, \tag{3.82}$$

and

$$D^2 v_m \geq -\frac{r_m^{1-\gamma_\varepsilon}}{\theta(r_m)} \quad \text{in} \quad \mathbb{R}^n, \qquad |(L + b\nabla)(\nabla v_m)| \leq \frac{r_m^{1-\gamma_\varepsilon}}{\theta(r_m)} \quad \text{in} \quad \{v_m > 0\}. \tag{3.83}$$

On the other hand,

$$\|\nabla v_m\|_{L^\infty(B_R)} = \frac{\|\nabla u_{i_m}\|_{L^\infty(B_{Rr_m})}}{r_m^{\gamma_\varepsilon}\theta(r_m)} \leq R^{\gamma_\varepsilon}\frac{\theta(Rr_m)}{\theta(r_m)} \leq R^{\gamma_\varepsilon} \quad \text{for} \quad R \geq 1. \tag{3.84}$$

Therefore, noticing that $r_m^{1-\gamma_\varepsilon}/\theta(r_m) \to 0$ as $m \to \infty$, we can apply Proposition 3.10 to deduce that, for some $\tau > 0$ independent of m,

$$\|v_m\|_{C^{1,\tau}(B_R)} \leq C(R),$$

for some constant depending on R, $C(R)$. Let us take limits as $m \to \infty$. By Arzelà-Ascoli, v_m converges, up to taking a subsequence, in $C^1_{\text{loc}}(\mathbb{R}^n)$ to some v_∞. By taking to the limit the properties (3.83)-(3.84) we reach that v_∞ should be a convex global solution. By the classification theorem, Theorem 3.11, we have that either $v \equiv 0$

$$v_\infty(x) = C(e \cdot x)_+^{1+\gamma(b\cdot e/\chi(e))} \quad \text{for some} \quad e \in \mathbb{S}^{n-1},$$

where γ and χ are given by (3.12)-(3.11). Notice, however, that taking (3.84) to the limit, v_∞ grows at most like γ_ε, and by definition $\gamma(b \cdot e/\chi(e)) > \gamma_\varepsilon$. Therefore, we must have $v_\infty \equiv 0$. But this is a contradiction with (3.82) in the limit. Therefore, we have proved (3.80).

Step 2: Conclusion. Let us combine the previous growth with interior estimates to obtain the desired result.

Let $x, y \in \mathbb{R}^n$, let $r = |x - y|$ and $R = \operatorname{dist}(x, \{u = 0\})$. We want to prove that for some constant C_ε then

$$|\nabla u(x) - \nabla u(y)| \leq C r^{\gamma_\varepsilon}.$$

Without loss of generality and by the growth found in the first step we can assume that $x, y \in \{u > 0\}$. Let $\bar{x} \in \partial\{u = 0\}$ be such that $\operatorname{dist}(\bar{x}, x) = R$. We separate two cases:

- If $4r > R$,

$$\begin{aligned} |\nabla u(x) - \nabla u(y)| &\leq |\nabla u(x) - \nabla u(\bar{x})| + |\nabla u(\bar{x}) - \nabla u(y)| \\ &\leq C\big(R^{\gamma_\varepsilon} + (R + r)^{\gamma_\varepsilon}\big) \leq C r^{\gamma_\varepsilon}, \end{aligned}$$

  where we have used the growth found in Step 1.

- If $4r \leq R$, then $x, y \in B_{R/2}(x)$, and $B_R(x) \subset \{u > 0\}$. Notice that we have

$$(-L + b \cdot \nabla)(\nabla u) = \nabla f \quad \text{in} \quad B_R(x).$$

  From the interior estimates in Proposition 3.4 rescaled, we have

$$R^{\gamma_\varepsilon}[\nabla u]_{C^{\gamma_\varepsilon}(B_{R/2}(x))} \leq C\left(R\|\nabla f\|_{L^\infty(B_R(x))} + \|\nabla u\|_{L^\infty(B_R(x))} + \int_{\mathbb{R}^n} \frac{|\nabla u(Rx)|}{1 + |x|^{n+1}}\right).$$

Now notice that thanks to the growth found in Step 1 we have, on the one hand,

$$\|\nabla u\|_{L^\infty(B_R(x))} \leq CR^{\gamma_\varepsilon},$$

and on the other hand,

$$\int_{\mathbb{R}^n} \frac{|\nabla u(Rx)|}{1+|x|^{n+1}} \leq R^{\gamma_\varepsilon} \int_{\mathbb{R}^n} \frac{|x|^{\gamma_\varepsilon}}{1+|x|^{n+1}} = CR^{\gamma_\varepsilon},$$

so that putting all together and using $\|\nabla f\|_{L^\infty(\mathbb{R}^n)} \leq 1$, it yields,

$$[\nabla u]_{C^{\gamma_\varepsilon}(B_{R/2}(x))} \leq C\left(1+R^{1-\gamma_\varepsilon}\right).$$

Thus, if $R \leq 4$ we are done. Now suppose $R > 4$. If $r < 1$, by applying interior estimates to $B_1(x)$ we are done. If $r \geq 1$, we are also done, because $|\nabla u(x) - \nabla u(y)| \leq 2\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \leq C$.

Thus, we have reached the desired result. □

As a consequence, we have the following immediate corollary.

Corollary 3.29. *Let L be an operator of the form* (3.7)-(3.8)*, and let $b \in \mathbb{R}^n$. Let u be the solution to* (3.9) *for a given obstacle φ of the form* (3.3)*. Then, for any $\varepsilon > 0$,*

$$\|u\|_{C^{1,\gamma^-_{L,b}-\varepsilon}(\mathbb{R}^n)} \leq C_\varepsilon,$$

where C_ε depends only on n, L, b, ε, and $\|\varphi\|_{C^{2,1}(\mathbb{R}^n)}$. The constant $\gamma^-_{L,b}$ is given by (3.79)*.*

Proof. After subtracting the obstacle and dividing by an appropriate constant, we can apply Proposition 3.28 and the result follows. □

Finally, we prove Corollary 3.2.

Proof of Corollary 3.2. After subtracting the obstacle and dividing by a constant, we get that this result is a particular case of Proposition 3.28, but the constant C_ε depends on b and not only on $\|b\|$.

To prove that C_ε actually depends on $\|b\|$, the proof of Proposition 3.28 can be rewritten by taking also sequences of vectors $b_k \in \mathbb{R}^n$ with $\|b_k\| = \|b\|$; by compactness, up to a subsequence they converge to some $\tilde{b}$ with $\|\tilde{b}\| = \|b\|$ and the rest of the proof is the same. □

3.9 A nondegeneracy property

In the obstacle problem for the fractional Laplacian (without drift), in [BFR18], Barrios, Figalli and the second author proved a non-degeneracy condition at all free boundary points for obstacles satisfying $\Delta\varphi \leq 0$. From this, and by means of a Monneau-type monotonicity formula, they establish a global regularity result for the free boundary.

In the obstacle problem with critical drift for the fractional Laplacian we can actually find a non-degeneracy result analogous to the one found in [BFR18]. In this case, however, we cannot establish regularity of the singular set, since we do not have (and do not expect) any monotonicity formula for this problem.

Proposition 3.30. *Let $b \in \mathbb{R}^n$, and suppose that $\varphi \in C^{1,1}(\mathbb{R}^n)$. Assume that φ is concave in $\{\varphi > 0\}$ or, more generally, that*

$$(\Delta + \partial^2_{bb})\,\varphi \le 0 \quad \textit{in} \quad \{\varphi > 0\}, \quad \varnothing \neq \{\varphi > 0\} \Subset \mathbb{R}^n.$$

Let u be a solution to the obstacle problem (3.2). Then, there exist constants $c, r_0 > 0$ such that for any x_0 a free boundary point then

$$\sup_{B_r(x_0)} (u - \varphi) \ge cr^2 \quad \textit{for all} \quad 0 < r < r_0.$$

Proof. Let $w := \big((-\Delta)^{1/2} + b\cdot\nabla\big)u$, so that $w \ge 0$. If $w \equiv 0$, by the interior estimates rescaled, and using that u is globally bounded, we reach u is constant. From $\lim_{|x|\to\infty} u(x) = 0$ we would get $u \equiv 0$, but this is a contradiction with $\varnothing \neq \{\varphi > 0\}$. Thus, $w \not\equiv 0$.

Notice, however, that $w \equiv 0$ in $\{u > \varphi\}$. In particular, given $\bar{x} \in \{u > \varphi\}$, then $\nabla w(\bar{x}) = 0$ and w has a global minimum at $\bar{x}$, so that

$$\big((-\Delta)^{1/2} - b\cdot\nabla\big)w(\bar{x}) = (-\Delta)^{1/2}w(\bar{x}) < 0.$$

Now, noticing that $\{\varphi > 0\} \Subset \mathbb{R}^n$, we get that by compactness there are some $\bar{c}, \bar{r} > 0$ such that for any $\bar{x} \in \{u > \varphi\}$ with $\operatorname{dist}(\bar{x}, \{u = \varphi\}) \le \bar{r}$ then

$$\big((-\Delta)^{1/2} - b\cdot\nabla\big)w(\bar{x}) \le -\bar{c} < 0.$$

Now, since $\big((-\Delta)^{1/2} + b\cdot\nabla\big)u = w$ in $\mathbb{R}^n$ and from the semigroup property of the fractional Laplacian,

$$-\Delta u - b_i b_j \partial_{ij} u = \big((-\Delta)^{1/2} - b\cdot\nabla\big)w \le -\bar{c} \quad \text{in} \quad \bar{U},$$

where $\bar{U} := \{u > \varphi\} \cap \{\operatorname{dist}(\cdot, \{u = \varphi\}) \le \bar{r}\}$. Note that the operator $\Delta + b_i b_j \partial_{ij}$ is uniformly elliptic, with ellipticity constants 1 and $1 + \|b\|^2$.

Since $u > 0$ on the contact set, by compactness there exists some $h > 0$ such that $\varphi \ge h$ in $\{u = \varphi\}$. By continuity, there exists some $0 < r_0 < \bar{r}/2$ such that

$$\varphi > 0 \quad \text{in} \quad U_0 := \{u > \varphi\} \cap \{\operatorname{dist}(\cdot, \{u = \varphi\}) \le 2r_0\}.$$

Now let $\bar{x} \in U_0$ with $\operatorname{dist}(\bar{x}, \{u = \varphi\}) \le r_0$, and consider $r \in (0, r_0)$. From the condition on φ, $(\Delta + \partial^2_{bb})\varphi \le 0$ in $\{\varphi > 0\}$, we get that if $\bar{u} := u - \varphi$ then

$$(\Delta + \partial^2_{bb})\,\bar{u} \ge \bar{c} > 0 \quad \text{in} \quad \{\bar{u} > 0\} \cap B_r(\bar{x}) \subset U_0.$$

Therefore, if we define

$$v := \bar{u} - \frac{\bar{c}}{2(n + \|b\|^2)}|x - \bar{x}|^2 \quad \text{in} \quad \{\bar{u} > 0\} \cap B_r(\bar{x}),$$

then

$$(\Delta + \partial^2_{bb})v \geq 0.$$

By the maximum principle, if $\Omega_r := \{\bar{u} > 0\} \cap B_r(\bar{x})$ then

$$0 < \bar{u}(x_1) \leq \sup_{\Omega_r} v = \sup_{\partial\Omega_r} v.$$

Since $v < 0$ in $\partial\{\bar{u} > 0\} \cap B_r(\bar{x})$,

$$0 < \sup_{\{\bar{u}>0\}\cap\partial B_r(\bar{x})} v \leq \sup_{\partial B_r(\bar{x})} \bar{u} - cr^2,$$

where $c = \frac{\bar{c}}{2(n+\|b\|^2)}$. Therefore, c is independent of $\bar{x}$, and we can let $\bar{x} \to x_0$, to obtain the desired result. □

Chapter 4

Regularity of minimal surfaces with lower dimensional obstacles

We study the Plateau problem with a lower dimensional obstacle in $\mathbb{R}^n$. Intuitively, in $\mathbb{R}^3$ this corresponds to a soap film (spanning a given contour) that is pushed from below by a "vertical" 2D half-space (or some smooth deformation of it). We establish almost optimal $C^{1,1/2-}$ estimates for the solutions near points on the free boundary of the contact set, in any dimension $n \geq 2$.

The $C^{1,1/2-}$ estimates follow from an ε-regularity result for minimal surfaces with thin obstacles in the spirit of the De Giorgi's improvement of flatness. To prove it, we follow Savin's small perturbations method. A nontrivial difficulty in using Savin's approach for minimal surfaces with thin obstacles is that near a typical contact point the solution consists of two smooth surfaces that intersect transversally, and hence it is not very flat at small scales. Via a new "dichotomy approach" based on barrier arguments we are able to overcome this difficulty and prove the desired result.

4.1 Introduction

4.1.1 Minimal surfaces with obstacles

In this paper we study the regularity of minimizers in the Plateau problem with a lower dimensional — or *thin* — obstacle. Before introducing the problem in further detail let us contextualize it by recalling five closely related classical problems and commenting on them.

- The Plateau problem:

$$\min\left\{P(E; B_1) \;:\; E \setminus B_1 = E_\circ \setminus B_1\right\}, \tag{4.1}$$

  where $E_\circ \subset \mathbb{R}^n$ (boundary condition), and B_1 denotes the unit ball of $\mathbb{R}^n$, $E \subset \mathbb{R}^n$, and $P(E; B_1)$ denotes the relative perimeter of the set E in B_1.

- The Plateau problem with an obstacle:

$$\min\left\{P(E; B_1) \;:\; E \supset \mathcal{O},\; E \setminus B_1 = E_\circ \setminus B_1\right\} \tag{4.2}$$

  where $E_\circ, E$ are as above and $\mathcal{O} \subset E_\circ$ (the obstacle) is given.

- The nonparametric obstacle problem:

$$\min_v \left\{ \int_{B_1'} \sqrt{1+|\nabla v|^2} \, : \, v \geq \psi \text{ in } B_1', \, v|_{\partial B_1'} = g \right\}, \tag{4.3}$$

  where B_1' denotes the unit ball of $\mathbb{R}^{n-1}$, $g : \partial B_1' \to \mathbb{R}$ (the boundary condition) is given, $v : B_1' \to \mathbb{R}$, and $\psi : B_1' \to \mathbb{R}$ is the obstacle satisfying $\psi|_{\partial B_1'} < g$.

- The obstacle problem:

$$\min_v \left\{ \int_{B_1'} \frac{|\nabla v|^2}{2} \, : \, v \geq \psi \text{ in } B_1', \, v|_{\partial B_1'} = g \right\}, \tag{4.4}$$

  where g, v, and ψ, are as above.

- The Signorini problem, or thin obstacle problem:

$$\min_v \left\{ \int_{B_1'} \frac{|\nabla v|^2}{2} \, : \, v \geq \psi \text{ in } B_1' \cap \{x_{n-1} = 0\}, \, v|_{\partial B_1'} = g \right\}, \tag{4.5}$$

  where g and v are as above, and now $\psi : B_1' \cap \{x_{n-1} = 0\} \to \mathbb{R}$ (the thin obstacle) acts only on $\{x_{n-1} = 0\}$.

Note that (4.3) is a particular case of (4.2), namely, when $\partial\mathcal{O}$ and ∂E are graphs. Also, (4.4) is, in turn, a limiting case of (4.3) — for ε-flat graphs, the area functional $\int \sqrt{1+|\varepsilon\nabla v|^2}$ becomes the Dirichlet energy $\int \frac{1}{2}|\varepsilon\nabla v|^2$ at leading order.

The regularity of solutions and free boundaries is nowadays well understood in both the classical obstacle problem (4.4) — see [Caf77, Caf98] — and in the Signorini problem — see [AC04, ACS08]. The case of minimal surfaces with thick obstacles (both in parametric and nonparametric form) is also well understood — see [Kin73, BK74, Jen80, Giu10].

This paper is concerned with the regularity of minimizers of the Plateau problem with lower dimensional, or thin, obstacles. Namely, we consider (4.2) with obstacle

$$\mathcal{O} := \Phi\big(\{x_{n-1} = 0, x_n \leq 0\}\big) \tag{4.6}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is some smooth ($C^{1,1}$) diffeomorphism. We denote

$$\partial\mathcal{O} := \Phi\big(\{x_{n-1} = 0, x_n = 0\}\big).$$

This problem (4.2)-(4.6) is the geometric version of the Signorini problem (4.5) in the same way that (4.2) with thick $\mathcal{O}$ is the geometric version of (4.4). To visualize a solution of this problem in $\mathbb{R}^3$, one can think of a soap film (spanning a given contour) that is pushed from below by a vertical 2D half-space, as depicted in Figure 4.1. Note that, in $\mathbb{R}^3$, we cannot use a "wire" (i.e. a one dimensional curve) as obstacle, since the surface will not "feel" it[1].

[1]More precisely, one can see that if $\mathcal{O}$ had codimension two, then solutions of (4.2) with an infinitesimal tubular neighbourhood of $\mathcal{O}$ as obstacle would become, in the limit, solutions of the Plateau problem (4.1) (without obstacle).

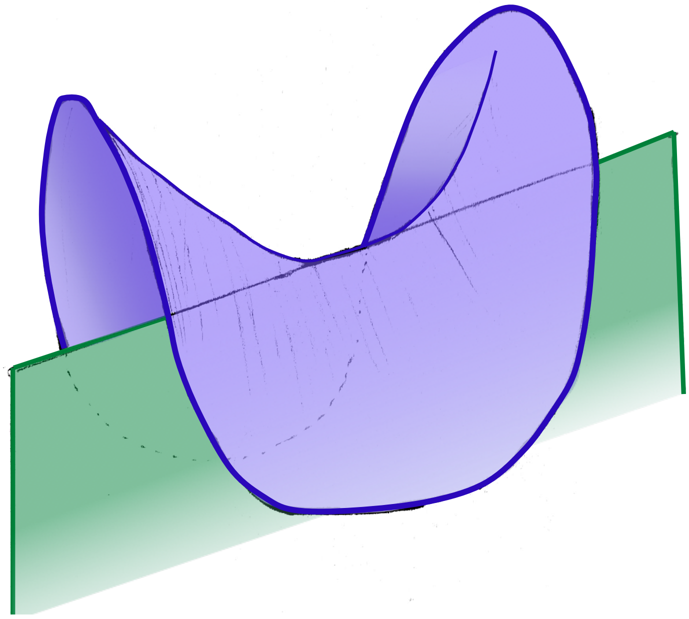

Figure 4.1: The "potato chip configuration", popularized by Caffarelli.

Although the problem of minimal surfaces with thin obstacles was introduced by De Giorgi [DeG73] already in 1973 (he established an existence result), very little was known on the regularity of its solutions. De Acutis in [DeA79] established C^1 regularity around points of the solution belonging to $\mathcal{O} \setminus \partial\mathcal{O}$. To our knowledge, the only known regularity results up to $\partial\mathcal{O}$ concern the nonparametric case — as in (4.3) but with ψ as in (4.5). They are due to Kinderlehrer [Kin71] who proved C^1 regularity estimates for the solution in two dimensions, and to Giusti [Giu72], who obtained Lipschitz estimates for the solution in every dimension.

The difficulty in studying (4.2)-(4.6) (with respect to the same problem with a thick obstacle) lies on the fact that near a typical point of the contact set the hypersurface ∂E consists of two surfaces that intersect transversally on $\partial\mathcal{O}$. Therefore, ∂E is typically not flat at small scales and thus (4.2) cannot be treated as a perturbation of (4.5). A more subtle dichotomy argument is needed: in Subsection 4.1.5 we outline the idea of this new approach that is tailored to overcome the previous difficulty.

Let us also point out that it is not completely obvious how to give a meaningful notion of solution to (4.2)-(4.6). The main issue is that with the Caccioppoli definition of relative perimeter P we have

$$P(E \cup \mathcal{O}; B_1) = P(E; B_1) \quad \text{for all measurable } E, \tag{4.7}$$

and thus the obstacle $\mathcal{O}$ seems to be ignored by P. This issue led De Giorgi [DeG73] to introduce a more appropriate notion of perimeter that is suitable for the study of thin obstacle problems (this is currently known as the De Giorgi measure). We choose the similar (and a posteriori equivalent) approach of looking at the thin obstacle as a limit of infinitesimaly thick neighbourhoods of it. See Subsection 4.1.4 for a more detailed discussion on this issue.

The goal of this paper is to address the question of the regularity of solutions to

(4.2)-(4.6). In particular, the main result of this paper is the proof of the following local almost optimal regularity result.

Theorem 4.1. *Let E be a solution to the thin obstacle problem* (4.2)-(4.6) *in the unit ball of $\mathbb{R}^n$, $n \geq 2$. Then, ∂E is $C^{1,1/2-}$ around contact points and up to the contact set.*

The appropriate notion of solution is discussed in Subsection 4.1.4. Let us emphasize here that this local regularity near contact points result holds in any dimension $n \geq 2$, in contrast to the classical regularity theory of minimal surfaces in which minimizers are regular only up to dimension 7. As we will see, this difference is due to the presence of the thin obstacle, which rules out solutions with singularities of the type of Simons and Lawson's cones like those appearing in dimension $n \geq 8$ in the Plateau problem without obstacles.

In the following subsections we recall the main steps in the regularity theory for sets of minimal perimeter and present the appropriate analogues for (4.2)-(4.6).

4.1.2 Improvement of flatness

For the classical Plateau problem De Giorgi [DeG61] established, in 1961, the following fundamental result:

Theorem 4.2 ([DeG61])**.** *Let $E \subset \mathbb{R}^n$ be a minimizer of the perimeter functional in B_1 and assume that $\partial E \cap B_1 \subset \{|e \cdot x| \leq \varepsilon_\circ\}$ for some $e \in \mathbb{S}^{n-1}$, where $\varepsilon_\circ = \varepsilon_\circ(n)$ is some positive dimensional constant. Then, $\partial E \cap B_{1/2}$ is a smooth hypersurface.*

This theorem follows from the following *improvement of flatness* property for minimizers E of the perimeter in B_1. Namely, given $\alpha \in (0,1)$ there exist positive constants $\varepsilon_\circ(n,\alpha)$ and $\rho_\circ(n,\alpha)$ such that, whenever $0 \in \partial E$ and $\varepsilon \in (0,\varepsilon_\circ)$ then the following implication holds:

$$\partial E \cap B_1 \subset \left\{|e \cdot x| \leq \varepsilon\right\} \quad \Rightarrow \quad \partial E \cap B_{\rho_\circ} \subset \left\{|\tilde{e} \cdot x| \leq \varepsilon \rho_\circ^{1+\alpha}\right\}. \tag{4.8}$$

Here, e and $\tilde{e}$ denote two possibly different unit vectors (in $\mathbb{S}^{n-1}$).

Combined with the classification of stable minimal cones by Simons [Sim68], Theorem 4.2 yields that minimizers of the perimeter in $\mathbb{R}^n$ are smooth for $3 \leq n \leq 7$. This result is optimal since, in dimensions $n \geq 8$, Bombieri, De Giorgi, and Giusti [BDG69] showed the existence of minimal boundaries with an $(n-8)$-dimensional linear space of cone-like singularities.

The philosophy of Theorem 4.2 is also shared by other key regularity results of nonlinear PDEs: *if a solution happens to be close enough to some special solution (e.g., the hyperplane), then it is regular.* These are the so-called "ε-regularity results".

The goal of the paper is to establish an ε-regularity result for (4.2)-(4.6), thus extending De Giorgi's improvement of flatness theorem to the setting of problem (4.2)-(4.6) — see Theorem 4.3 below. As a consequence, we will prove almost optimal $C^{1,1/2-}$ estimates for minimizers of (4.2)-(4.6) in $\mathbb{R}^n$ that are sufficiently close to a canonical blow-up solution (the *wedges* introduced in the following subsection). We will also see that these canonical blow-up solutions are the only possible blow-ups at any contact point, and then Theorem 4.1 will follow.

4.1.3 Blow-ups

An essential tool in the theory of minimal surfaces is the monotonicity formula. Namely, if ∂E is a minimal surface and $x_\circ \in \partial E$, then the function

$$\mathcal{A}(r) := \frac{1}{r^{n-1}}\mathcal{H}^{n-1}\big(\partial E \cap B_r(x_\circ)\big) \tag{4.9}$$

is monotone nondecreasing. In addition, $\mathcal{A}$ is constant if and only if E is a cone. A standard consequence of this monotonicity formula is that blow-ups of a minimizer of the perimeter $E \subset \mathbb{R}^n$ at any point $x_\circ \in \partial E$ are *minimizing cones*. Simons proved in [Sim68] that half-spaces are the only minimizing cones in dimensions $n \le 7$. As a consequence, one can always apply Theorem 4.2 near $x_\circ$ after zooming in enough — this gives the smoothness of perimeter minimizers for $n \le 7$.

For problem (4.2)-(4.6) we find several analogies with this theory. As we will prove in Lemma 4.27, if E is a minimizer of (4.2)-(4.6) and $x_\circ \in \partial E \cap \partial\mathcal{O}$ is a contact point, then the same function $\mathcal{A}(r)$ in (4.9) is still monotone when $\Phi = \mathrm{id}$ (and an approximate monotonicity formula is also available for general smooth Φ; see Lemma 4.27). As a consequence, blow-ups are also cones for (4.2)-(4.6). It is trivially false, however, that hyperplanes are the only possible blow-ups in low dimensions. Indeed, the *wedges* (see Figure 4.2)

$$\Lambda_{\gamma,\theta} := \big\{x \in \mathbb{R}^n \,:\, e_{\gamma+\theta}\cdot x \le 0 \;\text{ and }\; e_{\gamma-\theta}\cdot x \le 0\big\}, \tag{4.10}$$

for

$$e_\omega := \sin\omega\, \boldsymbol{e}_{n-1} + \cos\omega\, \boldsymbol{e}_n, \qquad -\frac{\pi}{2} \le \gamma \le \frac{\pi}{2}, \qquad 0 \le \theta \le \frac{\pi}{2} - |\gamma|. \tag{4.11}$$

are solutions to (4.2)-(4.6) for $\Phi = \mathrm{id}$. Thus, they are always possible blow-ups.

Being a wedge, $\Lambda_{\gamma,\theta}$ is the intersection of two semispaces with normal vectors contained in the plane generated by $\boldsymbol{e}_{n-1}$ and $\boldsymbol{e}_n$. The aperture angle of the wedge is given by $\pi - 2\theta$, while its rotation angle is given by γ with respect to $\boldsymbol{e}_n$ (we take the convention that $\boldsymbol{e}_{n-1} = e_{\pi/2}$). Note also that there is the restriction $0 \le \theta \le \frac{\pi}{2} - |\gamma|$ to guarantee that the obstacle $\{x_{n-1} = 0, x_n \le 0\}$ is contained in $\Lambda_{\gamma,\theta}$.

We will show that, in all dimensions, the wedges are the only possible blow-ups around contact points. More precisely, if E is a minimizer of (4.2)-(4.6) and $x_\circ \in \partial E \cap \partial\mathcal{O}$ (i.e. $x_\circ$ is a contact point) we have, in a suitable frame depending on $x_\circ$,

$$\frac{1}{r_k}\big(\mathcal{O} - x_\circ\big) \;\longrightarrow\; \{x_{n-1} = 0,\; x_n \le 0\} \tag{4.12}$$

and

$$\frac{1}{r_k}\big(E - x_\circ\big) \;\longrightarrow \Lambda_{\gamma,\theta}. \tag{4.13}$$

This will be a consequence of the the classification of conic solutions to the thin obstacle problem, given in Proposition 4.5.

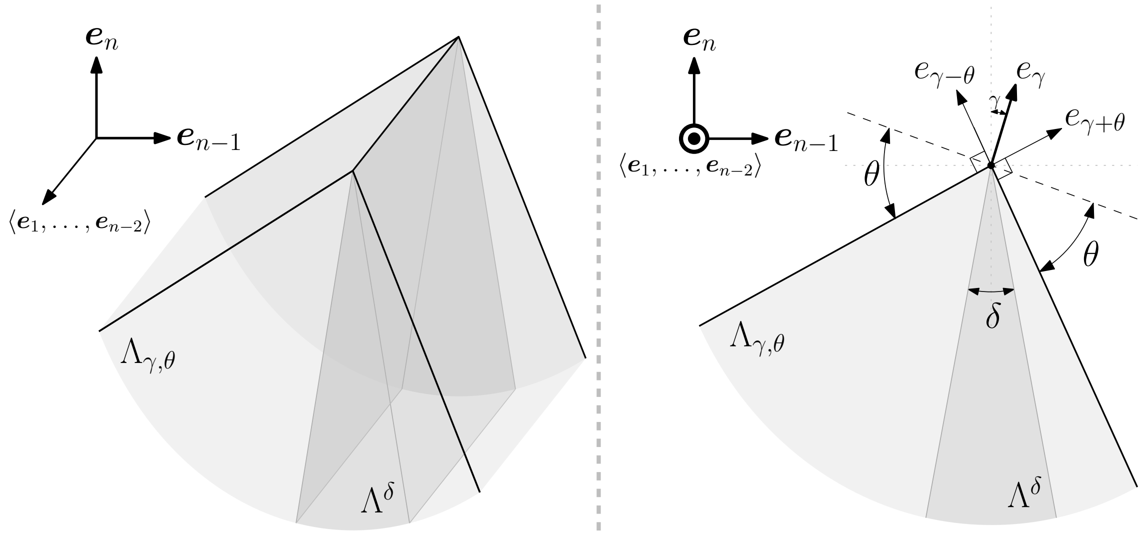

Figure 4.2: Representations of $\Lambda_{\gamma,\theta}$ and Λ^δ.

4.1.4 Rigorous notion of solution to (4.2)-(4.6)

Given a measurable set E and an open set $\Omega \subset \mathbb{R}^n$, we recall the standard definition of the relative perimeter of E in Ω as

$$P(E;\Omega) = \int_\Omega |\nabla \chi_E| = \sup_{g\in C_0^1(\Omega), \|g\|_{L^\infty}\le 1} \left| \int_E \operatorname{div} g \right|. \tag{4.14}$$

With this definition of perimeter (4.7) holds. Thus, unless we define the problem with further precision, minimizers of (4.2)-(4.6) will be — strictly speaking — just the ones of (4.1), ignoring $\mathcal{O}$.

This, of course, is not what we have in mind when we think of (4.2)-(4.6). Heuristically, we would like that if ∂E attaches from both sides to $\mathcal{O}$ in some region, then the area of it is counted twice in the computation of the perimeter of E instead of being ignored. To solve this issue De Giorgi introduced in [DeG73] a notion of perimeter that is suitable for the study of thin obstacle problems (the De Giorgi measure); see also [DeA79]. Here we will use the similar approach (that will be a posteriori equivalent) of considering a thin obstacle as a limit of thick obstacles.

Let us introduce the precise notion of (4.2)-(4.6) that will be used in this paper. For $\delta > 0$ small, let us denote

$$\Lambda^\delta := \Lambda_{0,\frac{\pi}{2}-\delta}. \tag{4.15}$$

(Note that Λ^δ is very sharp wedge, pointing in the $\boldsymbol{e}_n$ direction.)

Definition 4.1. We say that E is a *minimizer of* (4.2)-(4.6) in B_1 if E has positive density at some point of $\mathcal{O}$ and there exist $\delta_k \downarrow 0$, E_k minimizers of

$$\min \left\{ P(\tilde{E}; B_1) \ : \ \tilde{E} \setminus B_1 = \big(E_\circ \cup \Phi(\Lambda^{\delta_k})\big) \setminus B_1 \quad \text{and} \quad \Phi(\Lambda^{\delta_k}) \subset \tilde{E} \right\} \tag{4.16}$$

such that $\chi_{E_k} \to \chi_E$ in $L^1(B_1)$.

Note that $\Phi\big(\Lambda^{\delta_k}\big)$ are *thick* sets approximating $\mathcal{O}$. Now, minimizers of (4.16) "feel" the obstacle no matter how small δ_k is. The intuitive idea behind this definition is that a sequence E_k as in Definition 4.1 will not converge to a solution to the Plateau problem unless the obstacle $\mathcal{O}$ is "inactive" (i.e., the obstacle is contained in density one points for the solution to the Plateau problem). The philosophy of the paper will be to prove regularity estimates for problem (4.16) that are robust as $\delta_k \downarrow 0$. As a consequence, we will be able to show that the previous intuitive idea is actually fact. Namely, as it will be clear from the results of the paper, if the solution to the Plateau problem (with boundary data $E_\circ$) crosses $\mathcal{O}\setminus\partial\mathcal{O}$, then there exists a minimizer of (4.2)-(4.6) which is not a solution of Plateau problem (and therefore, the thin obstacle plays an active role).

We remark that any minimizer according to Definition 4.1 (up to replacing the complement of E by the zero density points of E) is a minimizer in the sense of De Giorgi by [DeA79] (see Remark 4.5). Conversely, it is not true a priori that any minimizer in the sense of De Giorgi can be recovered as a minimizer in the sense of Definition 4.1. Nonetheless, minimizers of the De Giorgi perimeter present *locally* an aperture around the obstacle by [DeA79] (and thus, a wedge fits within), and therefore, locally around contact points they are minimizers in the sense of Definition 4.1. In particular, since our regularity results are local, they apply to minimizers in the sense of De Giorgi. (See Remark 4.3.)

4.1.5 Regularity for solutions sufficiently close to a wedge

The first result of this paper is stated next, after introducing some notation and a definition. Throughout the paper we will denote

$$X \subset Y \text{ in } B \quad \Leftrightarrow \quad X\cap B \subset Y \cap B.$$

We also introduce the following

Definition 4.2. We say that E is *ε-close* to $\Lambda_{\gamma,\theta}$ in B if

$$\Lambda^{-\varepsilon}_{\gamma,\theta} \subset E \subset \Lambda^{\varepsilon}_{\gamma,\theta} \quad \text{in } B$$

where

$$\Lambda^{\varepsilon}_{\gamma,\theta} := \{x\in\mathbb{R}^n\,:\,\mathrm{dist}(x,\Lambda_{\gamma,\theta})\le\varepsilon\},\quad \Lambda^{-\varepsilon}_{\gamma,\theta} := \{x\in\mathbb{R}^n\,:\,\mathrm{dist}(x,\mathbb{R}^n\setminus\Lambda_{\gamma,\theta})\ge\varepsilon\}.$$

Here is our main result, which we call *improvement of closeness*:

Theorem 4.3 (Improvement of closeness)**.** *Given $\alpha\in\big(0,\tfrac12\big)$ there exist positive constants $\varepsilon_\circ$ and $\rho_\circ$ depending only on n and α such that the following holds:*

Assume that, for some $\delta>0$, a set $E\subset\mathbb{R}^n$ with $P(E;B_1)<\infty$ satisfies $\Phi(\Lambda^\delta)\cap B_1\subset E$ and

$$P(E;B_1)\le P(F;B_1)\quad \forall F \text{ such that } E\setminus B_1 = F\setminus B_1 \text{ and } \Phi(\Lambda^\delta)\cap B_1\subset F. \tag{4.17}$$

Suppose that $0\in\partial E\cap\partial\mathcal{O}$, $\varepsilon\in(0,\varepsilon_\circ)$, and

$$\Phi(0)=0,\quad D\Phi(0)=\mathrm{id},\quad |D^2\Phi|\le\varepsilon^{1+\frac12}. \tag{4.18}$$

Then,

$$E \textit{ is } \varepsilon\textit{-close to } \Lambda_{\gamma,\theta} \textit{ in } B_1 \quad \Rightarrow \quad E \textit{ is } \varepsilon\rho_\circ^{1+\alpha}\textit{-close to } \Lambda_{\tilde\gamma,\tilde\theta} \textit{ in } B_{\rho_\circ}, \tag{4.19}$$

where γ, $\tilde\gamma$, θ, *and* $\tilde\theta$, *are as in* (4.11).

Remark 4.1. Let us comment on the statement of Theorem 4.3:

(1) This result generalizes the classical De Giorgi's improvement of flatness theorem (4.8).

(2) Our estimate (4.19) is designed to be applied, iteratively in a sequence of dyadic balls, to a minimizer E of (4.16). It gives $C^{1,\alpha}$ regularity of ∂E at points of the contact set; see Theorem 4.4 below.

(3) An essential feature of our result is that the constant $\varepsilon_\circ$ is independent of δ. Thus (4.19) is stable as $\delta \downarrow 0$ and hence applies to solutions of (4.2)-(4.6); see Definition 4.1.

(4) The assumption $\alpha < 1/2$ is almost sharp. Indeed, one can easily see that the statement of the theorem cannot be true for $\alpha \in (\frac{1}{2}, 1)$ by using that the optimal regularity of solutions to the Signorini problem is $C^{1,\frac{1}{2}}$.

(5) If $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is any $C^{1,1}$ diffeomorphism and $x_\circ$ belongs to $\partial\mathcal{O} = \Phi(\{x_{n-1} = x_n = 0\})$, then for $\rho > 0$ and in some new coordinates $\bar x = \psi_{x_\circ}(x)$ with origin at $x_\circ$ such that

$$\psi_{x_\circ}(x) := \rho^{-1}R_{x_\circ}(x - x_\circ), \quad \text{where } R_{x_\circ} \text{ is an orthogonal matrix,}$$

the assumption (4.18) will be fulfiled by some new diffeomorphism $\bar\Phi$ satisfying $\bar\Phi(\Lambda^{\bar\delta}) = \psi(\Phi(\Lambda^\delta))$ — see Lemma 4.10. Hence, assumption (4.18) is always satisfied after a change of coordinates.

4.1.6 On the proof of Theorem 4.3

Let us now briefly comment on the proof of Theorem 4.3. Our main idea is to use a "dichotomy approach", which is combined with Savin's "small perturbation method". More precisely, we prove by a barrier argument that — if $\varepsilon_\circ$ is small enough — one of the following two alternatives must hold:

(a) ∂E is very flat in B_1.

(b) The contact set is full in $B_{3/4}$ (it contains $\partial\mathcal{O} \cap B_{3/4}$) and ∂E splits into two minimal surfaces that meet along $\partial\mathcal{O}$ with some angle.

Then, on the one hand, if (a) holds we can use that our problem is a perturbation of the Signorini problem (4.5) and exploit the $C^{1,1/2}$ regularity for (4.5) to prove (4.19). For this we use the "small perturbation method" pioneered by Savin — see [Sav09, Sav10, Sav10b].

On the other hand, if (b) holds then ∂E splits in $B_{3/4}$ into two minimal surfaces with boundary, each of them flat in a different direction. Since the contact set is full we can interpret it as a smooth "boundary condition". Then, using the $C^{1,1}$ regularity up to the boundary of flat minimal surfaces, we can improve the flatness of each of the two surfaces separately to prove (4.19).

4.1.7 Consequences

From our Theorem 4.3, as in the classical theory, we get that once the minimizer is sufficiently close to a "wedge" type set $\Lambda_{\gamma,\theta}$, then it has a local $C^{1,\alpha}$ structure.

Theorem 4.4. *Given $\alpha \in \left(0, \frac{1}{2}\right)$ there exists a positive constant $\varepsilon_\circ$ depending only on n and α such that the following holds:*

Assume that, for some $\delta > 0$, a set $E \subset \mathbb{R}^n$ with $P(E; B_1) < \infty$ satisfies $\Phi(\Lambda^\delta) \cap B_1 \subset E$ and (4.17). Suppose that $0 \in \partial E \cap \partial \mathcal{O}$, that

$$\Phi(0) = 0, \quad D\Phi(0) = \mathrm{id}, \quad |D^2\Phi| \le \varepsilon_\circ^{1+\frac{1}{2}}, \tag{4.20}$$

and that E is $\varepsilon_\circ$-close to $\Lambda_{\gamma,\theta}$ in B_1.

Then, ∂E has the following $C^{1,\alpha}$ structure in $B_{1/2}$. Either:

(a) In some appropriate coordinates $y = (y', y_n) = (y_1, \dots, y_n)$, $\Phi^{-1}(\partial E)$ is the graph $\{y_n = h(y')\}$ of a function $h \in C^0(\overline{B'_{1/2}})$ that belongs to $C^{1,\alpha}(\overline{B'^{+}_{1/2}}) \cap C^{1,\alpha}(\overline{B'^{-}_{1/2}})$, where $B'_{1/2}$ denotes the ball in $\mathbb{R}^{n-1}$ and $B'^{\pm}_{1/2}$ are the half-balls $B'_{1/2} \cap \{\pm y_{n-1} > 0\}$. Moreover, we have $h \ge 0$ on $y_{n-1} = 0$ and ∇h is continuous on $\{y_{n-1} = 0\} \cap \{h > 0\}$.

or

(b) $\partial E \cap B_{1/2}$ is the union of two $C^{1,1-}$ surfaces that meet on $\partial\mathcal{O}$ with full contact set in $B_{1/2}$.

In the previous statement $C^{1,1-} := \bigcap_{\beta \in (0,1)} C^{1,\beta}$.

Remark 4.2. It will be clear from the proofs that if $\mathcal{O}$ is a minimal surface (with boundary), then ∂E cannot stick to $\mathcal{O} \setminus \partial\mathcal{O}$ and (b) must hold with the same regularity as that of ∂O. Namely, if $\partial\mathcal{O}$ is a $C^{k,\beta}$ (resp. analytic) codimension two surface, then the two surfaces in (b) will also be $C^{k,\beta}$ (resp. analytic), and not just $C^{1,1-}$.

Theorem 4.4 requires the solution to be sufficiently close to a wedge-type set $\Lambda_{\gamma,\theta}$. Thanks to the following classification of global conical solutions to our problem, we will have that this is always the case (after rescaling) near any contact point.

Proposition 4.5 (Classification of minimal cones in $\mathbb{R}^n$)**.** *Let $\Sigma \subset \mathbb{R}^n$ be a cone, i.e. $t\Sigma = \Sigma$ for all $t > 0$, with $\partial\Sigma \neq \varnothing$. Suppose that Σ satisfies (4.17) with $\Phi \equiv \mathrm{id}$.*

Then, $\Sigma = \Lambda_{\gamma,\theta}$ for some γ and θ as in (4.11).

As a direct consequence of the combination of Theorem 4.4 and Proposition 4.5 we obtain the following result (which is just a more precise version of Theorem 4.1 above),

Corollary 4.6. *Let $n \ge 2$, and assume that $\mathcal{O}$ is a minimal surface and that $\Phi \in C^{k,\beta}$ for some $k \ge 2$ and $\beta \in (0,1)$ — or equivalently $\partial\mathcal{O}$ is of class $C^{k,\beta}$.*

Let E be a solution (in the sense of Definition 4.1) of (4.2)-(4.6) with $x_\circ \in \partial E \cap \partial\mathcal{O} \cap B_{1/2}$. Then, for all $\alpha \in \left(0, \frac{1}{2}\right)$, ∂E has the following $C^{1,\alpha}$ local structure near $x_\circ$. For $r > 0$ small enough, we have either:

(a) In some appropriate coordinates $y = (y', y_n) = (y_1, \dots, y_n)$, $\Phi^{-1}(\partial E)$ *is the graph* $\{y_n = h(y')\}$ *of a function* $h \in C^0(\overline{B'_r})$ *that belongs to* $C^{1,\alpha}(\overline{B'^+_r}) \cap C^{1,\alpha}(\overline{B'^-_r})$, *where* B'_r *denotes the ball in* $\mathbb{R}^{n-1}$ *and* $B'^{\pm}_r$ *are the half-balls* $B'_r \cap \{\pm y_{n-1} > 0\}$. *Moreover, we have* $h \geq 0$ *on* $y_{n-1} = 0$ *and* ∇h *is continuous on* $\{y_{n-1} = 0\} \cap \{h > 0\}$.

or

(b) $\partial E \cap B_r(x_\circ)$ *is the union of two* $C^{k,\beta}$ *minimal surfaces with boundary that meet on* $\partial\mathcal{O}$ *with full contact set in* $B_r(x_\circ)$.

Remark 4.3. By [DeA79, Theorem 2.1 and Theorem 2.2] (or by a standard barrier argument similar to that used in Hopf's lemma) if one considers a minimizer of the De Giorgi measure for obstacles as in Corollary 4.6, then its boundaries do not stick to the obstacle. More precisely, they present an aperture around the obstacle that allows, locally, a wedge contained in the minimizer.

As a consequence, minimizers of the De Giorgi measure are locally (in a neighborhood of any contact point) minimizers in the sense of Definition 4.1. Therefore, Corollary 4.6 above applies to minimizers in the sense of De Giorgi.

Remark 4.4. In the previous statement the condition that $\mathcal{O}$ is a minimal surface appears only to be able to apply Remark 4.2 and obtain (b). Otherwise, an analogous result with $C^{1,1-}$ regularity holds.

Remark 4.5. We observe that, as a consequence of our results,

$$E \text{ is a minimizer as in Definition 4.1} \quad \Rightarrow \quad P_{DG}(E; B_1) = P(E; B_1). \tag{4.21}$$

Indeed, let E be a minimizer as in Definition 4.1. First, as proven in [DeA79], since $\mathcal{O}$ is smooth, the De Giorgi perimeter P_{DG} of the minimizer can be expressed as

$$P_{DG}(F; B_1) = P(F; B_1) + 2\mathcal{H}^{n-1}((\mathcal{O} \setminus F) \cap B_1) \geq P(F; B_1) \quad \text{for any Borel set } F. \tag{4.22}$$

But note that ∂E cannot stick to the obstacle from both sides at any point of $\mathcal{O} \setminus \partial\mathcal{O}$ by the strong maximum principle. Hence,

$$\mathcal{H}^{n-1}((\mathcal{O} \setminus E) \cap B_1) = 0. \tag{4.23}$$

Using (4.22) and (4.23), E is therefore also a minimizer of P_{DG}, since $P_{DG}(F; B_1) \geq P(F; B_1) \geq P(E; B_1) = P_{DG}(E; B_1)$ for any competitor F.

Remark 4.6. Corollary 4.6 gives the regularity of the hypersurface around contact points. The regularity around other points follows from the classical theory for minimal surfaces (see for instance chapters 8 and 9 of the classical book of Giusti [Giu84]). Note that this is result only up to dimension 7 [Sim68] since nonsmooth minimizers exist in dimensions 8 and higher [BDG69]. In contrast, our regularity result holds around the contact set of the thin obstacle, in any dimension.

Remark 4.7. After a previous version of this manuscript, a preprint of Focardi and Spadaro [FoSp18b] appeared in which the authors establish optimal $C^{1,1/2}$ regularity estimates and rectifiability of the free boundary for minimal surfaces with flat thin obstacles in the nonparametric case (that is, in our notation, for the case $\Phi = \mathrm{id}$ and assuming that ∂E is a graph in the n-th direction). Interestingly, our Corollary (4.6) gives that (at least for flat obstacles) the assumptions of [FoSp18b] are always satisfied near any contact point by parametric minimal surfaces with thin obstacles. Thus, when combined with our results, the results in [FoSp18b] yield that solutions to parametric thin obstacle problems are $C^{1,1/2}$ near the obstacle and their free boundary is rectifiable.

4.1.8 Organization of the paper

The paper is organised as follows.

In Section 4.2 we introduce some notation, definitions, and preliminary results. In Section 4.3 we construct a barrier and prove the dichotomy presented in the introduction: if the solution is close to a wedge, then either ∂E is very flat or its contact set is full in a smaller ball. In Section 4.4 we focus on the flat configuration, showing the improvement of closeness result in this case (Proposition 4.14). In Section 4.5, instead, we focus on the full contact set configuration, which allows us to complete the proof of our first main result, Theorem 4.3. In Section 4.6 we prove Theorem 4.4 by iteratively applying Theorem 4.3. Finally, in Section 4.7 we discuss blow-ups (monotonicity formula and classification of minimal cones) and we complete the proofs of Proposition 4.5 and Corollary 4.6, thus obtaining Theorem 4.1.

4.2 Notation and preliminary results

4.2.1 Conventions and notation.

As it is standard, throughout the paper we will assume that the representative of E among sets that differ from it by a null set is such that topological and measure theoretic boundary agree. That is, given a set $E \subset \mathbb{R}^n$, we will say that $x \in \mathbb{R}^n$ belongs to the boundary of E, $x \in \partial E$, whenever

$$0 < |E \cap B_r(x)| < |B_r(x)|, \qquad \text{for all } r > 0.$$

Notice that, in general, this is not necessarily true. However, the set of points where this does not hold is of measure zero, and therefore we can consider instead the equivalent set $\tilde{E}$ that arises from removing all such points. Thus, without loss of generality, we will always assume that the measure theoretic and topological boundary agree.

The notation introduced in Subsections 4.1.3 and 4.1.4 will be recurrent throughout the work. In particular, the definitions of $\Lambda_{\gamma,\theta}$ and Λ^δ from (4.10)-(4.15) as well as the definition of e_w and the conditions on the constants θ and γ (see (4.11)). See also Figure 4.2.

On the other hand, when not stated otherwise, we add a superscript prime to an element or set in $\mathbb{R}^n$ to denote its projection to $\mathbb{R}^{n-1}$; and we proceed similarly with a double superscript prime projection to $\mathbb{R}^{n-2}$. Thus, if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we can also denote $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ or $x = (x'', x_{n-1}, x_n) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}$. Similarly, B_1 denotes the unit ball in $\mathbb{R}^n$, B_1' is the unit ball in $\mathbb{R}^{n-1}$ and B_1'' in $\mathbb{R}^{n-2}$. We may sometimes write $B_1' \subset \mathbb{R}^n$, or $x' \in \mathbb{R}^n$ as an abuse of notation, meaning $B_1' \times \{0\} \subset \mathbb{R}^n$ and $(x', 0) \in \mathbb{R}^n$ respectively.

4.2.2 Preliminary results

Definition 4.3. Let $E \subset \mathbb{R}^n$. We say that E is a *minimizer of the δ-thin obstacle problem in* $B_1 \subset \mathbb{R}^n$ if $\Phi(\Lambda^\delta) \cap B_1 \subset E$ and (4.17) holds.

We are also interested in the notion of super- and subsolutions to the minimal perimeter problem. Thus, the follow definition will also be useful.

In general terms, we say that a set E^+ is a supersolution to the minimal perimeter problem when compact additive perturbations to E^+ in B_1 produce sets of larger perimeter. Similarly, E^- is a subsolution to the minimal perimeter problem when compact subtractive perturbations to E^- in B_1 increase the perimeter.

Definition 4.4. Let $E^\pm \subset \mathbb{R}^n$. Then, E^+ is a *supersolution* in B if

$$P(F^+; B) \geq P(E^+; B),$$

for any F^+ with $E^+ \subset F^+$ and $\overline{F^+ \setminus E^+} \Subset B$.

Analogously, E^- is a *subsolution* in B if

$$P(F^-; B) \geq P(E^-; B),$$

for any F^- with $E^- \supset F^-$ and $\overline{E^- \setminus F^-} \Subset B$.

Notice that, in particular, a set satisfying (4.17) is a supersolution to the minimal perimeter problem.

Proposition 4.7. *Given $E_\circ \subset \mathbb{R}^n$ with $P(E_\circ; B_1) < \infty$, there exists E satisfying* (4.17) *with $E \setminus B_1 = E_\circ \setminus B_1$.*

Proof. The proof follows by classic methods in the calculus of variations. Lower semicontinuity and compactness in L^1 of BV functions directly yield the result (see [Giu84, Thm 1.9, Thm 1.19]). □

Proposition 4.8. *Let $E \subset \mathbb{R}^n$ satisfying* (4.17)*. Then, for any $B_r(x_\circ) \subset B_1$, E is a* supersolution *in $B_r(x_\circ)$. Moreover, if $B_r(x_\circ) \cap \Phi(\Lambda^\delta) = \varnothing$, then E is a set of minimal perimeter in $B_r(x_\circ)$.*

Proof. This just follows from the definitions of minimizer of the δ-thin obstacle problem (4.17) and supersolution. □

Lemma 4.9. *If E is a local minimizer of the perimeter around a point $x_\circ \in \partial E$, then ∂E satisfies the mean curvature equation*

$$M(D^2 v, \nabla v) := (1 + |\nabla v|^2)\Delta v - (\nabla v)^T D^2 v \nabla v = 0$$

in the viscosity sense. That is, if we define for any smooth $\varphi : B_1' \to \mathbb{R}$,

$$S_\varphi^\pm := \{\pm x_n < \varphi(x')\},$$

then, if $S_\varphi^\pm$ *is included in either* E *or* E^c *in some ball* $B_r(x_\circ)$ *and* $x_\circ \in \partial S_\varphi^\pm$, *we have that*

$$\pm\, M(D^2\varphi, \nabla\varphi) \le 0. \tag{4.24}$$

Moreover, if E *is a supersolution to the minimal perimeter problem around* $x_\circ \in \partial E$, *then if* $S_\varphi^\pm$ *is included in* E *in some ball* $B_r(x_\circ)$ *and* $x_\circ \in \partial S_\varphi^\pm$ *we have the same result,* (4.24).

Proof. The proof is very standard, just using the definitions of minimal perimeter and supersolution and noticing that we can decrease the perimeter if the conclusion does not hold. See, for example, [CC93]. □

Lemma 4.10. *Let* $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ *be any* $C^{1,1}$ *diffeomorphism and let* $x_\circ$ *belong to* $\partial\mathcal{O} = \Phi(\{x_{n-1} = x_n = 0\})$. *Assume that* $[\Phi]_{C^{1,1}} \le M$ *and* $|D(\Phi^{-1})(x_\circ)| \le M$. *Then, for* $\rho > 0$, *there are new coordinates* $\bar{x} = \psi_{x_\circ}(x)$

$$\psi_{x_\circ}(x) := \rho^{-1} R_{x_\circ}(x - x_\circ), \quad \textit{where } R_{x_\circ} \textit{ is an orthogonal matrix,}$$

and a new $C^{1,1}$ *diffeomorphism* $\bar\Phi$, *such that*

$$\bar\Phi(\Lambda^{\bar\delta}) = \psi_{x_\circ}(\Phi(\Lambda^\delta)) \quad \textit{for some } \bar\delta \in (0, C\delta)$$

and

$$\bar\Phi(0) = 0, \quad \bar\Phi(0) = \mathrm{id}, \quad \textit{and} \quad |D^2\bar\Phi| \le CM^3\rho,$$

where C *depends only on* n.

Proof. Let us choose $R_{x_\circ}$ to be some orthogonal matrix to be chosen and define

$$A_{x_\circ} := R_{x_\circ} D\Phi\big(\Phi^{-1}(x_\circ)\big).$$

Choose $R_{x_\circ}$ and $\bar\delta \in (0, C\delta)$ such that

$$A_{x_\circ}(\Lambda^\delta) = \Lambda^{\bar\delta}$$

as a consequence the set

$$\{x_{n-1} = 0, x_n \le 0\} \quad \text{is invariant under the linear map } A_{x_\circ}.$$

Now define

$$\Phi^{x_\circ} := R_{x_\circ}\big(\Phi(\Phi^{-1}(x_\circ) + A_{x_\circ}^{-1}x) - x_\circ\big) \quad \text{and} \quad \bar\Phi := \rho^{-1}\Phi^{x_\circ}(\rho x).$$

Note that since $\Phi^{-1}(x_\circ) \in \{x_{n-1} = x_n = 0\}$ we have $\Phi^{-1}(x_\circ) + A_{x_\circ}^{-1}\Lambda^{\bar\delta} = \Lambda^\delta$ and thus

$$\bar\Phi(\Lambda^{\bar\delta}) = \psi_{x_\circ}(\Phi(\Phi^{-1}(x_\circ) + A_{x_\circ}^{-1}\Lambda^{\bar\delta})) = \psi_{x_\circ}(\Phi(\Lambda^\delta)).$$

By construction, we have $\bar\Phi(0) = 0$, $D\bar\Phi(0) = \mathrm{id}$, and $[\bar\Phi]_{C^{1,1}} \le CM^3\rho$. □

4.3 Barriers and dichotomy

For this section let us start by defining the mean curvature operator H, on functions $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ as

$$H\varphi = \operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) = (1+|\nabla\varphi|^2)^{-\frac{3}{2}} M(D^2\phi, \nabla\varphi). \tag{4.25}$$

We start by introducing a supersolution that will be used as barrier.

Lemma 4.11 (Supersolution)**.** *Let $\beta \in \left(0, \frac{1}{10(n-2)}\right)$. Let*

$$S_\beta^+ := \big\{x = (x'', x_{n-1}, x_n) \in B_1 \subset \mathbb{R}^{n-2}\times\mathbb{R}\times\mathbb{R} : \\ x_n \le \varphi_\beta(x') := \beta\left(|x''|^2 - 2(n-2)x_{n-1}^2\right)\big\}$$

Then, S_β^+ is a strict supersolution to the equation of minimal graphs in B_1, and

$$H\varphi_\beta \le -c\beta, \quad in\ \ B_1',$$

for some positive constant c depending only on n.

Proof. Let us check that, given φ_β, then

$$H\varphi_\beta \le -c\beta.$$

Let us rewrite the operator H,

$$H\varphi_\beta(x') = \frac{1}{\sqrt{1+|\nabla\varphi_\beta|^2}}\left(\Delta\varphi_\beta - \frac{(\nabla\varphi_\beta)^T D^2\varphi_\beta\nabla\varphi_\beta}{1+|\nabla\varphi_\beta|^2}\right)(x') = \sum_{i,j} U_{ij}(x')\partial_{ij}\varphi_\beta(x'),$$

where

$$U_{ij}(x') := \frac{1}{\sqrt{1+|\nabla\varphi_\beta|^2}}\left(\delta_{ij} - \frac{\partial_i\varphi_\beta(x')\partial_j\varphi_\beta(x')}{1+|\nabla\varphi_\beta|^2}\right).$$

Let $S_\varphi(x') = \sqrt{1+|\nabla\varphi_\beta|^2}$. Note that, $U(x') = S_\varphi^{-1}(x')\left(\mathrm{Id} - \bar\varphi_\beta\bar\varphi_\beta^T\right)$, where $\bar\varphi_\beta(x') = \nabla\varphi_\beta(x')/S_\varphi(x')$. The only eigenvalue of $\mathrm{Id} - \bar\varphi_\beta\bar\varphi_\beta^T$ different from 1 is $1 - \|\bar\varphi_\beta\|^2$. Let $m_\varphi = \sup\{|\nabla\varphi_\beta|\}$, where the supremum is taken over the domain of definition of φ_β. Putting all together we have obtained that U is uniformly elliptic, with ellipticity constants $\lambda_\varphi = (1+m_\varphi^2)^{-3/2}$ and 1.

Notice then that

$$H\varphi_\beta(x') = \sum_{i,j} U_{ij}(x')\partial_{ij}\varphi_\beta(x') \le \beta\left(2(n-2) - 4(n-2)\lambda_\varphi\right), \quad \text{in } B_1'.$$

On the other hand, from the fact that $|\nabla\varphi| \le 4\beta(n-2)$ in B_1',

$$\lambda_\varphi = (1+m_\varphi^2)^{-3/2} \ge (1+16\beta^2(n-2)^2)^{-3/2}. \tag{4.26}$$

Putting all together, we get the desired result. □

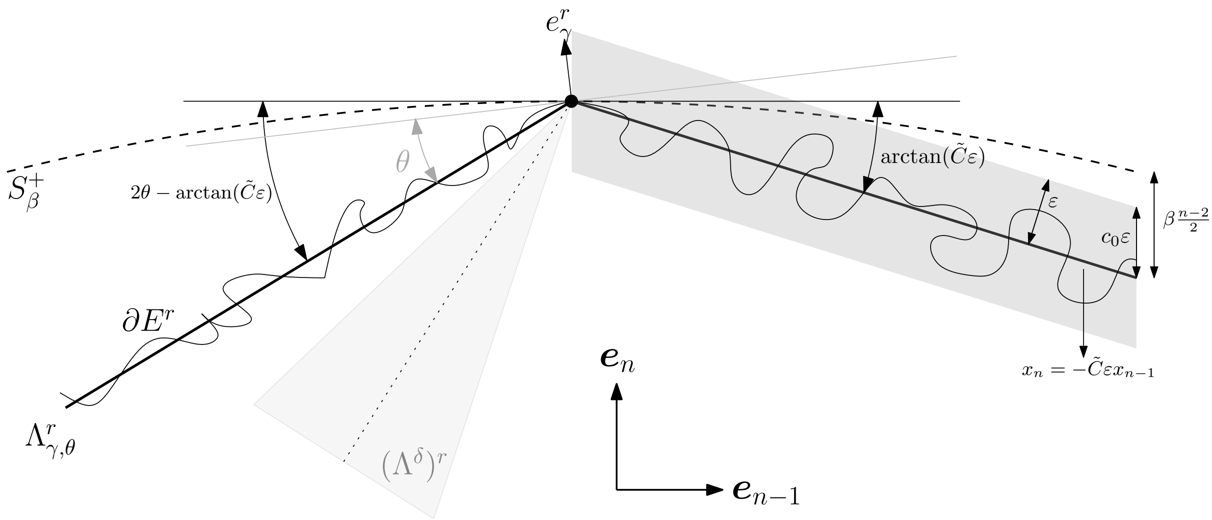

Figure 4.3: Representation of the setting in Lemma 4.12 after a rotation.

The following lemma shows that whenever the minimizer is not flat, then the contact set is full in the interior. The condition of flatness is used via the angle θ from the definition of the wedge $\Lambda_{\gamma,\theta}$: being flat means that θ is small, when compared to ε.

Lemma 4.12. *There exists $\varepsilon_\circ$ and $C_\circ$ depending only on n such that the following statement holds:*

Let $E \subset \mathbb{R}^n$ satisfying (4.17) *be such that it is ε-close to some $\Lambda_{\gamma,\theta}$ in B_1, for some $\varepsilon \in (0, \varepsilon_\circ)$, and* (4.18) *holds. Suppose that $\theta \in \left[C_\circ\varepsilon, \frac{\pi}{2}\right)$. Then*

$$E \subset \Phi(\Lambda_{\gamma,\theta - C_\circ\varepsilon}) \quad \text{in } B_{1/2}.$$

In particular, the contact set is full in $B_{1/2}$.

Proof. Let us prove this result, for simplicity, in the case $\Phi \equiv \mathrm{id}$, and at the end of the proof we discuss how to modify it in order to account for small second order perturbations.

We will slide an appropriate supersolution from above until we intersect with the surface ∂E.

Take $x_\circ \in B''_{1/2} \times \{0\} \times \{0\}$, and by making a translation let us assume $x_\circ$ is the origin. Let us also rotate the setting with respect to the last two coordinates so that the angle between e_γ and $\boldsymbol{e}_n$ is $\angle(e_\gamma, \boldsymbol{e}_n) = \theta - \arctan(\tilde C\varepsilon)$, for some constant $\tilde C$ depending only on n to be chosen, such that $\theta > \arctan(\tilde C\varepsilon)$. Let us denote e^r_γ, ∂E^r, $\partial\Lambda^r_{\gamma,\theta}$, and $(\Lambda^\delta)^r$, the corresponding rotated versions. The following argument can be done with both configurations that fulfil this property, so let us assume without loss of generality that we are in a situation where

$$\{x_n = -\tilde C\varepsilon x_{n-1}\} \cap \{x_{n-1} \geq 0\} \subset \partial\Lambda^r_{\gamma,\theta}, \quad \text{in } B_{1/2}. \tag{4.27}$$

See Figure 4.3 for a representation of this rotated situation, and the whole proof.

Take the supersolution S^+_β from Lemma 4.11. Slide ∂S^+_β from above until it touches the boundary of the minimizer of the δ-thin obstacle problem, ∂E^r. That is, define

$$S^t_\beta := \partial S^+_\beta + t\boldsymbol{e}_n,$$

and consider

$$m_\beta := \inf\{t > 0 : S^t_\beta \cap \partial E^r \cap B_{1/2} \neq \varnothing\}.$$

We recall that

$$\partial S^+_\beta = \left\{x = (x'', x_{n-1}, x_n) \in B_1 : x_n = \beta\left(|x''|^2 - 2(n-2)x^2_{n-1}\right)\right\}.$$

If $m_\beta > 0$ and $x^m = (x^m_1, \dots, x^m_n) \in B_{1/2}$ is such that $x_m \in S^{m_\beta}_\beta \cap \partial E^r \cap B_{1/2}$, then x^m cannot be an interior point to $S^{m_\beta}_\beta \cap B_{1/2}$. Indeed, since $S^{m_\beta}_\beta \cap B_{1/2} \cap \{x_{n-1} = 0\} \subset \{x_n \geq m_\beta > 0\}$ is strictly above zero, then thanks to Proposition 4.8 ∂E^r is a surface of minimal perimeter around x_m. On the other hand, $S^{m_\beta}_\beta$ is a supersolution, touching on an interior point with a surface of minimal perimeter locally, which is not possible.

We will show that the boundary $\partial B_{1/2} \cap S^{m_\beta}_\beta$ is always *above* ∂E^r in the $\boldsymbol{e}_n$ direction. From (4.27) and using that $\partial E^r \subset \Lambda^r_{\gamma,\theta} + B_\varepsilon$, it is enough to show that there exists $\tilde{C}$ depending only on n such that

$$\beta\left(|x''|^2 - 2(n-1)x^2_{n-1}\right) \geq -\tilde{C}\varepsilon x_{n-1} + c_0\varepsilon, \quad \text{for } x' = (x'', x_{n-1}) \in \partial B'_{1/2}, \quad (4.28)$$

for some constant c_0 depending only on n that accounts for the difference in distance between the Hausdorff distance and the distance in the $\boldsymbol{e}_n$-direction. For (4.28) to be satisfied, using $|x''|^2 = \frac{1}{4} - (x_{n-1})^2$, we want

$$-\beta(2n-1)x^2_{n-1} + \tilde{C}\varepsilon x_{n-1} \geq -\frac{\beta}{4} + c_0\varepsilon, \qquad \text{for } x_{n-1} \in [0, 1/2].$$

By taking $\beta = 4c_0\varepsilon$ and $\tilde{C} = 2c_0(2n-1)$ the previous condition holds, and notice that for ε small enough (depending only on n) S^+_β is a supersolution as wanted.

Thus, for $\beta = 4c_0\varepsilon$ and $\tilde{C} = 2c_0(2n-1)$, we can slide S^t_β until $t = 0$, where it touches ∂E^r at the origin (since it touches $(\Lambda^\delta)^r$ there). Therefore, the origin is a contact point, and moreover, ∂E^r is contained in $S^+_\beta \cap \{x_{n-1} \geq 0\}$. In particular, since the origin was a translation of any point in $B''_{1/2} \times \{0\} \times \{0\}$, we have that in $B''_{1/2} \times \{0\} \times \{0\} \cap \{x_{n-1} \geq 0\}$, ∂E^r is contained in $\{x_n \leq 0\}$.

Rotating back, and putting $\arctan(\tilde{C}\varepsilon) = C_\circ\varepsilon$ for some $C_\circ$ depending only on n, we obtain the desired result from one side. Doing the same on the other side completes the proof.

If $\Phi \not\equiv \text{id}$, we can proceed similarly using that $|D^2\Phi| \leq \varepsilon^{1+\frac{1}{2}}$. Indeed, if E is ε-close to $\Lambda_{\gamma,\theta}$, then $\Phi^{-1}(E)$ is 2ε-close to $\Lambda_{\gamma,\theta}$ for ε small enough depending only on n. Now we can repeat the previous argument with $\Phi^{-1}(E)$ instead of E. The only place where we used that E satisfies (4.17) is to check that we cannot touch at an interior point when sliding the supersolution (using the previous notation, to check that m_β cannot be strictly positive).

If we were touching at an interior point x_m in this case, then E would be a surface of minimal perimeter around $\Phi(x_m)$. Since we can choose $\beta = 4c_0\varepsilon$ to avoid contact in the boundary, thanks to Lemma 4.11 the mean curvature of $\partial S^{m_\beta}_\beta$ is below $-4c\varepsilon$. Consequently, the mean curvature of $\Phi(\partial S_\beta^{m_\beta})$ is below $-4c\varepsilon + c'\varepsilon^{1+\frac{1}{2}}$ and for ε small enough $\Phi(S_\beta^{m_\beta})$ is still a supersolution: there cannot be an interior tangential contact point. $\square$

Lemma 4.12 shows that if if E is ε-close to some wedge $\Lambda_{\gamma,\theta}$ in B_1 with $\theta \geq C_\circ\varepsilon$ then we have $E \subset \Phi(\Lambda_{\gamma,\theta-C_\circ\varepsilon})$. As a counterpart, the following lemma shows that $\Phi(\Lambda_{\gamma,\theta+C_\circ\varepsilon}) \subset E$ — even for $\theta < C_\circ\varepsilon$.

Lemma 4.13. *There exists $\varepsilon_\circ$ and $C_\circ$ depending only on n such that the following statement holds:*

Let $E \subset \mathbb{R}^n$ satisfying (4.17) *be such that it is ε-close to some $\Lambda_{\gamma,\theta}$ in B_1, for some $\varepsilon \in (0, \varepsilon_\circ)$ and $\theta \in \left[0, \frac{\pi}{2} - C_\circ\varepsilon\right)$. Suppose that Φ satisfies* (4.18)*. Then*

$$\Phi(\Lambda_{\gamma,\theta+C_\circ\varepsilon}) \subset E \quad \text{ in } B_{1/2}.$$

Proof. The proof follows very similarly to the previous result, Lemma 4.12. Again, as before, we assume $\Phi \equiv \text{id}$; and the proof can be adapted to the case $|D^2\Phi| \leq \varepsilon^{1+\frac{1}{2}}$ following analogously to the proof of Lemma 4.12.

We want to show that we can *open* Λ^δ up to being at an angle proportional to ε from $\Lambda_{\gamma,\theta}$. Let us show it for $x_{n-1} \geq 0$.

The fact that $\Lambda^\delta \subset E$ in B_1 allows us to establish a separation between $x_{n-1} \geq 0$ and $x_{n-1} \leq 0$.

Consider the surface $\partial E \cap \{x_{n-1} \geq 0\}$. Let θ_1 be the angle between $\partial\Lambda_{\gamma,\theta}$ and $\partial\Lambda^\delta$ in $\{x_{n-1} \geq 0\}$. If $\theta_1 \leq C_1\varepsilon$ for some C_1 depending only on n we are already done, since Λ^δ is already a barrier; so that we can suppose that $\theta_1 \geq C_1\varepsilon$ for some C_1 to be determined. We denote $\Gamma_{\gamma,\theta} = \partial\Lambda_{\gamma,\theta} \cap \{x_{n-1} \geq 0\}$.

Now, as in Lemma 4.12, we rotate the setting in the last two coordinates, so that $\Gamma^r_{\gamma,\theta} \subset \{x_n \geq 0\}$ at an angle $\arctan(\tilde{C}\varepsilon)$ from $\{x_n = 0\}$, for some constant $\tilde{C}$ to be chosen. See Figure 4.4 for a representation after the rotation.

Notice that $-S_\beta^+$ is a subsolution to the problem, where S_β^+ denotes the supersolution constructed in Lemma 4.11. Now the situation is the same as in Lemma 4.12 upside down. In the new coordinates after the rotation, since in $\{x_{n-1} > 0\}$ any point on ∂E^r is locally a supersolution, we will be able to slide up the subsolution up until the origin for the same constant $\tilde{C}$ as in Lemma 4.12 as long as we are are not touching with it in the region $\{x_{n-1} \leq 0\}$ after the rotation. But this can be avoided choosing C_1 such that $C_1\varepsilon \geq 3\arctan\tilde{C}\varepsilon$ for ε small. $\square$

4.4 Improvement of closeness in flat configuration

In this section we prove our main result, Theorem 4.3, in the flat configuration case in the case $\theta \in (0, C_\circ\varepsilon)$. Namely, we show:

Proposition 4.14. *For every $\alpha \in \left(0, \frac{1}{2}\right)$, there exist positive constants $\rho_\circ$ and $\varepsilon_\circ$ depending only on n and α, such that the following statement holds:*

Let $E \subset \mathbb{R}^n$ satisfying (4.17)*, with $0 \in \partial E$, be such that E is ε-close to $\Lambda_{\gamma,\theta}$ in B_1, for some $\theta \in (0, C_\circ\varepsilon)$ and $\varepsilon \in (0, \varepsilon_\circ)$, and* (4.18) *holds.*

Then,

$$E \text{ is } \rho_\circ^{1+\alpha}\varepsilon\text{-close to } \Lambda_{\tilde{\gamma},\tilde{\theta}} \text{ in } B_{\rho_\circ},$$

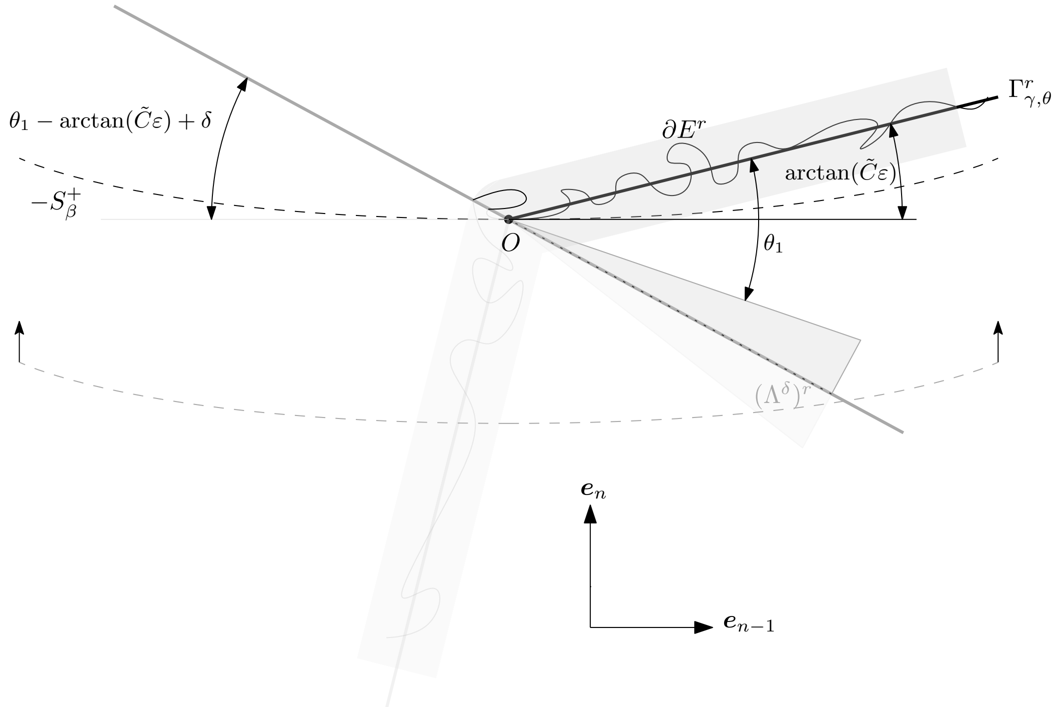

Figure 4.4: Representation of the setting in Lemma 4.13 after a rotation.

for some new $\tilde{\gamma}'$ and $\tilde{\theta}$ as in (4.11).

The proof of this proposition follows by compactness, using the $C^{1,1/2}$ regularity of the solutions to the classical thin obstacle problem with the Laplacian, Δ.

The following proposition will be used to show compactness of vertical rescalings $\big\{(x', x_n/\varepsilon) : (x', x_n) \in \partial E\big\}$ near a contact point.

Proposition 4.15. *There exist $h_\circ$ and $\tau_\circ$ depending only on n such that the following statement holds:*

Denote $Q_1 := B_1' \times (-1,1)$ Let $E \subset \mathbb{R}^n$ satisfying, for some $\boldsymbol{v} \in Q_1$,

$$P(E; Q_1) \le P(F; Q_1) \quad \forall F \ : \ E \setminus Q_1 = F \setminus Q_1 \text{ and } \big(\boldsymbol{v} + \Phi(\Lambda^\delta)\big) \cap Q_1 \subset F. \quad (4.29)$$

be such that for some $b \in (-1,1)$ and some $h \in (0, h_\circ)$, (4.18) *holds for $\varepsilon \in (0,h)$,*

$$\{x_n \le b - h\} \subset E \subset \{x_n \le b + h\}, \quad \text{in } B_1' \times (-1,1),$$

and

$$\big(\boldsymbol{v} + \Phi(\Lambda_{0,h})\big) \subset E, \quad \text{in } B_1' \times (-1,1).$$

Then,

- *either $\{x_n \le b - h(1-\tau_\circ)\} \subset E$, in $B'_{1/2} \times (-1,1)$;*
- *or $E \subset \{x_n \le b + h(1-\tau_\circ)\}$, in $B'_{1/2} \times (-1,1)$.*

To prove Proposition 4.15 we need the following half-Harnack for supersolutions; see [Sav10b, Section 2] or the proof of [Sav10, Thm 5.3].

Proposition 4.16 ([Sav10, Sav10b]). *Let $E \subset \mathbb{R}^n$ be a supersolution to the minimal perimeter problem in B_1, and suppose $\partial E \subset \{x_n \geq 0\}$. Then, for every $\eta_\circ > 0$, there exists some $\tau_\circ$ and C depending only on n and $\eta_\circ$ such that if $\tau < \tau_\circ$ and $\tau \boldsymbol{e}_n \in \partial E$, then*

$$|\Pi_{\boldsymbol{e}_n} (\partial E \cap \{x_n \leq C\tau\} \cap (B_1' \times (-1,1)))|_{\mathcal{H}^{n-1}} \geq (1-\eta_\circ)|B_1'|_{\mathcal{H}^{n-1}};$$

where $\Pi_{\boldsymbol{e}_n}$ denotes the projection of a set onto B_1' in the $\boldsymbol{e}_n$ direction.

Proof of Proposition 4.15. We separate the proof into two different scenarios.

The first possibility is $b \leq \varepsilon^{1+\frac{1}{4}}$. In this case, since $\Phi(\Lambda_{0,h}) \subset E$, it follows that

$$\left\{x_n \leq -\frac{\tan h}{2} - C\varepsilon^{1+\frac{1}{2}}\right\} \subset E, \quad \text{in } B_{1/2}' \times (-1,1),$$

for some C depending only on n. For $h_\circ$ small enough depending only on n, since $\varepsilon \leq h \leq h_\circ$ and $b \leq \varepsilon^{1+\frac{1}{4}}$,

$$\left\{x_n \leq b - \frac{3}{4}h\right\} \subset \left\{x_n \leq -\frac{\tan h}{2} - C\varepsilon^{1+\frac{1}{2}}\right\} \subset E, \quad \text{in } B_{1/2}' \times (-1,1).$$

This completes the case $b \leq \varepsilon^{1+\frac{1}{4}}$.

The second case is $b > \varepsilon^{1+\frac{1}{4}}$, and is less straight-forward. By Savin's half Harnack, Proposition 4.16, for every $\tau > 0$ small enough depending only on n, if there exists

$$z = (z', z_n) \in \partial E, \text{ with } |z'| \leq \frac{1}{2} \text{ and } z_n \leq b - h + \tau h, \tag{4.30}$$

then

$$\left|\Pi_{\boldsymbol{e}_n}\left(\partial E \cap B_1 \cap \left(B_{3/4}' \times (-1,1)\right) \cap \{x_n \leq b - h + C_1\tau h\}\right)\right|_{\mathcal{H}^{n-1}} \geq \frac{3}{4}|B_{3/4}'|_{\mathcal{H}^{n-1}}, \tag{4.31}$$

for some constant C_1 depending only on n.

On the other hand, notice that since we are in the case $b > \varepsilon^{1+\frac{1}{4}}$,

$$\tilde{E} := E \cup \{x_n \leq b\},$$

is a subsolution to the minimal perimeter problem in B_1 for h small enough. This follows since $\Phi(\Lambda^\delta) \subset \{x_n \leq \varepsilon^{1+\frac{1}{4}}\}$ for ε small enough, and ∂E is a surface of minimal perimeter whenever it does not touch $\Phi(\Lambda^\delta)$.

Take $\tilde{E}^c$, and apply again Proposition 4.16 to get that, for every $\tau > 0$ small enough depending only on n (take $\tau < C_1^{-1}$), if there exists

$$z = (z', z_n) \in \partial E, \text{ with } |z'| \leq \frac{1}{2} \text{ and } z_n \geq b + h - \tau h, \tag{4.32}$$

then

$$\left|\Pi_{\boldsymbol{e}_n}\left(\partial E \cap B_1 \cap \left(B_{3/4}' \times (-1,1)\right) \cap \{x_n \geq b + h - C_1\tau h\}\right)\right|_{\mathcal{H}^{n-1}} \geq \frac{3}{4}|B_{3/4}'|_{\mathcal{H}^{n-1}}. \tag{4.33}$$

Take $Q = B'_{3/4}\times(b-h,b+h)$ In particular, we must have that

$$P(E;Q)\ge \frac{3}{2}|B'_{3/4}|_{\mathcal{H}^{n-1}}.$$

Notice, on the other hand, that we can take h small enough so that the lateral perimeter of Q is less than $\frac{1}{2}|B'_{3/4}|_{\mathcal{H}^{n-1}}$. This yields a contradiction, since including Q to E gives a competitor for the minimizer of (4.17); and therefore either (4.30) or (4.32) does not hold. This completes the proof. □

We also need a similar improvement of oscillation *far away from contact points*. In such case, we can use the following classical Harnack inequality for minimal surfaces. The proof of this proposition is an straightforward application of Proposition 4.16.

Proposition 4.17 ([Sav10b]). *There exists $h_\circ$ and $\tau_\circ$ depending only on n such that the following statement holds:*

Let $E\subset\mathbb{R}^n$ be a set of minimal perimeter in $B'_1\times(-1,1)$, such that for some $b\in(-1,1)$ and some $h\in(0,h_\circ)$

$$\{x_n\le b-h\}\subset E\subset\{x_n\le b+h\},\qquad \text{in } B'_1\times(-1,1).$$

Then,

- *either $\{x_n\le b-h(1-\tau_\circ)\}\subset E$, in $B'_{1/2}\times(-1,1)$;*
- *or $E\subset\{x_n\le b+h(1-\tau_\circ)\}$, in $B'_{1/2}\times(-1,1)$.*

Actually, to account for situations in which ∂E may stick to $\partial\Phi(\Lambda_{\gamma,\theta})$, we need the following version of Proposition 4.17 for minimal surfaces with flat enough thin obstacles.

Proposition 4.18. *There exists $h_\circ$ and $\tau_\circ$ depending only on n such that the following statement holds:*

Assume that Φ satisfies (4.18) with $\varepsilon\in(0,h)$. Let $E\subset\mathbb{R}^n$, satisfying

$$\Phi\big(\{x_n\le 0\}\big)\cap Q_1\subset E$$

where we denote $Q_r := B'_r\times(-1,1)$, be a solution of

$$P(E;Q_1)\le P(F;Q_1)\quad \forall F \text{ such that } E\setminus Q_1 = F\setminus Q_1,\ \Phi\big(\{x_n\le 0\}\big)\cap Q_1\subset F.$$

Assume that for some $b\in(-1,1)$ and some $h\in(0,h_\circ)$

$$\{x_n\le b-h\}\subset E\subset\{x_n\le b+h\},\qquad \text{in } Q_1.$$

Then,

- *either $\{x_n\le b-h(1-\tau_\circ)\}\subset E$, in $Q_{1/2}$;*
- *or $E\subset\{x_n\le b+h(1-\tau_\circ)\}$, in $Q_{1/2}$.*

Proof. The proof is very similar to that of Proposition 4.17 in [Sav10b]. We sketch it.

Note that, by (4.18) we have

$$\Phi\big(\{x_n = 0\}\big) \subset \{|x_n| \le \varepsilon^{1+\frac{1}{2}}\} \quad \text{in } Q_1.$$

Now, if $b \le 0$, since ∂E is above $\Phi\big(\{x_n = 0\}\big)$ in Q_1, we have $\{x_n \le -\varepsilon^{1+\frac{1}{2}}\} \subset E$ in Q_1. Thus we obtain $\{x_n \le b - h(1-\tau_\circ)\} \subset E$ in Q_1 provided $\varepsilon^{1+\frac{1}{2}} \le h(1-\tau_\circ)$, which is trivially satisfied if $\tau_\circ \le 1/2$ and $\varepsilon < h < h_\circ \le 1/4$. In other words, the first alternative of the conclusion of the proposition holds whenever $b \le 0$.

Let us now consider the case $b \ge 0$. Note that we may suppose that the "coincidence set" $\partial E \cap \Phi\big(\{x_n = 0\}\big)$ is nonempty in $Q_{3/4}$ since otherwise the result follows immediately from Proposition 4.17, noting ∂E would be a minimal boundary in $Q_{3/4}$.

Since E is a supersolution in Q_1 satisfying $\{x_n \le -\varepsilon^{1+\frac{1}{2}}\} \subset E$ in Q_1 such that has some point $x_\circ = (x'_\circ, x_{\circ,n}) \in \partial E \cap Q_{3/4}$ with $x_{\circ,n} \in (-\varepsilon^{1+\frac{1}{2}}, \varepsilon^{1+\frac{1}{2}})$, Proposition 4.16 (with a standard covering argument) yields

$$\Big|\Pi_{\boldsymbol{e}_n}\Big(\partial E \cap \{x_n \le C\varepsilon^{1+\frac{1}{2}}\} \cap Q_{3/4}\Big)\Big|_{\mathcal{H}^{n-1}} \ge \frac{3}{4}|B'_{3/4}|_{\mathcal{H}^{n-1}}. \tag{4.34}$$

At the same time, the set $\tilde{E} := E \cup \{x_n \le b + h/2\})$ is a subsolution in Q_1 since the contact set $\partial E \cap \partial\Phi\big(\{x_n = 0\}\big) \cap Q_1$ is contained in $\{x_n \le \varepsilon^{1+\frac{1}{2}}\} \subset \{x_n \le b + h/2\}$ (recall $b \ge 0$ and $\varepsilon \le h$). Thus, either

$$E \subset \tilde{E} \subset \big\{x_n \le b + h(1-\tau_\circ)\big\} \quad \text{in } Q_{3/4} \tag{4.35}$$

or else, by Proposition 4.16 applied to $\tilde{E}^c$, we would have

$$\Big|\Pi_{\boldsymbol{e}_n}\Big(\partial\tilde{E} \cap \{x_n \ge b + h - C\tau_\circ h\} \cap Q_{3/4}\Big)\Big|_{\mathcal{H}^{n-1}} \ge \frac{3}{4}|B'_{3/4}|_{\mathcal{H}^{n-1}}. \tag{4.36}$$

Now (4.35) clearly implies the conclusion of the proposition (first alternative). On the other hand, should (4.36) hold then, by definition of $\tilde{E}$, (4.36) would also hold with $\partial\tilde{E}$ replaced by ∂E and thus we would find a contradiction with (4.34) when taking $\tau_\circ$ small enough so that $b + h - C\tau_\circ h > C\varepsilon^{1+\frac{1}{2}}$ (recall $\varepsilon < h < h_\circ$ small enough). Indeed, this contradiction argument — which uses the minimality of ∂E among boundaries of sets containing the obstacle — is identical to the one given in the proof of Proposition 4.15. □

At this point, combining Proposition 4.15 and Proposition 4.18 we obtain the following lemma regarding the convergence of vertical rescalings to a Hölder continuous function.

Lemma 4.19. *Let $(E_k)_{k\in\mathbb{N}}$ be a sequence such that $E_k \subset \mathbb{R}^n$ satisfy* (4.17)*, with $0 \in \partial E_k$, and with Φ_k such that* (4.18) *holds for $\varepsilon = \varepsilon_k$. Suppose E_k is ε_k-close to $\Lambda_{\gamma_k,\theta_k}$ in B_1, with $\theta_k \in (0, \varepsilon_k)$, and with $\varepsilon_k \to 0$ as $k \to \infty$. Suppose also that $\Phi_k(\Lambda_{\gamma_k,\theta_k+\varepsilon_k}) \subset E_k$ in B_1. Let*

$$E_k^{\varepsilon_k} := \left\{\left(x', \frac{x_n}{2\varepsilon_k}\right) : x = (x', x_n) \in E_k^r \cap B_1\right\}, \qquad \textit{for all } k \in \mathbb{N}, \tag{4.37}$$

where $E_k^r := R_{\gamma_k}(E_k)$*, and* R_{γ_k} *denotes the rotation of angle* γ_k *in the last two coordinates bringing* e_{γ_k} *to* $\boldsymbol{e}_n$*.*

Then, there exists $u \in C^{0,a}(\overline{B'_{1/2}})$ *with* $\|u\|_{C^{0,a}(\overline{B'_{1/2}})} \le C$*, for some* C *depending only on* n*, such that*

$$\{x_n \le u(x') - \varepsilon_k^\beta\} \subset E_k^{\varepsilon_k} \subset \{x_n \le u(x') + \varepsilon_k^\beta\}, \qquad in\ B'_{1/2} \times (-1,1), \tag{4.38}$$

for some $a > 0$ *and* $\beta > 0$ *depending only on* n*.*

Proof. Let us define the cylinder $Q_r(x_\circ) = (B'_r(x'_\circ) \times (-1,1)) \cap B_1$ for any $x_\circ = (x'_\circ, x_{\circ,n}) \in B_1$. Notice that, thanks to the hypotheses, for any $x_\circ \in \partial E_k^r \cap B_{1/2}$,

$$\partial E_k^r \cap Q_{1/2}(x_\circ^r) \subset \{x \in B_1 : |x_n - x_{\circ,n}| \le 2\varepsilon_k\},$$

where $x_\circ^r$ denotes the rotated version of r. That is, introducing a notation, we have

$$\operatorname*{osc}_{Q_{2^{-1}}(x_\circ^r)}{}_n\ \partial E_k^r \le 2\varepsilon_k;$$

the oscillation in the $\boldsymbol{e}_n$ direction of ∂E_k^r in the cylinder $Q_{2^{-1}}(x_\circ^r)$ is less than $2\varepsilon_k$. We would like to use that if ε_k is small enough, then either Proposition 4.15 or Proposition 4.18 improves the oscillation in the half cylinder, and proceed iteratively. In order to do that, we separate between four cases.

Case 1: $x_\circ = 0$. The first case we consider is $x_\circ = 0 \in \partial E_k$. By assumption, $\Phi_k(\Lambda_{\gamma_k,\theta_k+\varepsilon_k}) \subset E_k$ in B_1, and we have that

$$\operatorname*{osc}_{Q_{2^{-1}}(x_\circ^r)}{}_n\ \partial E_k^r \le 2\varepsilon_k.$$

If we denote as $h_\circ$ and $\tau_\circ$ the variables coming from Proposition 4.15; we have that if

$$4\varepsilon_k \le h_\circ, \tag{4.39}$$

then

$$\operatorname*{osc}_{Q_{2^{-2}}(x_\circ^r)}{}_n\ \partial E_k^r \le 2\varepsilon_k(1-\tau_\circ).$$

We are using here Proposition 4.15 with $h = \varepsilon_k$. Condition (4.39) is to ensure that $\theta_k + \varepsilon_k \le h_\circ$ [2]. If we rescale by a factor 2, we have

$$\operatorname*{osc}_{Q_{2^{-1}}(x_\circ^r)}{}_n\ 2\partial E_k^r \le 4\varepsilon_k(1-\tau_\circ),$$

so that, if we want to repeat the argument, hypothesis (4.39) becomes

$$8\varepsilon_k(1-\tau_\circ) \le h_\circ.$$

If we want to continue one next iteration, we can take $h = 2\varepsilon_k(1-\tau_\circ)$. Notice that, after the rescaling, the transformation Φ associated to $2\partial E_k$, is $\tilde{\Phi}_k(x) = 2\Phi_k(x/2)$,

[2] Notice that here we want to ensure that $\Phi(\Lambda_{0,h}) \subset E_k^r$ in order to apply Proposition 4.15. We actually have that $R_{\gamma_k}\Phi_k(\Lambda_{\gamma_k,\theta_k+\varepsilon_k}) \subset E_k^r$, but this is enough to use it as a barrier from below in the proof of Proposition 4.15.

so that $|D^2\tilde{\Phi}_k| \le 2^{-1}\varepsilon_k^{1+\frac{1}{2}}$, and the hypotheses of Proposition 4.15 are still fulfilled, with a better constant.

Rescaling and repeating this procedure iteratively, we have that as long as

$$2^m(1-\tau_\circ)^{m-2}\varepsilon_k \le h_\circ, \tag{4.40}$$

then

$$\operatorname*{osc}_{Q_{2^{-m}}(x_\circ^r)} {}_n\, \partial E_k^r \le 2\varepsilon_k(1-\tau_\circ)^{m-1}. \tag{4.41}$$

Case 2: $x_\circ \in \partial E_k \cap \partial \mathcal{O}_k \cap B_{1/2}$. The second case is when $x_\circ$ belongs to the contact set of the thin obstacle, $x_\circ \in \partial E_k \cap \partial \mathcal{O}_k$, where $\partial\mathcal{O}_k := \Phi(\{x_{n-1} = x_n = 0\})$. After a translation and a rotation, up to redefining Φ if necessary, we can put ourselves in Case 1 (see Lemma 4.10 with $\rho = 1$), so that

$$2^m(1-\tau_\circ)^{m-2}\varepsilon_k \le h_\circ \quad \Rightarrow \quad \operatorname*{osc}_{Q_{2^{-m}}(x_\circ^r)} {}_n\, \partial E_k^r \le 2\varepsilon_k(1-\tau_\circ)^{m-1}. \tag{4.42}$$

We must point out here that, a priori, the oscillation might be in a direction different from $\boldsymbol{e}_n$ due to the rotation coming from Lemma 4.10. However, since the rotation tends to the identity as $\varepsilon_k \downarrow 0$, we may also assume that for ε_k small enough, the previous also holds.

Case 3: $\operatorname{dist}(x_\circ, \partial E_k \cap \partial\mathcal{O}_k) \ge \frac{1}{8}$. Follows exactly as the two previous cases, using Proposition 4.18 instead of Proposition 4.15, yielding again (4.42).

Case 4: $2^{-p-1} \le \operatorname{dist}(x_\circ, \partial E_k \cap \partial\mathcal{O}_k) \le 2^{-p}$ *for* $p \ge 3$. This is a combination of Case 2 and Case 3. We apply Case 2 and rescale, until we can apply Case 3, so that (4.42) holds again.

That is, (4.42) holds for all $x_\circ \in \partial E_k \cap B_{1/2}$. Let m_k denote the largest m we can take for every ε_k such that (4.40) holds. Clearly, $m_k \to \infty$ as $k \to \infty$, since $\varepsilon_k \to 0$. If we consider the rescaled sets in the $\boldsymbol{e}_n$ direction, $E_k^{\varepsilon_k}$, we have that for every $m \le m_k$,

$$\operatorname*{osc}_{Q_{2^{-m}}(x_\circ)} {}_n\, \partial E_k^{\varepsilon_k} \le 2(1-\tau_\circ)^{m-1}. \tag{4.43}$$

In particular, there exists a Hölder modulus of continuity as $\varepsilon_k \to 0$ controlling the boundaries $\partial E_k^{\varepsilon_k}$. By Arzelà-Ascoli, up to subsequences, $\partial E_k^{\varepsilon_k}$ converges in the Hausdorff distance to the graph of some Hölder continuous function, u. □

Lemma 4.20. *The function* $u \in C^{0,a}(\overline{B'_{1/2}})$ *from the Lemma 4.19 is a viscosity solution to the classical thin obstacle problem with* $u(0) = 0$*. That is,* u *fulfils*

$$\begin{cases} \Delta u = 0 & \text{in } B'_{1/2} \setminus (\{x_{n-1} = 0\} \cap \{u = 0\}) \\ \Delta u \le 0 & \text{on } \{x_{n-1} = 0\} \cap \{u = 0\} \\ u \ge 0 & \text{on } \{x_{n-1} = 0\}, \end{cases} \tag{4.44}$$

in the viscosity sense. In particular,

$$\|u\|_{C^{1,1/2}\left(\overline{B'_{1/4}\cap\{x_{n-1}\ge 0\}}\right)} + \|u\|_{C^{1,1/2}\left(\overline{B'_{1/4}\cap\{x_{n-1}\le 0\}}\right)} \le C, \tag{4.45}$$

for some constant C *depending only on* n*. That is,* u *is* $C^{1,1/2}$ *up to* $\{x_{n-1} = 0\}$ *in either side.*

Proof. The proof follows along the lines of [Sav10].

Since $\partial E_k^{\varepsilon_k}$ converges uniformly to the graph of u, and $\partial E_k^{\varepsilon_k} \cap \{x_{n-1} = 0\} \subset \{x_n \geq -C\varepsilon_k\}$, we clearly have that $u \geq 0$ on $\{x_{n-1} = 0\}$. This follows since $\Phi(\Lambda_{\gamma_k,\theta_k+\varepsilon_k}) \subset E_k$. Similarly, $u(0) = 0$.

Now take any point $x'_\circ \in B'_{1/2}$. Consider $P(x')$ a quadratic polynomial in $B_{1/2'}$, with graph touching the graph of u from below at $(x'_\circ, u(x'_\circ))$. Since $\partial E_k^{\varepsilon_k}$ is converging uniformly to the graph of u, $P(x') - c_k$ touches from below $\partial E_k^{\varepsilon_k}$ at a point y_k such that $y_k \to (x'_\circ, u(x'_\circ))$ as $k \to \infty$. Rescaling back, $\varepsilon_k P(x') - \tilde{c}_k$ touches from below ∂E_k^r at $\tilde{y}_k$ such that $\tilde{y}'_k \to x'_\circ$ for some sequence $\tilde{c}_k$ bounded. Since ∂E_k^r is a supersolution being touched from below, by Lemma 4.9 we have

$$M(\varepsilon_k D^2 P, \varepsilon_k \nabla P) = \varepsilon_k \Delta P + \varepsilon_k^3 \left(\Delta P |\nabla P|^2 - \varepsilon_k (\nabla P)^T D^2 P \, \nabla P\right) \leq 0$$

at $\tilde{y}'_k$. By letting $\varepsilon_k \to 0$ we reach

$$\Delta P(x'_\circ) \leq 0,$$

so that u solves $\Delta u \leq 0$ in the viscosity sense.

On the other hand, suppose $x'_\circ \in B'_{1/2} \setminus (\{x_{n-1} = 0\} \cap \{u = 0\})$. Let $P(x')$ be a quadratic polynomial in $B_{1/2'}$, with graph touching the graph of u from above at $(x'_\circ, u(x'_\circ))$. Now, $P(x') + c_k$ touches from above $\partial E_k^{\varepsilon_k}$ at a point y_k such that $y_k \to (x'_\circ, u(x'_\circ))$ as $k \to \infty$. That is, $\varepsilon_k P(x') + \tilde{c}_k$ touches from above ∂E_k^r at $\tilde{y}_k$ such that $\tilde{y}'_k \to x'_\circ$ for some sequence $\tilde{c}_k$ bounded. If k large enough, $\tilde{y}'_k \in B'_{1/2} \setminus (\{x_{n-1} = 0\} \cap \{u = 0\})$. Therefore, either ∂E_k^r is a surface of minimal perimeter around $\tilde{y}_k$, or ∂E_k^r is touching $\Phi_k(\Lambda^\delta)$ at $\tilde{y}_k$. In the first case, we are already done proceeding as before, we get $M(\varepsilon_k D^2 P, \varepsilon_k \nabla P) \geq 0$.

Suppose then, that ∂E_k^r is touching $\Phi_k(\Lambda^\delta)$ at $\tilde{y}_k$. For this to happen, one must have that $\Phi_k(\Lambda^\delta)$ is a supersolution to the minimal perimeter problem around $\tilde{y}_k$, otherwise there could not be a contact point with a supersolution. However, notice that it is a supersolution with mean curvature around $\tilde{y}_k$ bounded from below by $-C\varepsilon_k^{1+\frac{1}{2}}$. Therefore, $M(\varepsilon_k D^2 P, \varepsilon_k \nabla P) \geq -C\varepsilon_k^{1+\frac{1}{2}}$ at $\tilde{y}_k$, and letting $k \to \infty$ we get $\Delta P(x'_\circ) \geq 0$. Thus, (4.44) holds in the viscosity sense.

Finally, the regularity of solution to the classical thin obstacle problem, (4.45), was first shown by Caffarelli in [Caf79]; and the optimal $C^{1,1/2}$ regularity here presented was obtained by Athanaopoulos and Caffarelli in [AC04]. □

We can now present the proof regarding the improvement of closeness to sets of the form $\Lambda_{\gamma,\theta}$, Proposition 4.14.

Proof of Proposition 4.14. Let us argue by contradiction, and suppose that the statement does not hold. Then, there exists some $\alpha_\star \in \left(0, \frac{1}{2}\right)$ and a sequence $E_k \subset \mathbb{R}^n$ satisfying (4.17), such that $0 \in \partial E_k$, E_k are ε_k-close to some $\Lambda_{\gamma_k,\theta_k}$ for $\theta_k \in (0, C_\circ \varepsilon_k)$, (4.18) holds for $\varepsilon = \varepsilon_k$ (and the transformation Φ_k), for some positive sequence $\varepsilon_k \to 0$ as $k \to \infty$, but such that the conclusion does not hold for any $\rho_\circ, \varepsilon_\circ > 0$.

By Lemma 4.13 we have that

$$\Phi_k(\Lambda_{\gamma_k,\theta_k+C_\circ\varepsilon_k}) \subset E_k, \qquad \text{in } B_{1/2}.$$

By rescaling and renaming the ε_k sequence if necessary, we can assume that $\theta_k \in (0, \varepsilon_k)$ and $\Phi_k(\Lambda_{\gamma_k,\theta_k+\varepsilon_k}) \subset E_k$ in B_1, so that we are in the same situation as in Lemma 4.19. In particular, due to Lemma 4.19, the sequence $\partial E_k^{\varepsilon_k}$ approaches (in Hausdorff distance) a function u in $B'_{1/2}\times(-1,1)$, which by Lemma 4.20 is a solution to a classical thin obstacle problem. Thanks to the regularity of u, and the fact that $u(0) = 0$ and $\nabla_{x''}u(0) = 0$, we have that

$$\left|u(x') - \partial^+_{n-1}u(0)(x'_{n-1})_+ - \partial^-_{n-1}u(0)(x'_{n-1})_-\right| \le C\rho^{3/2}, \quad \text{in } B'_{2\rho},$$

for any $\rho > 0$ and for some constant C depending only on n. Here, we have denoted $a_+ = \max\{a, 0\}$, $a_- = \min\{a, 0\}$, and

$$\partial^{\pm}_{n-1}u(0) := \lim_{\eta\downarrow 0} \frac{\partial u}{\partial x'_{n-1}}(0, \dots, 0, \pm\eta),$$

i.e., the limit of the derivative in the $\boldsymbol{e}_{n-1}$ direction coming from $\{x_{n-1} > 0\}$ or $\{x_{n-1} < 0\}$ (which exist by the regularity up to the contact set). Notice, moreover, that since $\Delta u \le 0$ around 0, we must have $\partial^-_{n-1}u(0) \ge \partial^+_{n-1}u(0)$. In particular, thanks to the closeness of $\partial E_k^{\varepsilon_k}$ to the graph of u, we have that

$$\partial E_k^{\varepsilon_k} \cap \left(B'_{3\rho/2} \times (-1,1)\right) \subset \left\{\left|x_n - \partial^+_{n-1}u(0)(x'_{n-1})_+ - \partial^-_{n-1}u(0)(x'_{n-1})_-\right| \le C\rho^{1/2}\right\},$$

which, after rescaling implies that ∂E_k^r is at distance at most $C\varepsilon_k\rho^{3/2}$ from some $\Lambda_{\tilde\gamma,\tilde\theta}$ in B_ρ, given by the graph of $\varepsilon_k\partial^+_{n-1}u(0)(x'_{n-1})_+ + \varepsilon_k\partial^-_{n-1}u(0)(x'_{n-1})_-$. Now, simply take ρ small enough depending only on n and $\alpha_\star$ such that $C\rho^{3/2} \le \rho^{1+\alpha_\star}$, and we reach a contradiction (notice that such ρ exists because $\alpha_\star < \frac{1}{2}$). □

4.5 Improvement of closeness in non-flat configuration

In this section we study the complementary case to the one in the previous section: the case where E is ε-close to a *non-flat* ($\theta \gtrsim \varepsilon$) wedge $\Lambda_{\gamma,\theta}$. Under this condition, thanks to Lemma 4.12, there exists a full contact set, so that the study of the regularity becomes a known matter.

We state and prove now the lemma that will allow us to conclude the proof of Theorem 4.3.

Lemma 4.21. *There exists $\varepsilon_\circ$ depending only on n such that the following statement holds:*

Let $E \subset \mathbb{R}^n$ satisfying (4.17) *with $0 \in \partial E$ be such that for some $\Lambda_{\gamma,\theta}$, and $\varepsilon \in (0, \varepsilon_\circ)$,*

$$\Phi(\Lambda_{\gamma,\theta+\varepsilon}) \subset E \subset \Phi(\Lambda_{\gamma,\theta-\varepsilon}), \quad \text{in } B_1, \tag{4.46}$$

where Φ satisfies (4.18)*.*

Then,

$$\partial E \cap \overline{B_{1/2}} = \overline{\Gamma_+} \cup \overline{\Gamma_-}, \tag{4.47}$$

where

$$\Gamma_\pm = \partial E \cap B_{1/2} \cap \Phi(\{\pm x_{n-1} > 0\}), \tag{4.48}$$

and

$$\overline{\Gamma_\pm} \cap \Phi(\{x_{n-1} = 0\}) \cap \overline{B_{1/2}} \subset \Phi(\{x_{n-1} = x_n = 0\}). \tag{4.49}$$

Moreover, for each $\beta \in (0,1)$, Γ_+ *and* Γ_- *are* $C^{1,\beta}$ *graphs up to the boundary in the* $e_{\gamma+\theta}$ *and* $e_{\gamma-\theta}$ *directions respectively, with* $C^{1,\beta}$*-norms bounded by* $C\varepsilon$*, where* C *depends only* n *and* β*.*

Remark 4.8. A a direct consequence of the $C^{1,\beta}$ estimates from Lemma 4.21 there exists $\Lambda_{\gamma_\star,\theta_\star}$ as in (4.11) such that for any $\bar\alpha \in (0, 1/2)$,

$$E \text{ is } C\varepsilon r^{1+\bar\alpha}\text{-close to } \Lambda_{\gamma_\star,\theta_\star} \text{ in } B_r, \qquad \text{for all } r \in (0, 1/2),$$

for some constant C depending only on n. Moreover,

$$|\bar\gamma - \gamma| + |\bar\theta - \theta| \le C\varepsilon,$$

for some constant C depending only on n. This will be useful later on in the paper. In fact, we could clearly take $\bar\alpha \in (0, 1)$ but we will only need $\bar\alpha < 1/2$ later on (see Proposition 4.24).

In order to prove Lemma 4.21 we need a version for thick smooth obstacles of the following standard result on regularity of flat minimizers of the perimeter.

Theorem 4.22 ([Giu84, Chapter 8])**.** *There exists* $\eta_\circ$ *small depending only on* n *such that the following statement holds:*

Let $E \subset \mathbb{R}^n$ *be a minimizer of the perimeter in* B_1 *such that*

$$\{x_n \le -\eta\} \subset E \subset \{x_n \le \eta\}, \quad \text{in } B_1,$$

for some $\eta \in (0, \eta_\circ)$*.*

Then, there exists a map $\varphi : B'_{1/2} \to \mathbb{R}$ *such that*

$$\partial E = \{x = (x', x_n) \subset \mathbb{R}^n : x_n = \varphi(x')\} \quad \text{ in } B'_{1/2} \times (-1/2, 1/2),$$

where $\|\varphi\|_{C^k(B'_{1/2})} \le C(n,k)\,\eta$*, for some constant* C *depending only on* n *and* k*.*

Let us comment on the standard proof of the previous theorem.

Remark 4.9. Theorem 4.22 is usually shown in two steps. First, one iterates (4.8) obtain

$$|\nu(x) - \nu(y)| \le C\eta|x - y|^\alpha,$$

for $\alpha > 0$, and where $\nu(x)$ for $x \in \partial E$ denotes the unit normal vector to ∂E pointing outwards E. This C^α estimate for the normal ν is a consequence of the improvement of flatness property (4.8).

Second, one improves this $C^{1,\alpha}$ estimate to obtain the C^k regularity using interior Schauder estimates for graphs.

Comparing normal vectors is like comparing the corresponding tangent hyperplanes (or half-spaces). A similar approach is what inspired part of this work, where we compare sets of the form $\Lambda_{\gamma,\theta}$ instead of half-spaces to get the regularity.

The version of the previous result we will need is the following

Theorem 4.23. *There exists $\eta_\circ$ small depending only on n such that the following statement holds:*

Assume $\eta \in (0, \eta_\circ)$ and that Φ satisfies (4.18) with $\varepsilon \in (0, \eta)$. Let $E \subset \mathbb{R}^n$, satisfying

$$\Phi\big(\{x_n \le 0\}\big) \cap B_1 \subset E,$$

$$P(E; B_1) \le P(F; B_1) \quad \forall F \text{ such that } E \setminus B_1 = F \setminus B_1,\ \Phi\big(\{x_n \le 0\}\big) \cap B_1 \subset F.$$

Assume that for some $b \in (-1/2, 1/2)$

$$\{x_n \le b - \eta\} \subset E \subset \{x_n \le b + \eta\}, \qquad \text{in } B_1.$$

Then, there exists a map $\varphi : B'_{1/2} \to \mathbb{R}$ such that

$$\partial E = \{x = (x', x_n) \subset \mathbb{R}^n : x_n = \varphi(x')\} \qquad \text{in } B'_{1/2} \times (b - 1/4, b + 1/4)\,, \tag{4.50}$$

where $\|\varphi\|_{C^{1,1}(B'_{1/2})} \le C\eta$, for some constant C depending only on n.

The proof of Theorem 4.23 is based on two steps as the proof of Theorem 4.22 (see Remark 4.9). First, we prove that ∂E is a $C^{1,\alpha}$ graph or, more precisely, (4.50) with $\|\varphi\|_{C^{1,\alpha}(B'_{1/2})} \le C\eta$. This can be done exactly by compactness of vertical rescaling, following the exact same strategy of Savin [Sav10, Sav10b].

Second, we can apply a theorem of Brézis and Kinderleher [BK74] to improve from this $C^{1,\alpha}$ estimate to the optimal $C^{1,1}$ estimate. By completeness we sketch the proof here.

Proof of Theorem 4.23. We do the argument in two steps.

Step 1. Fix some $\alpha \in (0,1)$, say $\alpha := 1/4$. Then, we claim that if $\eta_\circ$ is small enough then (4.50) holds with $\|\varphi\|_{C^{1,\alpha}(B'_{1/2})} \le C\eta$, where C depends only on n. Indeed, exactly as in the proof of Proposition 4.14, we establish by compactness the following improvement of flatness property, around $x_\circ \in B_{3/4} \cap \partial E$,

$$\partial E \subset \big\{|e\cdot(x - x_\circ)| \le \eta\big\} \text{ in } B_r(x_\circ) \quad \Rightarrow \quad \partial E \subset \big\{|\tilde e\cdot(x - x_\circ)| \le \rho_\circ^{1+\alpha}\eta\big\} \text{ in } B_{\rho_\circ r}(x_\circ). \tag{4.51}$$

for some $\rho_\circ \in (0,1)$ depending only on n. The proof of (4.51) is analogous to the Proof of Proposition 4.14. It is enough to do the case $r = 1$. To do it, we consider the vertical rescalings defined similarly as in (4.37) in Lemma 4.19. These vertical rescalings of ∂E are compact by Proposition 4.18 (similarly as in Lemma 4.19) and converge "uniformly" to a function $u \in C^a(B'_{1/2})$ which is harmonic. Indeed, the condition $|D^2\Phi| \le \eta^{1+\frac12}$ implies that the thick obstacle will be zero in the limit if we apply the vertical rescaling $(x', x_n) \mapsto (x', x_n/\eta)$ and let $\eta \downarrow 0$. Using the $C^{1,1}$ regularity of harmonic functions we establish (4.51).

With a standard iteration of (4.51) we establish that (4.50) holds with

$$\|\varphi\|_{C^{1,\alpha}(B'_{1/2})} \le C\eta \qquad (\alpha = 1/4),$$

as we wanted to show.

Step 2. We improve the previous $C^{1,1/4}$ estimate to the optimal estimate $\|\varphi\|_{C^{1,1}(B'_{1/2})} \le$

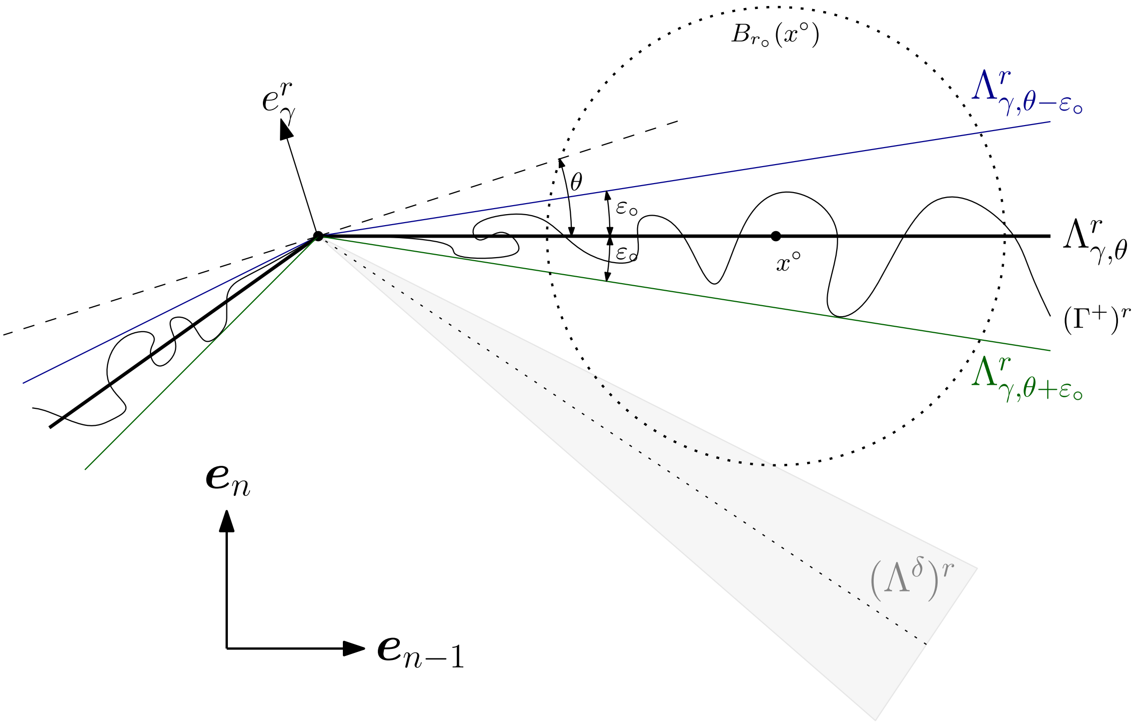

Figure 4.5: Representation of the setting after a rotation.

$C\eta$. This is a straightforward application of the results of Brézis and Kinderleher [BK74] of optimal $C^{1,1}$ regularity for obstacle problems with uniformly elliptic nonlinear operators. Indeed, once we have proved that ∂E is a graph and with bounded gradient, then it follows that the mean curvature operator H is uniformly elliptic and thus [BK74, Theorem 1] provides exactly the desired $C^{1,1}$ estimate. □

We can now prove Lemma 4.21.

Proof of Lemma 4.21. We divide the proof into two steps. In the first step we show that $\Gamma_\pm$ are a graphs, and in the second step we show their regularity.

Step 1: $\Gamma_\pm$ are graphs in an appropriate direction. The proof of the fact that $\Gamma_\pm$ are graphs is almost immediate, just noticing that (4.46) allows us to apply Theorem 4.23 at every scale.

Let us consider first the case $\Phi \equiv \mathrm{id}$, and let us rotate the setting with respect to the last two coordinates, in such a way that the normal vector to $\Lambda_{\gamma,\theta}$ for $\{x_{n-1} > 0\}$, $e_{\gamma+\theta}$, now becomes $\boldsymbol{e}_n$ (that is, rotate an angle $\gamma + \theta$). Let us denote as the corresponding rotated versions with superindex r, e.g. $\Lambda^r_{\gamma,\theta}$. See Figure 4.5 for a representation of the rotated setting.

Now take any point $x^\circ \in B_{1/2} \cap \{x_n = 0\}$, so that $x^\circ \in \Lambda^r_{\gamma,\theta}$. Denote $r_\circ = x^\circ_{n-1}/2$, and consider a ball $B_{r_\circ}(x^\circ)$. Notice that

$$\{x_n \le -3\tan(\varepsilon_\circ) r_\circ\} \subset E \subset \{x_n \le 3\tan(\varepsilon_\circ) r_\circ\}, \quad \text{in } B_{r_\circ}(x^\circ).$$

Thus, if $\varepsilon_\circ$ is small enough, we can apply Theorem 4.22 rescaled in the ball $B_{r_\circ}(x^\circ)$; which tells us that $(\Gamma^+)^r$ in $B_{r_\circ}(x^\circ)$ is the graph of a function in the $\boldsymbol{e}_n$

direction. Since we can cover all of $(\Gamma^+)^r$ with balls of this kind, we conclude that $(\Gamma^+)^r$ is the graph of a function in the $\boldsymbol{e}_n$ direction in $B_{1/2} \cap \{x_{n-1} \geq 0\}$.

The case $\Phi \not\equiv \mathrm{id}$ is a perturbation of the previous one, but we would need to use Theorem 4.23 instead of Theorem 4.22, since it is no longer true that we are necessarily a minimal surface in $B_{r_\circ}(x^\circ)$.

Step 2: $C^{1,1-}$ regularity of $\Gamma_\pm$. Let us first discuss the case $\Phi \equiv \mathrm{id}$. In this situation, using (4.46), we obtain that Γ^+ is a graph that is Lipschitz up to its boundary $\{x_{n-1} = x_n = 0\}$ and we may now consider the reflection Γ^+_* of Γ^+ under the transformation $(x'', x_{x-1}, x_n) \mapsto (x'', -x_{n-1}, -x_n)$. Since Γ^+ is a Lipschitz graph up to $\{x_{n-1} = x_n = 0\}$ the "odd reflection" $\overline{\Gamma^+ \cup \Gamma^+_*}$ is a Lipschitz graph which solves the equation of minimal graphs in the viscosity sense. It follows that $\overline{\Gamma^+ \cup \Gamma^+_*}$ is analytic.

In the case $\Phi \not\equiv \mathrm{id}$ we cannot use the reflection trick and the interior smoothness of minimal graph to conclude, but still using (4.46) and that $\Phi \in C^{1,1}$ we see that Γ^+ is a Lipschitz graph with now $C^{1,1}$ boundary datum solving a thick obstacle problem with the mean curvature operator H. It follows from standard perturbative methods and the boundary regularity theory for obstacle problems with elliptic operators (see, for instance, Jensen [Jen80]) that the Γ^+ is a $C^{1,\beta}$ graph up to its boundary $\Phi(\{x_{n-1} = x_n = 0\})$. □

With this, we can proceed and prove Theorem 4.3.

Proof of Theorem 4.3. If $\theta \in (0, C_\circ\varepsilon)$, then we can directly apply Proposition 4.14.

On the other hand, if $\theta \in \left[C_\circ\varepsilon, \frac{\pi}{2}\right)$, thanks to Lemmas 4.12 and 4.13 we have that

$$\Phi(\Lambda_{\gamma,\theta+C_\circ\varepsilon}) \subset E \subset \Phi(\Lambda_{\gamma,\theta-C_\circ\varepsilon}), \quad \text{in } B_{1/2}.$$

That is, by rescaling and taking ε smaller depending only on n if necessary, we have put ourselves in the situation to apply Lemma 4.21. We conclude the proof in this case by noticing Remark 4.8 and that we can take $\rho_\circ = \frac{1}{4}$. □

4.6 Regularity of solutions

In this section, in order to simplify the computations, we assume $\Phi \equiv \mathrm{id}$. All statements and proofs are done under this assumption. We leave to the interested reader the standard extension of this results to the cases $\Phi \in C^{k,\beta}$, $k \geq 2$ and $\beta \in (0,1)$ or Φ analytic.

Proposition 4.24. *There exists $\varepsilon_\circ$ depending only on n such that the following statement holds:*

Let $E \subset \mathbb{R}^n$ satisfying (4.17) *with $0 \in \partial E$, be such that E is ε-close to $\Lambda_{\gamma,\theta}$ in B_1, for some $\varepsilon \in (0, \varepsilon_\circ)$. Then, there exists some $\Lambda_{\overline{\gamma},\overline{\theta}}$ with $\overline{\gamma}$ and $\overline{\theta}$ as in* (4.11), *such that for $\alpha \in \left(0, \frac{1}{2}\right)$,*

$$E \text{ is } C_\alpha \varepsilon r^{1+\alpha}\text{-close to } \Lambda_{\overline{\gamma},\overline{\theta}} \text{ in } B_r, \quad \text{for all } r \in (0, 1/2),$$

for some constant C_α depending only on n and α.

Proof. We will suppose that $\varepsilon > 0$ is sufficiently small so that each of the results used can be applied.

We begin by noticing that there are two possible scenarios. Either $\theta \geq C_\circ\varepsilon$ or $\theta < C_\circ\varepsilon$, where $C_\circ$ is the constant given in Lemma 4.12 and in Proposition 4.14, depending only on n.

Notice that if $\theta \geq C_\circ\varepsilon$ we are already done. Indeed, in this case we can apply Lemma 4.12 and Lemma 4.13 to fulfill the hypotheses of Lemma 4.21; which at the same time yields the desired result, thanks to Remark 4.8.

Suppose otherwise that $\theta < C_\circ\varepsilon$. In this case we can apply the improvement of closeness in Proposition 4.14. That is, there exist some radius $\rho_\circ$, depending only on n and α, such that

$$E \text{ is } \rho_\circ^{1+\alpha}\varepsilon\text{-close to } \Lambda_{\gamma_2,\theta_2} \text{ in } B_{\rho_\circ},$$

for some γ_2 and θ_2 as in (4.11). Let us define $E_2 := \rho_\circ^{-1}E$, so that we have a set $E_2 \subset \mathbb{R}^n$, satisfying (4.17), with $0 \in \partial E_2$ and $\rho_\circ^\alpha\varepsilon$-close to $\Lambda_{\gamma_2,\theta_2}$ in B_1. We are now again presented with a dichotomy: either $\theta_2 \geq C_\circ\rho_\circ^\alpha\varepsilon$ or $\theta_2 \leq C_\circ\rho_\circ^\alpha\varepsilon$. In the former case, we can again apply Lemma 4.21 and Remark 4.8 to find that

$$E_2 \text{ is } C\varepsilon\rho_\circ^\alpha r^{1+\alpha}\text{-close to } \Lambda_{\bar\gamma_2,\bar\theta_2} \text{ in } B_r, \quad \text{for all } r \in (0, 1/2),$$

for some $\Lambda_{\bar\gamma_2,\bar\theta_2}$ (which is close to $\Lambda_{\gamma_2,\theta_2}$). Rescaling back, E is $C\varepsilon r^{1+\alpha}$-close to $\Lambda_{\bar\gamma_2,\bar\theta_2}$ in B_r for all $r \in (0, \rho_\circ/2)$. Using that E is ε-close to $\Lambda_{\gamma,\theta}$ in B_1 it follows that E is $C_\alpha\varepsilon r^{1+\alpha}$ close to $\Lambda_{\bar\gamma_2,\bar\theta_2}$ in B_r, for all $r \in (0, 1/2)$, and a constant C_α that depends on α and n, of the form $C_\alpha = C\rho_\circ^{-1-\alpha}$ for C depending only on n.

If $\theta_2 \leq C_\circ\rho_\circ^\alpha\varepsilon$, we can repeat the process iteratively. Suppose that for all $k < k_\circ \in \mathbb{N}$, we have $\theta_k \leq C_\circ\rho_\circ^{k\alpha}\varepsilon$, but $\theta_{k_\circ} \geq C_\circ\rho_\circ^{k_\circ\alpha}\varepsilon$. That is, there exist $E_k := \rho_\circ^{-k+1}E$, satisfying (4.17), with $0 \in \partial E_k$ such that it is $\rho_\circ^{\alpha(k-1)}\varepsilon$-close to $\Lambda_{\gamma_k,\theta_k}$ in B_1. By Lemma 4.21 and Remark 4.8,

$$E_{k_\circ} \text{ is } C\varepsilon\rho_\circ^{(k_\circ-1)\alpha}r^{1+\alpha}\text{-close to } \Lambda_{\bar\gamma_{k_\circ},\bar\theta_{k_\circ}} \text{ in } B_r, \quad \text{for all } r \in (0, 1/2), \tag{4.52}$$

for some $\Lambda_{\bar\gamma_{k_\circ},\bar\theta_{k_\circ}}$ (close to $\Lambda_{\gamma_{k_\circ},\theta_{k_\circ}}$) and for some constant C depending only on n. Alternatively, we can write

$$E \text{ is } C\varepsilon r^{1+\alpha}\text{-close to } \Lambda_{\bar\gamma_{k_\circ},\bar\theta_{k_\circ}} \text{ in } B_r, \quad \text{for all } r \in (0, \rho_\circ^{k-1}/2).$$

Let us redefine, from now on, and for convenience in the upcoming notation, $\Lambda_{\gamma_{k_\circ},\theta_{k_\circ}} := \Lambda_{\bar\gamma_{k_\circ},\bar\theta_{k_\circ}}$. Notice that E_k is $\rho_\circ^{\alpha(k-1)}\varepsilon$-close to $\Lambda_{\gamma_k,\theta_k}$ in B_1, but it is also $\rho_\circ^{\alpha(k-2)-1}\varepsilon$-close to $\Lambda_{\gamma_{k-1},\theta_{k-1}}$. Therefore,

$$|\theta_k - \theta_{k-1}| + |\gamma_k - \gamma_{k-1}| \leq C_\circ\rho_\circ^{\alpha(k-2)}\left(\rho_\circ^{-1} + \rho_\circ^\alpha\right)\varepsilon = C_{n,\alpha}\rho_\circ^{\alpha k}\varepsilon, \tag{4.53}$$

where the sub-indices denote the only dependences of the constants. In particular, by triangular inequality

$$|\theta_{k_\circ} - \theta_k| + |\gamma_{k_\circ} - \gamma_k| \leq C_{n,\alpha}\varepsilon\sum_{j=k+1}^{k_\circ}\rho_\circ^{\alpha j} \leq C_{n,\alpha}\varepsilon\frac{\rho_\circ^{\alpha(k+1)}}{1-\rho_\circ^\alpha} = C_{n,\alpha}\varepsilon\rho_\circ^{\alpha k}, \tag{4.54}$$

for a different constant $C_{n,\alpha}$, still depending only on n and α. Thus, since E_k is $\rho_\circ^{\alpha(k-1)}\varepsilon$-close to $\Lambda_{\gamma_k,\theta_k}$ in B_1, E is $\rho_\circ^{(1+\alpha)(k-1)}\varepsilon$-close to $\Lambda_{\gamma_k,\theta_k}$ in $B_{\rho_\circ^{k-1}}$.

Now, from (4.54), $\Lambda_{\gamma_k,\theta_k}$ is $C_{n,\alpha}\varepsilon\rho_\circ^{\alpha k}\rho_\circ^{k-1}$-close to $\Lambda_{\gamma_{k_\circ},\theta_{k_\circ}}$ in $B_{\rho_\circ^{k-1}}$. Putting all together, E is $C_{n,\alpha}\rho_\circ^{(1+\alpha)(k-1)}\varepsilon$-close to $\Lambda_{\gamma_{k_\circ},\theta_{k_\circ}}$ in $B_{\rho_\circ^{k-1}}$ for all $k < k_\circ$. This, combined with (4.52), yields the desired result.

Finally, if $\theta_k \le C_\circ \rho^{k\alpha}\varepsilon$ for all $k \in \mathbb{N}$, we can take $k_\circ = \infty$ and repeat the previous procedure. In this case, consider as e_∞ and θ_∞ the limits of the sequences $(e_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$, which exist by (4.53). Notice that $\theta_\infty = 0$. □

Remark 4.10. In the previous proof, notice that if $k_\circ < \infty$ we must be dealing with a point in the interior of the contact set. In particular, all points on the free boundary must have $k_\circ = \infty$, and since $\theta_\infty = 0$ there is a supporting plane at each of this points.

We now give a proposition on regularity of ∂E in the case that it is close enough to some $\Lambda_{\gamma,\theta}$ with θ small enough (the wedge is almost a half-space).

Proposition 4.25. *There exists $\varepsilon_\circ$ depending only on n such that the following statement holds:*

Let $E \subset \mathbb{R}^n$ satisfying (4.17), be such that E is ε-close to $\Lambda_{\gamma,\theta}$ in B_1, for $\varepsilon \in (0, \varepsilon_\circ)$, and $\theta \le C_\circ\varepsilon$ for a constant $C_\circ$ depending only on n. Then, after a rotation of angle γ, ∂E is the graph of a function $h : B'_{1/2} \to (-1, 1)$ in the $\boldsymbol{e}_n$ direction in $B_{1/2}$. Moreover,

$$\|h\|_{C^{1,\alpha}(\overline{B'_{1/2}\cap\{x_{n-1}\ge 0\}})} + \|h\|_{C^{1,\alpha}(\overline{B'_{1/2}\cap\{x_{n-1}\le 0\}})} \le C\varepsilon, \tag{4.55}$$

for any $\alpha \in \left(0, \frac{1}{2}\right)$, and some constant C depending only on n and α.

Proof. Let assume for simplicity that $\gamma = 0$, the other cases are analogous. We will assume that $\varepsilon_\circ$ is small enough so that the previous results can be applied. Let us also assume that the contact set, $\Delta_E := \partial E \cap \{x_{n-1} = x_n = 0\}$, is non-empty in $B_{1/2}$; $\Delta_E \cap B_{1/2} \ne \varnothing$. Otherwise we are already done by the classical improvement of flatness.

Step 1: ∂E is the graph of a function. Let us first show that indeed ∂E is the graph of a function. To do so, proceed as in the first part of Lemma 4.21, combined with Proposition 4.24 and the fact that $\theta \le C_\circ\varepsilon$:

Take any $x_\circ \in B_{1/2} \cap \partial E$ not belonging to the contact set Δ_E, and let $r := \operatorname{dist}(x_\circ, \Delta_E) = |x_\circ - z|$ for $z \in \Delta_E$. Applying Proposition 4.24 around z, we deduce that for some $\Lambda_{\bar\gamma,\bar\theta}$ (depending on z),

$$E \text{ is } C\varepsilon r\text{-close to } \Lambda_{\bar\gamma,\bar\theta}, \quad \text{in } B_{r/2}(x_\circ),$$

for some constant C depending only on n. If we rescale the space a factor $2r^{-1}$ with respect to z so that E becomes $\tilde E$ then

$$\tilde E \text{ is } C\varepsilon\text{-close to } \Lambda_{\bar\gamma,\bar\theta}, \quad \text{in } B_1(2r^{-1}\, x_\circ).$$

Notice that $\tilde E$ is a minimal surface in $B_1(2r^{-1}\, x_\circ)$, since E is a minimal surface in $B_{r/2}(x_\circ)$. Using that $|\bar\gamma - 0| + |\theta - \bar\theta| \le C\varepsilon$ for some C depending only on n, and that

$\theta \le C_\circ\varepsilon$, we get that $\Lambda_{\bar\gamma,\bar\theta}$ is $C\varepsilon r$-close to $\{x_n = 0\}$ in $B_{r/2}(x_\circ)$. After the rescaling, $\Lambda_{\bar\gamma,\bar\theta}$ is $C\varepsilon$-close to $\{x_n = 0\}$ in $B_1(2r^{-1}x_\circ)$, so that $\tilde E$ is $C\varepsilon$-close to $\{x_n = 0\}$ in $B_1(2r^{-1}x_\circ)$. Thanks to the classical improvement of flatness (Theorem 4.22) for ε small enough depending only on n, $\partial\tilde E$ is a graph in the $\boldsymbol{e}_n$ direction in $B_1(2r^{-1}x_\circ)$, and consequently the same occurs for ∂E in $B_{r/2}(x_\circ)$. Let us call h the function whose graph is defined on $B_{r/2}(x_\circ)$ in the $\boldsymbol{e}_n$ direction. In particular, applying Theorem 4.22 again, $h \in \mathrm{Lip}(B'_{r/4}(x'_\circ))$, with $[h]_{C^{0,1}(B'_{r/2})} \le C\varepsilon$; where $x'_\circ$ is the projection of $x_\circ$ to $\{x_n = 0\}$.

Now, by a standard covering argument together with the fact that ∂E is continuous and Δ_E has measure zero, u is defined in $B'_{1/2}$ with

$$[h]_{C^{0,1}(B'_{1/2})} \le C\varepsilon,$$

for some C depending only on n.

Step 2: Regularity bound. Let us now show (4.55). We will show that for any $y' \in B'_{1/4}\cap\{x_{n-1} \ge 0\}$ and any $\rho \in (0, 1/4)$, there exists some $p_{y'} \in \mathbb{R}^{n-1}$ depending only on y' such that for any $\alpha \in (0, 1/2)$,

$$|h(x') - h(y') - p_{y'}\cdot(x' - y')| \le C\varepsilon\rho^{1+\alpha} \quad \text{in } B'_\rho(y')\cap\{x'_{n-1} \ge 0\}, \tag{4.56}$$

for some constant C depending only on n and α. The other half, $\{x'_{n-1} \le 0\}$, follows by symmetry.

Throughout this second step we will be switching between the characterisation of the solution to our thin obstacle problem as a boundary, ∂E, and as the graph of a function u on $\mathbb{R}^{n-1}$. Thus, we can rewrite Proposition 4.24. That is, if $0 \in \partial E$, we know that

$$E \text{ is } C_\alpha\varepsilon r^{1+\alpha}\text{-close to } \Lambda_{\bar\gamma,\bar\theta} \text{ in } B_r, \quad \text{for all } r \in (0, 1/2), \tag{4.57}$$

for some constant C_α depending only on n and α, and for some $\Lambda_{\bar\gamma,\bar\theta}$. We want to rewrite it in terms of u. Note that $|\gamma| + \bar\theta \le C\varepsilon$ for some constant C depending only on n, since $\theta \le C_\circ\varepsilon$, and therefore, we have that (4.57) implies

$$|h(x') - A^+(x'_{n-1})_+ - A^-(x'_{n-1})_-| \le C_\alpha\varepsilon|x'|^{1+\alpha}, \quad \text{in } B'_{1/2}, \tag{4.58}$$

with $A^- \ge A^+$ and $|A^-| + |A^+| \le C\varepsilon$ for some C_α depending only on n and α. Notice that if 0 is in the free boundary of the contact set, $0 \in \partial\Delta'_E$, then $A^+ = A^-$, or equivalently $\bar\theta = 0$ (see Remark 4.10).

Let $y', z' \in B'_{1/4}\cap\{x'_{n-1} \ge 0\}$, and let $y'_\circ, z'_\circ \in \Delta'_E$ be such that $\mathrm{dist}(y', \Delta'_E) = |y' - y'_\circ|$ and $\mathrm{dist}(z', \Delta'_E) = |z' - z'_\circ|$. We denote by $y, z, y_\circ$, and $z_\circ$, the corresponding elements as seen in $\mathbb{R}^n$ (e.g. $y = (y', 0)$), and let $\bar y = (y', h(y')) \in \partial E$ and $\bar z = (z', h(z')) \in \partial E$. Suppose, without loss of generality, that $d = |y' - y'_\circ| \le |z' - z'_\circ|$, and we consider two different cases.

- *Case 1.* Suppose that $r = |z' - y'| \ge d/2$. Using (4.58) centered around $y'_\circ$instead of 0, we know that for some A^+ depending on $y'_\circ$,

$$|h(x') - A^+x'_{n-1}| \le C_\alpha\varepsilon|x' - y'_\circ|^{1+\alpha}, \quad \text{for } x' \in B'_{1/2}(y'_\circ)\cap\{x'_{n-1} \ge 0\}.$$

Putting y' and z' in the previous expression yields

$$|h(y') - A^+ y'_{n-1}| \le C_\alpha \varepsilon |y' - y'_\circ|^{1+\alpha} = d^{1+\alpha} \le C_\alpha \varepsilon r^{1+\alpha},$$
$$|h(z') - A^+ z'_{n-1}| \le C_\alpha \varepsilon |z' - y'_\circ|^{1+\alpha} \le C_\alpha \varepsilon \left(d + r\right)^{1+\alpha} \le C_\alpha \varepsilon r^{1+\alpha},$$

from which

$$|h(y') - h(z') - A^+(y'_{n-1} - z'_{n-1})| \le C_\alpha \varepsilon r^{1+\alpha},$$

and in particular, (4.56) holds with $p_{y'} = A^+$.

- *Case 2.* Suppose $r = |z' - y'| \le d/2$. If $B'_d(y') \not\subseteq \{x'_{n-1} \ge 0\}$, then $y'_\circ \in \Delta'_E$ belongs to the free boundary and the corresponding $\Lambda_{\gamma(y'_\circ),\theta(y'_\circ)}$ from Proposition 4.24 around $y_\circ$ is actually an hyperplane ($\theta(y'_\circ) = 0$) with normal vector $e_{\gamma(y'_\circ)}$ (see Remark 4.10). In particular, ∂E is $C\varepsilon d^{1+\alpha}$-flat in the $e_{\gamma(y'_\circ)}$ direction in the ball $B_d(y)$ thanks to Proposition 4.24. On the other hand, if $B'_d(y') \subset \{x'_{n-1} \ge 0\}$, we consider again the corresponding $\Lambda_{\gamma(y'_\circ),\theta(y'_\circ)}$ from Proposition 4.24 around $y_\circ$. Then ∂E is $C\varepsilon d^{1+\alpha}$-flat in the $e_{\gamma(y'_\circ)+\theta(y'_\circ)}$ direction in the ball $B_d(y)$ (recall that $e_{\gamma(y'_\circ)+\theta(y'_\circ)}$ is the normal vector to $\Lambda_{\gamma(y'_\circ),\theta(y'_\circ)}$ in $\{x_{n-1} \ge 0\}$). In any case, noting that E is a set of minimal perimeter in $B_d(y)$ we can apply the classical improvement of flatness (see Remark 4.9) in $B_d(y)$, to get
$$|\nu(y) - \nu(z)| \le C\varepsilon |y - z|^\alpha,$$
for some C depending only on n. We have denoted here by $\nu(x)$ for $x \in \partial E$ the unit normal vector to ∂E pointed outwards with respect to E at the point x.

  Now notice that if ε is small enough depending only on n, since $|\nabla h| \le C\varepsilon$, $|\nu(y) - \nu(z)| \ge |\nabla h(y') - \nabla h(z')|$, and on the other hand, $|y - z| \le |y' - z'| + |h(y') - h(z')| \le 2|y' - z'|$ so that
$$|\nabla h(y') - \nabla h(z')| \le C\varepsilon |y' - z'|^\alpha,$$
from which (4.56) follows.

From (4.56) the result (4.55) follows by a covering argument. □

With this, we can now prove Theorem 4.4.

Proof of Theorem 4.4. In the case $\Phi \equiv \mathrm{id}$ it is a direct consequence of Lemma 4.21 and Proposition 4.25, depending on whether the wedge $\Lambda_{\gamma,\theta}$ is ε-flat or not. The case $\Phi \not\equiv \mathrm{id}$ follows from standard perturbative arguments and is left to the interested reader. □

4.7 Monotonicity formula and blow-ups

In this section we prove Proposition 4.5 and Corollary 4.6.

Lemma 4.26 (Monotonicity formula for minimizers of (4.17))**.** *Let $E \subset \mathbb{R}^n$ satisfy (4.17) in B_2 (instead of B_1) and suppose $0 \in \partial E \cap \partial \mathcal{O}$. Let us define*

$$\mathcal{A}(r) := \frac{P\big(E; \Phi(B_r)\big)}{r^{n-1}}, \quad \textit{for} \quad r > 0. \tag{4.59}$$

Then,

(a) If $\Phi \equiv \mathrm{id}$ then $\mathcal{A}'(1) \geq 0$

(b) If $\Phi(0) = 0$, $D\Phi(0) = \mathrm{id}$, and $[\Phi]_{C^{1,1}} \leq \eta_\circ$ for $\eta_\circ \in (0,1)$ small enough depending only on n then

$$\mathcal{A}'(1) \geq -C\eta_\circ$$

for some C depending only on n.

Proof. (a) The proof is similar to that of the classical monotonicity formula for minimal surfaces. Indeed, we take as a competitor to E in B_1 the dilation of E to $B_{1-\varepsilon}$ and we extend it conically in the annulus. For simplicity in the following computations, from now on we rescale everything by a factor 2, so that we can deal with $r = 1$ and $\mathcal{A}'(1)$.

As in [Sav10b], we take F defined as

$$x \in F \Leftrightarrow \begin{cases} x \in E & \text{if } |x| > 1 \\ x/|x| \in E & \text{if } (1-\varepsilon) \leq |x| \leq 1 \\ (1-\varepsilon)^{-1}x \in E & \text{if } |x| < (1-\varepsilon), \end{cases} \tag{4.60}$$

that is, we first contract it by a factor $1-\varepsilon$ and then extend conically F in the annulus $B_1 \setminus B_{1-\varepsilon}$ to obtain a competitor for E in B_1.

Thus,

$$P_{B_1}(E) \leq P_{B_1}(F) = (1-\varepsilon)^{n-1} P_{B_1}(E) + P_{B_1 \setminus B_{1-\varepsilon}}(F). \tag{4.61}$$

Now, dividing by ε and letting $\varepsilon \downarrow 0$, we obtain

$$(n-1)P_{B_1}(E) \leq \mathcal{H}^{n-2}(\partial E \cap \partial B_1). \tag{4.62}$$

On the other hand, notice that

$$\mathcal{A}'(1) = \int \frac{1}{\sqrt{1-(x\cdot\nu(x))^2}} d\mathcal{H}^{n-2}_{\partial E \cap \partial B_1} - (n-1)P_{B_1}(E), \tag{4.63}$$

which combined with (4.62) yields the result in the case (a).

(b) The proof in this case is a perturbation of the proof in case (a). Now we have

$$\Phi(0) = 0, \quad D\Phi(0) \equiv \mathrm{id} \quad \text{and} \quad |D^2\Phi| \leq \eta_\circ \quad \text{in } B_1,$$

The observation that allows us to control the errors is that, for all $x_\circ \in B_1$.

$$\Phi(x) = \Phi(x_\circ) + D\Phi(x_\circ)(x - x_\circ) + O(\eta_\circ |x - x_\circ|^2), \tag{4.64}$$

$$D\Phi(x_\circ) = \mathrm{id} + O(\eta_\circ), \quad D\Phi(rx_\circ) = D\Phi(x_\circ) + O(\eta_\circ(1-r)), \quad \forall r \in (0,1). \tag{4.65}$$

As a consequence, for $r \in (0,1]$ the maps $\theta : (0,1] \times \Phi(B_1) \to \Phi(B_r)$ defined by

$$(r, x) \mapsto \Phi\big(r\Phi^{-1}(x)\big)$$

are bi-Lipschitz and are quasi-dilations with the estimate, for $r \in (1/2, 1)$

$$|\theta(r, x) - \theta(r, x_\circ)| \le r|x - x_\circ|\big(1 + C(1-r)\eta_\circ\big). \tag{4.66}$$

Indeed, (4.66) follows immediately from (4.64) and (4.65) if $|x_\circ - x| < (1 - r)$. For general $x_\circ, x$ we use the previous case and the triangle inequality.

Now, repeat the proof for the case (a) after applying Φ^{-1} and then check using (4.66) that the errors we make are small. Namely, we define F as in (4.60) but with E replaced by $\Phi^{-1}(E)$. Note that $\Phi(F)$ is a "competitor" of E in $\Phi(B_1)$, namely, $\Phi(\Lambda^\delta) \subset \Phi(F)$ and $\Phi(F) \setminus \Phi(B_1) = E \setminus \Phi(B_1)$.

Now (4.61) must be replaced by

$$P_{\Phi(B_1)}(E) \le P_{\Phi(B_1)}(\Phi(F)) = P_{\Phi(B_{1-\varepsilon})}(\Phi(F)) + P_{\Phi(B_1 \setminus B_{1-\varepsilon})}(\Phi(F)). \tag{4.67}$$

Now, using (4.66) and $\Phi(F) = \theta(1-\varepsilon, E)$ in $\Phi(B_{1-\varepsilon})$, we obtain

$$P_{\Phi(B_{1-\varepsilon})}(\Phi(F)) \le (1-\varepsilon)^{n-1} P_{\Phi(B_1)}(E) + O(\eta_\circ \varepsilon).$$

and

$$P_{\Phi(B_1 \setminus B_{1-\varepsilon})}(\Phi(F)) = \varepsilon \mathcal{H}^{n-2}\big(\Phi(F \cap \partial B_1)\big) + O(\eta_\circ \varepsilon).$$

So that,

$$P_{\Phi(B_1)}(E) \le (1-\varepsilon)^{n-1} P_{\Phi(B_1)}(E) + \varepsilon \mathcal{H}^{n-2}\big(\Phi(F \cap \partial B_1)\big) + O(\eta_\circ \varepsilon).$$

Dividing by ε and letting $\varepsilon \downarrow 0$ we obtain

$$(n-1) P_{\Phi(B_1)}(E) \le \mathcal{H}^{n-2}(\partial E \cap \Phi(\partial B_1)) + O(\eta_\circ).$$

Now we conclude the proof observing that

$$\mathcal{A}'(1) = \int \frac{\big|\partial_r \theta\big(1, \Phi^{-1}(x)\big)\big|}{\sqrt{1 - (x \cdot \nu(x))^2}} d\mathcal{H}^{n-2}_{\partial E \cap \Phi(\partial B_1)} - (n-1) P_{\Phi(B_1)}(E),$$

and that $\big|\partial_r \theta\big(1, \Phi^{-1}(x)\big)\big| = 1 + O(\eta_\circ)$. □

Lemma 4.27 (Monotonicity formula for minimizers of (4.17))**.** *Let $E \subset \mathbb{R}^n$ satisfy (4.17) and suppose $0 \in \partial E \cap \partial \mathcal{O}$. Let us define*

$$\mathcal{A}_E(r) := \frac{P\big(E; \Phi(B_r)\big)}{r^{n-1}}, \quad \textit{for} \quad r > 0. \tag{4.68}$$

Then,

(a) If $\Phi \equiv \mathrm{id}$ then $\mathcal{A}' \ge 0$ for $r \in (0,1)$. Moreover, $\mathcal{A}' \equiv 0$ (i.e., $\mathcal{A}$ constant) if and only if E is a cone ($tE = E$ for any $t > 0$).

(b) If $\Phi(0) = 0$, $D\Phi(0) = \mathrm{id}$, and $[\Phi]_{C^{1,1}} \le \eta_\circ$ for $\eta_\circ \in (0,1)$ small enough depending only on n then

$$\mathcal{A}_E'(r) \ge -C\eta_\circ$$

for some C depending only on n.

Proof. It follows by scaling Lemma 4.26. Part (a) is immediate, being the cone condition an immediate consequence of (4.63). For part (b), let us define, for any $\lambda > 0$, $\Phi^\lambda := \lambda\Phi\left(\frac{1}{\lambda}\,\cdot\right)$, and

$$\mathcal{A}_E^\lambda(r) := \frac{P\big(E;\Phi^\lambda(B_r)\big)}{r^{n-1}}, \quad \text{for} \quad r > 0. \tag{4.69}$$

Note now, that

$$\mathcal{A}_E(r) = \frac{P\big(\lambda E;\lambda\Phi(B_r)\big)}{\lambda^{n-1}r^{n-1}} = \frac{P\big(\lambda E;\Phi^\lambda(B_{\lambda r})\big)}{\lambda^{n-1}r^{n-1}} = \mathcal{A}_{\lambda E}^\lambda(\lambda r).$$

Differentiating both sides with respect to r we obtain

$$\mathcal{A}_E'(r) = \lambda\left(\mathcal{A}_{\lambda E}^\lambda\right)'(\lambda r). \tag{4.70}$$

On the other hand, applying Lemma 4.26 with λE and Φ^λ,

$$\left(\mathcal{A}_{\lambda E}^\lambda\right)'(1) \geq -C[\Phi^\lambda]_{C^{1,1}(B_1)} \geq -C\lambda^{-1}\eta_\circ.$$

Putting it together with (4.70) and fixing $\lambda = r^{-1}$ we obtain

$$\mathcal{A}_E'(r) = r^{-1}\left(\mathcal{A}_{\lambda E}^\lambda\right)'(1) \geq -C\eta_\circ,$$

as we wanted to see. □

We now recall the well-known density estimates lemma for perimeter minimizers. It is a very standard result in the theory of minimal surfaces which can be found extensively in the literature. We mention, for example, the survey [Sav10].

Lemma 4.28. *Let $E \subset \mathbb{R}^n$ be a minimizer of the perimeter in $B_{r_\circ}$ for some $r_\circ > 0$, such that $0 \in \partial E$. Then,*

$$\begin{aligned} &|E \cap B_r| \geq cr^n, \\ &|E^c \cap B_r| \geq cr^n, \quad \textit{for all} \quad r \in (0, r_\circ), \end{aligned}$$

for some c constant depending only on the dimension n.

We have a similar lemma for supersolutions to the minimal perimeter problem.

Lemma 4.29. *Let $E^+ \subset \mathbb{R}^n$ be a supersolution to the minimal perimeter problem in $B_{r_\circ}$ for some $r_\circ > 0$, such that $0 \in \partial E^+$. Then,*

$$|(E^+)^c \cap B_r| \geq cr^n, \quad \textit{for all} \quad r \in (0, r_\circ),$$

for some c constant depending only on the dimension n.

Proof. This is standard, and follows exactly the same as Lemma 4.28. □

Let us now prove the following proposition, stating that in order to prove that at some scale the solution is close enough to a wedge, it is enough to classify conical solutions.

Proposition 4.30. *Assume that in some dimension $n \geq 2$ the wedges $\Lambda_{\gamma,\theta}$ are the only cones $E \subset \mathbb{R}^n$ satisfying* (4.17) *with $\Phi = \mathrm{id}$ and any $\delta > 0$.*

Assume that, for some $\delta > 0$, the set $E \subset \mathbb{R}^n$ with $P(E; B_1) < \infty$ satisfies $\Phi(\Lambda^\delta) \cap B_1 \subset E$ and (4.17)*, with Φ a $C^{1,1}$ diffeomorphism.*

Then, for any $\varepsilon > 0$, there exists $\rho > 0$ depending only on n, ε, and $\|\Phi\|_{C^{1,1}}$, and $\|D\Phi^{-1}\|_{L^\infty}$, such that if $x_\circ \in \partial E \cap \partial \mathcal{O} \cap \overline{B_{1/2}}$, then

$$\rho^{-1}(R_{x_\circ} E - x_\circ) \quad \textit{is } \varepsilon\textit{-close to } \Lambda_{\gamma,\theta},$$

for some γ and θ as in (4.11) *and for some rotation $R_{x_\circ}$ depending only on $x_\circ$.*

Proof. After a translation, let us start by assuming that $x_\circ = 0$. Let us also take a rotation $R_{x_\circ}$ of the whole setting, in such a way that, if we denote $\Phi_k := k\Phi$, then $R_{x_\circ}\Phi_k(\Lambda^\delta)$ converges in Hausdorff distance locally to $\Lambda^{\delta'}$ as $k \to \infty$ for some $\delta' > 0$ (i.e., we take the blow-up of a Lipschitz boundary). Notice that the value δ' is determined only by δ and Φ. By redefining Φ if necessary, let us assume $R_{x_\circ} = \mathrm{id}$ for simplicity. (Note that we could also argue via Lemma 4.10.)

Let us argue by contradiction, and assume that the thesis does not hold.

Let $\rho_k = k^{-1}$, and consider the sequence of sets $E_k = \rho_k^{-1}E$. Notice that, for $\Phi_k := k\Phi$, each E_k fulfils $\Phi_k(\Lambda^\delta) \cap B_k \subset E_k$ and solves a thin obstacle problem of the type

$$P(E_k; B_k) \leq P(F; B_k) \quad \forall F \text{ such that } E_k \setminus B_k = F \setminus B_k \text{ and } \Phi_k(\Lambda^\delta) \cap B_k \subset F. \tag{4.71}$$

Recall that the set $\Phi_k(\Lambda^\delta)$ converges in Hausdorff distance to $\Lambda^{\delta'}$ as $k \to \infty$. From minimality, we have compactness in L^1_{loc} of E_k, so that, up to a subsequence, $E_k \xrightarrow{L^1_{\mathrm{loc}}} E_\infty$, for some global solution to the δ'-thin obstacle problem with $\Phi = \mathrm{id}$, E_∞, with $\Lambda^{\delta'} \subset E_\infty$. It immediately follows that $0 \in \overline{E_\infty}$.

On the other hand, by the density estimates in Lemma 4.29, since each E_k is a supersolution to the minimal perimeter problem in B_1 and $0 \in \partial E_k$ for all k, we have

$$|E_k^c \cap B_r| \geq cr^n, \quad \text{for all } r \in (0,1),$$

for some constant c. The convergence in L^1_{loc} implies that the limit also fulfils $|E_\infty^c \cap B_r| \geq cr^n$, and therefore $0 \in \partial E_\infty$.

Using the same notation as in the proof of Lemma 4.27 (see (4.69)), we know

$$\mathcal{A}_E(r) = \mathcal{A}^k_{E_k}(kr), \quad \text{for all } r > 0.$$

Notice, also, that

$$\mathcal{A}^k_{E_k}(r) \to \mathcal{A}_{E_\infty}(r) := \frac{P\big(E; B_r\big)}{r^{n-1}} \quad \text{locally as } k \to \infty,$$

where we are using the L^1_{loc} convergence of E_k to E_∞, and the fact that $\Phi^k = k\Phi(k^{-1}\,\cdot\,) \to \mathrm{id}$ as $k \to \infty$ in $C^{1,1}_{\mathrm{loc}}$. In particular, we have that

$$\lim_{\rho \downarrow 0} \mathcal{A}_E(\rho) = \mathcal{A}_{E_\infty}(r), \quad \text{for all } r > 0.$$

Thanks to Lemma 4.27 part (b), the left-hand side limit is well defined. That is, $\mathcal{A}_{E_\infty}(r)$ is bounded and constant for any $r > 0$, which, from Lemma 4.27 part (a) implies that E_∞ is a cone ($tE_\infty = E_\infty$ for any $t > 0$). By assumption, therefore, $E_\infty = \Lambda_{\gamma,\theta}$ for some γ and θ; and we have that E_k is converging in $L^1_{\rm loc}$ to some $\Lambda_{\gamma,\theta}$.

Finally, in order to reach the contradiction, let us show that the convergence of ∂E_k to ∂E_∞ is in Hausdorff distance locally, which will complete the proof.

Suppose that is is not. That is, after extracting a subsequence, we can assume that there exists some sequence of points $y_k \in \partial E_k$ such that $y_k \to y_\infty$ and $\mathrm{dist}(y_k, \partial E_\infty) > \varepsilon > 0$ for some $\varepsilon > 0$ and for all $1 \le k \le \infty$. We have a dichotomy, either $y_\infty \in E_\infty$ or $y_\infty \in E^c_\infty$.

Let us now use the density estimate in Lemma 4.29. If $y_\infty \in E_\infty$ then, after a subsequence if necessary, $|E^c_k \cap B_\varepsilon(y_k)| \ge c\varepsilon^n$ but $|E^c_\infty \cap B_\varepsilon(y_\infty)| = 0$, which is a contradiction with the $L^1_{\rm loc}$ convergence. On the other hand, if $y_\infty \in E^c_\infty$ assume that after a subsequence $y_k \in E^c_\infty$ for all $k > 0$. We have that for k large enough $y_k \in \partial E_k$ is a point around which E_k is a minimal surface (being E_∞ a barrier *from below*). That is, we can use the classical density estimates for minimal surfaces in Lemma 4.28 to reach that $|E_k \cap B_\varepsilon(y_k)| \ge c\varepsilon^n$ but $|E_\infty \cap B_\varepsilon(y_\infty)| = 0$, again, a contradiction. □

Thus, in order to prove Corollary 4.6, it will be enough to classify cones.

Proof of Proposition 4.5. The proof is by induction on the dimension n.

Step 1: Base case. Dimension $n = 2$.

Assume that $\Sigma^2 \subset \mathbb{R}^2$ is a cone satisfying (4.17), in other words, the boundary of Σ^2 in B_1 consists of radii of length one. By assumption, we have $(0, -1) \in \Sigma^2 \cap S^1$. Now, if Σ^2 were not a wedge (that is, if $\Sigma^2 \cap S^1$ were disconnected) then the convex hull of $\Sigma^2 \cap B_1$ would be a set containing the obstacle (it contains Σ^2) and having strictly less relative perimeter in B_1 than Σ^2. This would contradict the minimality of Σ^2 —i.e. (4.17).

Step 2: Induction step. Suppose that it holds up to dimension $n - 1 \ge 2$. Let us show it for dimension n.

Let us first prove regularity of the cone around contact points. Assume that we have, without loss of generality, $x_\circ = \boldsymbol{e}_1 = (1, 0, \ldots, 0) \in \partial\Sigma \cap \partial B_1$. The first thing to notice is that the blow up of Σ around $x_\circ$ is a wedge $\Lambda_{\gamma_1,\theta_1}$. Indeed, the blow-up is a cone by the monotonicity formula, and thanks to the fact that Σ is a cone and $x_\circ = \boldsymbol{e}_1$, we get that the blow up at $x_\circ$ must be of the form $\mathbb{R} \times \Sigma^{n-1}$; where now $\Sigma^{n-1} \subset \mathbb{R}^{n-1}$ is a cone in $n - 1$ dimensions such that satisfies (4.17) (also taking Λ^δ in $n - 1$ dimensions). In particular, by induction step, $\Sigma^{n-1} = \Lambda^{n-1}_{\gamma_1,\theta_1} \subset \mathbb{R}^{n-1}$, where $\Lambda^{n-1}_{\gamma_1,\theta_1}$ denotes $\Lambda_{\gamma_1,\theta_1}$ as seen in $n - 1$ dimensions. This immediately yields that the blow up at $x_\circ$ is a wedge of the form $\Lambda_{\gamma_1,\theta_1}$. By Proposition 4.30 and Theorem 4.4, $\partial\Sigma$ is a smooth minimal surface around any $x_\circ \in \partial\Sigma \cap \{x_{n-1} = x_n = 0\}$ in $\{\pm x_{n-1} \ge 0\}$ up to $\{x_{n-1} = 0\}$.

Let us separate the proof between both sides $\pm x_{n-1} \ge 0$, and let us focus first on $x_{n-1} \ge 0$ (the other side follows analogously). We can now take $s^* = \max\{s \ge \delta : \Lambda^s \subset \Sigma \text{ in } x_{n-1} \ge 0\}$. Notice that it is indeed a maximum, since it is enough to check that $\Lambda^s \cap S^{n-1} \subset \Sigma \cap S^{n-1}$, where $S^{n-1} \subset \mathbb{R}^n$ denotes the $(n-1)$-dimensional sphere.

The boundaries $\partial\Sigma \cap S^{n-1}$ and $\partial\Lambda^{s^*} \cap S^{n-1}$ must touch at a point $x_\circ \in \{x_{n-1} \geq 0\}$. If $x_\circ \in \{x_{n-1} > 0\}$, then by the strong maximum principle for minimal surfaces we must have $\Sigma_{\mathcal{O}} = \Lambda^{s^*}$ in $\{x_{n-1} \geq 0\}$, where $\Sigma_{\mathcal{O}}$ denotes the connected component of $\Sigma \setminus \{x_{n-1} = x_n = 0\}$ that contains the thin obstacle $\mathcal{O}$ (which, in this case, is flat). On the other hand, if $x_\circ \in \{x_{n-1} = x_n = 0\}$, then we have previously shown (by induction and dimension reduction) that $\partial\Sigma \cap \{x_{n-1} \geq 0\}$ is C^1 up to its boundary around the points $x_\circ$ and touches the half-plane of $\partial\Lambda^{s^*}$ tangentially at $x_\circ$. Using the boundary strong maximum principle (Hopf lemma) we obtain again that $\Sigma_{\mathcal{O}} = \Lambda^{s^*}$ in $\{x_{n-1} \geq 0\}$.

The same holds for the other side, $x_{n-1} \leq 0$, so that in all we have that

$$\Sigma_{\mathcal{O}} = \Lambda_{\gamma,\theta}$$

for some γ and θ as in (4.11).

We can now repeat the argument, but opening $\Lambda_{\gamma,\theta}$ instead, until we reach another connected component of $\Sigma \setminus \{x_{n-1} = x_n = 0\}$. Proceeding iteratively, this yields that Σ must be one dimensional; that is, Σ is the cone $\mathbb{R}^{n-2} \times \Sigma^2$ for some cone $\Sigma^2 \subset \mathbb{R}^2$. By the base case in Step 1 minimality implies that Σ^2 must be a convex angle and hence $\mathbb{R}^{n-2} \times \Sigma^2$ is a wedge. □

Once cones are classified, we can proceed with the proof of Corollary 4.6,

Proof of Corollary 4.6. We will apply Theorem 4.4 after an translation, rotation, and scaling. We have to check that the hypotheses are fulfilled.

By definition of minimizer of (4.2) (see Definition 4.1) there exist $\delta_k \downarrow 0$, E_k minimizers of (4.16) such that $\chi_{E_k} \to \chi_E$ in $L^1(B_1)$. For each E_k let $x_\circ$ be any point in $B_{1/2} \cap \partial E_k \cap \partial\mathcal{O}$. Let $E_k^{x_\circ,\rho} := \psi_{x_\circ}(E_k) = \rho^{-1}(R_{x_\circ}E_k - x_\circ)$, where $\psi_{x_\circ}$ denotes the change of coordinates from Lemma 4.10. Let us also denote $\Phi_\rho^{x_\circ} := \bar{\Phi}$ the new diffeomorphism (also from Lemma 4.10).

Thus, $E_k^{x_\circ,\rho}$ is a minimizer of the $\bar{\delta}$-thin obstacle problem around $x_\circ$ with diffeomorphism $\Phi_{x_\circ}^\rho$ such that $\Phi_\rho^{x_\circ}(0) = 0$, $D\Phi_\rho^{x_\circ}(0) = \mathrm{id}$, and $[\Phi_\rho^{x_\circ}]_{C^{1,1}(B_1)} \leq C\rho$ thanks to Lemma 4.10.

On the other hand, as a consequence of Proposition 4.5 and Proposition 4.30 in any dimension $n \geq 2$, we reach that, for ρ small enough, $E_k^{x_\circ,\rho}$ is $\varepsilon_\circ$-close to $\Lambda_{\gamma,\theta}$ for some γ and θ. Also, for ρ small enough, we will have $[\Phi_\rho^{x_\circ}]_{C^{1,1}(B_1)} \leq \varepsilon_\circ^{1+\frac{1}{2}}$ where $\varepsilon_\circ > 0$ is the constant in Theorem 4.4. Therefore, applying Theorem 4.4 to $E_k^{x_\circ,\rho}$ (and shrinking by a factor ρ) we obtain that ∂E_k has the following $C^{1,\alpha}$ structure in $B_{\rho/2}(x_\circ)$. Either:

(a) In appropriate coordinates y, $(\Phi^{x_\circ})^{-1}\big(R_{x_\circ}(\partial E_k - x_\circ)\big)$ is the graph $\{y_n = h(y')\}$ of a function $h \in C^0(\overline{B'_{\rho/2}})$ satisfying $h \in C^{1,\alpha}(\overline{B'^{+}_{\rho/2}}) \cap C^{1,\alpha}(\overline{B'^{-}_{\rho/2}})$. Moreover, we have $h \geq 0$ on $y_{n-1} = 0$ and ∇h is continuous on $\{y_{n-1} = 0\} \cap \{h > 0\}$.

or

(b) $R(\partial E_k - x_\circ) \cap B_{\rho/2}$ is the union of two $C^{1,1-}$ surfaces that meet on $\partial\mathcal{O}$ with full contact set in $B_{\rho/2}$.

Now we deduce in case (a) that in some new coordinates with origin at $x_\circ$ we have $\Phi^{-1}\big(\partial E_k\big)$ is the graph $\{z_n = \tilde{h}(z')\}$ of a function $\tilde{h} \in C^0(\overline{B'_{\tilde{\rho}}})$ satisfying $\tilde{h} \in C^{1,\alpha}(\overline{B'^{+}_{\tilde{\rho}}}) \cap C^{1,\alpha}(\overline{B'^{-}_{\tilde{\rho}}})$. Moreover, we have $\tilde{h} \geq 0$ on $z_{n-1} = 0$ and $\nabla\tilde{h}$ is continuous on $\{z_{n-1} = 0\} \cap \{\tilde{h} > 0\}$.

Since either (a) or (b) holds for E_k with estimates independent of k, we can pass to the limit and show that either (a) or (b) also holds for E.

Finally, if the alternative (b) near some point $x_\circ$ then using that $\partial\mathcal{O}$ is of class $C^{k,\beta}$ (and the classical $C^{k,\beta}$ regularity up to the boundary results for minimal surfaces [Jen80]) we obtain that ∂E splits into two $C^{k,\beta}$ minimal surfaces with boundary in a small ball around $x_\circ$. □

Proof of Theorem 4.1. After having introduced the appropriate notion of solution, we have that Theorem 4.1 corresponds to Corollary 4.6. □

Chapter 5

On the singular set in the thin obstacle problem: higher order blow-ups and the very thin obstacle problem

In this work, we consider the singular set in the thin obstacle problem with weight $|x_{n+1}|^a$ for $a \in (-1, 1)$, which arises as the local extension of the obstacle problem for the fractional Laplacian (a non-local problem). We develop a refined expansion of the solution around its singular points by building on the ideas introduced by Figalli and Serra to study the fine properties of the singular set in the classical obstacle problem. As a result, under a superharmonicity condition on the obstacle, we prove that each stratum of the singular set is locally contained in a single C^2 manifold, up to a lower dimensional subset, and the top stratum is locally contained in a $C^{1,\alpha}$ manifold for some $\alpha > 0$ if $a < 0$.

In studying the top stratum, we discover a dichotomy, until now unseen, in this problem (or, equivalently, the fractional obstacle problem). We find that second blow-ups at singular points in the top stratum are global, homogeneous solutions to a codimension two lower dimensional obstacle problem (or fractional thin obstacle problem) when $a < 0$, whereas second blow-ups at singular points in the top stratum are global, homogeneous, and a-harmonic polynomials when $a \geq 0$. To do so, we establish regularity results for this codimension two problem, what we call the very thin obstacle problem.

Our methods extend to the majority of the singular set even when no sign assumption on the Laplacian of the obstacle is made. In this general case, we are able to prove that the singular set can be covered by countably many C^2 manifolds, up to a lower dimensional subset.

5.1 Introduction

Lower dimensional obstacle problems are an important class of obstacle problems, arising in many areas of mathematics. For instance, they can be found in the theory of elasticity (see [Sig33, Sig59, KO88]), and they also appear in describing osmosis through semi-permeable membranes as well as boundary heat control (see, e.g., [DL76]). Moreover, they often are local formulations of fractional obstacle problems, another important class of obstacle problems. Fractional obstacle problems can be found in the optimal stopping problem for Lévy processes, and can be used to model American option prices (see [Mer76, CT04]). They also appear in the study of anomalous diffusion, [BG90], the study of quasi-geostrophic flows, [CV10], and in studies of the interaction energy of probability measures under singular potentials, [CDM16]. (We refer to [Ros18] for an extensive bibliography on the applications of obstacle-type problems.)

Broadly, lower dimensional obstacle problems are minimization problems for a given energy functional on class of functions constrained to sit above a given obstacle (function) defined on a lower dimensional manifold. Obstacle problems are free boundary problems: the principal part of their study is the structure and regularity of the boundary of the contact set of the solution and the obstacle, the free boundary. The lower dimensional obstacle problem we consider — the thin obstacle problem with weight $|x_{n+1}|^a$ — has garnered much interest and attention (see [AC04, CS07, ACS08, GP09, KRS19, FoSp18, CSV19, JN17]); it is a model setting, and has motivated the study of many other types of lower dimensional obstacle problems (see [MS08, AM11, Fer16, RS17, RuSh17, FS20, FoSp18b, GR19, BLOP19]).

Nevertheless, the study of the non-regular part of the free boundary has been rather limited. Only recently has significant progress been made (see [GP09, FoSp18, GR19, CSV19]). And many open questions still remain. In this work, we address some of these questions, focusing on the singular set (see Section 5.1.2). In particular, we explore the fine properties of the solution and its expansion around singular points, inspired by [FS18].

We note that the techniques of [FS18] have been further developed and improved in [FRS19], where the authors prove generic regularity (namely, the generic smoothness of the free boundary in the classical obstacle problem) in dimension three and the smoothness of the free boundary at almost every time for the three-dimensional Stefan problem. We expect the machinery built here to be useful in tackling genericness-type questions of this nature in the context of the thin/fractional obstacle problem, expanding on the very recent results by the first author and Ros-Oton in [FR19].

5.1.1 The Thin Obstacle Problem

In this paper, we consider a class of lower dimensional obstacle problems in $\mathbb{R}^{n+1} := \{X = (x, y) \in \mathbb{R}^n \times \mathbb{R}\}$ with weight $|y|^a$ where $\mathbb{R}^n \times \{0\}$ acts as the lower dimensional manifold. We will often refer to them as, simply, the thin obstacle problem, even though this name is usually reserved for the case $a = 0$. In particular, for an analytic *obstacle* $\varphi : B_1 \cap \{y = 0\} \to \mathbb{R}$, we look at the thin obstacle problem:

$$\min_{w\in\mathscr{A}} \left\{ \int_{B_1} |\nabla w|^2 |y|^a \, \mathrm{d}X \right\}, \quad \text{with} \quad a \in (-1, 1), \tag{5.1}$$

where $\mathscr{A}$ is the convex subset of the Sobolev space $W^{1,2}(B_1, |y|^a\,\mathrm{d}X)$ (which, for simplicity, we call $W^{1,2}(B_1, |y|^a)$) defined by

$$\mathscr{A} := \{w \in W_0^{1,2}(B_1, |y|^a) + g : w(x,0) \geq \varphi(x) \text{ and } w(x,-y) = w(x,y)\},$$

given some boundary data $g \in C(B_1)$ (even with respect to y) such that $g|_{\partial B_1 \cap \{y=0\}} \geq \varphi$. The condition that w sits above φ on the *thin space* $\mathbb{R}^n \times \{0\}$ needs to be understood in the trace sense, a priori.

If u is the (unique) solution to (5.1), then u satisfies the Euler–Lagrange equations

$$\begin{cases} u(x,y) & \geq & \varphi(x) & \text{on } B_1 \cap \{y=0\} \\ L_a u(x,y) & \leq & 0 & \text{in } B_1 \\ L_a u(x,y) & = & 0 & \text{in } B_1 \setminus \Lambda(u) \\ u(x,y) & = & u(x,-y) & \text{in } B_1 \\ u(x,y) & = & g(x,y) & \text{on } \partial B_1 \end{cases} \tag{5.2}$$

where

$$L_a u(x,y) := \operatorname{div}(|y|^a \nabla u(x,y))$$

and

$$\Lambda(u) := \{(x,0) : u(x,0) = \varphi(x)\}.$$

The set $\Lambda(u)$ is called the *contact set*, and is an unknown of the problem. Its topological boundary in $\mathbb{R}^n$

$$\Gamma(u) := \partial\Lambda(u) \subset \mathbb{R}^n \times \{0\}$$

is called the *free boundary*.

Remark 5.1. A useful equivalent characterization of the minimizer u of (5.1) is that u is the smallest super a-harmonic function in $\mathscr{A}$: $u \in \mathscr{A}$, $L_a u \leq 0$, and $u \leq w$ for all $w \in \mathscr{A}$ such that $L_a w \leq 0$.

Remark 5.2. In this work, we consider analytic obstacles. Clearly, this regularity restriction can be relaxed; the thin obstacle problem (5.1) can be well-formulated with significantly less regular obstacles (e.g., continuous obstacles). That said, the analytic setting allows us to understand the model behavior of $\Gamma(u)$, and for this reason, it deserves special consideration.

The Obstacle Problem for the Fractional Laplacian

As shown in [CSS08], the Euler–Lagrange equations (5.2) appear naturally in the context of the obstacle problem for the fractional Laplacian, or the fractional obstacle problem. Indeed, let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be an obstacle (with sufficient decay at infinity) and let $\bar{u}$ solve the fractional obstacle problem

$$\begin{cases} \bar{u} & \geq & \varphi & \text{in } \mathbb{R}^n \\ (-\Delta)^s \bar{u} & \geq & 0 & \text{in } \mathbb{R}^n \\ (-\Delta)^s \bar{u} & = & 0 & \text{in } \{\bar{u} > \varphi\} \\ \lim_{|x|\to\infty} \bar{u}(x) & = & 0 \end{cases} \quad \text{with} \quad s := \frac{1-a}{2} \in (0,1). \tag{5.3}$$

Then, the even in y, a-harmonic extension of $\bar{u}$ to $\mathbb{R}^{n+1}$ (i.e., $u : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $L_a u(x,y) = 0$ for $|y| > 0$, $u(x,0) = \bar{u}(x)$, $u(x,y) = u(x,-y)$, and

$\lim_{|(x,y)|\to\infty} u(x,y) = 0$) solves (5.2) in $\mathbb{R}^{n+1}$ (and, in particular, with its own boundary data, in B_1). Consequently, all of the results we prove in this work can be translated into statements regarding the fractional obstacle problem. We leave this translation to the interested reader.

5.1.2 Known Results

Let us briefly summarize some of the known properties of the solution to the thin obstacle problem and its free boundary. To do so, it will be useful to "normalize" φ, and it will be necessary to define a collection of rescalings of u.

Since $\varphi = \varphi(x)$ is analytic, we can extend it from a function defined on $B_1 \cap \{y = 0\}$ to an a-harmonic, even in y function defined on $\overline{B_1}$ (see [GR19, Lemma 5.1]). For simplicity, we still denote this extension by φ. So if we let

$$\tilde{u} := u - \varphi, \tag{5.4}$$

(5.2) becomes

$$\begin{cases} \tilde{u}(x,y) \geq 0 & \text{on } B_1 \cap \{y=0\} \\ L_a\tilde{u}(x,y) \leq 0 & \text{in } B_1 \\ L_a\tilde{u}(x,y) = 0 & \text{in } B_1 \setminus \Lambda(\tilde{u}) \\ \tilde{u}(x,y) = \tilde{u}(x,-y) & \text{in } B_1 \\ \tilde{u}(x,y) = \tilde{g}(x,y) & \text{on } \partial B_1, \end{cases} \tag{5.5}$$

with $\tilde{g} := g - \varphi$ and

$$\Lambda(\tilde{u}) := \{(x,0) : \tilde{u}(x,0) = 0\} = \Lambda(u).$$

Furthermore,

$$L_a\tilde{u} = 2\lim_{y\downarrow 0} y^a \partial_y \tilde{u}(x,y)\mathcal{H}^n \llcorner \Lambda(\tilde{u}). \tag{5.6}$$

Hence, considering (5.5),

$$\lim_{y\downarrow 0} y^a \partial_y \tilde{u}(x,y) \leq 0 \quad \text{for} \quad |x| < 1,$$

$$\lim_{y\downarrow 0} y^a \partial_y \tilde{u}(x,y) = 0 \quad \text{for} \quad |x| < 1 \text{ and } \tilde{u}(x,0) > 0,$$

and

$$\tilde{u}\, L_a\tilde{u} = 0 \quad \text{in} \quad B_1.$$

(See [CSS08, GP09, FoSp18, GR19].) All of the above expressions must be understood in a distributional sense.

As we have mentioned, we need to introduce a collection of rescalings of u around a free boundary point $X_\circ \in \Gamma(u)$ in order to outline the existing literature on (5.1). They are

$$\tilde{u}_{X_\circ,r}(X) := \frac{\tilde{u}_{X_\circ}(rX)}{\left(\frac{1}{r^{n+a}}\int_{\partial B_r} \tilde{u}_{X_\circ}^2 |y|^a\right)^{1/2}} \quad \text{where} \quad \tilde{u}_{X_\circ}(X) := \tilde{u}(X_\circ + X). \tag{5.7}$$

Blow-ups and Optimal Regularity

In [ACS08, CSS08], Athanasopoulos, Caffarelli, and Salsa and Caffarelli, Salsa, and Silvestre, for $a = 0$ and $a \in (-1, 1)$ respectively, proved that the set $\{\tilde{u}_{X_\circ,r}\}_{r>0}$ is weakly precompact in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}, |y|^a)$, and that the limit points of $\{\tilde{u}_{X_\circ,r}\}_{r>0}$ as $r \downarrow 0$ or *blow-ups of u at $X_\circ$* are global $\lambda_{X_\circ}$-homogeneous solutions to (5.5) with

$$\lambda_{X_\circ} \in [1 + s, \infty) \quad \text{for} \quad s := \frac{1-a}{2}.$$

It is important to note that the homogeneity of blow-ups depends only on the point $X_\circ \in \Gamma(u)$ at which they are taken, and is independent of the sequence along which the weak limit is produced.

Moreover, in [AC04, CSS08], it was shown that u is optimally $C^{1,s}$ on either side of the thin space (but only C^{2s} across, Lipschitz if $s = \frac{1}{2}$).

The Free Boundary

The free boundary $\Gamma(u)$ can be partitioned into three sets:

$$\Gamma(u) = \mathrm{Reg}(u) \cup \mathrm{Sing}(u) \cup \mathrm{Other}(u),$$

the set of *regular points*, the set of *singular points*, and set of *other points* (see [GP09, FoSp18, GR19]), and they can be characterized by the value of $\lambda_{X_\circ}$ with $X_\circ \in \Gamma(u)$.

$\mathrm{Reg}(u)$ is the set of free boundary points where blow-ups are $(1+s)$-homogeneous. In [ACS08, CSS08], it was proved that $\mathrm{Reg}(u)$ is relatively open, that blow-ups at points in $\mathrm{Reg}(u)$ are unique, and that $\mathrm{Reg}(u)$ is an $(n-1)$-dimensional $C^{1,\alpha}$ submanifold of the thin space (it is analytic, in fact, as proved in [KRS19]).

$\mathrm{Sing}(u)$ is the set of points in $\Gamma(u)$ where the contact set has zero $\mathcal{H}^n$-density,

$$\mathrm{Sing}(u) := \left\{ X_\circ \in \Gamma(u) : \lim_{r\downarrow 0} \frac{\mathcal{H}^n(\Lambda(u) \cap B_r(X_\circ))}{r^n} = 0 \right\}.$$

In [GP09, GR19], Garofalo and Petrosyan and Garofalo and Ros-Oton, for $a = 0$ and $a \in (-1, 1)$ respectively, proved that the points of $\mathrm{Sing}(u)$ are those at which blow-ups are evenly homogeneous and unique. In addition, they showed that $\mathrm{Sing}(u)$ is contained in the countable union of m-dimensional C^1 manifolds with m ranging from 0 to $n-1$. (The regularity of the covering manifolds was later improved to a more quantitative $C^{1,\log}$ in [CSV19] when $a = 0$.) The goal of this manuscript is to achieve a better understanding of singular points.

Finally, $\mathrm{Other}(u)$ is the remainder of the free boundary, and is not yet fully characterized. That said, in [FoSp18], Focardi and Spadaro proved that $\Gamma(u)$, in particular, $\mathrm{Other}(u)$, has finite $(n-1)$-dimensional Minkowski content, which implies that the free boundary is $\mathcal{H}^{n-1}$-rectifiable. Moreover, they showed that outside of an at most Hausdorff $(n-2)$-dimensional subset of $\Gamma(u)$, the possible homogeneities of blow-ups take values in $\{2k, 2k-1+s, 2k+2s\}_{k\in\mathbb{N}}$ (the same result was proved for $a = 0$ by Krummel and Wickramasekera in [KW13]).

The Non-degenerate Problem

We have already seen that the study of the thin obstacle problem for an analytic obstacle can be reduced to the study of the thin obstacle problem for the zero obstacle, (5.5). An alternative normalization is to reduce to the zero boundary data case by subtracting off the a-harmonic extension of g to B_1. Indeed, for simplicity, let g be its own a-harmonic extension to B_1, i.e., assume that g is defined on $\overline{B_1}$ and $L_a g = 0$ in B_1. Then, $u - g$ solves (5.1) with zero boundary data and obstacle $\varphi_g := (\varphi - g)|_{\{y=0\}}$. (This procedure does not require φ to be analytic.) Under this normalization, Barrios, Figalli, and Ros-Oton proved that if φ_g is strictly superharmonic, then

$$\lambda_{X_\circ} \in \{1+s, 2\},$$

for all $X_\circ \in \Gamma(u)$ (see [BFR18]). Consequently, we make the following definition.

Definition 5.1. We say that the thin obstacle problem (5.1) or, equivalently, (5.2) is non-degenerate if

$$\Delta_x \varphi_g \leq -c < 0 \quad \text{on} \quad B_1 \cap \{y = 0\}. \tag{5.8}$$

Analogously, we say the Euler–Lagrange equations (5.5) are non-degenerate if they arise from (5.1) or (5.2) satisfying (5.8); i.e., $\Delta_x \tilde{g} \geq c > 0$ on $B_1 \cap \{y = 0\}$, where $\tilde{g}$ denotes its own a-harmonic extension of $\tilde{g}$ to $\overline{B_1}$.

Remark 5.3. In the context of the obstacle problem for the fractional Laplacian in all of $\mathbb{R}^n$, (5.3), the problem is non-degenerate under the less restrictive assumption $\Delta\varphi \leq 0$ in $\{\varphi > 0\} \subset \mathbb{R}^n$.

5.1.3 Main Results

We are interested in studying the fine properties of u at points in $\mathrm{Sing}(u)$, in the spirit of the work of Figalli and Serra ([FS18]), wherein such a study is undertaken for the classical obstacle problem given obstacles with Laplacian identically equal to -1, i.e., under a non-degeneracy condition (cf. Definition 5.1). To do so, we establish a framework to better characterize the structure of singular points and the behavior of u around singular points: we develop a higher order expansion of u around singular points, which, up to lower dimensional sets, yields a more regular covering of $\mathrm{Sing}(u)$. Our approach and results are new even for the case $a = 0$.

Before stating our results, it will be convenient to expand our discussion of $\mathrm{Sing}(u)$ and the work of [GP09, GR19], and introduce some notation. Let

$$\Sigma_\kappa(u) := \{X_\circ \in \Gamma(u) : \lambda_{X_\circ} = \kappa\}$$

denote the set of free boundary points where the homogeneity of blow-ups is κ. Consequently,

$$\mathrm{Sing}(u) = \bigcup_{\kappa \in 2\mathbb{N}} \Sigma_\kappa(u). \tag{5.9}$$

As noted, in [GP09, GR19], the authors showed that one and only one blow-up exists, which is evenly homogeneous, at each singular point. In fact, they proved much more: the unique blow-up at a singular point is a non-trivial, a-harmonic, evenly homogeneous polynomial that is even in y and non-negative on the thin space. In other words, blow-ups at singular points belong to the set of polynomials

$$\mathscr{P}_\kappa := \{p : L_a p = 0,\ X \cdot \nabla p(X) = \kappa p(X),\ p(x,0) \geq 0,\ p(x,-y) = p(x,y)\}$$

for $\kappa \in 2\mathbb{N}$. Furthermore, they produce the first term in the expansion of u around $X_\circ \in \Sigma_\kappa(u) \subset \mathrm{Sing}(u)$; they show that

$$\frac{\tilde{u}(X_\circ + r\,\cdot\,)}{r^\kappa} \to p_{*,X_\circ} \in \mathscr{P}_\kappa \quad \text{locally uniformly as} \quad r \downarrow 0. \tag{5.10}$$

The polynomial $p_{*,X_\circ}$, which we call the *first blow-up of u at $X_\circ$*, is a constant (non-zero) multiple of the blow-up of u at $X_\circ$ given by the rescalings (5.7). With the rescalings (5.10), we have

$$\tilde{u}(X) = p_{*,X_\circ}(X - X_\circ) + o(|X - X_\circ|^\kappa). \tag{5.11}$$

Finally, consider

$$L(p_{*,X_\circ}) := \{\xi \in \mathbb{R}^n : \xi \cdot \nabla_x p_{*,X_\circ}(x,0) = 0 \text{ for all } x \in \mathbb{R}^n\}$$

the *invariant set* or *spine* of $p_{*,X_\circ}$ on $\{y = 0\}$ as well as

$$m_{X_\circ} := \dim L(p_{*,X_\circ}).$$

Observe that $L(p_{*,X_\circ})$ is a linear subspace of $\mathbb{R}^n$. Also, since $p_{*,X_\circ} \not\equiv 0$ on $\mathbb{R}^n \times \{0\}$,

$$m_{X_\circ} \in \{0, 1, \dots, n-1\},$$

and this number accounts for the dimension of the contact set around a singular point. Thus, the singular set can be further stratified:

$$\mathrm{Sing}(u) = \bigcup_{\kappa \in 2\mathbb{N}} \bigcup_{m=0}^{n-1} \Sigma_\kappa^m(u) \quad \text{where} \quad \Sigma_\kappa^m(u) := \{X_\circ \in \Sigma_\kappa(u) : m_{X_\circ} = m\}. \tag{5.12}$$

In particular, by [BFR18], if the problem is non-degenerate (see Definition 5.1), then

$$\Gamma(u) = \mathrm{Reg}(u) \cup \mathrm{Sing}(u) = \mathrm{Reg}(u) \cup \Sigma_2(u) = \mathrm{Reg}(u) \cup \bigcup_{m=0}^{n-1} \Sigma_2^m(u).$$

Now we are ready to present the main results of this work. First, given a non-degenerate obstacle, we prove that each m-dimensional component of $\mathrm{Sing}(u)$ can be locally covered by a single C^2 manifold outside a lower dimensional set:

Theorem 5.1. *Let u solve* (5.1) *in the non-degenerate case (see Definition 5.1). Then,*

(i) $\Sigma_2^0(u)$ *is isolated in* $\mathrm{Sing}(u) = \Sigma_2^0(u) \cup \cdots \cup \Sigma_2^{n-1}(u)$.

(ii) *There exists an at most countable set* $E_1 \subset \Sigma_2^1(u)$ *such that* $\Sigma_2^1(u) \setminus E_1$ *is locally contained in a single one-dimensional* C^2 *manifold.*

(iii) *For each* $m \in \{2, \ldots, n-1\}$, *there exists a set* $E_m \subset \Sigma_2^m(u)$ *of Hausdorff dimension at most* $m-1$ *such that* $\Sigma_2^m(u) \setminus E_m$ *is locally contained in a single* m*-dimensional* C^2 *manifold.*

(iv) *If* $a \in (-1, 0)$, $\Sigma_2^{n-1}(u)$ *is locally contained in a single* $(n-1)$*-dimensional* $C^{1,\alpha}$ *manifold, for some* $\alpha > 0$ *depending only on* n *and* a.

The framework we develop in order to prove Theorem 5.1 is rather robust, and only sees the non-degeneracy condition (5.8) superficially. As a result, we can suitably extend Theorem 5.1 to the bulk of $\mathrm{Sing}(u)$, the top stratum $\Sigma^{n-1}(u) := \bigcup_{\kappa \in 2\mathbb{N}} \Sigma_\kappa^{n-1}(u)$, in the general case. Recall that the lower stratum $\Sigma^{<n-1}(u) := \mathrm{Sing}(u) \setminus \Sigma^{n-1}(u)$ is strictly lower dimensional; it is contained in the countable union of $(n-2)$-dimensional C^1 manifolds. More precisely, we prove

Theorem 5.2. *Let* u *solve* (5.1). *Then,*

(i) $\Sigma_2^0(u)$ *is isolated in* $\mathrm{Sing}(u) = \bigcup_{\kappa \in 2\mathbb{N}} \bigcup_{m=0}^{n-1} \Sigma_\kappa^m(u)$.

(ii) *There exists an at most countable set* $E_{2,1} \subset \Sigma_2^1(u)$ *such that* $\Sigma_2^1(u) \setminus E_{2,1}$ *is contained in the countable union of one-dimensional* C^2 *manifolds.*

(iii) *For each* $m \in \{2, \ldots, n-1\}$, *there exists a set* $E_{2,m} \subset \Sigma_2^m(u)$ *of Hausdorff dimension at most* $m-1$ *such that* $\Sigma_2^m(u) \setminus E_{2,m}$ *is contained in the countable union of* m*-dimensional* C^2 *manifolds.*

Moreover, for each $\kappa \in 2\mathbb{N}$,

(iv) *If* $n = 2$, *there exists an at most countable set* $E_{\kappa,1} \subset \Sigma_\kappa^1(u)$ *such that* $\Sigma_\kappa^1(u) \setminus E_{\kappa,1}$ *is contained in the countable union of* 1*-dimensional* C^2 *manifolds.*

(v) *If* $n \geq 3$, *there exists a set* $E_{\kappa,n-1} \subset \Sigma_\kappa^{n-1}(u)$ *of Hausdorff dimension at most* $n-2$ *such that* $\Sigma_\kappa^{n-1}(u) \setminus E_{\kappa,n-1}$ *is contained in the countable union of* $(n-1)$*-dimensional* C^2 *manifolds.*

(vi) *If* $n \geq 2$ *and* $a \in (-1, 0)$, $\Sigma_\kappa^{n-1}(u)$ *can be covered by a countable union of* $(n-1)$*-dimensional* C^{1,α_κ} *manifolds, for some* $\alpha_\kappa > 0$ *depending only on* n, a, *and* κ.

Remark 5.4. Notice that from the lower-dimensionality of $\Sigma_\kappa^{<n-1}(u)$, by Theorem 5.2(iv) and (v), we find that the whole singular set can be covered by countably many $(n-1)$-dimensional C^2 manifolds up to a lower dimensional subset.

Remark 5.5. When $n = 1$, it is well-known that singular points are isolated. Recall that $\tilde{u}(X_\circ + \cdot) = p_{*,X_\circ} + o(|X|^\kappa)$ if $X_\circ \in \Sigma_\kappa(u)$. Since $n = 1$, $p_{*,X_\circ} > 0$ in a neighborhood of 0, so that $\tilde{u} > 0$ around $X_\circ$ and $X_\circ$ is isolated.

Before stating Theorem 5.2, we noted that our methods see the non-degeneracy of the problem superficially. Indeed, if we could show that $p_{*,X_\circ}$'s nodal set $\{(x,0) : p_{*,X_\circ}(x,0) = |\nabla_x p_{*,X_\circ}(x,0)| = 0\}$ and $p_{*,X_\circ}$'s spine align for every $X_\circ \in E_{\kappa,m}$ (see Section 5.7 (also 5.5) for a description of $E_{\kappa,m}$), then our analysis would immediately imply that $E_{\kappa,m}$ is lower dimensional, and $\Sigma_\kappa^m(u) \subset \mathrm{Sing}(u)$ is contained in a countable union of C^2 manifolds up to an $(m-1)$-dimensional subset for all $m \in \{0, \dots, n-1\}$, and not just when $m = n-1$.

We remark that due to potential accumulation of lower homogeneity singular points to higher homogeneity singular points, the countable covers of Theorem 5.2 cannot be improved to single covers, as done in the the non-degenerate setting, Theorem 5.1 (and also as done in [FS18]).

5.1.4 Strategy of the Proof

From this point forward, we do not distinguish u and $\tilde{u}$, as defined in (5.4) (or we assume that $\varphi \equiv 0$); we will always assume that we are in the normalized situation (5.5). Furthermore, in this section, whenever we discuss $\Sigma_\kappa(u)$, $\kappa \in 2\mathbb{N} = \{2, 4, 6, \dots\}$.

Theorems 5.1 and 5.2 are the culmination of a procedure that constructs the second term in the expansion of u at singular points, outside of a lower dimensional set. In order to study the higher infinitesimal behavior of u at $X_\circ \in \Sigma_\kappa(u)$, we, quite naturally, consider the rescalings

$$\tilde{v}_{X_\circ,r}(X) := \frac{v_{X_\circ}(rX)}{\left(\frac{1}{r^{n+a}}\int_{\partial B_r} v_{X_\circ}^2 |y|^a\right)^{1/2}} \quad \text{where} \quad v_{X_\circ}(X) := u(X_\circ + X) - p_{*,X_\circ}(X)$$

(cf. (5.7)).

First, we show that the set $\{\tilde{v}_{X_\circ,r}\}_{r>0}$ is weakly precompact in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}, |y|^a)$ and classify its limit points as $r \downarrow 0$ or blow-ups (see Sections 5.2 and 5.3):

Proposition 5.3. *Let u solve* (5.1), *and let $X_\circ \in \Sigma_\kappa^m(u)$ for $m \in \{0, \dots, n-1\}$.*

(i) If $a \in [0,1)$, the limit points of $\{\tilde{v}_{X_\circ,r}\}_{r>0}$ as $r \downarrow 0$ are $\lambda_{*,X_\circ}$-homogeneous, a-harmonic polynomials with $\lambda_{*,X_\circ} \geq \kappa$.

(ii) If $m < n-1$ and $\kappa = 2$, the limit points of $\{\tilde{v}_{X_\circ,r}\}_{r>0}$ as $r \downarrow 0$ are $\lambda_{*,X_\circ}$-homogeneous, a-harmonic polynomials with $\lambda_{*,X_\circ} \geq 2$.

(iii) If $m = n-1$ and $a \in (-1,0)$, the limit points of $\{\tilde{v}_{X_\circ,r}\}_{r>0}$ as $r \downarrow 0$ are $\lambda_{*,X_\circ}$-homogeneous, global solutions to the very thin obstacle problem (or fractional thin obstacle problem) (5.89) *on $L(p_{*,X_\circ}) \subset \mathbb{R}^n \times \{0\}$ with $\lambda_{*,X_\circ} \geq \kappa + \alpha_\kappa$, for some $\alpha_\kappa > 0$ depending only on n, a, and κ.*

As far as we know, Proposition 5.3 is the first instance of truly distinct behavior within our class of lower dimensional obstacle problems; in all previous studies of (5.1), the class parameterized by $a \in (-1,1)$ was treatable uniformly. The key difference is that if $a \geq 0$, subsets of the thin space $\{y = 0\}$ of Hausdorff dimension $n-1$ have zero $W^{1,2}(\mathbb{R}^{n+1}, |y|^a)$-capacity or a-harmonic capacity, while if $a < 0$, subsets of the thin space $\{y = 0\}$ of Hausdorff dimension $n-1$ have positive a-harmonic capacity. This capacitory distinction permits the formulation of, what we call, a very thin obstacle problem, i.e., a search for a weighted Dirichlet energy minimizer, as in (5.1), within a class of functions constrained to sit above a given function defined on an $(n-1)$-dimensional submanifold of $\mathbb{R}^n \times \{0\}$ (see Section 5.8), or, equivalently, a lower dimensional obstacle problem for the fractional Laplacian $(-\Delta)^s$ where $s > \frac{1}{2}$ (see Section 5.9 and cf. Section 5.1.1).

We remark that the above classification in the case $a < 0$ is analogous to the classification found in [FS18], wherein Figalli and Serra consider the classical obstacle problem. There, the analogous blow-ups in the top stratum of the singular set are global, homogeneous solutions to the thin obstacle problem (5.1) with zero obstacle and $a = 0$. And in the lower stratum of the singular set, the analogous blow-ups set are homogeneous, harmonic polynomials. That said, while Figalli and Serra could rely on developed theory (for the thin obstacle problem) for their analysis, we cannot; the very thin obstacle problem has, until now, been unstudied (Section 5.8).

Given Proposition 5.3 and our desire to produce the next term in the expansion of u at $X_\circ$, we then show that collection of points for which $\lambda_{*,X_\circ} \in [\kappa, \kappa+1)$ is lower dimensional (for $\kappa = 2$ or $m = n-1$). More specifically, if we define

$$\Sigma_\kappa^{m,\mathrm{a}}(u) := \{X_\circ \in \Sigma_\kappa^m(u) : \lambda_{*,X_\circ} \in [\kappa, \kappa+1)\},$$

then we have the following proposition.

Proposition 5.4. *Let u solve* (5.1)*. Then,*

(i) $\Sigma_2^{0,\mathrm{a}}(u)$ *is empty.*

(ii) For each $m \in \{1\dots, n-1\}$, $\Sigma_2^{m,\mathrm{a}}(u)$ *has Hausdorff dimension at most* $m-1$.

(iii) For each $\kappa \in 2\mathbb{N}$, $\Sigma_\kappa^{n-1,\mathrm{a}}(u)$ *has Hausdorff dimension at most* $n-2$.

Remark 5.6. In fact, we can show that for $n = 2$, if $a \in (-1,0)$, then $\Sigma_\kappa^{1,\mathrm{a}}(u)$ is countable; and if $a \in [0,1)$, then $\Sigma_\kappa^{1,\mathrm{a}}(u)$ is discrete. Moreover, for $n \geq 3$, $\Sigma_2^{1,\mathrm{a}}(u)$ is discrete.

In turn, we call $\Sigma_\kappa^{m,\mathrm{a}}(u)$ the set of *anomalous* points of $\Sigma_\kappa^m(u)$ and

$$\Sigma_\kappa^{m,\mathrm{g}}(u) := \Sigma_\kappa^m(u) \setminus \Sigma_\kappa^{m,\mathrm{a}}(u)$$

the *generic* points of $\Sigma_\kappa^m(u)$ (cf. [FS18]). (See Sections 5.4 and 5.5.) In order to prove Proposition 5.4, we use two Federer-type dimension reduction arguments. When $a \geq 0$ or $m < n-1$, we argue as in [FS18], while when $a < 0$ and $m = n-1$, we adopt the arguments pioneered in [FRS19].

After the statement of Theorem 5.2, we remarked that if the nodal set and spine of $p_{*,X_\circ}$ were aligned for each $X_\circ \in E_{\kappa,m}$, then Theorem 5.2 would immediately hold for all $m \in \{0,\dots,n-1\}$ and all $\kappa \in 2\mathbb{N}$. (Notice that this alignment is always true when $m \in \{0,\dots,n-1\}$ if $\kappa = 2$, but only when $m = n-1$ if $\kappa > 2$.) Another way to understand this remark is as follows. If the nodal set and spine of $p_{*,X_\circ}$ were aligned for each $X_\circ \in \Sigma_\kappa^{m,\mathrm{a}}(u)$, then our analysis would directly show that $\Sigma_\kappa^{m,\mathrm{a}}(u)$ is at most $(m-1)$-dimensional (in the Hausdorff sense), extending

Proposition 5.4 to every (κ, m) pair. Hence, Theorem 5.2 would immediately hold for all $m \in \{0, \ldots, n-1\}$ and all $\kappa \in 2\mathbb{N}$ since every other aspect of our analysis is indifferent to this issue. Nonetheless, it is unclear if such a statement is true; in fact, Remark 5.16 indicates (but does not prove) the opposite.

Thanks to Propositions 5.3 and 5.4, and Whitney's Extension Theorem, generic points are contained in the countable union of $C^{1,1}$ manifolds; and so, we have the following result, which is Theorem 5.2, but with $C^{1,1}$ coverings.

Theorem 5.5. *Let u solve* (5.1). *Then,*

(i) $\Sigma_2^0(u)$ *is isolated in* $\mathrm{Sing}(u) = \bigcup_{\kappa \in 2\mathbb{N}} \bigcup_{m=0}^{n-1} \Sigma_\kappa^m(u)$.

(ii) For each $m \in \{1, \ldots, n-1\}$, $\Sigma_2^m(u) \setminus \Sigma_2^{m,\mathrm{a}}(u)$ *is contained in the countable union of m-dimensional $C^{1,1}$ manifolds, where* $\dim_{\mathcal{H}} \Sigma_2^{m,\mathrm{a}}(u) \leq m-1$.

Moreover, for each $\kappa \in 2\mathbb{N}$,

(iii) $\Sigma_\kappa^{n-1}(u) \setminus \Sigma_\kappa^{n-1,\mathrm{a}}(u)$ *is contained in the countable union of $(n-1)$-dimensional $C^{1,1}$ manifolds, where* $\dim_{\mathcal{H}} \Sigma_\kappa^{n-1,\mathrm{a}}(u) \leq n-2$.

(iv) In addition, if $a \in (-1, 0)$, *each* $\Sigma_\kappa^{n-1}(u)$ *can be covered by a countable union of $(n-1)$-dimensional C^{1,α_κ} manifolds, for some $\alpha_\kappa > 0$ depending only on n, a, and κ.*

(See Section 5.6.) We refer to Remark 5.6 for the size of the anomalous set in the cases $n = 2$ and $m = 1$, which corresponds to parts (ii) and (iv) of Theorem 5.2. Just as Theorem 5.5 is a $C^{1,1}$ precursor to Theorem 5.2, we note that a $C^{1,1}$ precursor to Theorem 5.1 also holds.

To conclude the proofs of our main results and produce the next term in the expansion of u outside a lower dimensional set (and go from $C^{1,1}$ to C^2 covering manifolds), we prove that outside of an at most $(m-1)$-dimensional (in the Hausdorff sense) subset of $\Sigma_\kappa^{m,\mathrm{g}}(u)$, when $\kappa = 2$ and $m \in \{0, \ldots, n-1\}$ as well as when $\kappa > 2$ and $m = n-1$, the blow-ups classified in Proposition 5.3 are $(\kappa+1)$-homogeneous polynomials, and not just higher homogeneous, global solutions to a codimension two obstacle problem. In particular, we show that

$$\frac{v_{X_\circ}(r\,\cdot)}{r^{\kappa+1}} \to q_{*,X_\circ} \quad \text{locally uniformly as} \quad r \downarrow 0$$

where $q_{*,X_\circ}$ is a $(\kappa+1)$-homogeneous, a-harmonic polynomial at all but strictly lower dimensional set of $X_\circ \in \Sigma_\kappa^{m,\mathrm{g}}(u)$, again, when $\kappa = 2$ and $m \in \{0, \ldots, n-1\}$ as well as when $\kappa > 2$ and $m = n-1$. (See Section 5.7.)

5.1.5 Notation

We define the balls

$$\begin{aligned} B_r(X_\circ) &:= \{X \in \mathbb{R}^{n+1} : |X - X_\circ| < r\}, \\ B_r^*(x_\circ) &:= \{x \in \mathbb{R}^n : |x - x_\circ| < r\}, \\ B_r'(x_\circ') &:= \{x' \in \mathbb{R}^{n-1} : |x' - x_\circ'| < r\}, \end{aligned}$$

i.e., the balls of radius r centered at $X_\circ$, $x_\circ$, and $x'_\circ$ in $\mathbb{R}^{n+1}$, $\mathbb{R}^n$, and $\mathbb{R}^{n-1}$ respectively. We will also denote $B_r := B_r(0)$, $B^*_r := B^*_r(0)$, and $B'_r := B'_r(0)$. Similarly, we let

$$D_r \subset \mathbb{R}^2$$

be the disc of radius $r > 0$, centered at the origin.

For a polynomial $p : \mathbb{R}^n \to \mathbb{R}$, consider

$$\mathrm{Ext}_a(p) := p + \sum_{j=1}^{\infty} c_j \frac{(-1)^j}{(2j)!} y^{2j} \Delta_x^j p \quad \text{with} \quad c_j := \prod_{i=1}^{j} \frac{2i-1}{2i-1-a}. \tag{5.13}$$

Notice that $\mathrm{Ext}_a(p) : \mathbb{R}^{n+1} \to \mathbb{R}$ is the unique even in y, a-harmonic extension of p to $\mathbb{R}^{n+1}$ (see [GR19, Lemma 5.2]); $L_a(\mathrm{Ext}_a(p)) = 0$.

5.1.6 Structure of the Work

In Section 5.2, we introduce a collection of monotonicity formulae (in particular, Almgren's frequency function), and prove some basic but useful estimates. In Section 5.3, we start a blow-up analysis of the solution around singular points. We show the existence of second blow-ups and prove some facts about them. We also show Proposition 5.3 holds. In Section 5.4, we gather some important lemmas regarding the accumulation of singular points, which are then used to study the size of the anomalous set in Section 5.5. Whence, we prove Proposition 5.4 and Remark 5.6. In Section 5.6, we show that the set of generic points is contained in a countable union of $C^{1,1}$ manifolds, which combined with previous results yields the proof of Theorem 5.5. Finally, we conclude the proofs of our main results in Section 5.7, Theorems 5.1 and 5.2, by studying the case of $(\kappa+1)$-homogeneous, a-harmonic second blow-ups. Specifically, we show that those points at which the second-blow up is not the next order term in the expansion are collectively lower-dimensional. Finally, Section 5.8 is dedicated to studying the very thin obstacle problem. Here, we prove the estimates and claims on the very thin obstacle problem made use of throughout the work. In Section 5.9, we make a final remark on global obstacle problems.

5.2 Monotonicity Formulae and Preliminary Results

We recall that we will always assume that we are dealing with the zero obstacle case (5.5).

Let $X_\circ$ be a singular point for u of order $\kappa_{X_\circ} \in 2\mathbb{N} := \{2, 4, 6, \dots\}$, and let $p_{*,X_\circ}$ be the (unique) first blow-up of u at $X_\circ$,

$$p_{*,X_\circ}(X) := \lim_{r\downarrow 0} \frac{u(X_\circ + rX)}{r^{\kappa_{X_\circ}}} \tag{5.14}$$

(see (5.10)). Recall that $p_{*,X_\circ} \in \mathscr{P}_{\kappa_{X_\circ}}$, i.e., it is an a-harmonic, $\kappa_{X_\circ}$-homogeneous polynomial, non-negative on the thin space, and even in y, and $\kappa_{X_\circ}$ is equal to

Almgren's frequency of u at the $X_\circ$:

$$\kappa_{X_\circ} = N(0^+, u, X_\circ) := \lim_{r\downarrow 0} \frac{r \int_{B_r(X_\circ)} |\nabla u|^2 |y|^a}{\int_{\partial B_r(X_\circ)} u^2 |y|^a}$$

(see [ACS08, CSS08, GP09, GR19]).

We often assume that $X_\circ = 0$ (which we can do without loss of generality after a translation), and we let $p_* := p_{*,0}$. In particular, define

$$v_* := u - p_*,$$

and set

$$\kappa_* := \kappa_0, \quad L_* := L(p_*), \quad \text{and} \quad m_* := m_0, \tag{5.15}$$

so that m_* is the dimension of the spine of p_* in $\{y = 0\}$, L_*, which is κ_*-homogeneous.

Let, for $\kappa \in 2\mathbb{N}$,

$$p \in \mathscr{P}_\kappa \quad \text{and} \quad v = u - p,$$

and observe that

$$v\, L_a v = -p\, L_a u \geq 0. \tag{5.16}$$

Since $L_a u(x, y) = 2 \lim_{y\downarrow 0} y^a \partial_y u(x, y) \mathcal{H}^{n-1} \llcorner \Lambda(u) \leq 0$, $v\, L_a v$ is non-negative as soon as p is non-negative on $\Lambda(u) \setminus \mathcal{N}(u)$ where

$$\mathcal{N}(u) := \{(x, 0) : u(x, 0) = |\nabla_x u(x, 0)| = \lim_{y\downarrow 0} y^a \partial_y u(x, y) = 0\}. \tag{5.17}$$

The set $\mathcal{N}(u)$ is called the *nodal set* of u.

Remark 5.7. Notice that $v = u - p$ is a solution to the thin obstacle problem with obstacle $\varphi = -p|_{B_1 \cap \{y=0\}}$ and subject to its own boundary data. (This follows easily by Remark 5.1.)

The goal of this section is to prove monotonicity-type results and estimates for $v = u - p$ for any $p \in \mathscr{P}_\kappa$. We stress that κ might not be equal to κ_*, and so we will sometimes write $N(0^+, u) := N(0^+, u, 0)$ instead. Yet we will most often apply these results and estimates to v_*.

5.2.1 Monotonicity Formulae

To begin we study Almgren's frequency function on v at the origin, and prove that it is non-decreasing provided that $\kappa \leq \kappa_* = N(0^+, u)$.

Proposition 5.6. *Suppose that $\kappa \leq N(0^+, u)$, and let $v = u - p$ for $p \in \mathscr{P}_\kappa$. Then, Almgren's frequency function on v*

$$r \mapsto N(r, v) = \frac{r \int_{B_r} |\nabla v|^2 |y|^a}{\int_{\partial B_r} v^2 |y|^a}$$

is non-decreasing. Moreover, $N(0^+, v) \geq \kappa$.

Before proceeding with the proof of Proposition 5.6, let us recall a few definitions and facts. Let $W_\lambda(r,u)$ denote the λ-*Weiss energy of* u *at* r:

$$W_\lambda(r,u) := \frac{1}{r^{2\lambda}} D(r,u) - \frac{\lambda}{r^{2\lambda}} H(r,u) \tag{5.18}$$

where

$$D(r,u) := \frac{1}{r^{n+a-1}} \int_{B_r} |\nabla u|^2 |y|^a = r^2 \int_{B_1} |\nabla u(rX)|^2 |y|^a \tag{5.19}$$

and

$$H(r,u) := \frac{1}{r^{n+a}} \int_{\partial B_r} u^2 |y|^a = \int_{\partial B_1} u(rX)^2 |y|^a. \tag{5.20}$$

By [GR19, Theorem 2.11], we have that $N(r,u)$ is non-decreasing, from which, we immediately deduce that

$$W_\kappa(r,u) = \frac{H(r,u)}{r^{2\kappa}} (N(r,u) - \kappa) \geq 0 \tag{5.21}$$

(recall $N(0^+,u) \geq \kappa$). In turn, we have the following lemma:

Lemma 5.7. *Suppose that* $\kappa \leq N(0^+,u)$*, and let* $v = u - p$ *for* $p \in \mathscr{P}_\kappa$*. Then,*

$$\frac{1}{r^{n-1+a+2\kappa}} \int_{B_r} |\nabla v|^2 |y|^a \geq \frac{\kappa}{r^{n+a+2\kappa}} \int_{\partial B_r} v^2 |y|^a \tag{5.22}$$

and

$$\frac{1}{r^{n+a+2\kappa}} \int_{\partial B_r} v(X \cdot \nabla v - \kappa v) |y|^a \geq \frac{1}{r^{n-1+a+2\kappa}} \int_{B_r} v\, L_a v. \tag{5.23}$$

Proof. We proceed as in the proof of [GP09, Theorem 1.4.3]. By [GR19, Theorem 2.11], $N(r,p) \equiv \kappa$, from which it follows that $W_\kappa(r,p) \equiv 0$. Using (5.21) and integrating by parts, we immediately have that

$$\begin{aligned}
0 &\leq W_\kappa(r,u) - W_\kappa(r,p) \\
&= \frac{1}{r^{n-1+a+2\kappa}} \int_{B_r} \left(|\nabla v|^2 + 2\nabla v \cdot \nabla p\right) |y|^a - \frac{\kappa}{r^{n+a+2\kappa}} \int_{\partial B_r} \left(v^2 + 2vp\right) |y|^a \\
&= \frac{1}{r^{n-1+a+2\kappa}} \int_{B_r} |\nabla v|^2 |y|^a - \frac{\kappa}{r^{n+a+2\kappa}} \int_{\partial B_r} v^2 |y|^a + \frac{2}{r^{n+a+2\kappa}} \int_{\partial B_r} v(X \cdot \nabla p - \kappa p) |y|^a \\
&= \frac{1}{r^{n-1+a+2\kappa}} \int_{B_r} |\nabla v|^2 |y|^a - \frac{\kappa}{r^{n+a+2\kappa}} \int_{\partial B_r} v^2 |y|^a,
\end{aligned}$$

which directly yields (5.22). Continuing, integrating by parts again, we get

$$\begin{aligned}
&\frac{1}{r^{n-1+a+2\kappa}} \int_{B_r} |\nabla v|^2 |y|^a - \frac{\kappa}{r^{n+a+2\kappa}} \int_{\partial B_r} v^2 |y|^a \\
&\qquad = -\frac{1}{r^{n-1+a+2\kappa}} \int_{B_r} v\, L_a v + \frac{1}{r^{n+a+2\kappa}} \int_{\partial B_r} v(X \cdot \nabla v - \kappa v) |y|^a,
\end{aligned}$$

which implies (5.23). $\square$

With Lemma 5.7 in hand, we can now prove Proposition 5.6.

Proof of Proposition 5.6. Notice that

$$N(r, v) = \frac{D(r, v)}{H(r, v)},$$

where D and H are given by (5.19) and (5.20). By scaling (namely, $N(\rho, u_r) = N(r\rho, u)$, for the rescaling (5.7)), it is enough to show $N'(1, v) \geq 0$ or, equivalently, that

$$D'(1)H(1) - H'(1)D(1) \geq 0, \tag{5.24}$$

where we have let $D(1) = D(1, v)$ and $H(1) = H(1, v)$.

We compute $D'(1)$ and $H'(1)$. First,

$$\begin{aligned} D'(1) &= 2\int_{B_1} |\nabla v|^2|y|^a + 2\int_{B_1} \nabla v \cdot D^2 v \cdot X\,|y|^a \\ &= 2\int_{B_1} \nabla v \cdot \nabla(X \cdot \nabla v)|y|^a \\ &= 2\int_{\partial B_1} v_\nu^2|y|^a - 2\int_{B_1} L_a u\,(X \cdot \nabla u) + 2\int_{B_1} L_a u\,(X \cdot \nabla p), \end{aligned}$$

using integration by parts and that p is a-harmonic. Now notice that, by the regularity of the solution, $L_a u\,(X \cdot \nabla u) \equiv 0$. This, together with the fact that p is κ-homogeneous, yields

$$D'(1) = 2\int_{\partial B_1} v_\nu^2|y|^a + 2\kappa\int_{B_1} p\,L_a u = 2\int_{\partial B_1} v_\nu^2|y|^a - 2\kappa\int_{B_1} v\,L_a v,$$

where the last inequality follows by (5.16). On the other hand,

$$H'(1) = 2\int_{\partial B_1} v v_\nu|y|^a.$$

Now letting

$$I := \int_{B_1} v\,L_a v$$

and using

$$\int_{B_1} |\nabla v|^2|y|^a = \int_{\partial B_1} v v_\nu|y|^a - I,$$

in addition to the Cauchy–Schwarz inequality, we find that

$$\begin{aligned} &D'(1)H(1) - H'(1)D(1) \\ &\quad = \left(2\int_{\partial B_1} v_\nu^2|y|^a - 2\kappa I\right)\int_{\partial B_1} v^2|y|^a - 2\int_{\partial B_1} v v_\nu|y|^a\left(\int_{\partial B_1} v v_\nu|y|^a - I\right) \\ &\quad = 2\left(\int_{\partial B_1} v_\nu^2|y|^a\int_{\partial B_1} v^2|y|^a - \kappa I\int_{\partial B_1} v^2|y|^a - \left(\int_{\partial B_1} v v_\nu|y|^a\right)^2 + I\int_{\partial B_1} v v_\nu|y|^a\right) \\ &\quad \geq -2\kappa I\int_{\partial B_1} v^2|y|^a + 2I\int_{\partial B_1} v v_\nu|y|^a \\ &\quad = 2I\int_{\partial B_1} v(X \cdot \nabla v - \kappa v)|y|^a. \end{aligned}$$

Hence, by (5.16) and (5.23), we deduce that (5.24) holds, as desired. □

We end the subsection with a lemma on a Monneau-type monotonicity statement and Weiss-type monotonicity statement, arguing as in [FS18, Lemma 2.6 and 2.8], and a important Monneau-type limit.

Lemma 5.8. *Suppose that $\kappa \le N(0^+, u)$, and let $v = u - p$ for $p \in \mathscr{P}_\kappa$. Given $\lambda > 0$, define*

$$H_\lambda(r, v) := \frac{1}{r^{n+a+2\lambda}} \int_{\partial B_r} v^2 |y|^a = \frac{1}{r^{2\lambda}} H(r, v). \tag{5.25}$$

Then, $r \mapsto H_\lambda(r, v)$ is non-decreasing for all $0 \le \lambda \le N(0^+, v)$. Moreover, the λ-Weiss energy

$$r \mapsto W_\lambda(r, v)$$

on v is also non-decreasing for all $\lambda > 0$.

Proof. Let $v_r(X) := (u - p)(rX)$; then,

$$\frac{H'_\lambda}{H_\lambda}(r, v) = \frac{2r \int_{\partial B_1} v_r(X)(X \cdot \nabla v(rX))|y|^a - 2\lambda \int_{\partial B_1} v_r^2 |y|^a}{r \int_{\partial B_1} v_r^2 |y|^a}.$$

Notice also that

$$r \int_{\partial B_1} v_r(X)(X \cdot \nabla v(rX))|y|^a = \int_{\partial B_1} v_r (X \cdot \nabla v_r)|y|^a = \int_{B_1} |\nabla v_r|^2 |y|^a + \int_{B_1} v_r \, L_a v_r,$$

and $v_r \, L_a v_r \ge 0$ (see (5.16)). Hence, since $N(1, v_r) = N(r, v)$,

$$\frac{H'_\lambda}{H_\lambda}(r, v) \ge \frac{2}{r}\left(N(r, v) - \lambda\right). \tag{5.26}$$

Now using that $N(r, v) \ge N(0^+, v) \ge \lambda$, we reach the desired result, (5.25).

To see the monotonicity of $W_\lambda(r, v)$ for $0 \le \lambda \le N(0^+, v)$, we simply combine the expressions (5.21) and (5.25), so that $W_\lambda(r, v)$ is product of two non-decreasing non-negative functions.

On the other hand, if $\lambda > N(0^+, v)$, a simple manipulation (see the proof of Proposition 5.6) yields

$$\begin{aligned} W'_\lambda(1) &= D'(1) - \lambda H'(1) - 2\lambda(D(1) - \lambda H(1)) \\ &= 2 \int_{\partial B_1} (v_\nu - \lambda v)^2 |y|^a + 2(\lambda - \kappa) \int_{B_1} v \, L_a v. \end{aligned}$$

As $v \, L_a v \ge 0$ and $\lambda > N(0^+, v) \ge \kappa$ (by Proposition 5.6), we conclude. □

Notice also that if we set

$$\lambda_* := N(0^+, v_*) \ge \kappa_* = N(0^+, u),$$

then

$$\lim_{r \downarrow 0} H_\lambda(r, v_*) = \infty \quad \text{for all} \quad \lambda > \lambda_*,$$

which follows arguing exactly as in [FS18, Corollary 2.9].

5.2.2 Estimates

Let us define, for any function f, the positive and negative parts as

$$f^+ := \max\{f, 0\} \quad \text{and} \quad f^- := \max\{-f, 0\} = -\min\{f, 0\}.$$

Hence, $f = f^+ - f^-$.

We start with an L^∞–L^2 estimate on v.

Lemma 5.9. *Let $v = u - p$ for $p \in \mathscr{P}_\kappa$. Then,*

$$\|v\|_{L^\infty(B_{1/2})} \le C\|v\|_{L^2(B_1,|y|^a)}, \tag{5.27}$$

for some constant C depending only on n and a.

Proof. Observe that v^- is sub a-harmonic in B_1 as the maximum of two sub a-harmonic functions in B_1.

Let us show that v^+ is also sub a-harmonic in B_1. To this end, first, by Remark 5.7, recall that v is the solution to (5.1) with $\varphi = -p|_{B_1\cap\{y=0\}}$ and its own boundary data. Now let η be any smooth compactly supported function in B_1 such $0 \le \eta \le 1$. In addition, let h_δ be an approximation of the Heaviside function: $h_\delta(t) = 0$ for $t \le 0$, $h_\delta(t) = t/\delta$ for $t \in (0, \delta)$, and $h_\delta(t) = 1$ for $t \ge \delta$. Finally, for $0 < \varepsilon < \delta$, define $v_\varepsilon := v - \varepsilon\eta h_\delta(v)$.

Since $p(x, 0) \ge 0$, observe that $v_\varepsilon(x, 0) \ge -p(x, 0)$ and $v_\varepsilon|_{\partial B_1} = v|_{\partial B_1}$. Therefore,

$$\int_{B_1} |\nabla v - \varepsilon\nabla(\eta h_\delta(v))|^2|y|^a \ge \int_{B_1} |\nabla v|^2|y|^a,$$

which implies that, after dividing through by ε and letting $\varepsilon \downarrow 0$,

$$\int_{B_1} \nabla v \cdot \nabla(\eta h_\delta(v))|y|^a \le 0.$$

Expanding,

$$\int_{B_1} h_\delta(v)\nabla v \cdot \nabla\eta|y|^a \le -\int_{B_1} \eta|\nabla v|^2 h_\delta'(v)|y|^a \le 0.$$

In turn, if $H_\delta' = h_\delta$ with $H_\delta(0) = 0$, then

$$\int_{B_1} \nabla(H_\delta(v)) \cdot \nabla\eta|y|^a \le 0.$$

(Obviously, H_δ here is not the Monneau-type function from Lemma 5.8.) Because η was arbitrary, we find that $H_\delta(v)$ is sub a-harmonic in B_1. So letting $\delta \downarrow 0$, we determine that v^+ is sub a-harmonic in B_1 ($H_\delta(v)$ is an approximation of v^+).

To conclude, see that by the local boundedness of subsolutions for L_a (see, e.g., [JN17, Proposition 2.1]), we have that

$$\sup_{B_{1/2}} v^\pm \le C\left(\int_{B_1} |v^\pm|^2|y|^a\right)^{1/2},$$

and (5.27) holds. □

Next, we prove Lipschitz and semiconvexity estimates on v along the spine of p. But before doing so, we prove a characterization lemma on the spine of a generic κ-homogeneous polynomial.

Lemma 5.10. *Let $\kappa \in \mathbb{N}$, and let $p : \mathbb{R}^n \to \mathbb{R}$ be a κ-homogeneous polynomial. Then, the following sets are equal.*

(i) $L(p) := \{\xi \in \mathbb{R}^n : \xi \cdot \nabla p(x) = 0$ *for all* $x \in \mathbb{R}^n\}$.

(ii) $I(p) := \{\xi \in \mathbb{R}^n : p(x+\xi) = p(x)$ *for all* $x \in \mathbb{R}^n\}$.

(iii) $D_{\kappa-1}(p) := \{\xi \in \mathbb{R}^n : D^\alpha p(\xi) = 0$ *for all* $\alpha = (\alpha_1, \dots, \alpha_n) : |\alpha| = \kappa - 1\}$.

Proof. We prove that (i) and (ii) as well as (ii) and (iii) are equivalent.

– $L(p) \subset I(p)$: Let $\xi \in L(p)$. Then,

$$p(x+\xi) = p(x) + \int_0^1 \xi \cdot \nabla p(x + t\xi)\,\mathrm{d}t = p(x).$$

– $I(p) \subset L(p)$: We start by noticing that $I(p)$ is actually a linear space, thanks to the homogeneity of p. Indeed, the additive property is clear; it is also clear that $-\xi \in I(p)$ if $\xi \in I(p)$. Now suppose $\xi \in I(p)$ and consider $\beta\xi$ for some $\beta > 0$. Then, $p(x + \beta\xi) = \beta^\kappa p(\beta^{-1}x + \xi) = \beta^\kappa p(\beta^{-1}x) = p(x)$ for all $x \in \mathbb{R}^n$, so that $\beta\xi \in I(p)$.

Let $\xi \in I(p)$. Now for all $h > 0$ and for all $x \in \mathbb{R}^n$, $p(x + h\xi) = p(x)$. Hence,

$$\xi \cdot \nabla_x p(x) = \lim_{h \downarrow 0} \frac{p(x + h\xi, 0) - p(x)}{h} = 0,$$

that is, $\xi \in L(p)$.

– $I(p) \subset D_{\kappa-1}(p)$: Let $\xi \in I(p)$. Then, $p(\xi + x) = p(x)$ and $D^\alpha p(x+\xi) = D^\alpha p(x)$ for any $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ with $|\alpha| = \kappa - 1$. Taking $x = 0$, we conclude thanks to the κ-homogeneity of p.

– $D_{\kappa-1}(p) \subset I(p)$. Let $\xi \in D_{\kappa-1}(p)$. Consider the degree κ polynomial $q(x) := p(x+\xi)$. Notice that from the definition of $D_{\kappa-1}$, q is homogeneous. Now let $\beta > 0$. Using the homogeneity of q and p,

$$p(x+\xi) = q(x) = \beta^\kappa q(\beta^{-1}x) = \beta^\kappa p(\beta^{-1}x + \xi) = p(x + \beta\xi)$$

for all $\beta > 0$. Taking $\beta \downarrow 0$, we see that $\xi \in I(p)$.

This concludes the proof. □

Notice that the equivalence of (i) and (ii) also holds for general κ-homogeneous functions.

Remark 5.8. Lemma 5.10 will be applied to $p(x, 0)$ for $p \in \mathscr{P}_\kappa$.

The following lemma shows that derivatives of v along the invariant set of p are bounded. Recall that $L(p)$ denotes the invariant set of $p(x, 0)$. The lemma is proved by means of a Bernstein's technique for integro-differential equations, as introduced by Cabré, Dipierro, and Valdinoci, in [CDV20].

Lemma 5.11. *Let $v = u - p$ for $p \in \mathscr{P}_\kappa$. Then, for all $\boldsymbol{e} \in L(p) \cap \mathbb{S}^n$,*

$$\|\partial_{\boldsymbol{e}} v\|_{L^\infty(B_{1/2})} \leq C\|v\|_{L^2(B_1,|y|^a)},$$

for some constant C depending only on n and a.

Proof. We proceed by Bernstein's technique (see [CDV20]). Let $\eta \in C_c^\infty(B_{1/2})$ be even in y and such that $\eta \equiv 1$ in $B_{1/4}$. Consider the function,

$$\psi := \eta^2(\partial_{\boldsymbol{e}} v)^2 + \mu v^2,$$

for some $\mu > 0$ to be chosen.

Since v is a-harmonic outside $\Lambda(u)$, in $B_{1/2} \setminus \Lambda(u)$,

$$L_a(v^2) = 2v\, L_a v + 2|\nabla v|^2|y|^a \geq 2|\nabla v|^2|y|^a.$$

Similarly, because $\partial_{\boldsymbol{e}} v$ is a-harmonic outside $\Lambda(u)$, we have that in $B_{1/2} \setminus \Lambda(u)$, $L_a \partial_{\boldsymbol{e}} v = L_a \partial_{\boldsymbol{e}} u = 0$. Therefore, we find that in $B_{1/2} \setminus \Lambda(u)$,

$$\begin{aligned}
L_a(\eta^2(\partial_{\boldsymbol{e}} v)^2) &= (\partial_{\boldsymbol{e}} v)^2 L_a(\eta^2) + \eta^2 L_a((\partial_{\boldsymbol{e}} v)^2) + 2|y|^a \nabla(\partial_{\boldsymbol{e}} v)^2 \cdot \nabla \eta^2 \\
&= (\partial_{\boldsymbol{e}} v)^2 L_a(\eta^2) + 2\eta^2 |\nabla \partial_{\boldsymbol{e}} v|^2 |y|^a + 2|y|^a \nabla(\partial_{\boldsymbol{e}} v)^2 \cdot \nabla \eta^2 \\
&\geq (\partial_{\boldsymbol{e}} v)^2 L_a(\eta^2) + 2\eta^2 |\nabla(\partial_{\boldsymbol{e}} v)|^2 |y|^a - 8|y|^a |\nabla \partial_{\boldsymbol{e}} v||\partial_{\boldsymbol{e}} v||\nabla \eta|\eta \\
&\geq |y|^a |\partial_{\boldsymbol{e}} v|^2 (|y|^{-a} L_a(\eta^2) - 8|\nabla \eta|^2)
\end{aligned}$$

where there last inequality follows from

$$\eta^2 |\nabla \partial_{\boldsymbol{e}} v|^2 + 4|\partial_{\boldsymbol{e}} v|^2 |\nabla \eta|^2 \geq 4|\partial_{\boldsymbol{e}} v||\nabla \partial_{\boldsymbol{e}} v|\eta|\nabla \eta|.$$

So in $B_{1/2} \setminus \Lambda(u)$,

$$\begin{aligned}
L_a \psi &\geq |y|^a |\partial_{\boldsymbol{e}} v|^2 (|y|^{-a} L_a(\eta^2) - 8|\nabla \eta|^2) + |y|^a |\nabla v|^2 2\mu \\
&\geq |y|^a |\nabla v|^2 (2\mu - |y|^{-a}|L_a(\eta^2)| - 8|\nabla \eta|^2).
\end{aligned}$$

Now as η is even in y and smooth, $|y|^{-a}|L_a(\eta^2)| + 8|\nabla \eta|^2 \leq C_\eta$ in $B_{1/2}$, from which we deduce that

$$L_a \psi \geq 0 \quad \text{in} \quad B_{1/2} \setminus \Lambda(u)$$

provided $2\mu \geq C_\eta$.

By the maximum principle then, ψ must attain its maximum at the boundary of $B_{1/2} \setminus \Lambda(u)$. Being that $\partial_{\boldsymbol{e}} p = \partial_{\boldsymbol{e}} u = 0$ on $\Lambda(u)$ and $\eta|_{\partial B_{1/2}} = 0$, $\psi = \mu v^2$ on $\partial B_{1/2} \cup \Lambda(u)$. Hence,

$$\sup_{B_{1/2}} \psi \leq \mu \sup_{B_{1/2}} v^2.$$

In particular, as $\eta \equiv 1$ on $B_{1/4}$,

$$\|\partial_{\boldsymbol{e}} v\|_{L^\infty(B_{1/4})} \leq \mu^{1/2} \|v\|_{L^\infty(B_{1/2})}.$$

Thus, by Lemma 5.9 and a covering argument, we find the desired estimate. $\square$

Finally, we show that v is semiconvex along the spine of p. Naturally, for $h > 0$, let

$$\delta^2_{\boldsymbol{e},h} f := \frac{f(\,\cdot\, + h\boldsymbol{e}) + f(\,\cdot\, - h\boldsymbol{e}) - 2f}{h^2}$$

be the second order h-incremental quotient of the function f in the direction $\boldsymbol{e} \in \mathbb{S}^n$.

Lemma 5.12. *Let $v = u - p$ for $p \in \mathscr{P}_\kappa$. Then, for all $\boldsymbol{e} \in L(p) \cap \mathbb{S}^n$,*

$$\inf_{B_{1/2}} \partial_{\boldsymbol{ee}} v \geq -C\|v\|_{L^2(B_1,|y|^a)},$$

for some constant C depending only on n and a.

Proof. For any $\gamma > 0$, let u_γ be the solution to

$$\begin{cases} u_\gamma(x,y) & \geq & 0 & \text{on } B_{7/8} \cap \{y = 0\} \\ L_a u_\gamma(x,y) & \leq & 0 & \text{in } B_{7/8} \\ L_a u_\gamma(x,y) & = & 0 & \text{in } B_{7/8} \setminus \Lambda(u_\gamma) \\ u_\gamma(x,y) & = & u(x,y) + \gamma & \text{on } \partial B_{7/8}. \end{cases} \tag{5.28}$$

That is, in $B_{7/8}$, u_γ is the solution to the thin obstacle problem with zero obstacle and boundary data $u+\gamma$. Notice that since u is continuous in B_1, we have that $u_\gamma \downarrow u$ uniformly in $B_{7/8}$, as $\gamma \downarrow 0$. Also, $u_\gamma > 0$ in $B_{7/8} \setminus B_{7/8-\beta}$ for some $\beta = \beta(\gamma) > 0$, by the continuity of u_γ. In particular, u_γ is a-harmonic in the annulus $B_{7/8} \setminus B_{7/8-\beta}$.

Consider the function

$$f_\gamma(x) := (\partial_{\boldsymbol{ee}} u_\gamma(x))^-$$

as the pointwise limit of $(\delta^2_{\boldsymbol{e},h} u(x))^-$ as $h \downarrow 0$. To do so, we define

$$g^\gamma_{\varepsilon,h,\boldsymbol{e}}(x) := \min\{\delta^2_{\boldsymbol{e},h} u_\gamma(x), -\varepsilon\}.$$

Observe that $L_a(\delta^2_{\boldsymbol{e},h} u_\gamma) \leq 0$ in $B_{7/8} \setminus \Lambda(u_\gamma)$ (since $L_a u_\gamma \leq 0$ in $B_{7/8}$ and $L_a u_\gamma = 0$ in $B_{7/8} \setminus \Lambda(u_\gamma)$). Moreover, since u_γ is continuous and $\delta^2_{\boldsymbol{e},h} u_\gamma \geq 0$ on $\Lambda(u_\gamma)$, we have $g^\gamma_{\varepsilon,h,\boldsymbol{e}} = -\varepsilon$ in a neighbourhood of $\Lambda(u_\gamma)$. Thus, $L_a g^\gamma_{\varepsilon,h,\boldsymbol{e}} \leq 0$ in $B_{7/8}$.

We now want to let $\varepsilon \downarrow 0$ and then $h \downarrow 0$ to deduce that $L_a f_\gamma \geq 0$ in $B_{3/4}$ and $f_\gamma \equiv 0$ on $\Lambda(u_\gamma)$. In order to pass $L_a g^\gamma_{\varepsilon,h,\boldsymbol{e}} \leq 0$ to the limit (as $\varepsilon, h \downarrow 0$), it is enough to show that $|g^\gamma_{\varepsilon,h,\boldsymbol{e}}| \leq C$ for some C independent of ε and h (but possibly depending on γ). As $g^\gamma_{\varepsilon,h,\boldsymbol{e}}$ is super-a-harmonic in $B_{7/8}$, its minimum must be achieved on the boundary. In particular, since $g^\gamma_{\varepsilon,h,\boldsymbol{e}} \leq 0$,

$$\sup_{B_{3/4}} |g^\gamma_{\varepsilon,h,\boldsymbol{e}}| \leq \sup_{\partial B_{7/8-\beta/2}} |g^\gamma_{\varepsilon,h,\boldsymbol{e}}| \leq C(\beta),$$

where in the last inequality, we have used that $g^\gamma_{\varepsilon,h,\boldsymbol{e}}$ is a-harmonic in $B_{7/8} \setminus B_{7/8-\beta}$ and corresponding C^2 estimates in the tangential direction for a-harmonic functions. Hence, we can indeed pass $L_a g^\gamma_{\varepsilon,h,\boldsymbol{e}} \leq 0$ in $B_{3/4}$ to the limit and obtain that $L_a f_\gamma \geq 0$ in $B_{3/4}$ and $f_\gamma \equiv 0$ on $\Lambda(u_\gamma)$.

With the sub-a-harmonicity and nonnegativity of f_γ in hand, it is easy to see that f_γ is continuous in $B_{3/4}$. Indeed, sub-a-harmonic functions are upper semi-continuous (see [HKM93, Theorem 3.63]). So being that f_γ is continuous when $f_\gamma > 0$ and f_γ is nonnegative in general, we determine the continuity of f_γ, as desired.

To conclude, we again proceed by Bernstein's technique (see [CDV20]). Let $\eta \in C_c^\infty(B_{1/2})$ be even in y and such that $\eta \equiv 1$ in $B_{1/4}$, and set

$$\psi_\gamma := \eta^2 f_\gamma^2 + \mu(\partial_{\boldsymbol{e}} u_\gamma)^2,$$

where we recall $f_\gamma(x) := (\partial_{\boldsymbol{ee}} u_\gamma(x))^-$. By the discussion above, ψ_γ is continuous in $B_{1/2}$. Recall that $f_\gamma \equiv 0$ on $\Lambda(u_\gamma)$, and therefore, $\psi_\gamma \equiv 0$ on $\Lambda(u_\gamma)$. On the other hand, on the boundary of $B_{1/4}$, we have that $\psi_\gamma = \mu(\partial_{\boldsymbol{e}} u_\gamma)^2$. Following the proof of Lemma 5.11 exactly (and using that $L_a f_\gamma \geq 0$ in $B_{3/4}$), we see that $L_a \psi_\gamma \geq 0$ in $B_{1/2} \setminus \Lambda(u_\gamma)$ if μ is large enough, and so, its maximum must be achieved at the boundary. In turn,

$$\|f_\gamma\|_{L^\infty(B_{1/4})} \leq \mu^{1/2} \|\partial_{\boldsymbol{e}} u_\gamma\|_{L^\infty(B_{1/2})} = \mu^{1/2} \|\partial_{\boldsymbol{e}}(u_\gamma - p)\|_{L^\infty(B_{1/2})} \leq C \|u_\gamma - p\|_{L^2(B_1, |y|^a)},$$

where we have used Lemma 5.11 in the last inequality. This implies the family $\{u_\gamma\}$, for $0 < \gamma \leq 1$, is uniformly semiconvex. Letting $\gamma \downarrow 0$ then and applying a covering argument, we deduce the desired result (using that semiconvexity passes to the limit). □

Remark 5.9. Notice that p's polynomial nature plays no role in Lemmas 5.9, 5.11, and 5.12. We have only used that p is non-negative in the thin space and a-harmonic in Lemma 5.9, and that p is non-negative in the thin space, a-harmonic, and invariant in the $\boldsymbol{e}$ directions in Lemmas 5.11 and 5.12.

5.3 Blow-up Analysis

Recall, after a translation, we may assume that $0 \in \mathrm{Sing}(u)$ represents any singular point. And, as such, the first blow-up of u at 0 is an element of $\mathscr{P}_\kappa$ for some $\kappa \in 2\mathbb{N}$. As in Section 5.2, we let p_* denote the first blow-up of u at 0, and define

$$v_* := u - p_*, \quad \kappa_* := \kappa_0, \quad L_* := L(p_*), \quad m_* := m_0, \quad \text{and} \quad \lambda_* := N(0^+, v_*).$$

For notational simplicity, from this point forward, we often suppress the star subscript when denoting the homogeneity of p_*, and simply write κ instead of κ_*.

In this section, we are interested in classifying the *second blow-ups of* u *at* 0, that is, the limit points of the set $\{\tilde{v}_r\}_{r>0}$, which is weakly precompact by Proposition 5.6, as $r \downarrow 0$, with

$$\tilde{v}_r := \frac{v_r}{\|v_r\|_{L^2(\partial B_1, |y|^a)}} \quad \text{and} \quad v_r(X) := u(rX) - p_*(rX). \tag{5.29}$$

In turn, we will prove Proposition 5.3.

We will work according to two cases, determined by the value of a and the alignment of L_* and the nodal set of p_*,

$$\mathcal{N}_* := \mathcal{N}(p_*)$$

(see (5.17)). Notice that by Lemma 5.10, if we consider $L(p)$ as a subset of $\mathbb{R}^n \times \{0\}$, then

$$L(p) \subset \mathcal{N}(p)$$

for all $p \in \mathscr{P}_\kappa$; yet $L(p)$ may be smaller than $\mathcal{N}(p)$. In particular, we define Case 1 and Case 2 as follows.

$$\begin{array}{rl} \text{Either} & a \in [0,1) \\ \text{or} & a \in (-1,0) \text{ and } \dim_{\mathcal{H}} \mathcal{N}_* \le n-2 \end{array} \tag{Case 1}$$

and

$$a \in (-1,0) \text{ and } \dim_{\mathcal{H}} \mathcal{N}_* = \dim_{\mathcal{H}} L_* = n-1. \tag{Case 2}$$

Remark 5.10. We remark that Case 1 and Case 2, a priori, do not cover all possibilities. Indeed, the case when $a \in (-1,0)$ and $\dim_{\mathcal{H}} L_* < \dim_{\mathcal{H}} \mathcal{N}_* = n-1$ is missing. In fact, it is currently unknown if such a situation can occur when $u \not\equiv p_*$.

Before we proceed with our classification results, we make a pair of observations, the second of which will play a key feature in Case 2. Since $p_* \ge 0$ on $\mathbb{R}^{n+1} \cap \{y=0\}$, we have that

$$\{(x,0) : p_*(x,0) = 0\} = \{(x,0) : p_*(x,0) = |\nabla_x p_*(x,0)| = 0\} = \mathcal{N}_*. \tag{5.30}$$

Furthermore, if $L_* \cong \mathbb{R}^{n-1}$, as it is in Case 2, then $p_*|_{\mathbb{R}^n \times \{0\}}$ is a one-dimensional polynomial, and so we can identify L_* and $\mathcal{N}_*$ as the same subset of $\mathbb{R}^n \times \{0\}$.

Let us start by studying second blows-up in Case 1.

Proposition 5.13. *In Case 1, for every sequence $r_j \downarrow 0$, there is a subsequence $r_{j_\ell} \downarrow 0$ such that $\tilde v_{r_{j_\ell}} \rightharpoonup q$ weakly in $W^{1,2}(B_1, |y|^a)$ as $\ell \to \infty$, and $q \not\equiv 0$ is a λ_*-homogeneous, a-harmonic polynomial. In particular, $\lambda_* \in \{\kappa, \kappa+1, \kappa+2, \dots\}$.*

Proof. By Proposition 5.6, we see that given any sequence $r_j \downarrow 0$, the sequence $\tilde v_{r_j}$ is uniformly bounded in $W^{1,2}(B_1, |y|^a)$. Hence, there is a subsequence $r_{j_\ell} \downarrow 0$ such that

$$\tilde v_{r_{j_\ell}} \rightharpoonup q \quad \text{in} \quad W^{1,2}(B_1, |y|^a),$$

for some q, and as $\|\tilde v_{r_{j_\ell}}\|_{L^2(\partial B_1, |y|^a)} = 1$, we have that

$$\|q\|_{L^2(\partial B_1, |y|^a)} = 1.$$

Observe that $L_a \tilde v_r$ is a non-positive measure as

$$L_a v_r = 2r \lim_{y \downarrow 0} y^a \partial_y u_r \mathcal{H}^n \llcorner \Lambda(u_r) \le 0$$

in the sense of distributions. Furthermore, let $K \subset B_1$ be a any compact set and $\eta_K \in C_c^\infty(B_1)$ be such that $\eta_K \equiv 1$ on K and $0 \le \eta_K \le 1$ in B_1. By Hölder's inequality,

$$0 \le \int_K -L_a \tilde v_r \le \int_{B_1} -\eta_K L_a \tilde v_r = \int_{B_1} \nabla \eta_K \cdot \nabla \tilde v_r |y|^a \le C_K \|\nabla \tilde v_r\|_{L^2(B_1, |y|^a)}$$

Since the family $\tilde v_r$ is uniformly bounded in $W^{1,2}(B_1, |y|^a)$ by Proposition 5.6, it follows that the collection of measures $L_a \tilde v_r$ is tight. So, up to a further subsequence, which we still denote by r_{j_ℓ}, we have that $L_a q$ is a non-positive measure. Then, as

$r^{-\kappa}u_r \to p_*$ locally uniformly, with $u_r(X) := u(rX)$, the sets $\Lambda(u_r)$ converge to $\mathcal{N}_*$ in the Hausdorff sense (recall (5.30)). Therefore, the distribution $L_a q$ is supported on $\{(x,0) : p_*(x,0) = 0\}$. Yet we are in Case 1, and $\mathcal{N}_*$ is of zero a-harmonic capacity $\mathbb{R}^{n+1}$. Indeed, as $p_*|_{\mathbb{R}^n\times\{0\}} \not\equiv 0$, the set $\mathcal{N}_*$ has locally finite $\mathcal{H}^{n-1}$ measure. If $a \ge 0$, then the a-harmonic capacity of $\mathcal{N}_*$ is smaller than the harmonic capacity of $\mathcal{N}_*$, which is zero. If $a < 0$, then, by assumption, $\mathcal{N}_*$ has locally finite $\mathcal{H}^{n-2}$ measure, which implies that it is of zero a-harmonic capacity (see [Kil94, Corollary 2.12]). Thus, q is a-harmonic, i.e., $L_a q \equiv 0$.

Let us now show that q is homogeneous, arguing as in [FS18, Lemma 2.12], with homogeneity $\lambda_* := N(0^+, v_*)$. In order to do so, by [GR19, Theorem 2.11], it suffices to show that

$$\lambda_* = N(\rho, q) \quad \text{for all} \quad \rho \in (0,1). \tag{5.31}$$

Notice, first, that since q is a-harmonic, $N(\rho, q)$ is non-decreasing. On the other hand, by the lower semicontinuity of the weighted Dirichlet integral,

$$N(1,q) \le \liminf_{\ell\to\infty} N(1, \tilde{v}_{r_{j_\ell}}) = \liminf_{\ell\to\infty} N(1, v_{r_{j_\ell}}) = \liminf_{\ell\to\infty} N(r_{j_\ell}, v_*) = \lambda_*.$$

Also, by Lemma 5.8 applied to $\tilde{v}_{r_{j_\ell}}$, and taking $\ell \to \infty$,

$$\frac{1}{\rho^{n+a+2\lambda_*}} \int_{\partial B_\rho} q^2|y|^a \le \int_{\partial B_1} q^2|y|^a = 1. \tag{5.32}$$

However, because $L_a q = 0$ and by (5.26), we know that

$$\frac{H'_\lambda}{H_\lambda}(\rho, q) = \frac{2}{\rho}(N(\rho,q) - \lambda).$$

Suppose now that $N(\rho_\circ, q) = \lambda_\circ < \lambda_*$ for some $\rho_\circ \in (0,1)$. In particular, by the previous representation of H_λ, $H_{\lambda_\circ}$ is non-increasing for $\rho \in (0, \rho_\circ)$, so that

$$\frac{1}{\rho^{n+a+2\lambda_*}} \int_{\partial B_\rho} q^2|y|^a \ge \frac{\rho^{2(\lambda_\circ - \lambda_*)}}{\rho_\circ^{n+a+2\lambda_\circ}} \int_{\partial B_{\rho_\circ}} q^2|y|^a > 0 \quad \text{for all} \quad \rho \in (0, \rho_\circ).$$

But this contradicts (5.32) for ρ small enough. Therefore, (5.31) holds and q is homogeneous of degree λ_*. And by [CSS08, Lemma 2.7], we deduce that q is a polynomial. In particular, $\lambda_* \ge \kappa$ is an integer.

All in all, we have that $q \not\equiv 0$ is an a-harmonic, even in y, and λ_*-homogeneous polynomial with $\lambda_* \in \{\kappa, \kappa+1, \kappa+2, \dots\}$. In particular, $q\big|_{\mathbb{R}^n\times\{0\}} \not\equiv 0$. □

Before moving to Case 2, let us state and prove a lemma which will help us to compare p_* and q when working in Case 1. That said, this lemma is independent of Case 1 and Case 2, and holds generically.

Lemma 5.14. *Assume that $\tilde{v}_{r_\ell} \rightharpoonup q$ in $W^{1,2}(B_1, |y|^a)$ for some sequence $r_\ell \downarrow 0$. Then,*

$$\int_{\partial B_1} q p_* |y|^a = 0 \tag{5.33}$$

and

$$\int_{\partial B_1} q p |y|^a \le 0 \quad \textit{for all} \quad p \in \mathscr{P}_\kappa. \tag{5.34}$$

Proof. We proceed as in [FS18, Lemmas 2.11-2.12]. In order to see (5.33), we use $H_\lambda(r, u-p)$ is non-decreasing for $\lambda = \kappa \le N(0^+, u-p)$ (see Lemma 5.8), recalling that $\kappa = N(0^+, u)$, by assumption. In particular, we have

$$\begin{aligned}\frac{1}{r^{n+a+2\kappa}}\int_{\partial B_r}(u-p)^2|y|^a &\ge \lim_{\rho\downarrow 0}\frac{1}{\rho^{n+a+2\kappa}}\int_{\partial B_\rho}(u-p)^2|y|^a\\ &= \lim_{\rho\downarrow 0}\int_{\partial B_1}(\rho^{-\kappa}u(\rho X)-p)^2|y|^a \\ &= \int_{\partial B_1}(p_*-p)^2|y|^a,\end{aligned} \tag{5.35}$$

using the local uniform convergence of $r^{-\kappa}u_r$ to p_* as $r\downarrow 0$, with $u_r(X) := u(rX)$, and the κ-homogeneity of p. By the definition of p_*, notice that

$$h_r := \|v_r\|_{L^2(\partial B_1,|y|^a)} = o(r^\kappa) \quad \text{as} \quad r\downarrow 0 \qquad \text{and} \qquad \varepsilon_r := \frac{h_r}{r^\kappa} = o(1) \quad \text{as} \quad r\downarrow 0.$$

Furthermore, for some subsequence, which we still denote by r_ℓ, we have that $\tilde v_{r_\ell} = v_{r_\ell}/h_{r_\ell} \to q$ in $L^2(\partial B_1, |y|^a)$. Thus,

$$\int_{\partial B_1}\left(\frac{v_r}{r^\kappa}+p_*-p\right)^2|y|^a = \frac{1}{r^{n+a+2\kappa}}\int_{\partial B_r}(u-p)^2|y|^a \ge \int_{\partial B_1}(p_*-p)^2|y|^a \quad \text{for all} \quad r>0.$$

Since $r^{-\kappa}v_r = \tilde v_r\varepsilon_r$, taking the subsequence r_ℓ and expanding, we obtain

$$\varepsilon_{r_\ell}^2\int_{\partial B_1}\tilde v_{r_\ell}^2|y|^a + 2\varepsilon_{r_\ell}\int_{\partial B_1}\tilde v_{r_\ell}(p_*-p)|y|^a \ge 0 \quad \text{for all} \quad p\in\mathscr{P}_\kappa.$$

Dividing by ε_{r_ℓ} and taking the limit as $\ell\to\infty$,

$$\int_{\partial B_1} q(p_*-p)|y|^a \ge 0 \quad \text{for all} \quad p\in\mathscr{P}_\kappa.$$

Now taking $p = 2p_*$ and $p = 2^{-1}p_*$, which are both members of $\mathscr{P}_\kappa$, we deduce that

$$\int_{\partial B_1} qp_*|y|^a = 0,$$

from which (5.34) follows immediately. □

Let us now deal with Case 2. As we noted before, in this case, the spine and the nodal set of p_* can be identified: $L_* = \mathcal{N}_*$.

Proposition 5.15. *In Case 2, for every sequence $r_j\downarrow 0$, there is a subsequence $r_{j_\ell}\downarrow 0$ such that $\tilde v_{r_{j_\ell}} \rightharpoonup q$ weakly in $W^{1,2}(B_1,|y|^a)$ as $\ell\to\infty$, and $q\not\equiv 0$ is a λ_*-homogeneous solution to the very thin obstacle problem with zero obstacle on L_*,*

$$\begin{cases} q\ge 0 & \text{on } L_*\\ L_a q\le 0 & \text{in } \mathbb{R}^{n+1}\\ L_a q = 0 & \text{in } \mathbb{R}^{n+1}\setminus L_*\\ qL_a q = 0 & \text{in } \mathbb{R}^{n+1}.\end{cases} \tag{5.36}$$

Moreover, $\lambda_* \ge \kappa + \alpha_\kappa$, for some constant $\alpha_\kappa > 0$ depending only on n, a, and κ.

Proof. Without loss of generality, we will assume that $L_* = \{x_n = y = 0\}$. We divide the proof into several steps.

Step 1: Weak limit and non-negativity on L_*. As in the proof of Proposition 5.13, we have that

$$\tilde{v}_{r_\ell} \rightharpoonup q \quad \text{in} \quad W^{1,2}(B_1, |y|^a), \tag{5.37}$$

for some q, and $L_a\tilde{v}_r$ is converging weakly* as measures to a non-positive measure $L_a q$ supported on L_*. Unlike before, the set on which $L_a q$ is supported is now a set of strictly positive a-harmonic capacity (since $m = n-1$).

Consider the following trace operators

$$\gamma : W^{1,2}(B_1, |y|^a) \to W^{s,2}(B_1^*) \quad \text{and} \quad \tilde{\gamma} : W^{s,2}(B_1^*) \to W^{s-\frac{1}{2},2}(B_1').$$

By [NLM88] (see also [Kim07]), since $s > 1/2$, γ is continuous; and $\tilde{\gamma}$ is the standard continuous trace operator. (Recall that $a = 1-2s$.) The operator $\tau := \tilde{\gamma} \circ \gamma$ then is continuous. Hence, considering (5.37),

$$\tau(\tilde{v}_{r_\ell}) \rightharpoonup \tau(q) \quad \text{in} \quad W^{s-\frac{1}{2},2}(B_1') \qquad \text{and} \qquad \tau(\tilde{v}_{r_\ell}) \to \tau(q) \quad \text{in} \quad L^2(B_1').$$

Now $\tau(\tilde{v}_{r_\ell}) \geq 0$ on B_1' for all $\ell \in \mathbb{N}$, since $p_* \equiv 0$ and $u \geq 0$ on L_*. Thus, from the strong convergence above, $\tau(q) \geq 0$, or $q \geq 0$ on L_*.

Step 2: Semiconvexity in directions parallel to L_*. By Lemma 5.12,

$$\inf_{B_{1/2}} \partial_{\boldsymbol{ee}} \tilde{v}_r \geq -C \quad \text{for all} \quad \boldsymbol{e} \in L_* \cap \mathbb{S}^n, \tag{5.38}$$

for some constant C independent of r. Namely, the sequence of functions $\tilde{v}_r$ is locally uniformly semiconvex (and, therefore, locally uniformly Lipschitz) in the directions parallel to L_*.

Step 3: Strong convergence. We show that for every $0 < \varepsilon \ll 1$, there exists a constant $C_\varepsilon > 0$ independent of r_ℓ for which

$$[\tilde{v}_{r_\ell}]_{C^{-a-\varepsilon}(B_{1/2})} \leq C_\varepsilon. \tag{5.39}$$

Thus, by a covering argument, $\tilde{v}_{r_\ell} \to q$ locally uniformly in B_1, and, in fact, $q \in C^{-a-\varepsilon}_{\mathrm{loc}}(B_1)$.

Recall that $L_* = \{x_n = y = 0\}$ and $X = (x', x_n, y)$ for $x' \in \mathbb{R}^{n-1}$. For simplicity, in the following computations, set

$$w := \tilde{v}_r.$$

Let $Q_{r_\circ} := B'_{r_\circ} \times D_{r_\circ} \subset B_1$, for some $r_\circ > 0$. Recall that D_r denotes the disc of radius r in $\mathbb{R}^2$ centered at the origin. For convenience, rescale and assume $r_\circ = 1$. By Step 2, $\|w(x', \cdot, \cdot)\|^2_{L^2(D_1, |y|^a)}$ is Lipschitz, as a function of x'. Hence,

$$\operatorname*{osc}_{B_1'} \|w(x', \cdot, \cdot)\|^2_{L^2(D_1, |y|^a)} \leq C.$$

Recalling that $\|w\|_{L^2(B_1,|y|^a)} \leq C$ (we have rescaled to work in Q_1, else this bound would be 1), we have that

$$\int_{B_1'} \|w(x',\cdot,\cdot)\|^2_{L^2(D_1,|y|^a)}\,\mathrm{d}x' \leq C, \tag{5.40}$$

and so $\|w(x',\cdot,\cdot)\|_{L^2(D_1,|y|^a)}$ has bounded oscillation and integral. In turn,

$$\|w(x',\cdot,\cdot)\|_{L^2(D_1,|y|^a)} \leq C \quad \text{for all} \quad x' \in B_1'. \tag{5.41}$$

We also recall that

$$\lim_{y\downarrow 0} y^a \partial_y w \leq 0 \qquad \text{and} \qquad L_a w = 0 \quad \text{in} \quad B_1 \cap \{y > 0\}. \tag{5.42}$$

– *Step 3.1.* In this subset, we prove that the measure

$$n_w(x', x_n) := \lim_{y\downarrow 0} y^a \partial_y w \leq 0$$

is finite on each x' slice. Equivalently, we show that

$$0 \geq \int_{-1}^{1} \zeta(|(x',x_n)|) n_w(x',x_n)\,\mathrm{d}x_n \geq -C \quad \text{for all} \quad x' \in B_1' \tag{5.43}$$

where ζ is a smooth test function $\zeta = \zeta(r) : [0,\infty) \to [0,1]$ such that $\zeta \equiv 1$ in $[0,1/2]$ and $\zeta \equiv 0$ in $[3/4,\infty)$.

Let $\zeta = \zeta(|(x',x_n,y)|)$. By the divergence theorem,

$$\begin{aligned}\int_{-1}^{1} \zeta n_w\,\mathrm{d}x_n &= -\int_{D_1\cap\{y>0\}} \mathrm{div}_{x_n,y}(\zeta y^a \nabla_{x_n,y} w)\,\mathrm{d}x_n\,\mathrm{d}y \\ &= -\int_{D_1\cap\{y>0\}} \zeta L_a^{x_n,y} w - \int_{D_1\cap\{y>0\}} y^a \nabla_{x_n,y}\zeta \cdot \nabla_{x_n,y} w \\ &=: \mathrm{I} + \mathrm{II}\end{aligned} \tag{5.44}$$

where $L_a^{x_n,y} f := \mathrm{div}_{x_n,y}(|y|^a \nabla_{x_n,y} f)$. On one hand, observe that

$$L_a^{x_n,y} w = L_a w - y^a \Delta_{x'} w = -y^a \Delta_{x'} w \quad \text{in} \quad D_1 \cap \{y > 0\}$$

by (5.42). And so, by (5.38),

$$\mathrm{I} \geq -C. \tag{5.45}$$

On the other hand, by the symmetries of ζ (i.e., $\partial_y \zeta = O(y)$ as $\partial_y\zeta\big|_{y=0} = 0$ and ζ is smooth),

$$|w L_a^{x_n,y}\zeta| = |w y^a y^{-a} L_a^{x_n,y}\zeta| = |w| y^a |\partial_{nn}\zeta + \partial_{yy}\zeta + a y^{-1}\partial_y \zeta| \leq C|w| y^a.$$

So, by the symmetries of ζ again, Hölder's inequality, and (5.41), we deduce that

$$\mathrm{II} = -\int_{D_1\cap\{y>0\}} y^a \nabla_{x_n,y}\zeta \cdot \nabla_{x_n,y} w = \int_{D_1\cap\{y>0\}} w L_a^{x_n,y}\zeta \geq -C. \tag{5.46}$$

We have also used that the boundary term at $y = 0$ vanishes in the integration by parts, $y^a \partial_y \zeta \equiv 0$ on $\{y = 0\}$. Therefore, combining (5.44), (5.45), and (5.46), we see that (5.43) holds, as desired.

– *Step 3.2.* Now we conclude. Consider the fundamental solution for the operator L_a (see, e.g, [CS07]) given by

$$\Gamma_a(X) := C_{n,a}|X|^{-n+1-a}.$$

More precisely, Γ_a is such that $L_a\Gamma_a = 0$ in $\{|y| > 0\}$ and $\lim_{y\downarrow 0} y^a\partial_y\Gamma_a = \delta(x)$, the Dirac delta at x. Let

$$\bar{w}(x, y) := \Gamma_a(\,\cdot\,, y) *_x (\zeta n_w),$$

where ζ is the test function defined in Step 3.1, with $\zeta = \zeta(|x|)$ here. We have that $L_a\bar{w} = 0$ in $|y| > 0$, and $\lim_{y\downarrow 0} y^a\partial_y\bar{w} = \zeta n_w$. We claim that $\bar{w}$ is bounded. Indeed, by (5.43),

$$\begin{aligned}|\bar{w}(x', x_n, y)| &\le \int_{B_1'}\int_{-1}^{1} \frac{(\zeta n_w)(z', z_n)}{|(x' - z', x_n - z_n, y)|^{n-1+a}}\,\mathrm{d}z_n\,\mathrm{d}z' \\ &\le C\int_{B_1'} \frac{\mathrm{d}z'}{|(x' - z', 0, y)|^{n-1+a}} \le C.\end{aligned}$$

By means of the previous proof, $(-\Delta)^{\bar{s}}_X\bar{w} = ((-\Delta)^{\bar{s}}_X\Gamma_a *_x (\zeta n_w))$ is bounded as long as $2\bar{s} < -a$, since $(-\Delta)^{\bar{s}}_X|X|^{-n+1-a} = C|X|^{-n+1-a-2\bar{s}}$, and ζn_w does not depend on y. Thus, $(-\Delta)^{\bar{s}}_X\bar{w}$ is bounded as long so $2\bar{s} < -a$, and by interior regularity for the fractional Laplacian (suppose $\bar{s} \ne 1/2$), $\bar{w}$ is $C^{2\bar{s}}$ (see [RS16, Theorem 1.1]).

Finally, notice that $L_a(\bar{w} - w) = 0$ in $B_1 \cap \{|y| > 0\}$ and $\lim_{y\downarrow 0} y^a\partial_y(\bar{w} - w) = 0$ in $B_{1/2}\cap\{|y| > 0\}$. It follows that $L_a(\bar{w}-w) = 0$ in $B_{1/2}$, and then $\bar{w}-w \in C^1_{\mathrm{loc}}(B_{1/2})$ by interior estimates for a-harmonic functions (and recalling that $a \in (-1, 0)$). In turn, w inherits the regularity of $\bar{w}$; that is, w is $C^{2\bar{s}}$, so long as $2\bar{s} < -a$, and (5.39) is proved.

In particular, by Arzelà–Ascoli and a covering argument, we have that

$$\tilde{v}_{r_\ell} \to q \quad \text{in} \quad C^0_{\mathrm{loc}}(B_1), \tag{5.47}$$

and $q \in C^{-a-\varepsilon}_{\mathrm{loc}}(B_1)$ for any $\varepsilon > 0$.

Step 4: Homogeneous solution to the very thin obstacle problem in B_1. First, we show that q is a solution to the very thin obstacle problem, (5.36); the only condition that remains to be checked is that $q\,L_a q \equiv 0$.

By the proof of Proposition 5.6 and (5.23),

$$\frac{rN'(r, v_*)}{N(r, v_*)} = \frac{\mathrm{d}}{\mathrm{d}\rho}\log N(\rho, v_r)\bigg|_{\rho=1} \ge \frac{2\left(\int_{B_1} v_r L_a v_r\right)^2}{\int_{B_1}|\nabla v_r|^2|y|^a \int_{\partial B_1} v_r^2|y|^a} \ge 0.$$

Hence, by the definition of $\tilde{v}_r$,

$$rN'(r, v_*) \ge 2\left(\int_{B_1} \tilde{v}_r L_a\tilde{v}_r\right). \tag{5.48}$$

Furthermore, reasoning as in [FS18, Lemma 2.12], since $N(r,v)\downarrow\lambda_*$ as $r\downarrow 0$,

$$\fint_{r_{j_\ell}}^{2r_{j_\ell}} rN'(r,v_*)\,\mathrm{d}r \le 2(N(2r_{j_\ell},v_*)-N(r_{j_\ell},v_*))\to 0 \quad\text{as}\quad \ell\to\infty.$$

And so, by the mean value theorem, we can find $\bar r_{j_\ell}\in[r_{j_\ell},2r_{j_\ell}]$ with $\bar r_{j_\ell}N'(\bar r_{j_\ell},v_*)\to 0$ as $\ell\to\infty$. In turn, the non-negativity of $v_*\,L_a v_*$ and (5.48) then imply that

$$\int_{B_1}\tilde v_{r_{j_\ell}}\,L_a\tilde v_{r_{j_\ell}}\le\int_{B_{\bar\rho_{j_\ell}}}\tilde v_{r_{j_\ell}}\,L_a\tilde v_{r_{j_\ell}}\to 0$$

with $\bar\rho_{j_\ell}:=\bar r_{j_\ell}/r_{j_\ell}$. Therefore, since $L_a\tilde v_{r_{j_\ell}}\rightharpoonup L_a q$ weakly* as measures in B_1, $\tilde v_{r_{j_\ell}}\to q$ strongly in $C^0_{\rm loc}(B_1)$ by Step 3, (5.47), and $\tilde v_r\,L_a\tilde v_r\ge 0$, we obtain that

$$\int_{B_R} q\,L_a q = 0 \quad\text{for all}\quad R<1,$$

so that, in fact, $q\,L_a q\equiv 0$ in B_1.

Thus, q is a solution to the very thin obstacle problem (5.36) inside B_1.

To conclude, we show that q is homogeneous with homogeneity $\lambda_*:=N(0^+,v_*)$. Since q solves the very thin obstacle problem, by Lemma 5.52, it suffices to show that

$$\lambda_* = N(\rho,q)\quad\text{for all}\quad \rho\in(0,1). \tag{5.49}$$

But this follows from arguing exactly as in the proof of Proposition 5.13, where we obtained that q is homogeneous in Case 1, using Lemma 5.52, (5.108), and Lemma 5.53.

Step 5: $\lambda_*\ge\kappa+\alpha_\kappa$. We argue by contradiction (or compactness). Suppose, to the contrary, that there exists a bounded sequence of solutions u_ℓ such that $0\in\Sigma_\kappa(u_\ell)$, $\dim_{\mathcal H}L(p_{*,\ell})=n-1$, and $\lambda_{*,\ell}\le\kappa+\ell^{-1}$. Let $p_{*,\ell}$ be the first blow-up and q_ℓ be a second blow-up of u_ℓ at 0 (the homogeneity of q_ℓ is $\lambda_{*,\ell}$). Up to a subsequence (we can assume the sequences enjoy uniform bounds in appropriate Hölder spaces), taking ℓ to infinity, we find a solution u_∞ whose first blow-up at 0 is of order κ, whose spine has Hausdorff dimension equal to $n-1$, and whose second blow-up q_∞ is homogeneous of order κ.

Since q_∞ is a κ-homogeneous, global solution to the very thin obstacle problem, it is an a-harmonic polynomial. Indeed, by [GR19, Proposition 4.4], any global, evenly homogeneous function $u_\circ$ with $L_a u_\circ$ non-negative and supported on $\mathbb{R}^n\times\{0\}$ is actually an a-harmonic polynomial of degree κ. In particular, we have that $\|q_\infty\|_{{\rm Lip}(B_1)}\le C$ for some constant depending only on n, a, and κ. Also, by assumption, $q_\infty\ge 0$ on $L(p_{*,\infty})$, where $p_{*,\infty}$ is the first blow-up of u_∞ at 0.

For simplicity, let $q=q_\infty$ and $p_*=p_{*,\infty}$, and let us assume that $L(p_{*,\infty})=\{x_n=0\}$, so that p_* depends only on x_n in the thin space $\{y=0\}$. By Lemma 5.14,

$$\langle q,p\rangle_a:=\int_{\partial B_1} qp|y|^a\le 0\quad\text{for all}\quad p\in\mathscr{P}_\kappa\quad\text{and}\quad\langle q,p_*\rangle_a=0. \tag{5.50}$$

Since p_* is κ-homogeneous and depends only on x_n, a constant $c_*>0$ exists for which $p_*|_{B_1\cap\{y=0\}}=c_*|x_n|^\kappa$. Now for any $\varepsilon>0$, observe that

$$C_\varepsilon p_*+q\ge-\varepsilon\quad\text{on}\quad\partial B_1\cap\{y=0\}$$

with

$$C_\varepsilon := c_*^{-1}\varepsilon^{-\kappa}\|q\|_{L^\infty(B_1\cap\{y=0\})}\|q\|^\kappa_{\mathrm{Lip}(B_1\cap\{y=0\})}.$$

Indeed, if $|x_n| \geq \varepsilon/\|q\|_{\mathrm{Lip}(B_1\cap\{y=0\})}$, then $C_\varepsilon p_*|_{B_1\cap\{y=0\}} + q|_{B_1\cap\{y=0\}} \geq 0$, by the definition of C_ε. On the other hand, if $|x_n| \leq \varepsilon/\|q\|_{\mathrm{Lip}(B_1\cap\{y=0\})}$, then $q|_{B_1\cap\{y=0\}} \geq -\varepsilon$ since $q \geq 0$ on $\{x_n = 0\}$ (recall $p_* \geq 0$ on the thin space). Thus, $C_\varepsilon p_* + q + \varepsilon\,\mathrm{Ext}_a(|x|^\kappa) \in \mathscr{P}_\kappa$ for every $\varepsilon > 0$ (see (5.13)). So (5.50) implies that

$$\|q\|_{L^2(\partial B_1,|y|^a)} \leq -\varepsilon\langle \mathrm{Ext}_a(|x|^\kappa), q\rangle_a.$$

Taking $\varepsilon \downarrow 0$, we deduce that $q \equiv 0$, a contradiction. □

With Propositions 5.13 and 5.15 in hand, we can now prove Proposition 5.3.

Proof of Proposition 5.3. The proof is a simple consequence of Propositions 5.13 and 5.15. Without loss of generality, $X_\circ = 0$.

(i) If $a \in [0, 1)$, we are in Case 1. So by Proposition 5.13, our claim holds.

(ii) When $\kappa = 2$, since $p_* \geq 0$ on the thin space, we have that $L_* = \mathcal{N}_*$. Thus, since $m < n - 1$, we are again in Case 1, and we conclude by Proposition 5.13 once more.

(iii) Finally, if $m = n - 1$ and $a \in (-1, 0)$, we are in Case 2 (recall $L_* \subset \mathcal{N}_*$). Thus, applying Proposition 5.15, we arrive at our desired conclusion.

This completes the proof. □

5.4 Accumulation Lemmas

In this section, we gather some important lemmas concerning accumulation points of $\mathrm{Sing}(u)$. These lemmas are the key tools used in estimating the size of the points where we can construct the next term in the expansion of u. The lemmas of this section are analogous to the accumulation lemmas of [FS18], although several new, interesting technical challenges appear in our setting.

Let us start by proving an auxiliary lemma.

Lemma 5.16. *Let q be a κ-degree, a-harmonic polynomial, for $\kappa \geq 1$, and let $X_\circ \in \mathbb{R}^{n+1}$. Then,*

$$N(r, q, X_\circ) = \frac{r\int_{B_r(X_\circ)} |\nabla q|^2|y|^a}{\int_{\partial B_r(X_\circ)} q^2|y|^a} \leq \kappa \quad \textit{for all} \quad r > 0.$$

Moreover,

$$N(0^+, q, X_\circ) = m_\circ$$

where $m_\circ$ is the smallest integer for which the $m_\circ$-homogeneous part of $q(X_\circ + \cdot)$ is non-zero.

Proof. Without loss of generality, we assume that $X_\circ = 0$. Let

$$q = \sum_{m=0}^{\kappa} q_0$$

where q_m denotes the m-homogeneous part of q. Since q is a-harmonic and $a \in (-1,1)$, each of its homogeneous parts is a-harmonic. Notice that if p_1 and p_2 are homogeneous a-harmonic polynomials with non-zero homogeneities $m_1 \neq m_2$, then they are orthogonal in $L^1(\partial B_r, |y|^a)$. Indeed, using that $m_i p_i = x \cdot \nabla p_i = r\partial_\nu p_i$ on ∂B_r and integrating by parts,

$$\begin{aligned}(m_1 - m_2)\int_{\partial B_r} p_1 p_2 |y|^a &= r\int_{\partial B_r} p_2 \partial_\nu p_1 |y|^a - r\int_{\partial B_r} p_1 \partial_\nu p_2 |y|^a \\ &= -r\int_{B_r} \nabla p_1 \cdot \nabla p_2 |y|^a + r\int_{B_r} \nabla p_1 \cdot \nabla p_2 |y|^a = 0,\end{aligned}$$

where we have also used that $L_a p_i = 0$.

Now, by means of the m-homogeneity of q_m and the orthogonality in $L^2(\partial B_r, |y|^a)$ of homogeneous a-harmonic polynomials of different homogeneities, we find that

$$\int_{B_r} \nabla q \cdot \nabla q_m |y|^a = \frac{m}{r}\int_{\partial B_r} q_m^2 |y|^a.$$

Thus,

$$r\int_{B_r} |\nabla q|^2 |y|^a = \sum_{m=1}^{\kappa} m \int_{\partial B_r} q_m^2 |y|^a \le \kappa \sum_{m=1}^{\kappa} \int_{\partial B_r} q_m^2 |y|^a.$$

Pythagoras's theorem also implies that

$$\int_{\partial B_r} q^2 |y|^a = \sum_{m=0}^{\kappa} \int_{\partial B_r} q_m^2 |y|^a.$$

Hence,

$$r\int_{B_r} |\nabla q|^2 |y|^a \le \kappa \int_{\partial B_r} q^2 |y|^a,$$

as desired.

Now let $c_m := \int_{\partial B_1} q_m^2 |y|^a$, and set $m_\circ \ge 0$ to be the smallest integer so that $c_{m_\circ} \neq 0$. Then,

$$\frac{r\int_{B_r} |\nabla q|^2 |y|^a}{\int_{\partial B_r} q^2 |y|^a} = \frac{\sum_{m=m_\circ}^{\kappa} m c_m r^{2m}}{\sum_{m=m_\circ}^{\kappa} c_m r^{2m}} = m_\circ + O(r^2),$$

which concludes the proof. □

Just as in Section 5.3, we divide our attention between Case 1 and Case 2. Again, we begin with Case 1. Our accumulation lemma in this case is analogous to [FS18, Lemma 3.2]. We repeat the common parts for completeness.

We recall that, in the following lemmas, we are assuming that $0 \in \Sigma_\kappa$ is a singular point of order $\kappa \in 2\mathbb{N}$.

Lemma 5.17. *In Case 1, suppose that there exists a sequence of free boundary points* $\Sigma_{\geq\kappa} \ni X_\ell = (x_\ell, 0) \to 0$ *and radii* $r_\ell \downarrow 0$ *with* $|X_\ell| \leq r_\ell/2$ *such that* $\tilde v_{r_\ell} \rightharpoonup q$ *in* $W^{1,2}(B_1, |y|^a)$ *and* $Z_\ell := X_\ell/r_\ell \to Z_\infty$*. Then,*

$$Z_\infty = (z_\infty, 0) \in L_* \qquad \text{and} \qquad D^\alpha q(Z_\infty) = 0 \quad \text{for all} \quad \alpha = (\alpha', 0) \text{ and } |\alpha| \leq \kappa - 2.$$

Moreover, if $\lambda_* = \kappa$*, then* $q_{Z_\infty} := q(Z_\infty + \,\cdot\,) - q$ *is invariant under* $L_* + L(q)$*; that is,*

$$q_{Z_\infty}(x + \xi, 0) = q_{Z_\infty}(x, 0) \quad \text{for all pairs} \quad (\xi, x) \in (L_* + L(q)) \times \mathbb{R}^n.$$

Proof. From Proposition 5.6 applied at X_ℓ, the frequency of $u(X_\ell + \,\cdot\,) - p_*$ is at least κ. (Here, p_* is being considered as just an element of $\mathscr{P}_\kappa$. Recall, p_* is the blow-up at 0, not at X_ℓ.) Therefore,

$$N(\rho, u(X_\ell + r_\ell\,\cdot\,) - p_*(r_\ell\,\cdot\,)) = N(\rho r_\ell, u(X_\ell + \,\cdot\,) - p_*) \geq \kappa \quad \text{for all} \quad \rho \in (0, 1/2),$$

or, equivalently, for all $\rho \in (0, 1/2)$,

$$\frac{\rho \int_{B_\rho} |\nabla \tilde v_{r_\ell}(Z_\ell + \,\cdot\,) + h_{r_\ell}^{-1}\nabla(p_*(X_\ell + r_\ell\,\cdot\,) - p_*(r_\ell\,\cdot\,))|^2 |y|^a}{\int_{\partial B_\rho} |\tilde v_{r_\ell}(Z_\ell + \,\cdot\,) + h_{r_\ell}^{-1}(p_*(X_\ell + r_\ell\,\cdot\,) - p_*(r_\ell\,\cdot\,))|^2 |y|^a} \geq \kappa \tag{5.51}$$

with

$$h_{r_\ell} := \|v_{r_\ell}\|_{L^2(\partial B_1, |y|^a)}.$$

Now let

$$q_\ell(X) := \frac{p_*(X_\ell + r_\ell X) - p_*(r_\ell X)}{h_{r_\ell}},$$

which is a $(\kappa - 1)$-degree, a-harmonic polynomial. Also, observe that

$$\int_{B_{1/2}} |\tilde v_{r_\ell}(Z_\ell + \,\cdot\,)|^2 |y|^a + \int_{B_{1/2}} |\nabla \tilde v_{r_\ell}(Z_\ell + \,\cdot\,)|^2 |y|^a \leq \|\tilde v_{r_\ell}\|^2_{W^{1,2}(B_1, |y|^a)} \leq C. \tag{5.52}$$

We claim that the coefficients of q_ℓ are uniformly bounded with respect to ℓ, so that, up to subsequences, $q_\ell \to q_\infty$ locally uniformly where q_∞ is some a-harmonic polynomial of degree $\kappa - 1$. Indeed, suppose that this is not true. Then, letting $\{a^\ell\}_{i\in\mathscr{I}}$ denote the coefficients of q_ℓ and setting $\sigma_\ell := \sum_{i\in\mathscr{I}} |a_i^\ell|$, we have that $\sigma_\ell \to \infty$. Now set

$$\bar q_\ell := \frac{q_\ell}{\sigma_\ell},$$

which is a polynomial with coefficients bounded by 1, and let $\bar q_\infty$ denote its limit (up to a subsequence). Notice that $\bar q_\infty$ is an a-harmonic, $(\kappa - 1)$-degree polynomial as $\bar q_\ell$ are all a-harmonic, $(\kappa - 1)$-degree polynomials. So, from (5.51), dividing the numerator and denominator by σ_ℓ^2, and by Lemma 5.16, we deduce that

$$\kappa \leq \frac{\rho \int_{B_\rho} |\nabla \varepsilon_\ell + \nabla \bar q_\ell|^2 |y|^a}{\int_{\partial B_\rho} |\varepsilon_\ell + \bar q_\ell|^2 |y|^a} \to \frac{\rho \int_{B_\rho} |\nabla \bar q_\infty|^2 |y|^a}{\int_{\partial B_\rho} |\bar q_\infty|^2 |y|^a} = N(\rho, \bar q_\infty) \leq \kappa - 1$$

since, by (5.52),

$$\varepsilon_\ell := \frac{\tilde v_{r_\ell}(Z_\ell + \,\cdot\,)}{\sigma_\ell} \to 0 \quad \text{in} \quad W^{1,2}(B_{1/2}, |y|^a).$$

Impossible.

Since q_ℓ converges, up to subsequences, to some q_∞ uniformly in compact sets and by interior estimates for a-harmonic functions (see, e.g., [JN17, Propsition 2.3]), we have that $|D^\alpha q_\ell(0)| \le C$ for some C independently of ℓ for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n, 0)$. Then, from the κ-homogeneity of p_*, we have

$$D^\alpha q_\ell(0) = \frac{r_\ell^{|\alpha|}}{h_{r_\ell}} D^\alpha p_*(X_\ell) = \frac{r_\ell^\kappa}{h_{r_\ell}} D^\alpha p_*(Z_\ell) \tag{5.53}$$

for all $|\alpha| \le \kappa - 1$. Hence, using $|D^\alpha q_\ell(0)| \le C$ and $h_{r_\ell} = o(r_\ell^\kappa)$, we determine that

$$|D^\alpha p_*(Z_\ell)| = o(1) \to 0 \quad \text{as} \quad \ell \to \infty$$

when $|a| \le \kappa - 1$. That is, $D^\alpha p_*(Z_\infty) = 0$ for $|\alpha| \le \kappa - 1$. Thanks to Lemma 5.10,

$$Z_\infty \in L_*.$$

Proceeding as in [FS18, Lemma 3.2] by means of the Monneau-type monotonicity formula from Lemma 5.8, we obtain

$$\frac{1}{\rho^{a+2\kappa}} \fint_{\partial B_\rho} |q(Z_\infty + \cdot\,) + q_\infty|^2 |y|^a \le 2^{a+2\kappa} \fint_{\partial B_{1/2}} |q(Z_\infty + \cdot\,) + q_\infty|^2 |y|^a \tag{5.54}$$

for all $\rho \in (0, 1/2)$. Notice that, until now, we have not used any information on the second blow-up q. From Proposition 5.13, q is a λ_*-homogeneous, a-harmonic polynomial with $\lambda_* \ge \kappa$, since we are in Case 1. It follows that the polynomial $q(Z_\infty + \cdot\,) + q_\infty$ is only made up of monomials of degree greater than or equal to κ. Thus, recalling (5.53), we have that

$$\lambda_* q(Z_\infty) = Z_\infty \cdot \nabla_x q(Z_\infty) = -Z_\infty \cdot \nabla_x q_\infty(0) = -\lim_{\ell \to \infty} \frac{r_\ell}{h_{r_\ell}} (Z_\infty \cdot \nabla_x p_*(X_\ell)) = 0.$$

Here, we have also used that $Z_\infty \in L_*$, $X_\ell = (x_\ell, 0)$, and q is κ_*-homogeneous. Moreover, taking derivatives, we have

$$(\lambda_* - |\alpha|) D^\alpha q(Z_\infty) = -\lim_{\ell \to \infty} \frac{r_\ell^\kappa}{h_{r_\ell}} (Z_\infty \cdot \nabla_x D^\alpha p_*(Z_\ell)) = 0.$$

(By Lemma 5.10, $Z_\infty \cdot \nabla_x D^\alpha p_*(Z_\ell) = 0$.) Therefore,

$$D^\alpha q(Z_\infty) = 0 \quad \text{for all} \quad \alpha = (\alpha', 0) \text{ and } |\alpha| \le \kappa - 2.$$

In addition, notice that by construction, q_ℓ is invariant under L_*. Hence, so is q_∞.

Finally, suppose $\lambda_* = \kappa$. Then, $q(Z_\infty + \cdot\,) + q_\infty$ consists of only degree κ terms. In other words, it is κ-homogeneous. Now notice that $q(Z_\infty + \cdot\,) = q + s_\infty$ where s_∞ is a degree $\kappa - 1$ polynomial. Consequently, $q(Z_\infty + \cdot\,) + q_\infty - q = s_\infty + q_\infty$ is a κ-homogeneous polynomial. This is only possible if $s_\infty + q_\infty \equiv 0$ (recall, q_∞ is of degree $\kappa - 1$.) And so, it follows that

$$q_\infty = q - q(Z_\infty + \cdot\,),$$

from which we deduce that q_∞ is invariant under $L(q)$. Since the invariant set of a function is a linear space,

$$q_\infty(x+\xi,0) = q_\infty(x,0) \quad \text{for all pairs} \quad (\xi,x) \in (L_* + L(q)) \times \mathbb{R}^n.$$

Lastly, we find that

$$D^\alpha q_\infty(0) = 0 \quad \text{for all} \quad \alpha = (\alpha',0) \text{ and } |\alpha| \leq \kappa - 2,$$

making q_∞ a $(\kappa-1)$-homogeneous, even in y, a-harmonic polynomial. □

Notice that if $Z_\infty \in L(q)$, then $q_\infty \equiv 0$. Indeed, all of the derivatives of $q_\infty|_{\mathbb{R}^n\times\{0\}}$ up to order $\kappa-2$ vanish at the origin since $D^\alpha q(Z_\infty) = 0$ for all $\alpha = (\alpha',0)$ and $|\alpha| \leq \kappa-2$. So if $D^\alpha q(Z_\infty) = 0$ for all $\alpha = (\alpha',0)$ with $|\alpha| \leq \kappa-1$ too, then q_∞ would vanish up to infinite order at the origin, making it identically zero. In other words,

$$Z_\infty \in L(q) \text{ if and only if } q_\infty \equiv 0.$$

This also follows directly from the form q_∞ takes when $\lambda_* = \kappa$.

Before stating and proving a Case 2 accumulation lemma, we present a simple consequence of Lemma 5.17 and make a remark.

If $m_* = 0$, then $L_* = \{0\}$. Hence, from Lemma 5.17, we deduce that Σ_κ^0 is isolated in $\Sigma_{\geq\kappa}$.

Lemma 5.18. *Suppose Case 1 holds. Then,* 0 *is an isolated point of* $\Sigma_{\geq\kappa}$.

Proof. Suppose, to the contrary, that $\Sigma_{\geq\kappa} \ni X_\ell \to 0$ is a sequence of points ($X_\ell \neq 0$). Let $r_\ell := 2|X_\ell|$. By Lemma 5.17, we have that, up to a subsequence,

$$\tilde{v}_{r_\ell} \rightharpoonup q \quad \text{in} \quad W^{1,2}(B_1,|y|^a) \qquad \text{and} \qquad Z_\ell := \frac{X_\ell}{r_\ell} \to Z_\infty \in L_* \cap \partial B_{1/2}$$

where q is a κ^*-homogeneous harmonic polynomial with $\lambda_* \geq \kappa$. But, this is impossible, since $L_* = \{0\}$. □

Remark 5.11. In general, lower frequency singular points can accumulate to a higher frequency singular point. Take, for example, the harmonic extension of $x_1^2x_2^2$ to $\mathbb{R}^3$:

$$u(X) = x_1^2x_2^2 - (x_1^2+x_2^2)y^2 + \frac{1}{3}y^4.$$

This polynomial is a solution to the thin obstacle problem with $a = 0$, and has singular points of order 2 approaching a singular point of order 4. In particular, it is not true that Σ_κ^0 is isolated from $\Sigma_{<\kappa}$.

By the recent results of Colombo, Spolaor, and Velichkov, see [CSV19, Theorem 4], we know that the set of even frequencies ($\kappa = 2m$) is isolated from the set of all possible frequencies for the thin obstacle problem when $a = 0$. This, together with the upper semicontinuity of the frequency, implies that free boundary points of strictly higher order cannot accumulate to a singular point of lower order in this case. Therefore, the above hypothesis "$X_\ell \in \Sigma_{\geq\kappa}$ and $X_\ell \to 0 \in \Sigma_\kappa$" reduces to "$X_\ell \in \Sigma_\kappa$ and $X_\ell \to 0 \in \Sigma_\kappa$", at least when $a = 0$.

Now we prove a Case 2 accumulation lemma. It will only be applied when $\lambda_* < \kappa+1$ and $\lambda = \lambda_*$ (with λ as defined in the lemma). Nonetheless, we state it in more generality, for completeness.

We recall that Ext_a denotes the a-harmonic extension of a polynomial, see (5.13).

Lemma 5.19. *In Case 2, suppose that there exists a sequence of free boundary points $\Sigma_\kappa^{n-1} \ni X_\ell = (x_\ell, 0) \to 0$ and radii $r_\ell \downarrow 0$ with $|X_\ell| \le r_\ell/2$ such that $\tilde v_{r_\ell} \rightharpoonup q$ in $W^{1,2}(B_1, |y|^a)$ and $(z_\ell, 0) = Z_\ell := X_\ell/r_\ell \to Z_\infty$. Set*

$$\lambda_{*,X_\ell} := N(0^+, u(X_\ell + \cdot\,) - p_{*,X_\ell}),$$

where p_{*,X_ℓ} denotes the first blow-up of u at X_ℓ. Let $\boldsymbol{e}_* \in \mathbb{S}^n \cap \{y = 0\} \cong \mathbb{S}^{n-1}$ be such that $\boldsymbol{e}_* \perp L_*$, and let q^{even} and q^{odd} be the even and odd parts of q with respect to L_*,

$$q^{\mathrm{even}}(X) = \frac{1}{2}\left[q(X) + q(X - 2(\boldsymbol{e}_* \cdot X)\boldsymbol{e}_*)\right]$$

and

$$q^{\mathrm{odd}}(X) = \frac{1}{2}\left[q(X) - q(X - 2(\boldsymbol{e}_* \cdot X)\boldsymbol{e}_*)\right].$$

Let $\alpha_\kappa > 0$ be as in Proposition 5.15 and set $\lambda := \liminf_\ell\{\lambda_{*,X_\ell}\} \ge \kappa + \alpha_\kappa$. Then,

$$Z_\infty = (z_\infty, 0) \in L_*$$

and

$$\fint_{\partial B_\rho} |q^{\mathrm{even}}(Z_\infty + X) - c_\infty \,\mathrm{Ext}_a((\boldsymbol{e}_* \cdot x)^\kappa)|^2 |y|^a \le C\rho^{2\lambda+a} \quad \text{for all} \quad \rho \in (0, 1/2), \tag{5.55}$$

for some constants c_∞ and C independent of ρ. Moreover, if $\lambda_* < \kappa + 1$, then $q^{\mathrm{odd}} \equiv 0$. If, in addition, $\lambda = \lambda_*$, then $c_\infty = 0$ in (5.55), and q is invariant in the Z_∞ direction; that is, $q(Z_\infty + X) = q(X)$ for all $X \in \mathbb{R}^{n+1}$.

Proof. We divide the proof into two steps.

Step 1: We proceed using the ideas developed to prove [FS18, Lemma 3.3]. Recall that

$$p_{*,X_\ell}(X) := \lim_{r\downarrow 0} \frac{u(X_\ell + rX)}{r^\kappa}.$$

Define

$$q_\ell(X) := \frac{p_{*,X_\ell}(r_\ell X) - p_*(X_\ell + r_\ell X)}{h_{r_\ell}} \quad \text{with} \quad h_{r_\ell} := \|v_{r_\ell}\|_{L^2(\partial B_1, |y|^a)}.$$

By Proposition 5.15 and Proposition 5.6, for all $\rho \in (0, 1/2)$,

$$N(\rho r_\ell, u(X_\ell + \cdot\,) - p_{*,X_\ell}) \ge \lambda_{*,X_\ell} \ge \kappa + \alpha_\kappa > \kappa, \tag{5.56}$$

or, equivalently,

$$\frac{\rho \int_{B_\rho} |\nabla \tilde v_{r_\ell}(Z_\ell + \cdot\,) - \nabla q_\ell|^2 |y|^a}{\int_{\partial B_\rho} |\tilde v_{r_\ell}(Z_\ell + \cdot\,) - q_\ell|^2 |y|^a} \ge \kappa + \alpha_\kappa$$

(cf. (5.51)). Furthermore, arguing as in the proof of Lemma 5.17, we find that the family $\{q_\ell\}_{\ell\in\mathbb{N}}$ has uniformly bounded coefficients. This time, however, we use that q_ℓ is of degree κ and a-harmonic rather than of degree $\kappa-1$ and a-harmonic. Indeed, as in Lemma 5.17, suppose not. Then, dividing by the largest coefficient, we obtain uniformly bounded, a-harmonic polynomials $\bar{q}_\ell$ of degree κ and the inequality

$$\frac{1}{2}\frac{\int_{B_{1/2}}|\nabla\varepsilon_\ell-\nabla\bar{q}_\ell|^2|y|^a}{\int_{\partial B_{1/2}}|\varepsilon_\ell-\bar{q}_\ell|^2|y|^a}\geq\kappa+\alpha_\kappa\quad\text{for all}\quad\ell\in\mathbb{N}\tag{5.57}$$

and for some $\varepsilon_\ell\to0$ in $W^{1,2}(B_{1/2},|y|^a)$. Now notice that $\bar{q}_\ell$ are degree κ polynomials converging uniformly to some $\bar{q}_\infty$ (up to subsequences). Also, since the translations that define q_ℓ are in $\{y=0\}$, $\bar{q}_\ell$ are a-harmonic. In turn, the limit $\bar{q}_\infty$ is an a-harmonic, κ-degree polynomial. From (5.57) and Lemma 5.16, we obtain

$$\kappa\geq\frac{1}{2}\frac{\int_{B_{1/2}}|\nabla\bar{q}_\infty|^2|y|^a}{\int_{\partial B_{1/2}}|\bar{q}_\infty|^2|y|^a}\geq\kappa+\alpha_\kappa,$$

a contradiction, since $\alpha_\kappa>0$. Thus, q_ℓ converges, up to subsequences, locally uniformly to some q_∞, which is an a-harmonic polynomial of degree κ. So $|D^\alpha q_\ell(0)|\leq C$ for some C independently of ℓ for any multiindex $\alpha=(\alpha_1,\dots,\alpha_n,0)$, and for $|a|\leq\kappa-1$,

$$D^\alpha q_\ell(0)=\frac{r_\ell^{|\alpha|}}{h_{r_\ell}}D^\alpha p_*(X_\ell)=\frac{r_\ell^\kappa}{h_{r_\ell}}D^\alpha p_*(Z_\ell).\tag{5.58}$$

Then, as $h_{r_\ell}=o(r_\ell^\kappa)$, we determine that

$$|D^\alpha p_*(Z_\ell)|=o(1)\to0\quad\text{as}\quad\ell\to\infty$$

when $|a|\leq\kappa-1$. That is, $D^\alpha p_*(Z_\infty)=0$ for $|\alpha|\leq\kappa-1$. Thanks to Lemma 5.10,

$$Z_\infty\in L(p_*)\in L_*.$$

Now, by assumption, for some $\boldsymbol{e}_\ell\in\mathbb{S}^{n-1}$ and $c_\ell,c_*>0$,

$$p_{*,X_\ell}(x,0)=c_\ell(\boldsymbol{e}_\ell\cdot x)^\kappa\quad\text{and}\quad p_*(x,0)=c_*(\boldsymbol{e}_*\cdot x)^\kappa.$$

Also, setting $a_\ell:=\boldsymbol{e}_*\cdot z_\ell$, we see that

$$\begin{aligned}q_\ell(x,0)&=h_{r_\ell}^{-1}\left(p_{*,X_\ell}(r_\ell x,0)-p_*(x_\ell+r_\ell x,0)\right)\\&=r_\ell^\kappa h_{r_\ell}^{-1}\left(c_\ell(\boldsymbol{e}_\ell\cdot x)^\kappa-c_*(\boldsymbol{e}_*\cdot(z_\ell+x))^\kappa\right)\\&=r_\ell^\kappa h_{r_\ell}^{-1}\Bigg(c_\ell(\boldsymbol{e}_\ell\cdot x)^\kappa-c_*(\boldsymbol{e}_*\cdot x)^\kappa-c_*\kappa a_\ell(\boldsymbol{e}_*\cdot x)^{\kappa-1}\\&\qquad\qquad+c_*a_\ell^2\sum_{j=2}^{\kappa}\binom{\kappa}{j}\alpha_\ell^{j-2}(\boldsymbol{e}_*\cdot x)^{\kappa-j}\Bigg).\end{aligned}$$

Since $p_{*,X_\ell}\to p_*$, we have that $c_\ell\to c_*$ and $\boldsymbol{e}_\ell\to\boldsymbol{e}_*$ (up to a sign). Moreover, as $Z_\ell\to Z_\infty\in L_*$ and $\boldsymbol{e}_*\perp L_*$, $a_\ell\to0$. Therefore, by the uniform boundedness in ℓ of the coefficients of $q_\ell(x,0)$, we immediately find that

$$\begin{aligned}q_\ell(x,0)&=r_\ell^\kappa h_{r_\ell}^{-1}\big(c_\ell(\boldsymbol{e}_\ell\cdot x)^\kappa-c_*(\boldsymbol{e}_*\cdot x)^\kappa-c_*\kappa a_\ell(\boldsymbol{e}_*\cdot x)^{\kappa-1}\big)+O(a_\ell)\\&=r_\ell^\kappa h_{r_\ell}^{-1}\big((c_\ell-c_*)(\boldsymbol{e}_\ell\cdot x)^\kappa+c_*((\boldsymbol{e}_\ell\cdot x)^\kappa-(\boldsymbol{e}_*\cdot x)^\kappa)-c_*\kappa a_\ell(\boldsymbol{e}_*\cdot x)^{\kappa-1}\big)+O(a_\ell).\end{aligned}$$

Set $\boldsymbol{e}'_\ell := \frac{\boldsymbol{e}_\ell - \boldsymbol{e}_*}{|\boldsymbol{e}_\ell - \boldsymbol{e}_*|}$. Then,

$$\frac{(\boldsymbol{e}_\ell \cdot x)^\kappa - (\boldsymbol{e}_* \cdot x)^\kappa}{|\boldsymbol{e}_\ell - \boldsymbol{e}_*|} = \left(\frac{\boldsymbol{e}_\ell - \boldsymbol{e}_*}{|\boldsymbol{e}_\ell - \boldsymbol{e}_*|} \cdot x\right) \sum_{i=1}^{\kappa-1} (\boldsymbol{e}_* \cdot x)^i (\boldsymbol{e}_\ell \cdot x)^{\kappa-1-i} =: (\boldsymbol{e}'_\ell \cdot x)\, Q_\ell(x).$$

In addition, as $\ell \to \infty$,

$$\boldsymbol{e}'_\ell \to \boldsymbol{e}'_\infty \in \mathbb{S}^{n-1} \quad \text{and} \quad Q_\ell \to (\kappa-1)(\boldsymbol{e}_* \cdot x)^{\kappa-1},$$

and $\boldsymbol{e}'_\infty \perp \boldsymbol{e}_*$. Thus,

$$q_\infty(x,0) = c_1(\boldsymbol{e}_* \cdot x)^\kappa + c_2(\boldsymbol{e}'_\infty \cdot x)(\boldsymbol{e}_* \cdot x)^{\kappa-1} + c_3(\boldsymbol{e}_* \cdot x)^{\kappa-1}, \tag{5.59}$$

for some constants c_1, c_2, and c_3. So q_∞ vanishes on L_*.

Thanks to Lemma 5.8 applied to $u(X_\ell + r_\ell\,\cdot\,) - p_{*,X_\ell}$, denoting $\lambda_\ell := \lambda_{*,X_\ell}$, for all $\rho \in (0,1/2)$,

$$\frac{1}{\rho^{2\lambda_\ell+a}} \fint_{\partial B_\rho} |\tilde v_{r_\ell}(Z_\ell + \,\cdot\,) - q_\ell|^2 |y|^a \le 2^{2\lambda_\ell - a} \fint_{\partial B_{1/2}} |\tilde v_{r_\ell}(Z_\ell + \,\cdot\,) - q_\ell|^2 |y|^a,$$

from which we deduce that, taking $\ell \to \infty$,

$$\frac{1}{\rho^{2\lambda+a}} \fint_{\partial B_\rho} |q(Z_\infty + \,\cdot\,) - q_\infty|^2 |y|^a \le C \fint_{\partial B_{1/2}} |q(Z_\infty + \,\cdot\,) - q_\infty|^2 |y|^a. \tag{5.60}$$

In turn, because $q_\infty(X) = \operatorname{Ext}_a(q_\infty(x,0))$ and by (5.59),

$$\begin{aligned}
\fint_{\partial B_\rho} |q^{\text{even}}(Z_\infty + \,\cdot\,) - \operatorname{Ext}_a(c_1(\boldsymbol{e}_* \cdot x))^\kappa|^2 |y|^a &= \fint_{\partial B_\rho} |(q(Z_\infty + \,\cdot\,) - q_\infty)^{\text{even}}|^2 |y|^a \\
&\le \fint_{\partial B_\rho} |q(Z_\infty + \,\cdot\,) - q_\infty|^2 |y|^a \\
&\le C\rho^{2\lambda+a} \fint_{\partial B_{1/2}} |q(Z_\infty + \,\cdot\,) - q_\infty|^2 |y|^a,
\end{aligned}$$

from which, taking $c_\infty = c_1$, we find (5.55). (Here, we have used that taking the even part of a function with respect to L_*, i.e., $f \mapsto f^{\text{even}}$, is an orthogonal projection in $L^2(\partial B_\rho, |y|^a)$.)

Step 2: Let us now show that if $\lambda_* < \kappa + 1$, then $q^{\text{odd}} \equiv 0$; and if, in addition, $\lambda = \lambda_*$, then $c_\infty = 0$ in (5.55). We remark that the fact that $q^{\text{odd}} \equiv 0$ if $\lambda_* \notin \mathbb{N}$ is independent of Step 1.

If $X \in \mathbb{R}^{n+1} \setminus L_*$, then $X - 2(\boldsymbol{e}_* \cdot X)\boldsymbol{e}_* \in \mathbb{R}^{n+1} \setminus L_*$; so

$$L_a q^{\text{odd}}(X) = L_a q(X) - L_a q(X - 2(\boldsymbol{e}_* \cdot X)\boldsymbol{e}_*) = 0 \quad \text{for} \quad X \in \mathbb{R}^{n+1} \setminus L_*$$

(by Proposition 5.15, q solves the very thin obstacle problem and is a-harmonic outside of L_*). On the other hand, if $X \in L_*$, then we have that $X - 2(\boldsymbol{e}_* \cdot X)\boldsymbol{e}_* = X$. And so,

$$L_a q^{\text{odd}}(X) = L_a q(X) - L_a q(X) = 0 \quad \text{for} \quad X \in L_*.$$

Therefore, q^{odd} is a-harmonic in $\mathbb{R}^{n+1}$. This, together with the fact that q^{odd} is λ_*-homogeneous (again, by Proposition 5.15) and even in y, yields that, by Liouville's theorem for a-harmonic functions, q^{odd} is a λ_*-homogeneous polynomial (see, e.g., [CSS08, Lemma 2.7]). Hence, if $\kappa < \lambda_* < \kappa + 1$, then $q^{\text{odd}} \equiv 0$, and $q = q^{\text{even}}$.

Finally, let us now show that if $\lambda = \lambda_* < \kappa + 1$, then $c_\infty = 0$. Let

$$q_{Z_\infty}(X) := q(Z_\infty + X) - c_\infty \operatorname{Ext}_a((\boldsymbol{e}_* \cdot x)^\kappa),$$

which is a solution to the very thin obstacle problem with zero obstacle on L_*. If (5.55) holds with $\lambda = \lambda_*$, then from Lemma 5.55 and recalling that $q = q^{\text{even}}$, we deduce that

$$N(0^+, q_{Z_\infty}) \geq \lambda_*.$$

In turn, q_{Z_∞} is λ_*-homogeneous. Indeed, for all $r > 0$, by Lemma 5.52,

$$\lambda_* \leq N(r, q_{Z_\infty}) \leq N(+\infty, q_{Z_\infty}) = N(+\infty, q(X) - c_\infty \operatorname{Ext}_a((\boldsymbol{e}_* \cdot x)^\kappa)) = \lambda_*.$$

The penultimate equality holds since the limit as $r \to +\infty$ of Almgren's frequency function is independent of the point at which it is centered, and the last equality holds because q is λ_*-homogeneous with $\lambda_* > \kappa$, and thus q out-scales a κ-homogeneous polynomial.

Since q_{Z_∞} is λ_*-homogeneous, we deduce that

$$q(X + Z_\infty) = \frac{q(X) + q(X + 2Z_\infty)}{2}. \tag{5.61}$$

To see this, first, observe that

$$\tau^{\lambda_*} q(X + \tau^{-1} Z_\infty) = q(\tau X + Z_\infty) = \tau^{\lambda_*} q_{Z_\infty}(X) + \tau^\kappa c_\infty \operatorname{Ext}_a((\boldsymbol{e}_* \cdot x)^\kappa),$$

for all $\tau > 0$. The first equality follows from the λ_*-homogeneity of q, while the second follows from the λ_*-homogeneity of q_{Z_∞}. So

$$q(X + \tau^{-1} Z_\infty) - q_{Z_\infty}(X) = \tau^{\kappa - \lambda_*} c_\infty \operatorname{Ext}_a((\boldsymbol{e}_* \cdot x)^\kappa),$$

for all $\tau > 0$. Taking the limit as $\tau \to +\infty$ yields

$$q_{Z_\infty} = q. \tag{5.62}$$

(Recall, $\lambda_* > \kappa$.) That is,

$$c_\infty \operatorname{Ext}_a((\boldsymbol{e}_* \cdot x)^\kappa) = q(X + Z_\infty) - q(X). \tag{5.63}$$

And because $e_* \perp Z_\infty$,

$$c_\infty \operatorname{Ext}_a((\boldsymbol{e}_* \cdot x)^\kappa) = q(X) - q(X - Z_\infty).$$

Hence, (5.61) holds, as desired.

To conclude, from the λ_*-homogeneity of q and (5.63), observe that

$$\partial^{(\kappa)}_{\boldsymbol{e}_*} q(Z_\infty) = \kappa! c_\infty.$$

On the other hand, (5.61) implies

$$\partial^{(\kappa)}_{\boldsymbol{e}_*} q(Z_\infty) = \frac{\partial^{(\kappa)}_{\boldsymbol{e}_*} q(2Z_\infty)}{2} = 2^{\lambda_*-\kappa-1}\partial^{(\kappa)}_{\boldsymbol{e}_*} q(Z_\infty).$$

Thus,

$$(1 - 2^{\lambda_*-\kappa-1})\kappa! c_\infty = 0.$$

Yet $\lambda_* - \kappa - 1 \neq 0$, by assumption. Consequently, $c_\infty = 0$.

Therefore, we have that if $\lambda_* < \kappa + 1$ and $\lambda = \lambda_*$,

$$\frac{1}{\rho^{n+2\lambda_*+a}} \int_{\partial B_\rho} |q(Z_\infty + \,\cdot\,)|^2 |y|^a \leq C. \tag{5.64}$$

By Lemma 5.55, $N(0^+, q(Z_\infty + \cdot\,)) = \lambda_*$ as $q(Z_\infty + \cdot\,)$ is a solution to the very thin obstacle problem. On the other hand, since q is λ_*-homogeneous, $N(+\infty, q(Z_\infty + \cdot\,)) = \lambda_*$, and from the monotonicity formula in Lemma 5.52, we deduce that $q(Z_\infty + \cdot\,)$ is λ_*-homogeneous. Then,

$$q(X + Z_\infty) = \tau^{\lambda_*} q(\tau^{-1}X + Z_\infty) = q(X + \tau Z_\infty) \quad \text{for all} \quad X \in \mathbb{R}^n \text{ and } \tau > 0;$$

that is, q is invariant in the Z_∞ direction. □

We close this section with a pair of remarks and a Case 2 version of Lemma 5.18. The observations made in these remarks are crucial to our analysis of when we can produce the next term in the expansion of u around a singular point.

Remark 5.12. In Lemma 5.19, as in Lemma 5.17, if q is an a-harmonic, $(\kappa + 1)$-homogeneous polynomial and $\lambda = \lambda_* = \kappa + 1$, we also have that

$$D^\alpha q(Z_\infty) = 0 \quad \text{for all} \quad \alpha = (\alpha', 0) \text{ and } |\alpha| \leq \kappa - 2. \tag{5.65}$$

Indeed, observe that (5.60) becomes

$$⨍_{\partial B_\rho} |q(Z_\infty + \,\cdot\,) - q_\infty|^2 |y|^a \leq C\rho^{2(\kappa+1)+a},$$

for all $\rho \in (0, 1/2)$. Hence, the polynomial $q(Z_\infty + \,\cdot\,) + q_\infty$ is only made up of monomials of degree $\kappa + 1$. In particular, since q is $(\kappa + 1)$-homogeneous and q_∞ is of degree κ, $q(Z_\infty + \,\cdot\,) + q_\infty$ is a $(\kappa + 1)$-homogeneous polynomial. So, for all multiindices $|\alpha| \leq \kappa$,

$$D^\alpha q(Z_\infty) = D^\alpha q_\infty(0),$$

which, by (5.59), implies (5.65) holds, as desired.

Remark 5.13. The last part of the proof of Lemma 5.19 fails to show that q^{even} is invariant in the Z_∞ direction when $\lambda = \lambda_* = \kappa + 1$. In this case, however, we find that

$$q^{\text{e}}_{Z_\infty}(X) := q^{\text{even}}(Z_\infty + X) - c_\infty \operatorname{Ext}_a((\boldsymbol{e}_* \cdot x)^\kappa),$$

is λ_*-homogeneous. Hence,

$$q^{\text{even}}(X + \tau^{-1}Z_\infty) - q^{\text{e}}_{Z_\infty}(X) = \tau^{-1} c_\infty \operatorname{Ext}_a((\boldsymbol{e}_* \cdot x)^\kappa),$$

as before, for all $\tau > 0$. By letting $\tau \to \infty$, we deduce that $q^{\mathrm{e}}_{Z_\infty} = q^{\mathrm{even}}$, which substituting back yields

$$q^{\mathrm{even}}(X + \tau Z_\infty) = q^{\mathrm{even}}(X) + \tau c_\infty \operatorname{Ext}_a((\boldsymbol{e}_* \cdot x)^\kappa) \quad \text{for all} \quad X \in \mathbb{R}^n \text{ and } \tau > 0. \tag{5.66}$$

Moreover, considering $X - Z_\infty$, we find that $q^{\mathrm{even}}(X) = q^{\mathrm{even}}(X - Z_\infty) + c_\infty \operatorname{Ext}_a((\boldsymbol{e}_* \cdot x)^\kappa)$ for all $X \in \mathbb{R}^{n+1}$. Hence, (5.66) for all $\tau \in \mathbb{R}$.

Lemma 5.20. *Suppose Case 2 holds. Then,* 0 *is an isolated point of* $\Sigma_{\geq\kappa}$.

Proof. The proof is identical to that of Lemma 5.18, but using Lemma 5.19 instead of Lemma 5.17. □

5.5 The Size of the Anomalous Set

The goal of this section is to further stratify the set of singular points and prove Proposition 5.4 and Remark 5.6. Proposition 5.4 (and Remark 5.6) is a statement regarding the Hausdorff dimension of the anomalous singular points of order 2 and $(n-1)$-dimensional singular points of arbitrary order (i.e., singular points whose first blow-up has $(n-1)$-dimensional spine and is κ-homogeneous). As such, let us recall the definition of anomalous singular points, and generic singular points, as well as some measure theoretic facts.

5.5.1 Singular Points Revisited

Given the set of singular points of order κ and dimension m (i.e., whose first blow-up has m-dimensional spine), we recall that the anomalous points are those for which the homogeneity of second blow-ups is strictly less than $\kappa + 1$:

$$\Sigma^{m,\mathrm{a}}_\kappa := \{X_\circ \in \Sigma^m_\kappa : N(0^+, u(X_\circ + \cdot\,) - p_{*,X_\circ}) < \kappa + 1\}.$$

The generic points, on the other hand, are those for which the homogeneity of second blow-ups jumps by at least one:

$$\Sigma^{m,\mathrm{g}}_\kappa := \{X_\circ \in \Sigma^m_\kappa : N(0^+, u(X_\circ + \cdot\,) - p_{*,X_\circ}) \geq \kappa + 1\}.$$

In turn, $\Sigma^{m,\mathrm{g}}_\kappa = \Sigma^m_\kappa \setminus \Sigma^{m,\mathrm{a}}_\kappa$.

While Proposition 5.4 and Remark 5.6 ignore higher order (greater than two) and lower dimensional (less than $n-1$) singular points, our analysis, in a sense, does not. In particular, our results rely on the alignment of the nodal set and the spine of first blow-ups at anomalous singular points. And so, we set

$$\tilde{\Sigma}_\kappa := \{X_\circ \in \Sigma_\kappa : \mathcal{N}(p_{*,X_\circ}) = L(p_{*,X_\circ})\}$$

and define

$$\tilde{\Sigma}^{m,\mathrm{a}}_\kappa := \tilde{\Sigma}_\kappa \cap \Sigma^{m,\mathrm{a}}_\kappa \quad \text{and} \quad \tilde{\Sigma}^{m,\mathrm{g}}_\kappa := \tilde{\Sigma}_\kappa \cap \Sigma^{m,\mathrm{g}}_\kappa.$$

Remark 5.14. A key consequence of the coincidence of $\mathcal{N}(p_{*,X_\circ})$ and $L(p_{*,X_\circ})$ is that $p_{*,X_\circ}|_{\mathbb{R}^n \times \{0\}}$ is positive away from its spine, i.e., if $\mathcal{N}(p_{*,X_\circ}) = L(p_{*,X_\circ})$, then $p_{*,X_\circ}(x, 0) > 0$ for any $x \in \mathbb{R}^n$ such that $x \notin L(p_{*,X_\circ})$.

Remark 5.15. Notice that if $m = n-1$ or if $\kappa = 2$, then $\tilde{\Sigma}_\kappa^{m,\mathrm{a}} = \Sigma_\kappa^{m,\mathrm{a}}$. Moreover, if the spine and the nodal set of the first blow-up at anomalous points coincide, Case 1 and Case 2 exhaust all possibilities (cf. Remark 5.10).

5.5.2 Some Measure Theory

Given $\beta > 0$ and $\delta \in (0, \infty]$, we define the Hausdorff premeasures

$$\mathcal{H}^\beta_\delta(E) := \inf\left\{\sum_i \omega_\beta \left(\frac{\operatorname{diam} E_i}{2}\right)^\beta : E \subset \bigcup_i E_i \text{ with } \operatorname{diam} E_i < \delta\right\},$$

so that the β-dimensional Hausdorff measure of a set E is

$$\mathcal{H}^\beta(E) := \lim_{\delta \downarrow 0} \mathcal{H}^\beta_\delta(E).$$

(Here, ω_β is the volume of the β-dimensional unit ball.) The Hausdorff dimension of a set can then be defined as

$$\dim_{\mathcal{H}} E := \inf\{\beta > 0 : \mathcal{H}^\beta_\infty(E) = 0\}. \tag{5.67}$$

(See, e.g., [Sim83].)

Lemma 5.21. *Let $E \subset \mathbb{R}^{n+1}$ be a set with $\mathcal{H}^\beta_\infty(E) > 0$ for some $\beta \in (0, n+1]$. The following holds:*

(i) For $\mathcal{H}^\beta$-almost every point $X_\circ \in E$, there is a sequence $r_j \downarrow 0$ such that

$$\lim_{k \to \infty} \frac{\mathcal{H}^\beta_\infty(E \cap B_{r_j}(X_\circ))}{r_j^\beta} \geq c_{n,\beta} > 0, \tag{5.68}$$

where $c_{n,\beta}$ is a constant depending only on n and β. We call these points "density points".

(ii) Assume that $0 \in E$ is a "density point", let $r_j \downarrow 0$ be a sequence along which (5.68) *holds, and define the "accumulation set" for E at 0 as*

$$\mathcal{A}_E := \{Z \in \overline{B_{1/2}} : \exists \{Z_\ell\}_{\ell \in \mathbb{N}}, \{j_\ell\}_{\ell \in \mathbb{N}} \text{ such that } Z_\ell \in r_{j_\ell}^{-1} E \cap B_{1/2} \text{ and } Z_\ell \to Z\}.$$

Then,

$$\mathcal{H}^\beta_\infty(\mathcal{A}) > 0.$$

Proof. See [FS18, Lemma 3.5]. □

In order to prove that anomalous points form a small set in Case 2, we will focus on "almost continuity" points of the frequency, in the spirit of [FRS19]. More precisely, as shown in [FRS19], points where the frequency is discontinuous along "too many" sets of converging sequences have small Hausdorff measure. This fact, which plays a crucial role in [FRS19], allows us to use Lemma 5.19 to show that second blow-ups are translation invariant in directions of "almost continuity" of the frequency.

Lemma 5.22. *Let $E \subset \mathbb{R}^{n+1}$ and $f : E \to \mathbb{R}$ be any function. The set*

$$\{X \in E : \text{for all } \{X_\ell\}_{\ell\in\mathbb{N}} \text{ such that } X_\ell \neq X \text{ and } X_\ell \to X,\ f(X_\ell) \not\to f(X)\}$$

is at most countable.

Proof. If $X_\circ$ is an element of the set in question, then $(X_\circ, f(X_\circ))$ is an isolated point of $\{(X, f(X)) : X \in E\}$. Since collection of isolated points of any subset of $\mathbb{R}^{n+2}$ is at most countable, the lemma follows. □

Lemma 5.23. *Let $E \subset \mathbb{R}^n$ and $f : E \to [0,\infty)$ be any function. Suppose for any $x \in E$ and any $\varepsilon > 0$, there exists a $\rho > 0$ such that for all $r \in (0, \rho)$,*

$$E \cap \overline{B_r(x)} \cap f^{-1}\left([f(x) - \rho, f(x) + \rho]\right) \subset \{z : \mathrm{dist}(z, \Pi_{x,r}) \le \varepsilon r\} \tag{5.69}$$

for some m-dimensional plane $\Pi_{x,r}$ passing through x, possibly depending on r. Then,

$$\dim_{\mathcal{H}} E \le m.$$

Proof. See [FRS19]. □

5.5.3 Proofs of Proposition 5.4 and Remark 5.6

We now move to the goal of this section. We start with a set of results which pertain to Case 1.

Proposition 5.24. *Assume $n \ge 2$.*

(i) If $a \in [0,1)$, $\dim_{\mathcal{H}} \tilde{\Sigma}^{m,\mathrm{a}}_\kappa \le m - 1$ for any $1 \le m \le n - 1$.

(ii) If $a \in (-1, 0)$, $\dim_{\mathcal{H}} \tilde{\Sigma}^{m,\mathrm{a}}_\kappa \le m - 1$ for any $1 \le m \le n - 2$.

Proof. The first part of the proof follows the steps of [FS18, Lemma 3.6]. Set $\Sigma := \tilde{\Sigma}^{m,\mathrm{a}}_\kappa$.

Step 1: We argue by contradiction. Suppose that $\mathcal{H}^\beta_\infty(\Sigma) > 0$ for some $\beta > m - 1$. Then, there is a point $X_\circ \in \Sigma$ and sequence $r_j \downarrow 0$ such that

$$\frac{\mathcal{H}^\beta_\infty(\Sigma \cap B_{r_j}(X_\circ))}{r_j^\beta} \ge c_{n,\beta} > 0.$$

Up to a translation, $X_\circ = 0$. By definition, we have that

$$\lambda_* := N(0^+, v_*) < \kappa + 1,$$

and that, after extracting a subsequence,

$$\tilde{v}_{r_j} \to q \quad \text{in} \quad L^2(B_1, |y|^a).$$

Additionally, from Lemma 5.21(ii), we have that the accumulation set $\mathcal{A} := \mathcal{A}_\Sigma$ satisfies

$$\mathcal{H}^\beta_\infty(\mathcal{A}) > 0. \tag{5.70}$$

By the definition of $\mathcal{A}$, $Z \in \mathcal{A}$ if there are sequences $X_\ell \in \Sigma$ and $r_{j_\ell} \downarrow 0$ such that $|X_\ell| \le r_{j_\ell}$ and $X_\ell/r_{j_\ell} \to Z$. Thus, $X_\ell/2r_{j_\ell} \to Z/2$, and by Lemma 5.17 (notice that we are in Case 1), $Z \in L_*$ and $Z \in T(q) := \{X = (x, 0) \in \mathbb{R}^{n+1} : q(x + \cdot\,, 0) - q(\cdot\,, 0)$ is invariant under $L_*\}$. That is,

$$\mathcal{A} \subset \overline{B_1} \cap L_* \cap T(q).$$

Notice that by assumption, q is κ-homogeneous and $\dim_{\mathcal{H}} L_* = m$. Also, $T(q)$ is a linear space. We will, therefore, reach a contradiction if we can show that $L_* \not\equiv T(q) \cap L_*$; since then, $\dim_{\mathcal{H}} L_* \cap T(q) \le m - 1$, which contradicts (5.70).

Step 2: Let $\bar{p} := p|_{\mathbb{R}^n \times \{0\}}$ for any a-harmonic, even in y polynomial p, and recall that $\bar{p}$ uniquely determines p (see the lines after (5.13)). Suppose, again, to the contrary, that $L_* \equiv T(q) \cap L_*$. After a change of variables, since L_* has dimension m, we can assume that $\bar{p}_* = \bar{p}_*(x_1, \dots, x_l)$ for $l = n - m$. Set $x^l = (x_1, \dots, x_l)$. Notice that $L_* = \{(0^l, x^m, 0) : x^m \in \mathbb{R}^m\}$, where $0^l \in \mathbb{R}^l$ denotes the vector 0 in l dimensions. The inclusion $L_* \subset T(q)$ implies that $\bar{q}((0^l, x^m) + \cdot\,) - \bar{q}$ can only depend on x^l for any $x^m \in \mathbb{R}^m$. This, together with the homogeneity of q, directly yields that

$$\bar{q}(x) = q_1(x^l) + \sum_{j=l+1}^{n} q_j(x^l)x_j =: \bar{q}_1(x^l) + \bar{q}_2(x),$$

where q_1 and q_j are κ-homogeneous and $(\kappa-1)$-homogeneous polynomial respectively depending only on $x_1, \dots, x_l$.

Now recall Lemma 5.14:

$$\int_{B_1} qp|y|^a \le 0 \quad \text{for all} \quad p \in \mathscr{P}_\kappa. \tag{5.71}$$

Moreover, (5.71) is an equality if $p = p_*$ (this is (5.33)). Notice, first, that (recall (5.13))

$$\int_{B_1} \mathrm{Ext}_a(\bar{q}_1)\, \mathrm{Ext}_a(\bar{q}_2)|y|^a = 0. \tag{5.72}$$

Indeed, $\bar{q}_1$ does not depend on $x_{l+1}, \dots, x_n$, whereas the terms of $\bar{q}_2$ are sums of linear terms in one of $x_{l+1}, \dots, x_n$; thus, odd in one of the last variables.

Since $0 \in \Sigma = \tilde{\Sigma}_\kappa^{m,\mathrm{a}}$, $\bar{p}_*(x^l, 0^m) > 0$ for all $x^l \in \mathbb{R}^l \setminus \{0\}$. In particular, we can choose $C \gg 1$ such that $C\bar{p}_* + \bar{q}_1 \ge 0$ ($\bar{p}_*$ and $\bar{q}_1$ have the same homogeneity and depend on the same variables). That is, $\mathrm{Ext}_a(C\bar{p}_* + \bar{q}_1) = Cp_* + \mathrm{Ext}_a(\bar{q}_1) \in \mathscr{P}_\kappa$, from which it follows that

$$0 \ge \int_{B_1} (Cp_* + \mathrm{Ext}_a(\bar{q}_1))q|y|^a = \int_{B_1} \mathrm{Ext}_a(\bar{q}_1)^2|y|^a,$$

using the equality in (5.71) and (5.72). Hence,

$$\bar{q}_1 \equiv 0.$$

Finally, fix $l + 1 \le j \le n$, and take $\bar{p}_j := C(|x^l|^\kappa + x_j^\kappa) + q_j(x^l)x_j$ for some $C \gg 1$ so that $\bar{p}_j \ge 0$. (The fact that such a constant $C > 0$ exists is straight-forward.

Indeed, it suffices to show that $x_1^\kappa + x_2^\kappa - x_1^{\kappa-1}x_2 \geq 0$, which after dividing by x_2^κ is analogous to showing that $\xi^\kappa \geq \xi - 1$ for all $\xi \in \mathbb{R}$; this is immediate.) Arguing as before, by odd/even symmetry, we find that

$$\int_{B_1} \operatorname{Ext}_a(q_j x_j) \operatorname{Ext}_a(q_i x_i)|y|^a = 0 \quad \text{for all} \quad l+1 \leq i \neq j \leq n$$

and

$$\int_{B_1} \operatorname{Ext}_a(|x^l|^\kappa + x_j^\kappa) \operatorname{Ext}_a(q_i x_i)|y|^a = 0 \quad \text{for all} \quad l+1 \leq i, j \leq n.$$

And so, as $\bar{q}_1 \equiv 0$ and $\operatorname{Ext}_a(\bar{p}_j) \in \mathscr{P}_\kappa$,

$$0 \geq \int_{B_1} \operatorname{Ext}_a(\bar{p}_j) q|y|^a = \int_{B_1} \operatorname{Ext}_a(q_j x_j)^2 |y|^a,$$

which is only true if $q_j \equiv 0$. Because j was fixed arbitrarily, we deduce that

$$\bar{q}_2 \equiv 0.$$

A contradiction. □

Lemma 5.25. *Let $n \geq 2$. Then, $\tilde{\Sigma}_\kappa^{0,\mathrm{a}}$ is empty.*

Proof. Suppose $0 \in \tilde{\Sigma}_\kappa^{0,\mathrm{a}}$. Then, $\mathcal{N}_* = L_* = \{0\}$ and $\bar{p}_* := p_*|_{\mathbb{R}^n \times \{0\}} > 0$ outside of the origin. Hence, there exists a $C \gg 1$ so that $\operatorname{Ext}_a(C\bar{p}_* + \bar{q}) = Cp_* + q \in \mathscr{P}_\kappa$ (cf. Step 2 of the proof of Proposition 5.24). Here, $\bar{q}$ is the restriction of any second blow-up of u at 0. So, by (5.34) and (5.33), we find that

$$0 \geq \int_{\partial B_1} q(Cp_* + q)|y|^a = \int_{\partial B_1} q^2 |y|^a,$$

which cannot be: $q \not\equiv 0$. □

Lemma 5.26. *Let $n \geq 2$ and $a \in [0,1)$ or $n \geq 3$ and $a \in (-1,1)$. Then, $\tilde{\Sigma}_\kappa^{1,\mathrm{a}}$ is isolated in $\Sigma_{\geq\kappa}$.*

Proof. Suppose not and assume that $0 \in \tilde{\Sigma}_\kappa^{1,\mathrm{a}}$. Then, there exists a sequence $X_\ell \in \Sigma_{\geq\kappa}$ with $X_\ell \to 0$. Let $r_\ell := 2|X_\ell|$, and notice that $\dim_{\mathcal{H}} L_* = \dim_{\mathcal{H}}\{p_* = 0\} = 1$, by assumption. On the other hand, up to a subsequence, we can assume that $\tilde{v}_{r_\ell} \to q$ in $L^2(B_1, |y|^a)$, which is κ-homogeneous.

The proof now follows exactly as in Step 2 of the proof of Proposition 5.24. □

Remark 5.16. In all of the above results, Proposition 5.24, Lemma 5.25, and Lemma 5.26, the coincidence of the nodal set and the spine of p_* is crucial. To illustrate how much, let us consider $\Sigma_4^{0,\mathrm{a}}$, which we would like to say is empty. (Notice that $\Sigma_2^{0,\mathrm{a}}$ is empty; in this case, the nodal set and spine of the first blow-up at any point are aligned.)

In Proposition 5.24, in order to rule out a κ-homogeneous, a-harmonic q as a second blow-up at anomalous points, we have used Lemmas 5.17 and 5.14. Since we are dealing with $\Sigma_4^{0,\mathrm{a}}$, Lemma 5.17 provides no new information on q. Also, Lemma 5.14 is too weak to rule out that q is a 4-homogeneous, harmonic (assume $a = 0$, for simplicity) polynomial.

Indeed, in $\mathbb{R}^3 = \{(x_1, x_2, y)\}$, consider the harmonic extensions

$$p_* = \mathrm{Ext}_0(x_1^2 x_2^2)$$

and

$$q = \mathrm{Ext}_0\left(bx_1^4 - \left(\frac{11}{24} + b\right)x_2^4 + x_1^2x_2^2\right) \quad \text{with} \quad b \in \left[-\frac{1}{3}, -\frac{1}{8}\right].$$

Notice that the spine and nodal set of p_* are different:

$$L_* = \{(0,0)\} \quad \text{while} \quad \mathcal{N}_* = \{x_1 = 0\} \cup \{x_2 = 0\}.$$

Moreover, by direct (but tedious) computations, the pair (p_*, q) is such that

$$\langle q, p_* \rangle_0 = 0 \qquad \text{and} \qquad \langle q, p \rangle_0 \le 0 \quad \text{for all} \quad p \in \mathscr{P}_4.$$

In turn, this pair could be a first and second blow-up pair at 0 for a solution u for which $\Sigma_4^0(u) = \{0\}$, leaving open the possibility that $\Sigma_4^{0,\mathrm{a}}$ is not only not lower dimensional, but all of Σ_4^0.

Now we study of the size of the anomalous set in Case 2.

Lemma 5.27. *Let $n = 2$ and $a \in (-1, 0)$. Then, $\Sigma_\kappa^{1,\mathrm{a}}$ is at most countable.*

Proof. Assume that $0 \in \Sigma_\kappa^{1,\mathrm{a}}$, which holds up to a translation, and that there exists a sequence $X_\ell \in \Sigma_\kappa^1$ such that $X_\ell \to 0$ and $N(0^+, u(X_\ell + \cdot\,) - p_{*,X_\ell}) =: \lambda_{*,X_\ell} \to \lambda_* = N(0^+, v_*)$. By Proposition 5.15 and the definition of anomalous set, $\Sigma_\kappa^{1,\mathrm{a}}$, we have that

$$\lambda_* \in [\kappa + \alpha_\kappa, \kappa + 1). \tag{5.73}$$

Moreover, up to a subsequence, if $r_\ell := 2|X_\ell|$,

$$\tilde{v}_{r_\ell} \rightharpoonup q \quad \text{in} \quad W^{1,2}(B_1, |y|^a)$$

where q is a global λ_*-homogeneous solution to the very thin obstacle problem with zero obstacle on L_*. In addition, by Lemma 5.19, $X_\ell/r_\ell \to Z_\infty \in L_* \cap \partial B_{1/2}$, $q = q^{\mathrm{even}}$ ($\lambda_* < \kappa + 1$ by assumption, forcing $q^{\mathrm{odd}} \equiv 0$), and q is invariant in the Z_∞ direction (i.e., in the L_* direction). That is, q is two-dimensional. So by Lemma 5.58, since $\lambda_* > 2$, q is a polynomial and, in particular, $\lambda_* \ge \kappa + 1$. But this contradicts (5.73).

In turn, by Lemma 5.22 applied to $E = \Sigma_\kappa^{n-1}$ and $f(X) = N(0^+, u(X + \cdot\,) - p_{*,X})$, we conclude. □

Proposition 5.28. *Let $n \ge 3$ and $a \in (-1, 0)$. Then, $\dim_{\mathcal{H}} \Sigma_\kappa^{n-1,\mathrm{a}} \le n - 2$.*

Proof. Let $\Sigma := \Sigma_\kappa^{n-1,\mathrm{a}}$. We will show that Σ fulfills the hypotheses of Lemma 5.23 with $m = n - 2$ and

$$f(X) := \begin{cases} N(0^+, u(X + \cdot\,) - p_{*,X}) & \text{if } X \in \mathbb{R}^n \times \{0\} \\ 0 & \text{otherwise.} \end{cases} \tag{5.74}$$

Then, by Lemma 5.23, $\dim_{\mathcal{H}} \Sigma \le n - 2$.

Suppose, to the contrary, that (5.69) does not hold, that is, in particular, there exists some $X_\circ \in \Sigma$, $\varepsilon_\circ > 0$, $\rho_\ell \downarrow 0$ as $\ell \to \infty$, and $0 < r_\ell < \rho_\ell$ for which

$$\inf_{\Pi \in \Pi_{X_\circ}} \operatorname{dist}\left(\Sigma \cap \overline{B_{r_\ell}(X_\circ)} \cap f^{-1}\left([f(X_\circ) - \rho_\ell, f(X_\circ) + \rho_\ell]\right), \Pi\right) \geq \varepsilon_\circ r_\ell, \tag{5.75}$$

where $\Pi_{X_\circ}$ denotes the set of $(n-2)$-dimensional planes passing through $X_\circ$, and we denote $\operatorname{dist}(A, B) := \sup_{x \in A} \inf_{y \in B} |x - y|$. Up to a translation, assume $X_\circ = 0$.

We claim (and prove later) that thanks to (5.75), for any $\ell \in \mathbb{N}$, there exist $n-1$ points

$$X_1^\ell, \ldots, X_{n-1}^\ell \in \Sigma \cap \overline{B_{r_\ell}} \cap f^{-1}\left([f(0) - \rho_\ell, f(0) + \rho_\ell]\right) \tag{5.76}$$

such that

$$|Y_1^\ell \wedge \cdots \wedge Y_{n-1}^\ell| \geq \varepsilon_\circ^{n-1} \quad \text{where} \quad Y_i^\ell := r_\ell^{-1} X_i^\ell \in B_1 \setminus B_{\varepsilon_\circ}, \tag{5.77}$$

for all $i \in \{1, \ldots, n-1\}$. In particular, up to subsequences, $Y_i^\ell \to Y_i^\infty \in B_1 \setminus B_{\varepsilon_\circ}$ for all $i \in \{1, \ldots, n-1\}$, and passing to the limit, in (5.77),

$$|Y_1^\infty \wedge \cdots \wedge Y_{n-1}^\infty| \geq \varepsilon_\circ^{n-1}. \tag{5.78}$$

On the other hand, from (5.76), we have that

$$f(X_i^\ell) \to f(0) \quad \text{as} \quad \ell \to \infty \quad \text{for all} \quad i \in \{1, \ldots, n-1\}. \tag{5.79}$$

Up to subsequences, by Proposition 5.15,

$$\frac{v_{r_\ell}}{\|v_{r_\ell}\|_{L^2(\partial B_1, |y|^a)}} \rightharpoonup q \quad \text{in} \quad W^{1,2}(B_1, |y|^a),$$

and q is some λ_*-homogeneous solution to the very thin obstacle problem. Moreover, since $0 \in \Sigma$,

$$N(0^+, v_*) \in [\kappa + \alpha_\kappa, \kappa + 1) \tag{5.80}$$

Thus, for each $i \in \{1, \ldots, n-1\}$, we can apply Lemma 5.19 with the sequence of radii $2r_\ell$. By definition and using the notation of Lemma 5.19, we are in the case $\lambda_* < \kappa + 1$ (see (5.80)) and $\lambda = \lambda_*$ (thanks to (5.79)). So by Lemma 5.19, q is invariant in the directions Y_i^∞ for all $i \in \{1, \ldots, n-1\}$.

From (5.78), the set $\{Y_1^\infty, \ldots, Y_{n-1}^\infty\}_{i \in \mathbb{N}} \subset L_* \times \{0\}$ is linearly independent. That is, q is independent of the $n-1$ directions determined by this linearly independent set. Therefore, it is a two-dimensional solution to the very thin obstacle problem. Hence, by Lemma 5.58, q is a polynomial, and $N(0^+, q) \geq \kappa + 1$. But this runs contrary to 0 living in $\Sigma = \Sigma_\kappa^{n-1,\mathrm{a}}$, (5.80).

In turn, Σ meets the hypotheses of Lemma 5.23 with $m = n-2$ and f as above. And so, $\dim_{\mathcal{H}} \Sigma \leq n-2$.

We now prove (5.76) and (5.77).

After a dilation, it suffices to show that if $S \subset B_1$ is such that

$$\inf_{\Pi \in \Pi_0} \operatorname{dist}(S, \Pi) \geq \varepsilon,$$

then there exist points $X_1, \dots, X_{n-1} \in S$ such that

$$|X_1 \wedge \cdots \wedge X_{n-1}| \geq \varepsilon^{n-1}.$$

But this follows from a simple construction.

Take any $X_1 \in S \cap (B_1 \setminus B_\varepsilon)$, and let $\boldsymbol{e}_1 := X_1/|X_1|$ be the first element of our orthogonal $n-1$ dimensional basis on which we will compute the $(n-1)$-determinant. Notice $X_1 = a_{1,1}\boldsymbol{e}_1$ for some $|a_{1,1}| \geq \varepsilon$. Now take any $(n-2)$-dimensional plane passing through X_1 (and 0), Π_1^X, take any $X_2 \in S \cap (B_1 \setminus B_\varepsilon)$ that is ε far from Π_1^X, and choose $\boldsymbol{e}_2 \perp \boldsymbol{e}_1$ and such that $\mathrm{span}(\boldsymbol{e}_1, \boldsymbol{e}_2) = \mathrm{span}(X_1, X_2)$. Then, $X_2 = a_{2,1}\boldsymbol{e}_1 + a_{2,2}\boldsymbol{e}_2$, and since X_2 is ε far from $\Pi_1^X \supset \mathrm{span}(\boldsymbol{e}_1)$, $|a_{2,2}| \geq \varepsilon$.

Proceed recursively until $m = n-1$: let $X_m \in S \cap (B_1 \setminus B_\varepsilon)$ be ε far from Π_{m-1}^X, where Π_{m-1}^X is any $(n-2)$-dimensional plane containing $\{0, X_1, \dots, X_{m-1}\}$ (such a plane always exists since $m \leq n-1$). Choose $\boldsymbol{e}_m \perp \mathrm{span}(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{m-1})$ and such that $\mathrm{span}(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m) = \mathrm{span}(X_1, \dots, X_m)$. Then,

$$X_m = a_{m,1}\boldsymbol{e}_1 + a_{m,2}\boldsymbol{e}_2 + \cdots + a_{m,m-1}\boldsymbol{e}_{m-1} + a_{m,m}\boldsymbol{e}_m,$$

and since X_m is ε far from $\Pi_{m-1}^X \supset \mathrm{span}(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{m-1})$, $|a_{m,m}| \geq \varepsilon$. Therefore,

$$\begin{aligned} |X_1 \wedge \cdots \wedge X_{n-1}| &= |a_{1,1}\boldsymbol{e}_1 \wedge a_{2,1}\boldsymbol{e}_1 + a_{2,2}\boldsymbol{e}_2 \wedge \cdots \wedge a_{n-1,1}\boldsymbol{e}_1 + \cdots + a_{n-1,n-1}\boldsymbol{e}_{n-1}| \\ &= |a_{1,1}a_{2,2}\cdots a_{n-1,n-1}||\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \cdots \wedge \boldsymbol{e}_{n-1}| \\ &\geq \varepsilon^{n-1}, \end{aligned}$$

as desired. □

We close this section by collecting the results we have proved to understand the size of $\Sigma_\kappa^{m,\mathrm{a}}$ when $\kappa = 2$ and $m \leq n-1$ and when $\kappa \in 2\mathbb{N}$ and $m = n-1$.

Proofs of Proposition 5.4 and Remark 5.6. We separate each case.

(i) This follows by Lemma 5.25, noting that $\tilde{\Sigma}_2^{0,\mathrm{a}} = \Sigma_2^{0,\mathrm{a}}$.

(ii) If $a \in [0,1)$ or $a \in (-1,0)$ and $m < n-1$, this follows from Proposition 5.24 by noting that $\tilde{\Sigma}_2^{m,\mathrm{a}} = \Sigma_2^{m,\mathrm{a}}$. If $a \in (-1,0)$ and $m = n-1$, this is due to Proposition 5.28.

(iii) If $a \in [0,1)$, we use Proposition 5.24, noticing that $\tilde{\Sigma}_\kappa^{n-1,\mathrm{a}} = \Sigma_\kappa^{n-1,\mathrm{a}}$. If $a \in (-1,0)$, we use Proposition 5.28.

Finally, regarding Remark 5.6, if $n = 2$ and $a \in (-1,0)$, $\Sigma_\kappa^{1,\mathrm{a}}$ is countable by Lemma 5.27, and if $n = 2$ and $a \in [0,1)$, $\Sigma_\kappa^{1,\mathrm{a}}$ is discrete by Lemma 5.26. If $n \geq 3$, $\Sigma_2^{1,\mathrm{a}}$ is discrete by Lemma 5.26, as well. □

5.6 Whitney's Extension Theorem and the Proof of Theorem 5.5

In this section, we prove our first higher regularity result Theorem 5.5. The proof of Theorem 5.5 is a model for the proofs of our main results, and utilizes an implicit function theorem argument and the following generalized Whitney's extension theorem. (See [Fef09] and the references therein.)

Lemma 5.29 (Whitney's Extension Theorem)**.** *Let $\beta \in (0,1]$, $\ell \in \mathbb{N}$, $K \subset \mathbb{R}^{n+1}$ be compact, and $f : K \to \mathbb{R}$. For every $Z_\circ \in K$, suppose that there exits a degree ℓ polynomial $P_{Z_\circ}$ for which*

(i) $P_{Z_\circ}(Z_\circ) = f(Z_\circ)$*; and*

(ii) $|D^\alpha P_{Z_\circ}(X_\circ) - D^\alpha P_{X_\circ}(X_\circ)| \leq C|Z_\circ - X_\circ|^{\ell+\beta-|\alpha|}$ *for all* $|\alpha| \leq \ell$ *and* $X_\circ \in K$*, where* $C > 0$ *is independent of* $Z_\circ$

hold. Then, there exists a function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ of class $C^{\ell,\beta}$ and constant $C_{\ell,n} > 0$ for which

$$F|_K \equiv f \quad \text{and} \quad |F(X) - P_{X_\circ}(X)| \leq C_{\ell,n}|X - X_\circ|^{\ell+\beta} \quad \text{for all} \quad X_0 \in K.$$

Now we state and prove a collection of results that, in aggregate, prove Theorem 5.5.

Theorem 5.30. *The set $\Sigma_\kappa^{m,g}$ is contained in the countable union of m-dimensional $C^{1,1}$ manifolds.*

Proof. Let us define

$$E_h := \{X_\circ \in \Sigma_\kappa \cap \overline{B_{1-1/h}} : h^{-1}\rho^\kappa \leq \sup_{|X-X_\circ|=\rho} |u(X)| < h\rho^\kappa, 0 < \rho < 1-|X_\circ|\}. \quad (5.81)$$

From the continuity of the map

$$\Sigma_\kappa \ni X_\circ \mapsto p_{*,X_\circ}$$

(see [GR19, Proposition 4.6]), we find that the map

$$E_h \ni X_\circ \mapsto N(0^+, u(X_\circ + \,\cdot\,) - p_{*,X_\circ})$$

is upper semicontinuous (it is the pointwise monotone decreasing limit of a sequence of continuous maps). Here, the sets $E_h \subset \overline{B_{(h-1)/h}}$ are closed and decompose Σ_κ as follows:

$$\Sigma_\kappa = \bigcup_{h=1}^{\infty} E_h$$

(this follows arguing exactly as in the proof of [GP09, Lemma 1.5.3], using [GR19, Lemma 4.5]). In turn, the set

$$S_{\kappa,\lambda}^h := \{X_\circ \in E_h : N(0^+, u(X_\circ + \,\cdot\,) - p_{*,X_\circ}) \geq \lambda\}$$

is compact in $\mathbb{R}^{n+1}$.

For each $X_\circ \in S_{\kappa,\lambda}^h$, define

$$P_{X_\circ}(X) := p_{*,X_\circ}(X - X_\circ).$$

We claim that $f \equiv 0, K = S_{\kappa,\lambda}^h$, and $\{P_{X_\circ}\}_{X_\circ \in K}$ satisfy the hypotheses of Whitney's Extension Theorem, Lemma 5.29, with $\ell + \beta = \lambda$.

Clearly, (i) holds.

To show (ii) holds, first observe that Lemma 5.8 implies that for all $X_\circ \in S^h_{\kappa,\lambda}$,

$$\|u(X_\circ+r\,\cdot\,)-p_{*,X_\circ}(r\,\cdot\,)\|_{L^2(B_1,|y|^a)} \le \frac{1}{n+1+a+2\lambda}\|u(X_\circ+r\,\cdot\,)-p_{*,X_\circ}(r\,\cdot\,)\|_{L^2(\partial B_1,|y|^a)}$$

and

$$\|u(X_\circ+r\,\cdot\,)-p_{*,X_\circ}(r\,\cdot\,)\|_{L^2(\partial B_1,|y|^a)} \le C_h r^\lambda.$$

So

$$\|u(X_\circ+r\,\cdot\,)-p_{*,X_\circ}(r\,\cdot\,)\|_{L^2(B_1,|y|^a)} \le C_h r^\lambda \quad \text{for all} \quad X_\circ \in S^h_{\kappa,\lambda}. \tag{5.82}$$

(Of course, $r < 1-|X_\circ|$.) Now for any pair $Z_\circ, X_\circ \in S^h_{\kappa,\lambda}$,

$$\|[P_{Z_\circ}-P_{X_\circ}](r\,\cdot\,)\|_{L^2(B_{1/2}(r^{-1}X_\circ),|y|^a)} \le C_h r^\lambda \tag{5.83}$$

where $r := 2|X_\circ - Z_\circ|$. Indeed,

$$\|[P_{Z_\circ}-P_{X_\circ}](r\,\cdot\,)\|_{L^2(B_{1/2}(r^{-1}X_\circ),|y|^a)} \le \mathrm{I}+\mathrm{II}.$$

with

$$\mathrm{I}+\mathrm{II} := \|u(r\,\cdot\,)-P_{X_\circ}(r\,\cdot\,)\|_{L^2(B_{1/2}(r^{-1}X_\circ),|y|^a)} + \|u(r\,\cdot\,)-P_{Z_\circ}(r\,\cdot\,)\|_{L^2(B_{1/2}(r^{-1}X_\circ),|y|^a)}.$$

Now assume that $r < h^{-1}$. Then, by (5.82) applied at $X_\circ$ and $Z_\circ$,

$$\mathrm{I} = \|u(X_\circ+r\,\cdot\,)-p_{*,X_\circ}(r\,\cdot\,)\|_{L^2(B_{1/2},|y|^a)} \le \|u(X_\circ+r\,\cdot\,)-p_{*,X_\circ}(r\,\cdot\,)\|_{L^2(B_1,|y|^a)} \le C_h r^\lambda$$

and

$$\begin{aligned}\mathrm{II} &= \|u(Z_\circ+r\,\cdot\,)-p_{*,Z_\circ}(r\,\cdot\,)\|_{L^2(B_{1/2}(r^{-1}(X_\circ-Z_\circ)),|y|^a)} \\ &\le \|u(Z_\circ+r\,\cdot\,)-p_{*,Z_\circ}(r\,\cdot\,)\|_{L^2(B_1,|y|^a)} \le C_h r^\lambda.\end{aligned}$$

When $h^{-1} \le r < 4$, (5.83) is true by the triangle inequality, using that $p_{*,X_\circ}$ and $p_{*,Z_\circ}$ are homogeneous, and the bound $\|p_{*,X_\circ}\|_{L^2(B_1,|y|^a)}, \|p_{*,Z_\circ}\|_{L^2(B_1,|y|^a)} \le C$. Finally, since all norms are equivalent on the finite dimensional space of κ-homogeneous polynomials, (5.83) implies that

$$\|[P_{Z_\circ}-P_{X_\circ}](r\,\cdot\,)\|_{C^\ell(B_{1/2}(X_\circ/r))} \le C_h r^\lambda$$

for any $X_\circ, Z_\circ \in S^h_{\kappa,\lambda}$ with $r = 2|X_\circ - Z_\circ|$. In turn,

$$|D^\alpha P_{Z_\circ}(rX)-D^\alpha P_{X_\circ}(rX)| \le C_h r^{\ell+\beta-|\alpha|} = C_h|X_\circ-Z_\circ|^{\ell+\beta-|\alpha|} \quad \text{for all} \quad X \in B_{1/2}(r^{-1}X_\circ).$$

Taking $X = X_\circ/r$, we see that (ii) holds.

With our claim justified, applying Whitney's Extension Theorem, we find an $F \in C^{\ell,\beta}(\mathbb{R}^{n+1})$ such that

$$|F(X) - P_{X_\circ}(X)| \le C_h|X-X_\circ|^{\ell+\beta} \quad \text{for all} \quad X_\circ \in S^h_{\kappa,\lambda}.$$

If $X_\circ \in S^h_{\kappa,\lambda} \cap \Sigma^m_\kappa$, by definition, there exist $n-m$ linearly independent unit vectors $\boldsymbol{e}_i \in \mathbb{R}^n$ and points $(x^i, 0)$, $i = 1, \ldots, n-m$, such that

$$\partial_{\boldsymbol{e}_i} p_{*,X_\circ}(x^i,0) = \boldsymbol{e}_i \cdot \nabla_x p_{*,X_\circ}(x^i,0) \ne 0.$$

Let $\boldsymbol{v}_i$ be the unit vector parallel to $(x^i, 0)$ and oriented so that $\boldsymbol{v}_i \cdot x^i > 0$. Then, we deduce that

$$\partial_{\boldsymbol{e}_i}\partial_{\boldsymbol{v}_i}^{(\kappa-1)}F(X_\circ) = \partial_{\boldsymbol{v}_i}^{(\kappa-1)}\partial_{\boldsymbol{e}_i}p_{*,X_\circ}(0) \neq 0 \quad \text{for all} \quad i = 1, \dots n-m.$$

On the other hand,

$$\Sigma_\kappa^m \cap S_{\kappa,\lambda}^h \subset \bigcap_{i=1}^{n-m}\{\partial_{\boldsymbol{v}_i}^{(\kappa-1)}F = 0\}.$$

Notice that $\partial_{\boldsymbol{v}_i}^{(\kappa-1)}F \in C^{\ell-\kappa+1,\beta}(\mathbb{R}^{n+1})$. In turn, by the implicit function theorem, $\Sigma_\kappa^m \cap S_{\kappa,\lambda}^h$ is contained in an m-dimensional manifold of class $C^{\ell-\kappa+1,\beta}$.

The theorem then follows by the definition of $\Sigma_\kappa^{m,\mathrm{g}}$, which implies that $\ell = \kappa$ and $\beta = 1$. □

Remark 5.17. In contrast to the classical (non-degenerate) obstacle problem, studied in [FS18], in the thin obstacle problem, singular points of many different orders may exist. Their interaction (see Remark 5.11) makes it impossible to prove that $\Sigma_\kappa^{m,\mathrm{g}}$ is contained in a single m-dimensional manifold. But in the non-degenerate setting, this is ruled out, and only singular points of order 2 exist.

Theorem 5.31. *In the non-degenerate case, $\Sigma_2^{m,\mathrm{g}}$ is contained in a single m-dimensional $C^{1,1}$ manifold.*

Proof. In this setting, the singular set is closed. Consider $B_{1-\eta} \subset B_1$, for any $\eta \in (0,1)$. Thanks to the non-degeneracy condition (see Definition 5.1), there exists a constant $c_\circ > 0$, depending only on n, a, the non-degeneracy constant c, and η, such that

$$\sup_{B_r(X_\circ)} u \geq c_\circ r^2,$$

for all small $r > 0$ and all $X_\circ \in \Sigma_2(u) \cap B_{1-\eta}$ (see [BFR18, Lemma 3.1]). In particular, using the notation from the proof of Theorem 5.30, there exists some $h_\circ \geq \max\{c_\circ^{-1}, \eta^{-1}\}$ such that $\Sigma_2 \cap B_{1-\eta} \subset E_{h_\circ}$. Thus, by the proof of Theorem 5.30, $\Sigma_2^m \cap S_{2,3}^{h_\circ}$ is contained in a single m-dimensional manifold of class $C^{1,1}$, and since this can be done for any $\eta > 0$, we obtain that Σ_2^m is locally contained in a single m-dimensional manifold. This concludes the proof. □

Proposition 5.32. *If $a \in (-1,0)$, the set Σ_κ^{n-1} is contained in a countable union of $(n-1)$-dimensional C^{1,α_κ} manifolds. Moreover, in the non-degenerate case, it is contained in a single $(n-1)$-dimensional $C^{1,\alpha}$ manifold, for some $\alpha > 0$ depending only on n and a.*

Proof. The proof follows that of Theorem 5.30 exactly, replacing $\beta = 1$ with $\beta = \alpha_\kappa$; when $a \in (-1,0)$ and $m = n-1$, second blow-ups always have higher homogeneity: $\lambda_* \geq \kappa + \alpha_\kappa > \kappa$ (see Proposition 5.15). In the non-degenerate case, we can proceed as in Theorem 5.31 instead. □

Finally, we can proceed with the proof of one of our main results, Theorem 5.5.

Proof of Theorem 5.5. We separate each case.

(i) Notice that we are in Case 1 (since $\kappa = 2$). So apply Lemma 5.18.

(ii) $\Sigma_2^{m,\mathrm{a}}$ is lower dimensional by Proposition 5.4, while $\Sigma_2^{m,\mathrm{g}}$ is covered by a countable union of m-dimensional $C^{1,1}$ manifolds by Theorem 5.30.

(iii) Again, $\Sigma_\kappa^{n-1,\mathrm{a}}$ is lower dimensional by Proposition 5.4. And $\Sigma_\kappa^{n-1,\mathrm{g}}$ is covered by a countable union of $(n-1)$-dimensional $C^{1,1}$ manifolds by Theorem 5.30.

(iv) This follows by Proposition 5.32.

This completes the proof. □

5.7 The Main Results

In this section, we construct the second term in the expansion of u at singular points, up to a lower dimensional set. We start by defining a specific subset of the generic singular points at which the nodal set and spine of the first blow-up align. These points will be those at which we produce the next term (of order at least $\kappa + 1$) in the expansion of u at a order κ singular point, the goal of this work.

Definition 5.2. Let $n \geq 2$ and $0 \leq m \leq n-1$. We define the set $\Sigma_\kappa^{m,\mathrm{nxt}}$ as the set of singular points $X_\circ \in \tilde{\Sigma}_\kappa^{m,\mathrm{g}}$ for which there exists a sequence $r_\ell \downarrow 0$ as $\ell \to \infty$ such that the following holds: there exists a $(\kappa+1)$-homogeneous, a-harmonic polynomial $q_\circ$ (possibly $q_\circ \equiv 0$) such that

(i)

$$v_{r_\ell,\kappa+1} := \frac{u(X_\circ + r_\ell\,\cdot\,) - p_{*,X_\circ}(r_\ell\,\cdot\,)}{r_\ell^{\kappa+1}} \rightharpoonup q_\circ \quad \text{in} \quad W^{1,2}(B_1, |y|^a);$$

(ii) $D^\alpha q_\circ$ vanishes on $L(p_{*,X_\circ})$ for all $\alpha = (\alpha_1, \dots, \alpha_n, 0)$ and $|\alpha| \leq \kappa - 2$; and

(iii)

$$\|q_\circ\|^2_{L^2(\partial B_1, |y|^a)} = H_{\kappa+1}(0^+, u(X_\circ + \,\cdot\,) - p_{*,X_\circ}).$$

In the first set of results of this section, we estimate the size of $\Sigma_\kappa^{m,\mathrm{nxt}}$ for certain pairs of κ and m.

Lemma 5.33. *Let $n \geq 2$ and $a \in [0,1)$. Then,* $\dim_{\mathcal{H}} \Sigma_\kappa^{n-1} \setminus \Sigma_\kappa^{n-1,\mathrm{nxt}} \leq n-2$.

Proof. We proceed as in Proposition 5.24. Notice that by Proposition 5.24, $\Sigma_\kappa^{n-1,\mathrm{a}}$ is already lower dimensional. So we restrict our attention to $\Sigma_\kappa^{n-1,\mathrm{g}}$. Let

$$\Sigma := \Sigma_\kappa^{n-1,\mathrm{g}} \setminus \Sigma_\kappa^{n-1,\mathrm{nxt}}$$

and suppose that $\mathcal{H}^\beta_\infty(\Sigma) > 0$ for some $\beta > n-2$. Then, there exists a point $X_\circ \in \Sigma$ and a sequence $r_j \downarrow 0$ such that

$$\frac{\mathcal{H}^\beta_\infty(\Sigma \cap B_{r_j}(X_\circ))}{r_j^\beta} \geq c_{n,\beta} > 0.$$

Up to a translation, assume that $X_\circ = 0$. By assumption,

$$\lambda_* := N(0^+, v_*) \geq \kappa + 1,$$

and, up to a subsequence $r_\ell = r_{j_\ell}$,

$$\tilde{v}_{r_\ell} \rightharpoonup q \quad \text{in} \quad W^{1,2}(B_1, |y|^a),$$

where $\tilde{v}_{r_\ell}$ is defined as in (5.29), and q is a-harmonic and at least (κ+1)-homogeneous. Moreover, by Lemma 5.21(ii),

$$\mathcal{H}^\beta_\infty(\mathcal{A}) > 0,$$

where $\mathcal{A} = \mathcal{A}_\Sigma$. Now if $Z \in \mathcal{A}$, then there are sequences $X_\ell \in \Sigma$ and $r_\ell \downarrow 0$ such that $|X_\ell| \leq r_\ell$ and $X_\ell/2r_\ell \to Z/2$. By Lemma 5.17, if we denote

$$D_{\kappa-2}(q) := \{X = (x, 0) : D^\alpha q(X) = 0 \text{ for all } \alpha = (\alpha_1, \ldots, \alpha_n, 0) \text{ with } |\alpha| \leq \kappa - 2\},$$

then $Z \in L_* \cap D_{\kappa-2}(q)$, so that

$$\mathcal{A} \subset \overline{B_1} \cap L_* \cap D_{\kappa-2}(q).$$

Now, using the monotonicity of $H_{\kappa+1}(r_\ell, v_*)$ (see Lemma 5.8), we have that $H^{1/2}_{\kappa+1}(0^+, v_*)$ exists. So let

$$q_\circ := H^{1/2}_{\kappa+1}(0^+, v_*)q$$

and notice that

$$v_{r_\ell,\kappa+1} := \frac{u(r_\ell\,\cdot\,) - p_*(r_\ell\,\cdot\,)}{r_\ell^{\kappa+1}} = \tilde{v}_{r_\ell}\frac{\|v_{r_\ell}\|_{L^2(\partial B_1)}}{r_\ell^{\kappa+1}} = \tilde{v}_{r_\ell} H^{1/2}_{\kappa+1}(r_\ell, v_*).$$

In turn,

$$v_{r_\ell,\kappa+1} \rightharpoonup q_\circ \quad \text{in} \quad W^{1,2}(B_1, |y|^a).$$

Additionally,

$$\|q_\circ\|^2_{L^2(\partial B_1, |y|^a)} = H_{\kappa+1}(0^+, v_*) \tag{5.84}$$

since $\|q\|_{L^2(\partial B_1, |y|^a)} = 1$.

If $\lambda_* > \kappa + 1$, then $\|v_*\|^2_{L^2(\partial B_r, |y|^a)} = o(r^{2(\kappa+1)+n+a})$. And so $H_{\kappa+1}(0^+, v_*) = 0$, which, by (5.84), implies that $q_\circ \equiv 0$. But this is impossible:

$$0 \notin \Sigma^{n-1,\text{nxt}}_\kappa. \tag{5.85}$$

(In this case, $q_\circ$ is trivially $(\kappa+1)$-homogeneous and $D_{\kappa-2}(q_\circ) = \mathbb{R}^n \times \{0\}$.) In turn, $\lambda_* = \kappa + 1$ and $q_\circ \not\equiv 0$. Thus, by (5.85),

$$D_{\kappa-2}(q) \cap L_* = D_{\kappa-2}(q_\circ) \cap L_* \subsetneq L_*.$$

Hence, by the analyticity of q,

$$\dim_\mathcal{H} D_{\kappa-2}(q) \cap L_* \leq n - 2.$$

But then, $\mathcal{H}^\beta_\infty(\mathcal{A}) = 0$, a contradiction. □

Notice that $\Sigma_2^0 = \Sigma_2^{0,\mathrm{nxt}}$ since $\Sigma_2^{0,\mathrm{a}}$ is empty and $\tilde{\Sigma}_2 = \Sigma_2$.

Lemma 5.34. *Let $n \geq 3$. Suppose that $1 \leq m \leq n-2$. Then, $\dim_{\mathcal{H}} \Sigma_2^m \setminus \Sigma_2^{m,\mathrm{nxt}} \leq m-1$.*

Proof. The proof is identical to the proof of Lemma 5.33. Notice that $\Sigma_2^{m,\mathrm{a}}$ is lower dimensional by Proposition 5.24, and $\tilde{\Sigma}_2^{m,\mathrm{a}} = \Sigma_2^{m,\mathrm{a}}$. □

Lemma 5.35. *Let $n \geq 2$. Suppose that $a \in (-1,0)$. Then, $\dim_{\mathcal{H}} \Sigma_\kappa^{n-1} \setminus \Sigma_\kappa^{n-1,\mathrm{nxt}} \leq n-2$.*

Proof. We proceed as in Lemma 5.33. By Proposition 5.28, $\Sigma_\kappa^{n-1,\mathrm{a}}$ is lower dimensional, so we define

$$\Sigma := \Sigma_\kappa^{n-1,\mathrm{g}} \setminus \Sigma_\kappa^{n-1,\mathrm{nxt}}$$

and suppose that $\mathcal{H}^\beta_\infty(\Sigma) > 0$ for some $\beta > n-2$. We can assume that at 0 and for some $r_j \downarrow 0$,

$$\frac{\mathcal{H}^\beta_\infty(\Sigma \cap B_{r_j}(X_\circ))}{r_j^\beta} \geq c_{n,\beta} > 0 \quad \text{and} \quad \lambda_* = N(0^+, v_*) \geq \kappa + 1.$$

Furthermore, up to a subsequence $r_\ell = r_{j_\ell}$,

$$\tilde{v}_{r_\ell} \rightharpoonup q \quad \text{in} \quad W^{1,2}(B_1, |y|^a),$$

where $\tilde{v}_{r_\ell}$ is defined as in (5.29), and q is a global homogeneous solution to the very thin obstacle problem with homogeneity $\lambda_* \geq \kappa + 1$. Moreover, by Lemma 5.21(ii),

$$\mathcal{H}^\beta_\infty(\mathcal{A}) > 0,$$

where $\mathcal{A} = \mathcal{A}_\Sigma$, and

$$\mathcal{A} \subset \overline{B_1} \cap L_*$$

by Lemma 5.19.

Arguing as in Lemma 5.33, since $0 \in \Sigma$, if we set

$$q_\circ := H^{1/2}_{\kappa+1}(0^+, v_*) q,$$

we find that $q_\circ$ and q are $(\kappa+1)$-homogeneous and non-zero.

Let us decompose q into its odd and even parts with respect to L_* as defined in Lemma 5.19: $q = q^{\mathrm{odd}} + q^{\mathrm{even}}$. Without loss of generality and for simplicity, assume that

$$L_* = \{x_n = y = 0\}.$$

On one hand, by the proof of Lemma 5.19, q^{odd} is an a-harmonic, $(\kappa+1)$-homogeneous function, which by Liouville's theorem ([CSS08, Lemma 2.7]), is a polynomial. On the other hand, since $\mathcal{H}^\beta_\infty(\mathcal{A}) > 0$, there are $n-1$ elements in $\mathcal{A}$, $Y_1, \dots, Y_{n-1}$, such that $\mathrm{span}(Y_1, \dots, Y_{n-1}) = L_*$. By Remark 5.13, q is then a polynomial. Indeed, for each Y_i, there exists a sequence $\{X_i^\ell\}_{\ell \in \mathbb{N}}$ with $X_i^\ell \in \Sigma$ such that $|X_i^\ell| \leq r_\ell$ and $Y_i^\ell := X_i^\ell / r_\ell \to Y_i$ as $\ell \to \infty$. In addition, if we let

$$f(X) := N(0^+, u(X + \cdot\,) - p_{*,X}) \quad \text{for} \quad X \in \mathbb{R}^n \times \{0\},$$

then $\kappa+1 \leq f(X_i^\ell)$ (since $X_i^\ell \in \Sigma$). Also, $\limsup_{\ell\to\infty} f(X_i^\ell) \leq \lambda_* = \kappa+1$ (f is upper semicontinuous). So Almgren's frequency at 0^+ is continuous along the sequences $\{X_i^\ell\}_{\ell\in\mathbb{N}}$ and $i \in \{1,\dots,n-1\}$. Therefore, the hypotheses of Remark 5.13 hold, and we have that

$$q^{\text{even}}(X+\tau Y_i) = q^{\text{even}}(X) + \tau c_i \operatorname{Ext}_a(x_n^\kappa), \tag{5.86}$$

(recall, $L_* = \{x_n = y = 0\}$). Since $\{Y_i\}_{1\leq i\leq n-1}$ spans L_*, for any $X = (x', x_n, x_{n+1}) \in \mathbb{R}^{n+1}$,

$$X = (0,\dots,0,x_n,x_{n+1}) + (\boldsymbol{a}_1 \cdot x')Y_1 + \cdots + (\boldsymbol{a}_{n-1}\cdot x')Y_{n-1},$$

for some fixed vectors $\boldsymbol{a}_j' \in \mathbb{R}^{n-1}$ for $j \in \{1,\dots,n-1\}$. Now applying (5.86) iteratively, we deduce that

$$\begin{aligned} q^{\text{even}}(X) &= (c_1(\boldsymbol{a}_1'\cdot x') + \dots c_{n-1}(\boldsymbol{a}_{n-1}'\cdot x')) \operatorname{Ext}_a(x_n^\kappa) + \bar{q}(x_n,x_{n+1}) \\ &= (\boldsymbol{a}'\cdot x') \operatorname{Ext}_a(x_n^\kappa) + \bar{q}(x_n,x_{n+1}). \end{aligned}$$

Here, $\bar{q}(x_n,x_{n+1}) = q^{\text{even}}(0,\dots,0,x_n,x_{n+1})$ and $\boldsymbol{a}' \in \mathbb{R}^{n-1}$. As q^{even} is a solution to the very thin obstacle problem which is $(\kappa+1)$-homogeneous and $(\boldsymbol{a}'\cdot x')\operatorname{Ext}_a(x_n^\kappa)$ is $(\kappa+1)$-homogeneous, a-harmonic, and vanishes on L_*, we find that $\bar{q}$ is a $(\kappa+1)$-homogeneous solution to the very thin obstacle problem. By Lemma 5.58, since $\bar{q}$ is two-dimensional and $(\kappa+1)$-homogeneous, it is a polynomial. But $\bar{q}$ is also even in both x_n and y, which implies $\bar{q}\equiv 0$. Therefore, q^{even} is also a polynomial. That is, q is a $(\kappa+1)$-homogeneous polynomial since both q^{odd} and q^{even} are polynomials.

To conclude, observe that because q is a $(\kappa+1)$-homogeneous polynomial, for any $Z \in \mathcal{A}$, there are sequences $X_\ell \in \Sigma$ and $r_\ell \downarrow 0$ such that $|X_\ell| \leq r_\ell$ and $X_\ell/r_\ell \to Z$. By Remark 5.12, $D^\alpha q(Z) = 0$ for all $\alpha = (\alpha_1,\dots,\alpha_n,0)$ such that $|\alpha| \leq \kappa-2$, i.e.,

$$\mathcal{A} \subset \overline{B_1} \cap L_* \cap D_{\kappa-2}(q) = \overline{B_1} \cap L_* \cap D_{\kappa-2}(q_\circ).$$

Since $0 \notin \Sigma_\kappa^{n-1,\text{nxt}}$,

$$D_{\kappa-2}(q) \cap L_* = D_{\kappa-2}(q_\circ) \cap L_* \subsetneq L_*,$$

and by the analyticity of q,

$$\dim_{\mathcal{H}} D_{\kappa-2}(q) \cap L_* \leq n-2.$$

But then, $\mathcal{H}_\infty^\beta(\mathcal{A}) = 0$, a contradiction. □

In some of the end point cases, we can say more.

Corollary 5.36.

(i) If $n = 2$ and $a \in [0,1)$, then $\Sigma_\kappa^1 \setminus \Sigma_\kappa^{1,\text{nxt}}$ is countable.

(ii) If $n = 2$ and $a \in (-1,0)$, then $\Sigma_\kappa^1 \setminus \Sigma_\kappa^{1,\text{nxt}}$ is countable.

(iii) If $n \geq 3$ and $m = 1$, then $\Sigma_2^1 \setminus \Sigma_2^{1,\text{nxt}}$ is countable.

Proof. We separate each case.

(i) Notice that if $n = 2$ and $a \in [0, 1)$, then $\Sigma_\kappa^{1,\mathrm{a}}$ is discrete by Lemma 5.26. Repeating the proof of Lemma 5.33, but assuming, to the contrary, that $\Sigma_\kappa^{1,\mathrm{g}} \setminus \Sigma_\kappa^{1,\mathrm{nxt}}$ has accumulation points, we deduce that $\Sigma_\kappa^{1,\mathrm{g}} \setminus \Sigma_\kappa^{1,\mathrm{nxt}}$ is discrete as well. The result follows.

(ii) By Lemma 5.27, we see that $\Sigma_\kappa^{1,\mathrm{a}}$ is countable. In addition, repeating the arguments used to prove Lemma 5.35, we deduce that $\Sigma_\kappa^{1,\mathrm{g}} \setminus \Sigma_\kappa^{1,\mathrm{nxt}}$ cannot have accumulation points.

(iii) Following the proof of (i), but using Lemma 5.34, we conclude.

This completes the proof. □

The next pair of statement concern the almost monotonicity of a Monneau-type energy and the uniqueness and continuity of second blow-ups at points in $\Sigma_\kappa^{m,\mathrm{nxt}}$.

Lemma 5.37. *Let $X_\circ \in \Sigma_\kappa^{m,\mathrm{nxt}} \cap K$ for some compact set $K \subset B_1 \cap \{y = 0\}$, $q_\circ$ be as in Definition 5.2, and H_λ be as in* (5.25). *Then,*

$$\frac{\mathrm{d}}{\mathrm{d}r} H_{\kappa+1}(r, u(X_\circ + \cdot\,) - p_{*,X_\circ} - q_\circ) \geq -C_K \left\| \frac{q_\circ^\kappa}{p_{*,X_\circ}^{\kappa-1}} \right\|_{L^\infty(B_1 \cap \{y=0\})}.$$

Proof. Without loss of generality, assume that $X_\circ = 0 \in \Sigma_\kappa^{m,\mathrm{nxt}}$. Set

$$w := v_* - q_\circ.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}r} H_\lambda(r, w) = \frac{2}{r^{n+a+2\lambda+1}} \int_{\partial B_r} w(\nabla w \cdot X - \lambda w)|y|^a,$$

arguing as we did to show (5.23), we find that

$$\frac{\mathrm{d}}{\mathrm{d}r} H_{\kappa+1}(r, w) \geq \frac{2}{r^{n+a+2\kappa+2}} \int_{B_r} w\, L_a w.$$

Now observe that

$$w\, L_a w = -(p_* + q_\circ)\, L_a u.$$

From the numerical inequality $1 - \xi + \xi^\kappa \geq 0$ for all $\xi \geq 0$ and as $q_\circ|_{\mathbb{R}^n \times \{0\}} = 0$ when $p_*|_{\mathbb{R}^n \times \{0\}} = 0$, we see that

$$p_* + q_\circ \geq p_* - |q_\circ| \geq -\frac{q_\circ^\kappa}{p_*^{\kappa-1}} \quad \text{on} \quad \mathbb{R}^n \times \{0\}$$

(recall that κ is even and $p_* \geq 0$ on $\mathbb{R}^n \times \{0\}$). Therefore, using that $L_a u$ is a non-positive measure supported on $B_1 \cap \{y = 0\}$, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}r} H_{\kappa+1}(r, w) \geq -\frac{1}{r^{n+a+2\kappa+2}} \int_{B_r} \frac{q_\circ^\kappa}{p_*^{\kappa-1}}\, L_a u.$$

Because $D^\alpha q_\circ$ vanishes for all $\alpha = (\alpha_1, \ldots, \alpha_n, 0)$ with $|\alpha| \le \kappa - 2$ on L_*, we have that $q_\circ^\kappa / p_*^{\kappa-1}$ is locally bounded on $\mathbb{R}^n \times \{0\}$. Moreover, $q_\circ^\kappa / p_*^{\kappa-1}$ is a 2κ-homogeneous polynomial. Thus,

$$-\frac{1}{r^{n+a+2\kappa+2}} \int_{B_r} \frac{q_\circ^\kappa}{p_*^{\kappa-1}} L_a u \ge -\frac{C}{r^3} \left\| \frac{q_\circ^\kappa}{p_*^{\kappa-1}} \right\|_{L^\infty(B_1 \cap \{y=0\})} \int_{B_{1/2}} |L_a u_{2r}|,$$

as $r^{2-a} L_a u(rX) = L_a u_r(X)$. Now from the proof of Proposition 5.13, we know that

$$-\int_{B_{1/2}} |L_a u_{2r}| = -\|v_{2r}\|_{L^2(\partial B_1)} \int_{B_{1/2}} |L_a \tilde{v}_{2r}| \ge -C \|v_{2r}\|_{L^2(\partial B_1)}.$$

Moreover, thanks to Lemma 5.8,

$$-\|v_{2r}\|_{L^2(B_1)} \ge -C \|v_{2r}\|_{L^2(\partial B_1)} \ge -C r^{\lambda_*}.$$

In turn,

$$\frac{\mathrm{d}}{\mathrm{d}r} H_{\kappa+1}(r, w) \ge -C r^{\lambda_* - 3} \left\| \frac{q_\circ^\kappa}{p_*^{\kappa-1}} \right\|_{L^\infty(B_1 \cap \{y=0\})},$$

which, after recalling that $\lambda_* \ge \kappa + 1 \ge 3$, proves the lemma. □

Proposition 5.38. *For every $X_\circ \in \Sigma_\kappa^{m,\mathrm{nxt}}$, there exists a unique $(\kappa+1)$-homogeneous, a-harmonic polynomial $q_{*,X_\circ}$ such that*

$$\frac{u(X_\circ + r\,\cdot\,) - p_{*,X_\circ}(r\,\cdot\,)}{r^{\kappa+1}} \rightharpoonup q_{*,X_\circ} \quad \textit{in} \quad W^{1,2}(B_1) \ \textit{as} \ r \downarrow 0, \tag{5.87}$$

$D^\alpha q_{*,X_\circ}$ vanishes on $L(p_{*,X_\circ})$ for any $\alpha = (\alpha_1, \ldots, \alpha_n, 0)$ with $|\alpha| \le \kappa - 2$, and

$$\|q_{*,X_\circ}\|^2_{L^2(\partial B_1, |y|^a)} = H_{\kappa+1}(0^+, u(X_\circ + \,\cdot\,) - p_{*,X_\circ}).$$

Moreover, the convergence in (5.87) *is uniform on compact subsets of $B_1 \cap \{y = 0\}$, and the map*

$$\Sigma_\kappa^{m,\mathrm{nxt}} \ni X_\circ \mapsto q_{*,X_\circ}$$

is continuous.

Proof. Without loss of generality, we take $X_\circ = 0$. Let $q_\circ$ denote the limit along the sequence r_ℓ given by Definition 5.2. Let $\tilde{q}_\circ$ be another limit taken through another sequence, $\tilde{r}_\ell$, such that (after relabelling if necessary) $\tilde{r}_\ell \le r_\ell$. Then, by Lemma 5.37, we have that

$$H_{\kappa+1}(r_\ell, v_* - q_\circ) \ge H_{\kappa+1}(\tilde{r}_\ell, v_* - q_\circ) - C \left\| \frac{q_\circ^\kappa}{p_*^{\kappa-1}} \right\|_{L^\infty(B_1 \cap \{y=0\})} |r_\ell - \tilde{r}_\ell| \quad \text{for all} \quad \ell \in \mathbb{N}.$$

Thus, using that $r_\ell^{-\kappa-1} v_{r_\ell} \to q_\circ$ strongly in $L^2(\partial B_1, |y|^a)$,

$$\begin{aligned} 0 &= \lim_{\ell \to \infty} \int_{\partial B_1} (r_\ell^{-\kappa-1} v_{r_\ell} - q_\circ)^2 |y|^a \\ &\ge \lim_{\ell \to \infty} \left(\int_{\partial B_1} (\tilde{r}_\ell^{-\kappa-1} v_{\tilde{r}_\ell} - q_\circ)^2 |y|^a - C \left\| \frac{q^\kappa}{p_*^{\kappa-1}} \right\|_{L^\infty(B_1 \cap \{y=0\})} |\tilde{r}_\ell - r_\ell| \right) \\ &= \int_{\partial B_1} (\tilde{q}_\circ - q_\circ)^2 |y|^a. \end{aligned}$$

And so, $\tilde{q}_\circ = q_\circ$, and the limit is unique. The remainder of the proof follows the proof of [FS18, Proposition 4.5]. □

Remark 5.18. Thanks to Proposition 5.38, Definition 5.2 can be amended to say *for every sequence* $r_\ell \downarrow 0$, instead of just *a sequence.*

An important consequence of Proposition 5.38, particularly, the uniform convergence in compact sets of the limit (5.87), is the following: for each compact set $K \subset B_1 \cap \{y = 0\}$, we have a modulus of continuity ω_K such that

$$H_{\kappa+1}(r, u(X_\circ + \cdot\,) - p_{*,X_\circ} - q_{*,X_\circ}) \le \omega_K(r) \quad \text{for all} \quad X_\circ \in K \cap \Sigma_\kappa^{m,\mathrm{nxt}}.$$

This modulus of continuity allows us to prove the following regularity result, a precursor to our main results.

Theorem 5.39. *The set* $\Sigma_\kappa^{m,\mathrm{nxt}}$ *is contained in the countable union of* m*-dimensional* C^2 *manifolds.*

Proof. The proof will be completed in two steps.

Step 1: Let E_h be the compact sets defined in the proof of Theorem 5.30, and set

$$E_{h,m} := \Sigma_\kappa^m \cap E_h \quad \text{and} \quad E_{h,m}^{\mathrm{nxt}} := \Sigma_\kappa^{m,\mathrm{nxt}} \cap E_h.$$

Observe that by Lemma 5.33 and Lemma 5.34,

$$\dim_{\mathcal{H}} E_{h,m} \setminus E_{h,m}^{\mathrm{nxt}} \le m - 1$$

when $m \ge 1$. Hence, for any $j \in \mathbb{N}$, we can find a family of balls $\{\hat{B}_i\}_{i=1}^\infty$ such that

$$E_{h,m} \setminus E_{h,m}^{\mathrm{nxt}} \subset \mathcal{O}_j := \bigcup_{i=1}^\infty \hat{B}_i \quad \text{and} \quad \sum_{i=1}^\infty \mathrm{diam}(\hat{B}_i)^{m-1+\frac{1}{j}} < \frac{1}{j}.$$

In particular,

$$\mathcal{H}_\infty^{m-1+\frac{1}{j}}(\mathcal{O}_j) < \frac{1}{j}.$$

Now define

$$\mathcal{U}_j := \left\{ X \in \mathbb{R}^{n+1} : \mathrm{dist}(X, \overline{E}_{h,m} \setminus E_{h,m}) < \frac{1}{j} \right\} \quad \text{and} \quad K_j := E_{h,m} \setminus (\mathcal{O}_j \cup \mathcal{U}_j).$$

Notice that $\mathcal{O}_j$ and $\mathcal{U}_j$ are open, K_j is closed, and

$$K_j \subset E_{h,m}^{\mathrm{nxt}}.$$

Moreover, we have that

$$\bigcup_{j=1}^\infty K_j = E_{h,m} \setminus \bigcap_{j=1}^\infty \mathcal{O}_j.$$

Indeed, using the continuity of the map $\Sigma_\kappa \ni X_\circ \mapsto p_{*,X_\circ}$ (see [GR19, Proposition 4.6]) and that E_h is closed and contained in Σ_κ, we find that

$$\overline{\overline{E}_{h,m} \setminus E_{h,m}} \subset \bigcup_{d \ge m+1} E_{h,d},$$

so that $\bigcap_{j=1}^{\infty} \mathcal{U}_j$ is disjoint from $E_{h,m}$. Finally, by construction, $\mathcal{H}^{\beta}_{\infty}(\bigcap_{j=1}^{\infty} \mathcal{O}_j) = 0$ for all $\beta > m-1$, which implies that $\dim_{\mathcal{H}} \bigcap_{j=1}^{\infty} \mathcal{O}_j \leq m-1$. In turn, if we can show that $E_{h,m}$ is contained in a m-dimensional C^2 manifold, then Σ^m_κ can be covered by a countable collection of m-dimensional C^2 manifolds along with a set of Hausdorff dimension at most $m-1$.

Step 2: This step is essentially identical to the second half of the proof of Theorem 5.30. For completeness, however, we provide some details regarding the justification of hypothesis (ii) in the statement of Whitney's Extension Theorem.

For $X_\circ \in K_j$, define

$$P_{X_\circ}(X) := p_{*,X_\circ}(X - X_\circ) + q_{*,X_\circ}(X - X_\circ).$$

Now let $X_\circ, Z_\circ \in K_j$ and $r := 2|X_\circ - Z_\circ|$. Arguing as we did in the proof of Theorem 5.30, but using Proposition 5.38, we see that there exists a modulus of continuity ω_{K_j} such that

$$\frac{1}{r^{\kappa+1}} \|[P_{Z_\circ} - P_{X_\circ}](r\,\cdot\,)\|_{L^2(B_{1/2}(r^{-1}X_\circ),|y|^a)} \leq 2\omega_{K_j}(r).$$

So since all norms are equivalent on the finite dimensional space of polynomials of degree less than or equal to $\kappa+1$,

$$\|[P_{Z_\circ} - P_{X_\circ}](r\,\cdot\,)\|_{C^{\kappa+1}(B_{1/2}(r^{-1}X_\circ))} \leq 2\omega_{K_j}(r) r^{\kappa+1},$$

for any $X_\circ, Z_\circ \in K_j$ with $r = 2|X_\circ - Z_\circ|$. In turn, given $|\alpha| \in \{0, \ldots, \kappa+1\}$,

$$|D^\alpha P_{Z_\circ}(X_\circ) - D^\alpha P_{X_\circ}(X_\circ)| \leq 2\omega_{K_j}(|X_\circ - Z_\circ|)|X_\circ - Z_\circ|^{\kappa+1-|\alpha|} \quad \text{for all} \quad X_\circ \in K_j.$$

Thus, thanks to the Whitney's Extension Theorem, Lemma 5.29, we conclude. □

Theorem 5.40. *In the non-degenerate case, the set $\Sigma^{m,\mathrm{nxt}}_\kappa$ is contained in a single m-dimensional C^2 manifold.*

Proof. The proof follows the proof of Theorem 5.39, but with the same modifications as the proof of Theorem 5.31. □

To finish, we present the proofs of our two main results.

Proof of Theorem 5.1. We prove each case separately.

(i) As in the proof of Theorem 5.5(i), this case holds by Lemma 5.18.

(ii) $\Sigma^1_2 \setminus \Sigma^{1,\mathrm{nxt}}_2$ is countable by Corollary 5.36. So the proof follows from Theorem 5.40.

(iii) $\Sigma^{m,\mathrm{nxt}}_2$ is contained in an m-dimensional C^2 manifold by Theorem 5.40. On the other hand, $\dim_{\mathcal{H}} \Sigma^m_2 \setminus \Sigma^{m,\mathrm{nxt}}_2 \leq m-1$ by Lemmas 5.33, 5.34, and 5.35.

(iv) See Proposition 5.32.

This concludes the proof. □

Proof of Theorem 5.2. We consider each case separately.

(i) See Lemma 5.18.

(ii) See the proof of Theorem 5.1(ii), but consider Theorem 5.39 instead of Theorem 5.40.

(iii) See the proof of Theorem 5.1(iii), but consider Theorem 5.39 instead of Theorem 5.40.

(iv) $\Sigma_\kappa^1 \setminus \Sigma_\kappa^{1,\mathrm{nxt}}$ is countable by Corollary 5.36, and $\Sigma_\kappa^{1,\mathrm{nxt}}$ is contained in the countable union of one-dimensional C^2 manifolds by Theorem 5.39.

(v) $\Sigma_\kappa^{n-1,\mathrm{nxt}}$ is contained in the countable union of $(n-1)$-dimensional C^2 manifolds by Theorem 5.39. On the other hand, $\dim_{\mathcal{H}} \Sigma_\kappa^{n-1} \setminus \Sigma_\kappa^{n-1,\mathrm{nxt}} \le n-2$ by Lemmas 5.33 and 5.35.

(vi) See Theorem 5.5(iv).

This conclude the proof. □

5.8 The Very Thin Obstacle Problem

This section is dedicated to studying, what we have called, the very thin obstacle problem for L_a when $a < 0$: a minimization problem like (5.1), but for $a \in (-1,0)$ and subject to a codimension two obstacle constraint. Alternatively (see Section 5.1.1), this problem corresponds to the fractional thin obstacle problem. Namely, we consider

$$\min_{w\in\mathscr{C}} \left\{ \int_{B_1} |\nabla w|^2 |y|^a \right\}, \quad \text{with} \quad a \in (-1,0), \tag{5.88}$$

where $\mathscr{C}$ is the convex subset of the Sobolev space $W^{1,2}(B_1, |y|^a)$ defined by

$$\mathscr{C} := \{w \in W_0^{1,2}(B_1, |y|^a) + g : w(x',0,0) \ge 0 \text{ and } w(x,-y) = w(x,y)\},$$

given some boundary data $g \in C(B_1)$ (even with respect to y) such that $g|_{\partial B_1 \cap \{x_n = y = 0\}} \ge 0$. The condition that w is non-negative on the *very thin space* $\mathbb{R}^{n-1} \times \{0\} \times \{0\}$ needs to be understood in the trace sense, a priori. Notice that since $a < 0$, the condition $w \ge 0$ on $B_1' \times \{0\} \times \{0\}$ is relevant; the very thin space has non-zero a-harmonic capacity if and only if $a \in (-1,0)$. Indeed, recalling the proof of Proposition 5.15, the (“double”) trace operator $\tau : W^{1,2}(B_1, |y|^a) \to W^{s-\frac{1}{2},2}(B_1') \subset L^2(B_1')$ is well-defined and continuous.

In this setting, the Euler–Lagrange equations characterizing the unique solution u to (5.88) are

$$\begin{cases} u(x', x_n, y) & \ge & 0 & \text{in } B_1 \cap \{x_n = y = 0\} \\ L_a u & \le & 0 & \text{in } B_1 \\ u\, L_a u & = & 0 & \text{in } B_1 \\ L_a u & = & 0 & \text{in } B_1 \setminus \Lambda(u) \\ u(x,y) & = & u(x,-y) & \text{in } B_1 \end{cases} \tag{5.89}$$

where, as expected,

$$\Lambda(u) := \{(x', 0, 0) : u(x', 0, 0) = 0\}$$

is the the *contact set*. The *free boundary* here is the topological boundary in $\mathbb{R}^{n-1}$ of $\Lambda(u)$:

$$\Gamma(u) := \partial\Lambda(u) \subset \mathbb{R}^{n-1} \times \{0\} \times \{0\}.$$

We close this introduction with a lemma that proves an analogous representation of u to that in (5.6) for the solution to the thin obstacle problem. Recall that (as defined in Subsection 5.1.5) D_r denotes the two-dimensional disc centered at the origin of radius $r > 0$.

Lemma 5.41. *Let u be such that $L_a u = 0$ in $B_1 \setminus \{x_n = y = 0\}$. Then,*

$$L_a u(x', x_n, y) = f_a(x')\mathcal{H}^{n-1} \llcorner B_1',$$

where

$$\begin{aligned} f_a(x') &:= \lim_{\varepsilon\downarrow 0} \int_{\partial D_\varepsilon} u_\nu |y|^a \,\mathrm{d}\sigma(x_n, y) \\ &= \lim_{\varepsilon\downarrow 0} \int_{\partial D_\varepsilon} \left(\frac{x_n}{\varepsilon}\partial_n u(x', x_n, y) + \frac{y}{\varepsilon}\partial_y u(x', x_n, y)\right) |y|^a \,\mathrm{d}\sigma(x_n, y). \end{aligned}$$

In particular, if u is the solution to (5.89)*, then*

$$L_a u(x', x_n, y) = f_a(x')\mathcal{H}^{n-1} \llcorner \Lambda(u).$$

Proof. For every $\varphi \in C_c^\infty(B_1)$,

$$\begin{aligned} \langle L_a u, \varphi\rangle &:= -\int_{B_1} \nabla u \cdot \nabla\varphi |y|^a \\ &= -\lim_{\varepsilon\downarrow 0} \int_{B_1\cap\{x_n^2+y^2\geq\varepsilon^2\}} \nabla u \cdot \nabla\varphi |y|^a \\ &= \lim_{\varepsilon\downarrow 0} \int_{B_1\cap\{x_n^2+y^2=\varepsilon^2\}} u_\nu\, \varphi |y|^a \\ &= \int_{B_1'} f_a(x')\varphi(x', 0, 0)\,\mathrm{d}x', \end{aligned}$$

recalling that $L_a u = 0$ in $B_1 \setminus \{x_n = y = 0\}$. □

In the following subsections, we prove a collection of results on the very thin obstacle problem, (5.88) or, equivalently, (5.89).

5.8.1 A Non-local Operator

It is now well-known that the fractional Laplacian, or s-Laplacian, of a function v defined on $\mathbb{R}^n$ can be reinterpreted as a weighted normal derivative of the a-harmonic extension of v to the upper half-space $\mathbb{R}^{n+1}_+$ (see [MO69, CSS08]). In particular, if we let $\bar{v}$ denote this extension,

$$-c_{n,s}(-\Delta)^s v(x) = \lim_{y\downarrow 0} y^a \partial_y \bar{v}(x, y).$$

This reinterpretation has been extremely useful in studying the thin obstacle problem (see [CS07] and cf. (5.6)).

In this subsection, we show that an analogous reinterpretation exists for a non-local operator of a function v defined on $\mathbb{R}^{n-1}$ as a weighted normal derivative of an a-harmonic extension of v to $\mathbb{R}^{n+1}$, and in the next subsection, we will use it to help us prove a collection of regularity results on the solution to (5.88). For a given (sufficiency smooth) function $u : \mathbb{R}^{n+1} \to \mathbb{R}$, define

$$\mathcal{F}_a(u)(x') := \lim_{\varepsilon \downarrow 0} \int_{\partial D_\varepsilon} u_\nu(x', x_n, y)|y|^a \, \mathrm{d}\sigma(x_n, y). \tag{5.90}$$

Hence, if $\bar{v} : \mathbb{R}^{n+1} \to \mathbb{R}$ is the unique a-harmonic extension that vanishes at infinity to $\mathbb{R}^{n+1}$ of a given function $v : \mathbb{R}^{n-1} \to \mathbb{R}$ that vanishes at infinity:

$$\begin{cases} L_a \bar{v} &= 0 \quad \text{in } \mathbb{R}^{n+1} \setminus \{x_n = y = 0\} \\ \bar{v}(x', 0, 0) &= v(x') \quad \text{on } \mathbb{R}^{n-1} \\ \lim_{|X|\to\infty} \bar{v}(X) &= 0, \end{cases} \tag{5.91}$$

then we can define the non-local operator $\mathcal{I}_a$ on $v : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$\mathcal{I}_a(v) := \mathcal{F}_a(\bar{v}). \tag{5.92}$$

Notice that $\bar{v}$ can be constructed as the unique solution to the following minimization problem:

$$\min_{\mathscr{K}} \left\{ \int_{\mathbb{R}^{n+1}} |\nabla w|^2 |y|^a \right\}, \quad \text{with} \quad a \in (-1, 0),$$

where

$$\mathscr{K} := \{ w \in W^{1,2}(\mathbb{R}^{n+1}, |y|^a) : w = v \text{ on } \{x_n = y = 0\}, \lim_{|X|\to\infty} w(X) = 0 \}.$$

An important and interesting fact is $\mathcal{I}_a$ is nothing but the $-\frac{a}{2}$-Laplacian:

Proposition 5.42. *Let $\mathcal{I}_a$ be defined as in* (5.92)*. Then,*

$$\mathcal{I}_a = c_{n,a}(-\Delta)^{-\frac{a}{2}} \equiv c_{n,a}(-\Delta)^{s-\frac{1}{2}},$$

for some positive constant $c_{n,a}$ depending only on n and a.

Before proving Proposition 5.42, notice that the Poisson kernel associated to (5.91) is

$$P_a(x', x_n, y) = C_{n,a} \frac{(x_n^2 + y^2)^{-\frac{a}{2}}}{(|x'|^2 + x_n^2 + y^2)^{\frac{n-1-a}{2}}}. \tag{5.93}$$

That is, if $v : \mathbb{R}^{n-1} \to \mathbb{R}$, then $v *_{x'} P(\cdot, x_n, y) = \bar{v}$. Indeed, it is easy to see that $L_a P_a(x', x_n, y) = 0$ when $x_n^2 + y^2 > 0$ and $P_a(x', 0, 0)$ is concentrated at $x' = 0$. Furthermore, since $P_a(x', r\cos\theta, r\sin\theta) = r^{-n+1} P_a(r^{-1}x', \cos\theta, \sin\theta)$, we deduce that P_a is a multiple of the Dirac delta of the right dimensionality as $x_n^2 + y^2 \downarrow 0$.

The intuition behind (5.93) is as follows: the Poisson kernel for the fractional Laplacian can be thought as the Poisson kernel regular Laplacian extended to a fractional number of additional dimensions, $+a$ dimensions. In our case, we extend an additional dimension, not only in y, but also in x_n. So we are considering an $(1+a)$-dimensional extension starting from $n-1$ dimensions. That is, (5.93) can be recovered from the Poisson kernel for the fractional Laplacian (see [CS07]) by renaming the variable y to $|(x_n, y)|$ (by Pythagoras) and replacing a with $1+a$ and n with $n-1$.

Proof of Proposition 5.42. Thanks to (5.93), we have that

$$\bar{v}(x', x_n, y) = \int_{\mathbb{R}^{n-1}} v(z') P_a(x'-z', x_n, y)\,\mathrm{d}z'.$$

In turn,

$$\mathcal{I}_a(v)(x') = \lim_{\varepsilon\downarrow 0} \int_{\partial D_\varepsilon} \partial_\nu \left(\int_{\mathbb{R}^{n-1}} v(z') P_a(x'-z', x_n, y)\,\mathrm{d}z' \right) |y|^a \,\mathrm{d}\sigma(x_n, y).$$

Now since P_a is radially symmetric in the (x_n, y) variables,

$$\begin{aligned} \mathcal{I}_a(v)(x') &= \lim_{\varepsilon\downarrow 0} \partial_y \left(\int_{\mathbb{R}^{n-1}} v(z') P_a(x'-z', 0, y)\,\mathrm{d}z' \right) \Big|_{y=\varepsilon} \int_{\partial D_\varepsilon} |y|^a \,\mathrm{d}\sigma(x_n, y) \\ &= C \lim_{\varepsilon\downarrow 0} \varepsilon^{1+a} \partial_y \left(\int_{\mathbb{R}^{n-1}} v(z') P_a(x'-z', 0, y)\,\mathrm{d}z' \right) \Big|_{y=\varepsilon} \\ &= C(-\Delta)^{-\frac{a}{2}} v(x'), \end{aligned}$$

where, in the last step, we have used that $P_a(x', 0, y)$ is the Poisson kernel for the fractional Laplacian of order $1+a$ in $n-1$ dimensions (see [CS07, Sections 1 and 2]). □

Thanks to Proposition 5.42, we can construct some useful Hölder regular barriers.

Lemma 5.43 (Hölder Barriers)**.** *Let $\zeta(r) : [0,\infty) \to [0,1]$ be a smooth function with the following properties: $\zeta(r) \equiv 1$ for $0 \le r \le 2$, $\zeta(r) \equiv 0$ for $r > 3$, and $\zeta' \le 0$. Set $h_\beta(x') := |x'|^\beta \zeta(|x'|)$ and define*

$$\bar{h}_\beta(x', x_n, y) := \int_{\mathbb{R}^{n-1}} h_\beta(z') P_a(x'-z', x_n, y). \tag{5.94}$$

Then,

$$\begin{cases} L_a \bar{h}_\beta = 0 & \text{in } \mathbb{R}^{n+1} \setminus \{x_n = y = 0\} \\ \bar{h}_\beta(x', 0, 0) = |x'|^\beta & \text{on } B_1' \\ \bar{h}_\beta \ge c & \text{in } \partial B_1, \end{cases}$$

for some constant c depending only on n, a. Moreover, $\bar{h}_\beta \in C^\gamma(B_1)$ for $\gamma := \min\{-a, \beta\}$.

Proof. First, observe that by the definition of P_a, $L_a \bar{h}_\beta = 0$ in $\mathbb{R}^{n+1} \setminus \{x_n = y = 0\}$ and $\bar{h}_\beta(x', 0, 0) = |x'|^\beta$ in B_1'. Second, by continuity, $\bar{h}_\beta \ge c > 0$ on ∂B_1 since $\bar{h}_\beta > 0$ on $\partial B_1 \cap \{x_n^2 + y^2 > 0\}$ and $\bar{h}_\beta = 1$ on $\partial B_1 \cap \{x_n^2 + y^2 = 0\}$. Finally, notice that if we fix $x_n = 0$, then $\bar{h}_\beta(x', 0, y)$ is the $(1+a)$-harmonic extension of h_β to $\mathbb{R}^n$ (see the proof of Proposition 5.42); note that $1+a \in (0,1)$. Namely, $\bar{h}_\beta(x',0,y)$ is such that

$$\begin{cases} L_{1+a}\bar{h}_\beta(x',0,y) &= 0 \quad \text{in } \mathbb{R}^n \setminus \{y=0\} \\ \bar{h}_\beta(x',0,0) &= h_\beta(x') \quad \text{on } \mathbb{R}^{n-1} \\ \lim_{|(x',y)|\to\infty} \bar{h}_\beta(x',0,y) &= 0. \end{cases} \tag{5.95}$$

Now by [JN17, Proposition 2.3], we have that $\bar{h}_\beta$ is (locally) smooth in x' and is (locally) $(-a)$-Hölder in the y up to $\{y=0\}$. Therefore, since $\bar{h}_\beta$ is radially symmetric in the (x_n, y) variables, $\bar{h}_\beta \in C^\gamma(B_1)$, as desired. □

We conclude this subsection with a higher regularity result.

Lemma 5.44. *If $u \in L^\infty(B_1)$ is such that $L_a u = 0$ in $B_1 \setminus \{x_n = y = 0\}$ and $u(\,\cdot\,,0,0) \in C^{k+\beta}(B_1')$ for $k \in \mathbb{N}\cup\{0\}$ and $\beta \in (0,1]$, then for $\gamma := \min\{-a,\beta\}$,*

$$[D^k_{x'}u]_{C^\gamma(B_{1/2})} \le C\left(\|u\|_{L^\infty(B_1)} + \|u(\cdot,0,0)\|_{C^{k+\beta}(B_1')}\right),$$

for some constant C depending only on n, a, k, and β. Moreover, if $u(\cdot,0,0)$ is continuous, then u is continuous.

Proof. The proof follows simply by combining interior estimates for the operator L_a and a barrier argument on $\{x_n = y = 0\}$, with the barrier $\bar{h}_\beta$ constructed in Lemma 5.43.

Suppose $k = 0$ and let $\bar{C}$ be a constant such that

$$\bar{C} \ge [u(\cdot,0,0)]_{C^\beta(B_1')} \qquad \text{and} \qquad \bar{C}\bar{h}_\beta \ge \|u\|_{L^\infty(B_1)} \quad \text{on} \quad \partial B_{1/2}.$$

Then, $\bar{C}\bar{h}_\beta$ serves both as a barrier from above and from below at any point $x' \in B'_{1/2}$. This barrier combined with interior estimates for a-harmonic functions (see, e.g, [JN17, Proposition 2.3]) directly yields the desired estimate (as in [MS06], for instance).

If $k \ge 1$, we apply the previous result iteratively, starting with $\beta = 1$, to the derivatives $D^\alpha_{x'}u$, up to a ball $B_{2^{-k-1}}$, and finish by a covering lemma.

To prove the last part, let us suppose that $u(\cdot,0,0)$ is continuous. We want to show that u is continuous as well. Let us extend u to the whole space with any cutoff function and consider $v(x',x_n,y) := u(\cdot,0,0) * P_a(\cdot,x_n,y)$. Notice that since $u(\cdot,0,0)$ is continuous, v is continuous as well. Then, $u = v + w$ where w satisfies $w(\cdot,0,0) \equiv 0$ and $L_a w = 0$ in $B_1 \setminus \{x_n = y = 0\}$. Thus, by the above result, w is smooth and therefore, u is continuous. □

Corollary 5.45. *Let $u \in L^\infty(B_1)$ be a solution to* (5.89)*. Then, u is continuous in B_1.*

Proof. The continuity on the very thin space follows from a standard argument in obstacle type problems (see [Caf98, Theorem 1]) using super-a-harmonicity of the solution and the mean value formula on the thin space for the operator L_a. The continuity in B_1 then follows from Lemma 5.44. □

5.8.2 Basic Estimates

In this subsection, we prove some regularity properties of solutions to (5.88). Our first result contains two classical estimates: an energy estimate and an L^∞ estimate.

Lemma 5.46. *Let u be a solution to* (5.88) *and* (5.89). *Then,*

$$\|u\|_{W^{1,2}(B_{1/2},|y|^a)} \le C\|u\|_{L^2(B_1,|y|^a)} \tag{5.96}$$

and

$$\|u\|_{L^\infty(B_{1/2})} \le C\|u\|_{L^2(B_1,|y|^a)}, \tag{5.97}$$

for some constant C depending only on n and a.

Proof. This is standard (see [AC04] or Lemma 5.9). □

Next, we prove the solutions are Lipschitz and semiconvex in the directions parallel to the very thin space.

Lemma 5.47. *Let u be a solution to* (5.89). *Then, for all $\boldsymbol{e} \in \{x_n = y = 0\} \cap \mathbb{S}^n$,*

$$\|\partial_{\boldsymbol{e}} u\|_{L^\infty(B_{1/4})} \le C\|u\|_{L^2(B_1,|y|^a)} \tag{5.98}$$

and

$$\inf_{B_{1/8}} \partial_{\boldsymbol{ee}} u \ge -C\|u\|_{L^2(B_1,|y|^a)}, \tag{5.99}$$

for some constant C depending only on n and a.

Proof. The proofs of these estimates are identical to the proofs of Lemmas 5.11 and 5.12. That said, to get (5.98), we need to use the incremental quotients $((u(x+h\boldsymbol{e}) - u(x))/h)^-$ and $((u(x-h\boldsymbol{e}) - u(x))/h)^-$, in the spirit of Lemma 5.12, and the continuity of u (proved in Corollary 5.45). □

An easy corollary of Lemma 5.47 is that u is C^{-a}.

Corollary 5.48. *Let u be the solution to* (5.89). *Then,*

$$[u]_{C^{-a}(B_{1/2})} \le C\|u\|_{L^\infty(B_1)}, \tag{5.100}$$

for some constant C depending only on n and a.

Proof. This is an immediate consequence of Lemmas 5.47 and 5.44. □

Using Corollary 5.48, we now prove an L^∞ estimate on $\mathcal{F}_a(u)$.

Lemma 5.49. *Let u be the solution to* (5.89) *and $\mathcal{F}_a$ be as in* (5.90). *Then,*

$$\|\mathcal{F}_a(u)\|_{L^\infty(B'_{1/2})} \le C\|u\|_{L^\infty(B_1)},$$

for some constant C depending only on n and a. That is, $L_a u$ is a locally bounded, absolutely continuous measure, with respect to $\mathcal{H}^{n-1}$, supported on $\{x_n = y = 0\}$.

Proof. Recall, if $L_a u = 0$ in $B_r(X_\circ)$, then

$$\|\nabla_x u\|_{L^\infty(B_{r/2}(X_\circ))} \le C r^{-1} \operatorname*{osc}_{B_r(X_\circ)} u \tag{5.101}$$

and

$$\||y|^a \partial_y u\|_{L^\infty(B_{r/2}(X_\circ))} \le C r^{a-1} \operatorname*{osc}_{B_r(X_\circ)} u. \tag{5.102}$$

(See, e.g., [JN17, Proposition 2.3].) Now let $x' \in B'_{1/2}$. And assume that $\mathcal{F}_a(u)(x') < 0$, so that $u(x') = 0$ (otherwise, there is nothing to prove). We claim that

$$\lim_{\varepsilon \downarrow 0} \left| \int_{\partial D_\varepsilon} \left(\frac{x_n}{\varepsilon} \partial_n u(x', x_n, y) + \frac{y}{\varepsilon} \partial_y u(x', x_n, y) \right) |y|^a \, \mathrm{d}\sigma(x_n, y) \right| \le C \|u\|_{L^\infty(B_1)}. \tag{5.103}$$

From (5.101) and (5.102) and by Corollary 5.48, rescaled to $B_\varepsilon(x', x_n, y)$, we have that

$$\sup_{\partial D_\varepsilon} |\partial_n u| \le C\varepsilon^{-1-a} \|u\|_{L^\infty(B_1)} \quad \text{and} \quad \sup_{\partial D_\varepsilon} ||y|^a \partial_y u| \le C\varepsilon^{-1} \|u\|_{L^\infty(B_1)}.$$

Hence, (5.103), as desired. □

The following theorem proves that u is $C^{1,\tau}$ in the directions parallel to $\{x_n = y = 0\}$.

Theorem 5.50. *Let u be the solution to* (5.89)*. Then, for all $\boldsymbol{e}' \in \{x_n = y = 0\} \cap \mathbb{S}^n$,*

$$[\partial_{\boldsymbol{e}'} u]_{C^\tau(B_{1/2})} \le C \|u\|_{L^2(B_1, |y|^a)},$$

for some constants $\tau > 0$ small and C depending only on n and a.

Proof. Define the cut-off function $\xi(X) := \zeta(|x'|^2)\zeta(x_n^2 + y^2)$ where

$$\zeta : [0, \infty) \to [0, 1], \quad \zeta' \le 0, \quad \zeta \equiv 1 \text{ in } [0, 1/8], \quad \text{and} \quad \zeta \equiv 0 \text{ in } [1/4, \infty),$$

and set $\hat{u}(X) := u(X)\xi(X)$ in B_1 and $\hat{u}(X) \equiv 0$ outside of B_1. Notice that

$$L_a \hat{u} = u L_a \xi + |y|^a \nabla\xi \cdot \nabla u =: |y|^a \hat{f}(X) \quad \text{in} \quad \mathbb{R}^{n+1} \setminus \{x_n = y = 0\}.$$

Now let $\hat{w}$ be such that

$$\begin{cases} L_a \hat{w} &= |y|^a \hat{f} \quad \text{in } \mathbb{R}^{n+1} \setminus \{x_n = y = 0\} \\ \hat{w}(x', 0, 0) &= 0 \qquad \text{on } \mathbb{R}^{n-1} \\ \lim_{|X| \to \infty} \hat{w}(X) &= 0. \end{cases} \tag{5.104}$$

Clearly, $L_a \hat{w} = 0$ in $B_{1/8} \setminus \{x_n = y = 0\}$, so that by Lemma 5.44, $\hat{w}$ is smooth in $B_{1/16}$. Hence, $\mathcal{F}_a(\hat{w})$ is smooth in $B'_{1/16}$.

Observe that

$$\begin{cases} L_a(\hat{u} - \hat{w}) &= 0 \quad \text{in } \mathbb{R}^{n+1} \setminus \{x_n = y = 0\} \\ \hat{u} - \hat{w} &\ge 0 \quad \text{on } \mathbb{R}^{n-1} \times \{0\} \times \{0\}. \end{cases}$$

Moreover, by the symmetries of ξ in the (x_n, y) directions, we have that

$$\begin{cases} \mathcal{F}_a(u\xi)(x') &= 0 \quad \text{if } u(x',0,0) > 0 \\ \mathcal{F}_a(u\xi)(x') &\leq 0 \quad \text{if } u(x',0,0) = 0; \end{cases}$$

so

$$\begin{cases} \mathcal{F}_a(\hat{u}-\hat{w})(x') &= -\mathcal{F}_a(\hat{w})(x') \quad \text{if } (\hat{u}-\hat{w})(x',0,0) > 0 \\ \mathcal{F}_a(\hat{u}-\hat{w})(x') &\leq -\mathcal{F}_a(\hat{w})(x') \quad \text{if } (\hat{u}-\hat{w})(x',0,0) = 0. \end{cases}$$

Alternatively, thanks to Proposition 5.42, $U(x') := (\hat{u}-\hat{w})(x',0,0)$ solves the following obstacle problem

$$\begin{cases} U &\geq 0 & \text{in } \mathbb{R}^{n-1}, \\ (-\Delta)^{-\frac{a}{2}} U &= -C\mathcal{F}_a(\hat{w}) & \text{in } \{x' : U(x') > 0\}, \\ (-\Delta)^{-\frac{a}{2}} U &\leq -C\mathcal{F}_a(\hat{w}) & \text{in } \mathbb{R}^{n-1} \\ \lim_{|x'|\to\infty} U(x') &= 0. \end{cases} \tag{5.105}$$

By [CRS17, Proposition 2.2], recalling that $\mathcal{F}_a(\hat{w})$ is smooth in $B'_{1/16}$ and that u is Lipschitz (5.98) and semiconvex (5.99), we deduce that $U \in C^{1,\tau}(B'_{1/32})$. And via a simple covering argument, $U \in C^{1,\tau}(B'_{3/4})$.

The theorem now follows from Lemma 5.44. □

The last result of this subsection is a Hölder regularity result for the X-directional derivative of u for $X \in B_1$.

Corollary 5.51. *Let u be the solution to* (5.89)*. Then, $X \cdot \nabla u$ is continuous in B_1. In particular,*

$$\|X \cdot \nabla u\|_{C^{\bar{\tau}}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)},$$

for some constants $\bar{\tau} > 0$ small and C, depending only on n and a.

Proof. Let $X_\circ \in \Lambda(u)$. By (5.101), (5.102), and Corollary 5.48,

$$\sup_{B_{r/2}(X_\circ)} |x_n \partial_n u| + |y \partial_y u| \leq C r^{-a}.$$

This, Theorem 5.50, the $C^{1,\tau}$ regularity of u in x', and interior estimates for a-harmonic functions in $B_1 \setminus \Lambda(u)$ (see, e.g., [JN17]) yield the desired result (again, as in [MS06], for instance). □

5.8.3 Monotonicity Formulae

In this subsection, we prove that u has the same monotonicity properties as its cousin, the solution to the thin obstacle problem. We start with Almgren's frequency function.

Lemma 5.52. *Let u be the solution to* (5.89) *and $0 \in \Lambda(u)$. Then, Almgren's frequency function on u*

$$r \mapsto N(r,u) := \frac{r \int_{B_r} |\nabla u|^2 |y|^a}{\int_{\partial B_r} u^2 |y|^a}$$

is non-decreasing for $0 < r < 1$. Moreover, $N(u,r) \equiv \lambda$ if and only if u is homogeneous of degree λ in B_1, i.e., $x \cdot \nabla u - \lambda u = 0$ in B_1.

Proof. The proof of this lemma is standard, and follows the lines of the proof that Almgren's frequency function is monotone on solutions the thin obstacle problem. Nonetheless, some of the steps now require justification because of the inherent lower regularity of the very thin obstacle problem. Justifying these steps is where Theorem 5.50 — more precisely, Corollary 5.51 — comes into play.

Set, for $0 < r < 1$,

$$D(r) = D(r,u) := \int_{B_r} |\nabla u|^2|y|^a \quad \text{and} \quad H(r) = H(r,u) := \int_{\partial B_r} u^2|y|^2,$$

so that $N(r) := N(r,u) = rD(r)/H(r)$. Notice that both quantities are pointwise defined, since $u \in W^{1,2}(B_1,|y|^a) \cap C^{-a}_{\rm loc}(B_1)$, and in particular, $N(r)$ is continuous. Following the proof of Proposition 5.6 (where we remark that D and H were defined differently), we immediately find that

$$H'(r) = \frac{n+a}{r}H(r) + 2\int_{\partial B_r} uu_\nu|y|^a$$

and

$$D'(r) = \frac{n+a-1}{r}D(r) + \frac{2}{r}\int_{B_r} \nabla u \cdot \nabla(X \cdot \nabla u)|y|^a. \tag{5.106}$$

By Corollary 5.51, the quantity $H'(r)$ is well-defined pointwise (and finite). On the other hand, $D(r)$ is absolutely continuous, being the integral in B_r of an integrable function, so that its derivative is well-defined pointwise and almost everywhere finite (and non-negative). Thus, $N(r,u)$ is locally absolutely continuous.

Integrating by parts in the second term of (5.106), we deduce that

$$\frac{1}{r}\int_{B_r} \nabla u \cdot \nabla(X \cdot \nabla u)|y|^a = \int_{\partial B_r} u_\nu^2|y|^a - \frac{1}{r}\int_{B_r}(X \cdot \nabla u)L_a u.$$

Now notice that $L_a u$ is a finite measure concentrated on $\{x_n = y = 0\}$ (see Lemma 5.49), and $X \cdot \nabla u$ is continuous (see Corollary 5.51). Moreover, by the proof of Corollary 5.51, $X \cdot \nabla u = 0$ whenever $L_a u < 0$. In turn, the second term above vanishes. On the other hand, by the continuity of $X \cdot \nabla u$, the first term is well-defined pointwise. Hence,

$$D'(r) = \frac{n+a-1}{r}D(r) + 2\int_{\partial B_r} u_\nu^2|y|^a.$$

Integrating by parts again, observe that

$$D(r) = \int_{B_r} |\nabla u|^2|y|^a = \int_{\partial B_r} uu_\nu|y|^a - \int_{B_r} uL_a u = \int_{\partial B_r} uu_\nu|y|^a,$$

where the term $\int_{B_r} uL_a u = 0$ arguing as before: u is continuous (Corollary 5.48) and vanishes whenever $L_a u < 0$, and $L_a u$ is a finite measure concentrated on $\{x_n = y = 0\}$ (Lemma 5.49).

Combing the above estimates, we determine that

$$\frac{N'(r)}{N(r)} = \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} + \frac{1}{r} = 2\left(\frac{\int_{\partial B_r} u_\nu^2|y|^a}{\int_{\partial B_r} uu_\nu|y|^a} - \frac{\int_{\partial B_r} uu_\nu|y|^a}{\int_{\partial B_r} u^2|y|^a}\right) \geq 0,$$

by the Cauchy–Schwarz inequality, which yields the monotonicity of $N(r,u)$. Analyzing the equality case, we see that if $N(r)$ is constant, then u is homogeneous of degree $N(r)$ (see, e.g., [ACS08, Lemma 1]). $\square$

Next we prove a Monneau-type monotonicity formula.

Lemma 5.53. *Let u be the solution to* (5.89) *and $0 \in \Lambda(u)$. Given $\lambda \geq 0$, define*

$$H_\lambda(r,u) := \frac{1}{r^{n+a+2\lambda}} \int_{\partial B_r} u^2 |y|^a. \tag{5.107}$$

For all $0 \leq \lambda \leq N(0^+,u)$, the map $r \mapsto H_\lambda(r,u)$ is non-decreasing.

Proof. Arguing as in the proof of Lemma 5.52, using $\int_{B_r} uL_a u = 0$, we compute that

$$\frac{H_\lambda'}{H_\lambda}(r,u) = \frac{2}{r}(N(r,u) - \lambda). \tag{5.108}$$

(See, also, the proof of Lemma 5.8.) The lemma then follows from Lemma 5.52: $N(r,u) \geq N(0^+,u) \geq \lambda$. $\square$

Now we move to the Weiss energies.

Lemma 5.54. *Let u be the solution to* (5.89) *and $0 \in \Lambda(u)$. Given $\lambda \geq 0$, define*

$$W_\lambda(r,u) := H_\lambda(r,u)(N(r,u) - \lambda). \tag{5.109}$$

For all $\lambda \geq 0$, the map $r \mapsto W_\lambda(r,u)$ is non-decreasing.

Proof. Arguing as in the proof of Lemma 5.52, using $\int_{B_r} uL_a u = 0$, an explicit computation directly yields

$$\frac{\mathrm{d}}{\mathrm{d}r} W_\lambda(r,u) = \frac{2}{r^{n+1+a+2\lambda}} \int_{\partial B_r} (X \cdot \nabla u - \lambda u)^2 |y|^a \geq 0,$$

as desired. $\square$

We close this subsection with a useful limit.

Lemma 5.55. *Let u be the solution to* (5.89) *and $0 \in \Lambda(u)$. Suppose that $N(0^+,u) = \lambda^*$. Given $\lambda > \lambda^*$,*

$$\lim_{r \downarrow 0} H_\lambda(r,u) = +\infty.$$

Proof. Suppose, to the contrary, we can find a sequence of radii $r_\ell \downarrow 0$ such that $H_\lambda(r_\ell,u) \leq C$ for all $\ell \in \mathbb{N}$. Then, for $\mu \in (\lambda^*, \lambda)$, $H_\mu(r_\ell,u) \to 0$ as $\ell \to \infty$. Hence, as $W_\mu(r,u) \geq -\mu H(r,u)$ for all $r > 0$,

$$\liminf_{\ell \to \infty} W_\mu(r_\ell,u) \geq \liminf_{\ell \to \infty} -\mu H_\mu(r_\ell,u) = 0.$$

By the monotonicity of $r \mapsto W_\mu(r,u)$, Lemma 5.54, we find that

$$N(r_\ell,u) \geq \mu,$$

for all $\ell \in \mathbb{N}$. But this is impossible: $\mu > \lambda^* := N(0^+,u)$. $\square$

5.8.4 Blow-up Analysis and Consequences

This subsection is dedicated to the analysis of blow-ups of u at points $X_\circ \in \Lambda(u)$. As such, for $X_\circ \in \Lambda(u)$, define

$$u_{X_\circ,r}(X) := u(X_\circ + rX) \quad \text{and} \quad \tilde{u}_{X_\circ,r} := \frac{u_{X_\circ,r}}{\|u_{X_\circ,r}\|_{L^2(\partial B_1,|y|^a)}}. \tag{5.110}$$

We start by showing that blow-ups exists and are global, homogeneous solutions to (5.89).

Lemma 5.56. *Let u be the solution to* (5.89) *and suppose that $X_\circ \in \Lambda(u)$. Let $\tilde{u}_{X_\circ,r}$ be as in* (5.110)*. Then, for every sequence $r_j \downarrow 0$, there exists a subsequence $r_{j_\ell} \downarrow 0$ such that*

$$\tilde{u}_{X_\circ,r_{j_\ell}} \rightharpoonup \tilde{u}_{X_\circ,0} \quad \text{in} \quad W^{1,2}(B_1,|y|^a) \quad \text{as} \quad \ell \to \infty \tag{5.111}$$

for some $\tilde{u}_{X_\circ,0} \in W^{1,2}(B_1,|y|^a)$. Moreover, $\tilde{u}_{X_\circ,0} \not\equiv 0$ is a global, homogeneous solution to a very thin obstacle problem with zero obstacle. If, in addition, u is homogeneous, then $\tilde{u}_{X_\circ,0}$ is translation invariant with respect to $X_\circ$.

Proof. By Lemma 5.52, we see that given any sequence $r_j \downarrow 0$, the family $\{\tilde{u}_{X_\circ,r_j}\}_{j\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(B_1,|y|^a)$. Hence, there is a subsequence $r_{j_\ell} \downarrow 0$ such that

$$\tilde{u}_{X_\circ,r_{j_\ell}} \rightharpoonup \tilde{u}_{X_\circ,0} \quad \text{in} \quad W^{1,2}(B_1,|y|^a).$$

As $\|\tilde{u}_{X_\circ,r_{j_\ell}}\|_{L^2(\partial B_1,|y|^a)} = 1$,

$$\|\tilde{u}_{X_\circ,0}\|_{L^2(\partial B_1,|y|^a)} = 1.$$

Clearly, $\tilde{u}_{X_\circ,0} \not\equiv 0$.

Since the family of functions $\{\tilde{u}_{X_\circ,r_{j_\ell}}\}_{\ell\in\mathbb{N}}$ is locally uniformly Hölder continuous (by Corollary 5.48), we have that $\tilde{u}_{X_\circ,r_{j_\ell}} \to \tilde{u}_{X_\circ,0}$ locally uniformly. Moreover, $L_a\tilde{u}_{X_\circ,r_{j_\ell}} \rightharpoonup L_a\tilde{u}_{X_\circ,0}$ (which is non-positive) weakly* as measures (see, e.g., the proof of Proposition 5.13). Therefore, for every $\rho > 0$,

$$0 = \int_{B_\rho} \tilde{u}_{X_\circ,r_{j_\ell}} L_a\tilde{u}_{X_\circ,r_{j_\ell}} |y|^a \to \int_{B_\rho} \tilde{u}_{X_\circ,0} L_a\tilde{u}_{X_\circ,0} |y|^a \quad \text{as} \quad \ell \to \infty, \tag{5.112}$$

so that, since $\tilde{u}_{X_\circ,0} L_a\tilde{u}_{X_\circ,0} \le 0$,

$$\tilde{u}_{X_\circ,0} L_a\tilde{u}_{X_\circ,0} = 0 \quad \text{in} \quad \mathbb{R}^{n+1}.$$

This, together with the uniform convergence of $\tilde{u}_{X_\circ,r_{j_\ell}}$ and the weak* convergence of $L_a\tilde{u}_{X_\circ,r_{j_\ell}}$ to $L_a\tilde{u}_{X_\circ,0}$ directly yields that $\tilde{u}_{X_\circ,0}$ is a global solution to the very thin obstacle problem with zero obstacle.

Furthermore, from the local uniform continuity of $X \cdot \nabla\tilde{u}_{x_\circ,r_\ell}$ given by Corollary 5.51,

$$\int_{\partial B_\rho} \tilde{u}_{X_\circ,r_{j_\ell}} (X \cdot \nabla\tilde{u}_{X_\circ,r_{j_\ell}}) |y|^a \to \int_{\partial B_\rho} \tilde{u}_{X_\circ,0} (X \cdot \nabla\tilde{u}_{X_\circ,0}) |y|^a \quad \text{as} \quad \ell \to \infty.$$

Consequently, for all $\rho > 0$,

$$N(\rho, \tilde{u}_{X_\circ,0}) = \lim_{r_{j_\ell}\downarrow 0} N(\rho, \tilde{u}_{X_\circ,r_{j_\ell}}),$$

and, in particular,

$$N(\rho, \tilde{u}_{X_\circ,0}) = N(0^+, u(X_\circ + \cdot\,)) =: \lambda_{X_\circ}$$

for all $\rho > 0$. (By scaling, $\lim_{r_{j_\ell}\downarrow 0} N(\rho, \tilde{u}_{X_\circ,r_{j_\ell}}) = \lim_{r_{j_\ell}\downarrow 0} N(\rho r_{j_\ell}, u(X_\circ + \cdot\,))$.) Hence, by Lemma 5.52, $\tilde{u}_{X_\circ,0}$ is $\lambda_{X_\circ}$-homogeneous, and the first part of the proof is complete.

Now assume that u is λ-homogeneous. Then,

$$\int_E \nabla \tilde{u}_{X_\circ,r_{j_\ell}}(X) \cdot (X_\circ + r_{j_\ell} X)|y|^a = \int_E \lambda r_{j_\ell} \tilde{u}_{X_\circ,r_{j_\ell}}(X)|y|^a$$

for any compact set $E \subset B_1$. In turn, as $\tilde{u}_{X_\circ,r_{j_\ell}} \rightharpoonup \tilde{u}_{X_\circ,0}$ weakly in $W^{1,2}(B_1, |y|^a)$, taking $r_{j_\ell} \downarrow 0$, we find that

$$X_\circ \cdot \nabla \tilde{u}_{X_\circ,0}(X) = 0 \tag{5.113}$$

for almost every $X \in B_1$. Finally, by Corollary 5.51 and the $\lambda_{X_\circ}$-homogeneity of $\tilde{u}_{X_\circ,0}$ established above, we see that (5.113) holds for all $X \in \mathbb{R}^{n+1}$. □

Just as we did in the thin obstacle setting, we define the nodal set of a solution u to (5.89):

$$\mathcal{N}(u) := \{(x', 0, 0) : u(x', 0, 0) = |\nabla_{x'} u(x', 0, 0)| = f_a(x') = 0\} \tag{5.114}$$

where f_a is defined as in Lemma 5.41.

In the following result, we prove an estimate on the size of the points whose blow-ups have spines

$$L(\tilde{u}_{X_\circ,0}) := \{\xi' \in \mathbb{R}^{n-1} : \xi' \cdot \nabla_{x'} \tilde{u}_{X_\circ,0}(x', 0, 0) = 0 \text{ for all } x' \in \mathbb{R}^{n-1}\}$$

with a certain dimensional bound.

Proposition 5.57. *Let u be a solution to* (5.89). *Then,*

$$\dim_{\mathcal{H}}(\{X_\circ \in \mathcal{N}(u) : \dim L(\tilde{u}_{X_\circ,0}) \le d \text{ for all blow-ups } \tilde{u}_{X_\circ,0}\}) \le d, \tag{5.115}$$

for any $d \in \{0, \dots, n-1\}$. Moreover, if $d = 0$, the previous set is countable.

Proof. The proof follows the first half of the proof of [FoSp18, Theorem 1.3]; and so, we have to check that the assumptions of [Whi97, Theorem 3.2] are fulfilled. In particular, we argue in parallel to [FoSp18, Section 8.1].

Define the upper semicontinuous function $f : B_1' \to \mathbb{R}^+$ by

$$f(x_\circ') := \begin{cases} N(0^+, u(X_\circ + \cdot\,)) & \text{if } X_\circ \in \mathcal{N}(u) \\ 0 & \text{if } X_\circ \notin \mathcal{N}(u), \end{cases}$$

and for any $x'_\circ \in B'_1$, let $\mathscr{G}_{x'_\circ}$ be the family of upper semicontinuous functions $g : \mathbb{R}^{n-1} \to \mathbb{R}^+$ given by

$$g(z') := \begin{cases} N(0^+, \tilde{u}_{X_\circ,0}(Z + \cdot\,)) & \text{if } X_\circ \in \mathcal{N}(u) \\ 0 & \text{if } X_\circ \notin \mathcal{N}(u) \end{cases}$$

where $\tilde{u}_{X_\circ,0}$ is a possible blow-up limit of u at $X_\circ = (x'_\circ, 0, 0)$ (as produced in Lemma 5.56), and of course, $Z = (z', 0, 0)$. Observe, arguing as in [FoSp18, Lemma 5.2], that for all $g \in \mathscr{G}_{x'_\circ}$,

$$\text{if } g(z') = g(0), \text{ then } g(z' + \tau x') = g(z' + x') \text{ for all } x' \in \mathbb{R}^{n-1} \text{ and } \tau > 0;$$

that is, g is *conical*, following the definitions used in [FoSp18, Section 8.1] and [FMS15].

Furthermore, let $\{g_j\}_{j\in\mathbb{N}} \subset \mathscr{G}_{x'_\circ}$. For each g_j, we have an associated blow-up $\tilde{u}_{X_\circ,0,j}$ which has $L^2(\partial B_1, |y|^a)$-norm equal to 1. And arguing as in Lemma 5.56 and then applying a diagonal argument, we can find a subsequence $\{\tilde{u}_{X_\circ,0,j_\ell}\}_{\ell\in\mathbb{N}}$ that converges weakly in $W^{1,2}(B_1, |y|^a)$ and locally uniformly in $C^{-a}(B_1)$ to a blow-up of u at $X_\circ$. Call $\tilde{u}^{(\infty)}_{X_\circ,0}$ this blow-up and define

$$g_\infty(z') := \begin{cases} N(0^+, \tilde{u}^{(\infty)}_{X_\circ,0}(Z + \cdot\,)) & \text{if } X_\circ \in \mathcal{N}(u) \\ 0 & \text{if } X_\circ \notin \mathcal{N}(u). \end{cases}$$

By construction, $g_\infty \in \mathscr{G}_{x'_\circ}$. Now given any convergent sequence $x'_\ell \to x'_\infty \in \mathbb{R}^{n-1}$ as $\ell \to \infty$, by Lemma 5.52 and the upper semicontinuity of the frequency,

$$\begin{aligned} \limsup_{\ell\to\infty} N(0^+, \tilde{u}_{X_\circ,0,j_\ell}(X_\ell + \cdot\,)) &\le \inf_{\rho>0} \limsup_{\ell\to\infty} N(\rho, \tilde{u}_{X_\circ,0,j_\ell}(X_\ell + \cdot\,)) \\ &= \inf_{\rho>0} N(\rho, \tilde{u}^{(\infty)}_{X_\circ,0}(X_\infty + \cdot\,)) \\ &= N(0^+, \tilde{u}^{(\infty)}_{X_\circ,0}(X_\infty + \cdot\,)). \end{aligned}$$

In turn,

$$\limsup_{\ell\to\infty} g_{j_\ell}(x'_\ell) \le g_\infty(x'_\infty),$$

and $\mathscr{G}_{x'_\circ}$ is a class of *compact conical* functions (see [FoSp18, Section 8.1] and [FMS15, Definition 3.3]). Like before, $X_\ell = (x'_\ell, 0, 0)$ and $X_\infty = (x'_\infty, 0, 0)$.

In addition, we need to check the structural hypotheses of [Whi97, Theorem 3.2], which we do as in [FoSp18, Section 8.1(i) and (ii)]. For all $g \in \mathscr{G}_{x'_\circ}$, from the proof of Lemma 5.56,

$$g(0) = f(x'_\circ).$$

Moreover, suppose $r_j \downarrow 0$. By Lemma 5.56, we can find a subsequence $r_{j_\ell} \downarrow 0$ and element $g_\infty \in \mathscr{G}_{x'_\circ}$ so that for any convergent sequence $x'_\ell \to x'_\infty \in B'_1$ as $\ell \to \infty$,

$$\limsup_{\ell\to\infty} f(x'_\circ + r_{j_\ell} x'_\ell) \le g_\infty(x'_\infty).$$

In particular,

$$g_\infty(z') := \begin{cases} N(0^+, \tilde{u}_{X_\circ,0}(Z + \cdot\,)) & \text{if } X_\circ \in \mathcal{N}(u) \\ 0 & \text{if } X_\circ \notin \mathcal{N}(u) \end{cases}$$

with $\tilde{u}_{X_\circ,0}$ being the weak $W^{1,2}(B_1,|y|^a)$ limit of $\tilde{u}_{X_\circ,r_{j_\ell}}$ (it is also the limit in $C^{-a}_{\rm loc}(B_1)$ of $\tilde{u}_{X_\circ,r_{j_\ell}}$). Indeed,

$$\begin{aligned}\limsup_{\ell\to\infty} N(0^+,u(X_\circ+r_{j_\ell}X_\ell+\,\cdot\,)) &\le \inf_{\rho>0}\limsup_{\ell\to\infty} N(r_{j_\ell}\rho,u(X_\circ+r_{j_\ell}X_\ell+\,\cdot\,))\\ &= \inf_{\rho>0}\limsup_{\ell\to\infty} N(\rho,\tilde{u}_{X_\circ,r_{j_\ell}}(X_\ell+\,\cdot\,))\\ &= \inf_{\rho>0} N(\rho,\tilde{u}_{X_\circ,0}(X_\infty+\,\cdot\,))\\ &= N(0^+,\tilde{u}_{X_\circ,0}(X_\infty+\,\cdot\,)).\end{aligned}$$

(Again, $X_\ell=(x'_\ell,0,0)$ and $X_\infty=(x'_\infty,0,0)$.) Hence, applying [Whi97, Theorem 3.2] (or see [FoSp18, Section 8.1]), we prove (5.115). □

We close this section recalling the classification of two-dimensional homogeneous solutions to (5.89), which was proved in [FoSp18, Proposition A.1(i)], and an important consequence.

Lemma 5.58. *Let $n=1$. Let u be a λ-homogeneous solution to* (5.89)*, subject to its own boundary data. Then,*

$$\lambda\in\{-a,1,2,3,\dots\}.$$

In addition, when $\lambda\in\mathbb{N}$, u is an a-harmonic polynomial in $\mathbb{R}^2$.

Proof. The possible values of λ are classified in [FoSp18, Proposition A.1(i)], whence $\lambda\in\mathbb{N}$. Moreover, these integrally homogeneous solutions are polynomials; in particular, they are a-harmonic. That said, in [FoSp18], only homogeneities greater or equal than $1+s$ are considered. Within the proof of [FoSp18, Proposition A.1(i)], however, if homogeneities in $(0,1)$ are also considered, then only one extra homogeneity appears: $-a$, by taking $\nu=-1+s$ (using the notation of [FoSp18]). □

Corollary 5.59. *Let $n\ge 2$ and u be the solution to* (5.89)*. Then,*

$$\dim_{\mathcal{H}}(\{X_\circ\in\Lambda(u):N(0^+,u(X_\circ+\cdot))\notin\mathbb{N}\cup\{-a\}\})\le n-2.$$

Proof. If $Z_\circ\in\Lambda(u)\setminus\{X_\circ\in\mathcal{N}(u):\dim L(u_{X_\circ,0})\le n-2$ for all blow-ups $\tilde{u}_{X_\circ,0}\}$, then there exists a blow-up $\tilde{u}_{Z_\circ,0}$ such that $\dim L(\tilde{u}_{Z_\circ,0})=n-1$. In turn, since two-dimensional homogeneous solutions to the very thin obstacle problem with zero obstacle are polynomials or a multiple of $|X|^{-a}$ (by Lemma 5.58), we deduce that $N(0^+,u(Z_\circ+\,\cdot\,))\in\mathbb{N}\cup\{-a\}$. Hence, from Proposition 5.57, we conclude. □

5.9 Final Remark: Global Problems

In this final section, we state three global obstacle problems — all equivalent — to provide some additional perspective on the very thin obstacle problem. Let

$$\psi\in C^{1,1}(\mathbb{R}^{n-1}) \tag{5.116}$$

be our obstacle, which we assume decays rapidly at infinity.

The very thin obstacle problem for L_a in $\mathbb{R}^{n+1}$ with $a \in (-1, 0)$. Our first problem is a global version of the very thin obstacle problem for L_a with obstacle ψ on $\{x_n = y = 0\}$. Namely, we can consider either the global minimizer of the energy (5.88) among those functions that sit above the obstacle ψ on $\{x_n = y = 0\}$ and go to zero at infinity or, equivalently, the solution to Euler–Lagrange equations

$$\begin{cases} w_1(x', 0, 0) & \geq & \psi(x') & \text{in } \mathbb{R}^{n-1} \\ L_a w_1 & = & 0 & \text{in } \mathbb{R}^{n+1} \setminus \{(x', 0, 0) : w_1(x', 0, 0) = \psi(x')\} \\ L_a w_1 & \leq & 0 & \text{in } \mathbb{R}^{n+1} \\ \lim_{|X|\to\infty} w_1(X) & = & 0. \end{cases} \tag{5.117}$$

Since $a \in (-1, 0)$, it makes sense to say that the solution sits above the ψ on the set $\{x_n = y = 0\}$.

The thin obstacle problem for $(-\Delta)^s$ in $\mathbb{R}^n$ with $s \in (1/2, 1)$ Our second problem is the fractional thin obstacle problem. That is, we consider

$$\begin{cases} w_2(x', 0) & \geq & \psi(x') & \text{in } \mathbb{R}^{n-1} \\ (-\Delta)^s w_2 & = & 0 & \text{in } \mathbb{R}^{n} \setminus \{(x', 0) : w_2(x', 0) = \psi(x')\} \\ (-\Delta)^s w_2 & \leq & 0 & \text{in } \mathbb{R}^{n} \\ \lim_{|x|\to\infty} w_2(x) & = & 0. \end{cases} \tag{5.118}$$

The obstacle problem for $(-\Delta)^{s-\frac{1}{2}}$ in $\mathbb{R}^{n-1}$ with $s \in (1/2, 1)$. Our third and final problem is the obstacle problem for the fractional Laplacian $(-\Delta)^{s-\frac{1}{2}}$ in $\mathbb{R}^{n-1}$. This problem is classical already, and its Euler–Lagrange equations are

$$\begin{cases} w_3(x') & \geq & \psi(x') & \text{in } \mathbb{R}^{n-1} \\ (-\Delta)^{s-\frac{1}{2}} w_3 & = & 0 & \text{in } \mathbb{R}^{n-1} \setminus \{x' : w_3(x') = \psi(x')\} \\ (-\Delta)^{s-\frac{1}{2}} w_3 & \leq & 0 & \text{in } \mathbb{R}^{n-1} \\ \lim_{|x'|\to\infty} w_3(x') & = & 0. \end{cases} \tag{5.119}$$

Proposition 5.60. *If $w_1(x', x_n, y)$ is the solution to (5.117), then $w_2(x', x_n) = w_1(x', x_n, 0)$ is the solution to (5.118), and $w_3(x') = w_2(x', 0) = w_1(x', 0, 0)$ is the solution to (5.119).*

Proof. The fact that $w_2(x', x_n)$ is a solution to (5.118) comes from the extension problem for the fractional Laplacian (see [CS07]). The fact that $w_3(x')$ solves (5.119) is due to Lemma 5.41 and Proposition 5.42. □

Chapter 6

Free boundary regularity for almost every solution to the Signorini problem

We investigate the regularity of the free boundary for the Signorini problem in $\mathbb{R}^{n+1}$. It is known that regular points are $(n-1)$-dimensional and C^∞. However, even for C^∞ obstacles φ, the set of non-regular (or degenerate) points could be very large — e.g. with infinite $\mathcal{H}^{n-1}$ measure.

The only two assumptions under which a nice structure result for degenerate points has been established are: when φ is analytic, and when $\Delta\varphi < 0$. However, even in these cases, the set of degenerate points is in general $(n-1)$-dimensional — as large as the set of regular points.

In this work, we show for the first time that, "usually", the set of degenerate points is *small*. Namely, we prove that, given any C^∞ obstacle, for *almost every* solution the non-regular part of the free boundary is at most $(n-2)$-dimensional. This is the first result in this direction for the Signorini problem.

Furthermore, we prove analogous results for the obstacle problem for the fractional Laplacian $(-\Delta)^s$, and for the parabolic Signorini problem. In the parabolic Signorini problem, our main result establishes that the non-regular part of the free boundary is $(n-1-\alpha_\circ)$-dimensional for almost all times t, for some $\alpha_\circ > 0$.

Finally, we construct some new examples of free boundaries with degenerate points.

6.1 Introduction

The Signorini problem (also known as the thin or boundary obstacle problem) is a classical free boundary problem that was originally studied by Antonio Signorini in connection with linear elasticity [Sig33, Sig59, KO88]. The problem gained further attention in the seventies due to its connection to mechanics, biology, and even finance — see [DL76], [Mer76, CT04], and [Ros18] —, and since then it has been widely studied in the mathematical community; see [Caf79, AC04, CS07, ACS08, GP09, PSU12, KPS15, KRS19, DGPT17, FoSp18, CSV19, JN17, FJ20, Shi18] and references therein.

The main goal of this work is to better understand the size and structure of the non-regular part of the free boundary for such problem.

In particular, our goal is to prove for the first time that, for *almost every* solution (see Remark 6.1), the set of non-regular points is *small*. As explained in detail below, this is completely new even when the obstacle φ is analytic or when it satisfies $\Delta\varphi < 0$.

6.1.1 The Signorini problem

Let us denote $x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ and $B_1^+ = B_1 \cap \{x_{n+1} > 0\}$. We say that $u \in H^1(B_1^+)$ is a solution to the Signorini problem with a smooth obstacle φ defined on $B_1' := B_1 \cap \{x_{n+1} = 0\}$ if u solves

$$\left\{ \begin{array}{rcll} \Delta u & = & 0 & \text{in } B_1^+ \\ \min\{-\partial_{x_{n+1}} u, u - \varphi\} & = & 0 & \text{on } B_1 \cap \{x_{n+1} = 0\}, \end{array} \right. \tag{6.1}$$

in the weak sense, for some boundary data $g \in C^0(\partial B_1 \cap \{x_{n+1} \geq 0\})$. Solutions to the Signorini problem are minimizers of the Dirichlet energy

$$\int_{B_1^+} |\nabla u|^2,$$

under the constrain $u \geq \varphi$ on $\{x_{n+1} = 0\}$, and with boundary conditions $u = g$ on $\partial B_1 \cap \{x_{n+1} > 0\}$.

Problem (6.1) is a *free boundary problem*, i.e., the unknowns of the problem are the solution itself, and the contact set

$$\Lambda(u) := \big\{x' \in \mathbb{R}^n : u(x', 0) = \varphi(x')\big\} \times \{0\} \subset \mathbb{R}^{n+1},$$

whose topological boundary in the relative topology of $\mathbb{R}^n$, which we denote $\Gamma(u) = \partial\Lambda(u) = \partial\{x' \in \mathbb{R}^n : u(x', 0) = \varphi(x')\} \times \{0\}$, is known as the *free boundary*.

Solutions to (6.1) are known to be $C^{1,\frac{1}{2}}$ (see [AC04]), and this is optimal.

6.1.2 The free boundary

While the optimal regularity of the solution is already known, the structure and regularity of the free boundary is still not completely understood. The main known results are the following.

The free boundary can be divided into two sets,

$$\Gamma(u) = \mathrm{Reg}(u) \cup \mathrm{Deg}(u),$$

the set of *regular points*,

$$\mathrm{Reg}(u) := \left\{ x = (x', 0) \in \Gamma(u) : 0 < cr^{3/2} \leq \sup_{B_r'(x')} (u - \varphi) \leq Cr^{3/2}, \quad \forall r \in (0, r_\circ) \right\},$$

and the set of non-regular points or *degenerate points*

$$\mathrm{Deg}(u) := \left\{ x = (x', 0) \in \Gamma(u) : 0 \le \sup_{B_r'(x')} (u - \varphi) \le Cr^2, \quad \forall r \in (0, r_\circ) \right\}, \qquad (6.2)$$

(see [ACS08]). Alternatively, each of the subsets can be defined according to the order of the blow-up at that point. Namely, the set of regular points are those whose blow-up is of order $\frac{3}{2}$, and the set of degenerate points are those whose blow-up is of order κ for some $\kappa \in [2, \infty]$.

Let us denote Γ_κ the set of free boundary points of order κ. That is, those points whose blow-up is homogeneous of order κ (we will be more precise about it later on, in Section 6.2; the definition of Γ_∞ is slightly different). Then, it is well known that the free boundary can be divided as

$$\Gamma(u) = \Gamma_{3/2} \cup \Gamma_{\mathrm{even}} \cup \Gamma_{\mathrm{odd}} \cup \Gamma_{\mathrm{half}} \cup \Gamma_* \cup \Gamma_\infty, \qquad (6.3)$$

where:

- $\Gamma_{3/2} = \mathrm{Reg}(u)$ is the set of regular points. They are an open $(n-1)$-dimensional subset of $\Gamma(u)$, and it is C^∞ (see [ACS08, KPS15, DS16]).

- $\Gamma_{\mathrm{even}} = \bigcup_{m \ge 1} \Gamma_{2m}(u)$ denotes the set of points whose blow-ups have even homogeneity. Equivalently, they can also be characterised as those points of the free boundary where the contact set has zero density, and they are often called singular points. They are contained in the countable union of C^1 $(n-1)$-dimensional manifolds; see [GP09].

- $\Gamma_{\mathrm{odd}} = \bigcup_{m \ge 1} \Gamma_{2m+1}(u)$ is, a priori, also an at most $(n-1)$-dimensional subset of the free boundary and it is $(n-1)$-rectifiable (see [FoSp18, KW13, FoSp19, FRS19]), although it is not actually known whether it exists.

- $\Gamma_{\mathrm{half}} = \bigcup_{m \ge 1} \Gamma_{2m+3/2}(u)$ corresponds to those points with blow-up of order $\frac{7}{2}$, $\frac{11}{2}$, etc. They are much less understood than regular points. The set Γ_{half} is an $(n-1)$-dimensional subset of the free boundary and it is $(n-1)$-rectifiable (see [FoSp18, KW13, FoSp19]).

- Γ_* is the set of all points with homogeneities $\kappa \in (2, \infty)$, with $\kappa \notin \mathbb{N}$ and $\kappa \notin 2\mathbb{N} - \frac{1}{2}$. This set has Hausdorff dimension at most $n-2$, so it is always *small*, see [FoSp18, KW13, FoSp19].

- Γ_∞ is the set of points with infinite order (namely, those points at which $u - \varphi$ vanishes at infinite order, see (6.24)). For general C^∞ obstacles it could be a huge set, even a fractal set of infinite perimeter with dimension exceeding $n-1$. When φ is analytic, instead, Γ_∞ is empty.

Overall, we see that, for general C^∞ obstacles, the free boundary could be really irregular.

The only two assumptions under which a better regularity is known are:

- $\Delta\varphi < 0$ on B_1' and $u = 0$ on $\partial B_1 \cap \{x_{n+1} > 0\}$. In this case, $\Gamma(u) = \Gamma_{3/2} \cup \Gamma_2$ and the set of degenerate points is locally contained in a C^1 manifold; see [BFR18].
- φ is analytic. In this case, $\Gamma_\infty = \varnothing$ and Γ is $(n-1)$-rectifiable, in the sense that it is contained in a countable union of C^1 manifolds, up to a set of zero $\mathcal{H}^{n-1}$-measure, see [FoSp18, KW13].

The goal of this paper is to show that, actually, for *most* solutions, *all* the sets $\Gamma_{\rm even}$, $\Gamma_{\rm odd}$, $\Gamma_{\rm half}$, and Γ_∞ are *small*, namely, of dimension at most $n-2$. This is new even in case that φ is analytic and $\Delta\varphi < 0$.

6.1.3 Our results

We will prove here that, even if degenerate points could potentially constitute a large part of the free boundary (of the same dimension as the regular part, or even higher), they are not *common*. More precisely, for almost every obstacle (or for almost every boundary datum), the set of degenerate points is *small*. This is the first result in this direction for the Signorini problem, even for zero obstacle.

Let $g_\lambda \in C^0(\partial B_1)$ for $\lambda \in [0,1]$, and let us denote by u_λ the family of solutions to (6.1), satisfying

$$u_\lambda = g_\lambda, \quad \text{on} \quad \partial B_1 \cap \{x_{n+1} > 0\}, \tag{6.4}$$

with g_λ satisfying

$$\begin{array}{rcll} g_{\lambda+\varepsilon} & \geq & g_\lambda, & \text{on} \quad \partial B_1 \cap \{x_{n+1} > 0\} \\ g_{\lambda+\varepsilon} & \geq & g_\lambda + \varepsilon & \text{on} \quad \partial B_1 \cap \{x_{n+1} \geq \frac{1}{2}\}, \end{array} \tag{6.5}$$

for all $\lambda \in [0,1)$, $\varepsilon \in (0, 1-\lambda)$.

Our main result reads as follows.

Theorem 6.1. *Let u_λ be any family of solutions of* (6.1) *satisfying* (6.4)-(6.5)*, for some obstacle $\varphi \in C^\infty$. Then, we have*

$$\dim_{\mathcal{H}}\big(\mathrm{Deg}(u_\lambda)\big) \leq n-2 \quad \textit{for a.e.} \quad \lambda \in [0,1],$$

where $\mathrm{Deg}(u_\lambda)$ *is defined by* (6.2)*.*

In other words, for a.e. $\lambda \in [0,1]$, the free boundary $\Gamma(u_\lambda)$ is a C^∞ $(n-1)$-dimensional manifold, up to a closed subset of Hausdorff dimension $n-2$.

This result is completely new even for analytic obstacles, or for $\varphi = 0$. No result of this type was known for the Signorini problem.

The results we prove (see Theorem 6.21 and Proposition 6.25) are actually more precise and concern the Hausdorff dimension of $\Gamma_{\geq\kappa}(u_\lambda)$, the set of points of order greater or equal than κ. We will show that, if $3 \leq \kappa \leq n+1$, then $\Gamma_{\geq\kappa}(u_\lambda)$ has dimension $n-\kappa+1$, while for $\kappa > n+1$, then $\Gamma_{\geq\kappa}(u_\lambda)$ is empty for almost every $\lambda \in [0,1]$. We refer to [Mat95, Chapter 4] for the definition of Hausdorff dimension.

Theorem 6.1 also holds true for non-smooth obstacles. Namely, we will prove that for $\varphi \in C^{3,1}$ we have $\dim_{\mathcal{H}}(\mathrm{Deg}(u_\lambda)) \leq n-2$ for a.e. $\lambda \in [0,1]$. In particular, the free boundary $\Gamma(u_\lambda)$ is $C^{2,\alpha}$ up to a subset of dimension $n-2$ for a.e. $\lambda \in [0,1]$; see [JN17, KPS15, AR19].

Remark 6.1. In the context of the theory of prevalence, [HSY92] (see also [OY05]), Theorem 6.1 says that the set of solutions satisfying that the free boundary has a small degenerate set is *prevalent* within the set of solutions (say, given by C^0 or L^∞ boundary data). Alternatively, the set of solutions whose degenerate set is not lower dimensional is *shy*.

In particular, we can say that for *almost every* boundary data (see [OY05, Definition 3.1]) the corresponding solution has a lower dimensional degenerate set. This is because adding a constant as in (6.5) is a *1-probe* (see [OY05, Definition 3.5]) for the set of boundary data, thanks to Theorem 6.1.

We will establish the following finer result regarding the set $\Gamma_\infty(u_\lambda)$. While it is known that it can certainly exist for some solutions u_λ (see Proposition 6.8), we show that it will be empty for almost every $\lambda \in [0, 1]$:

Theorem 6.2. *Let* u_λ *be any family of solutions of* (6.1) *satisfying* (6.4)-(6.5)*, for some obstacle* $\varphi \in C^\infty$*. Then, there exists* $\mathcal{E} \subset [0, 1]$ *such that* $\dim_{\mathcal{H}} \mathcal{E} = 0$ *and*

$$\Gamma_\infty(u_\lambda) = \varnothing,$$

for every $\lambda \in [0, 1] \setminus \mathcal{E}$*.*

Furthermore, for every $h > 0$*, there exists some* $\mathcal{E}_h \subset [0, 1]$ *such that* $\dim_{\mathcal{M}} \mathcal{E}_h = 0$ *and*

$$\Gamma_\infty(u_\lambda) \cap B_{1-h} = \varnothing,$$

for every $\lambda \in [0, 1] \setminus \mathcal{E}_h$*.*

We remark that in the previous result, $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension, whereas $\dim_{\mathcal{M}}$ denotes the Minkowski dimension (we refer to [Mat95, Chapters 4 and 5]). As such, the second part of the result is much stronger than the first one (e.g., $0 = \dim_{\mathcal{H}} \big(\mathbb{Q} \cap [0, 1]\big) < \dim_{\mathcal{M}} \big(\mathbb{Q} \cap [0, 1]\big) = 1$).

Let us briefly comment on the condition (6.5). Notice that such condition can be reformulated in many ways. In the simplest case, one could simply take $g_\lambda = g_0 \pm \lambda$. Alternatively, one could take a family of obstacles $\varphi_\lambda = \varphi_0 \pm \lambda$ (with fixed boundary conditions); this is equivalent to fixing the obstacle φ_0 and moving the boundary data $g_\lambda = g \mp \lambda$. Furthermore, one could also consider $g_\lambda = g_0 + \lambda\Psi$ for any $\Psi \geq 0$, $\Psi \not\equiv 0$. Then, even if the second condition in (6.5) is not directly fulfilled, a simple use of strong maximum principle makes it true in some smaller ball $B_{1-\rho}$, so that $g_{\lambda+\varepsilon} \geq g_\lambda + c(\rho)\varepsilon$ on $\partial B_{1-\rho} \cap \{x_{n+1} \geq \frac{1}{2} - \rho/2\}$. By rescaling the function and the domain, we can rewrite it as (6.5).

Regularity results for almost every solution have been established before in the context of the classical obstacle problem by Monneau in [Mon03]. In such problem, however, all free boundary points have homogeneity 2, and non-regular points are characterised by the density of the contact set around them: non-regular points are those at which the contact set has density zero. In the Signorini problem, instead, the structure of non-regular points is quite different, and they are characterised by the growth of u around them (recall (6.2) and the definition of Γ_{even}, Γ_{odd}, Γ_{half}, and Γ_∞). This is why the approach of [Mon03] cannot work in the present context.

More recently, the results of Monneau for the classical obstacle problem have been widely improved by Figalli, the second author, and Serra in [FRS19]. The results in [FRS19] are based on very fine higher order expansions at singular points, which then lead to a better understanding of solutions around them, combined with new dimension reduction arguments and a cleaning lemma to get improved bounds on higher order expansions.

Here, due to the different nature of the problem, we do not need any fine expansion at non-regular points nor any dimension reduction. Most of our arguments require only the growth of solutions at different types of degenerate points, combined with appropriate barriers, and Harnack-type inequalities. The starting point of our results is to use a simple (but key) GMT lemma from [FRS19] (see Lemma 6.19 below).

6.1.4 Parabolic Signorini problem

The previous results use rather general techniques that suitably modified can be applied to other situations. We show here that using a similar approach as in the elliptic case, one can deduce results regarding the size of the non-regular part of the free boundary for the parabolic version of the Signorini problem, for almost every time t.

We say that a function $u = u(x,t) \in H^{1,0}(B_1^+ \times (-1,0])$ (see [DGPT17, Chapter 2]) solves the parabolic Signorini problem with stationary obstacle $\varphi = \varphi(x)$ if u solves

$$\begin{cases} \partial_t u - \Delta u &= 0 \quad \text{in } B_1^+ \times (-1,0] \\ \min\{-\partial_{x_{n+1}} u, u - \varphi\} &= 0 \quad \text{on } B_1 \cap \{x_{n+1} = 0\} \times (-1,0], \end{cases} \tag{6.6}$$

in the weak sense (cf. (6.1)). A thorough study of the parabolic Signorini problem was made by Danielli, Garofalo, Petrosyan, and To, in [DGPT17].

The parabolic Signorini problem is a free boundary problem, where the free boundary belongs to $B_1' \times (-1,0]$ and is defined by

$$\Gamma(u) := \partial_{B_1' \times (-1,0]} \big\{ (x',t) \in B_1' \times (-1,0] : u(x',0,t) > \varphi(x') \big\},$$

where $\partial_{B_1' \times (-1,0]}$ denotes the boundary in the relative topology of $B_1' \times (-1,0]$. Analogously to the elliptic Signorini problem, the free boundary can be divided into regular points and degenerate (or non-regular) points:

$$\Gamma(u) = \mathrm{Reg}(u) \cup \mathrm{Deg}(u).$$

The set of regular points are those where parabolic blow-ups are parabolically $\frac{3}{2}$-homogeneous. On the other hand, degenerate points are those where parabolic blow-ups of the solution are parabolically κ-homogeneous, with $\kappa \geq 2$ (alternatively, the solution detaches at most quadratically from the obstacle in parabolic cylinders, $B_r \times (-r^2, 0]$). Further stratifications according to the homogeneity of the parabolic blow-ups can be done in an analogous way to the elliptic problem, see [DGPT17].

The set of regular points $\mathrm{Reg}(u)$ is a relatively open subset of $\Gamma(u)$ and the free boundary is smooth ($C^{1,\alpha}$) around them (see [DGPT17, Chapter 11]). The set of degenerate points, however, could be even larger than the set of regular points.

In this manuscript we show that, under the appropriate conditions, for a.e. time $t \in (-1,0]$ the set of degenerate points has dimension $(n - 1 - \alpha_\circ)$ for some $\alpha_\circ > 0$

depending only on n. That is, for a.e. time, the free boundary is mostly comprised of regular points, and therefore, it is smooth almost everywhere.

In order to be able to get results of this type we must impose some conditions on the solution. We will assume that

$$u_t > 0 \quad \text{in} \quad B_1^+ \cup [(B_1' \times (-1,0]) \cap \{u > \varphi\}], \tag{6.7}$$

that is, wherever the solution u is not in contact with the obstacle φ, it is strictly monotone. Alternatively, by the strong maximum principle, the condition can be rewritten as

$$\begin{aligned} u_t &\geq 0, \quad \text{in } \overline{B_1^+} \times (-1,0], \\ u_t &\geq 1, \quad \text{in } \big(B_1^+ \cap \{x_{n+1} \geq 1/2\}\big) \times (-1,0], \end{aligned}$$

up to a constant multiplicative factor.

Condition (6.7) is somewhat necessary. If the strict monotonicity was not required, we could be dealing with a *bad* solution (with large non-regular set) of the elliptic problem for a set of times of positive measure, and therefore, we could not expect a result like the one we prove. On the other hand, if one allowed changes in the sign of u_t (alternatively, one allowed *non-stationary* obstacles), then the result is also not true (see, for instance, the example discussed in [DGPT17, Figure 12.1]).

Condition (6.7) is actually quite natural. One of the main applications of the parabolic Signorini problem is the study of semi-permeable membranes (see [DL76, Section 2.2]):

We consider a domain (B_1^+) and a thin membrane (B_1'), which is semi-permeable: that is, a fluid can pass through B_1' into B_1^+ freely, but outflow of the fluid is prevented by the membrane. If we suppose that there is a given liquid pressure applied to the membrane B_1' given by φ, and we denote $u(x,t)$ the inside pressure of the liquid in B_1^+, then the parabolic Signorini problem (6.6) describes the evolution of the inside pressure with time. In particular, since liquid can only enter B_1^+ (and we assume no liquid can leave from the other parts of the boundary), pressure inside the domain can only become higher, and the solution will be such that $u_t > 0$. The same condition also appears in volume injection through a semi-permeable wall ([DL76, subsections 2.2.3 and 2.2.4]).

Our result reads as follows.

Theorem 6.3. *Let $\varphi \in C^\infty$ and let u be a solution to* (6.6) *satisfying* (6.7)*. Then,*

$$\dim_{\mathcal{H}} \big(\mathrm{Deg}(u) \cap \{t = t_\circ\}\big) \leq n - 1 - \alpha_\circ \quad \textit{for a.e.} \quad t_\circ \in (-1, 0],$$

for some $\alpha_\circ > 0$ depending only on n.

In particular, for a.e. $t_\circ \in (-1,0]$ the free boundary $\Gamma(u) \cap \{t = t_\circ\}$ is a $C^{1,\alpha}$ $(n-1)$-dimensional manifold, up to a closed subset of Hausdorff dimension $n-1-\alpha_0$.

When φ is analytic, then the free boundary is actually C^∞ around regular points. Higher regularity of the free boundary is also expected for smooth obstacles, but so far it is only known when φ is analytic; see [BSZ17].

It is important to remark that the parabolic case presents some extra difficulties with respect to the elliptic one, and in fact we do not know if a result analogous to Theorem 6.2 holds in this context. This means that points of order ∞ could a priori still appear for all times (even though by Theorem 6.3 they are lower-dimensional for almost every time).

6.1.5 The fractional obstacle problem

The Signorini problem in $\mathbb{R}^{n+1}$ can be reformulated in terms of a fractional obstacle problem with operator $(-\Delta)^{\frac{1}{2}}$ in $\mathbb{R}^n$. Conversely, fractional obstacle problems (with the operator $(-\Delta)^s$, $s \in (0,1)$) can also be reformulated in terms of thin obstacle problems with weights. In this work we will generally deal with the thin obstacle problem with a weight, so that the results from subsection 6.1.3 can also be formulated for the fractional obstacle problem.

Given an obstacle $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\{\varphi > 0\} \subset\subset \mathbb{R}^n, \tag{6.8}$$

the fractional obstacle problem with obstacle φ in $\mathbb{R}^n$ $(n \geq 2)$ is

$$\begin{cases} (-\Delta)^s v &= 0 \quad \text{in } \mathbb{R}^n \setminus \{v = \varphi\} \\ (-\Delta)^s v &\geq 0 \quad \text{in } \mathbb{R}^n \\ v &\geq \varphi \quad \text{in } \mathbb{R}^n \\ v(x) &\to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{6.9}$$

Solutions to the fractional obstacle problem are $C^{1,s}$ (see [CSS08]). We denote $\Lambda(v) = \{v = \varphi\}$ the contact set, and $\Gamma(v) = \partial\Lambda(v)$ the free boundary. As in the Signorini problem (which corresponds to $s = \frac{1}{2}$) the free boundary can be partitioned into regular points

$$\mathrm{Reg}(v) := \left\{ x' \in \Gamma(v) : 0 < cr^{1+s} \leq \sup_{B'_r(x')} (v - \varphi) \leq Cr^{1+s}, \quad \forall r \in (0, r_\circ) \right\},$$

and non-regular (or degenerate) points,

$$\mathrm{Deg}(v) := \left\{ x' \in \Gamma(v) : 0 \leq \sup_{B'_r(x')} (v - \varphi) \leq Cr^2, \quad \forall r \in (0, r_\circ) \right\}. \tag{6.10}$$

More precisely, if we denote by $\Gamma_\kappa(v)$ the free boundary points of order κ, then the free boundary $\Gamma(v)$ can be further stratified analogously to (6.3) as

$$\Gamma(v) = \Gamma_{1+s} \cup \left(\bigcup_{m \geq 1} \Gamma_{2m} \right) \cup \left(\bigcup_{m \geq 1} \Gamma_{2m+2s} \right) \cup \left(\bigcup_{m \geq 1} \Gamma_{2m+1+s} \right) \cup \Gamma_* \cup \Gamma_\infty. \tag{6.11}$$

Here, $\Gamma_{1+s} = \mathrm{Reg}(v)$ is the set of regular points ([CSS08, Sil07]). Again, it is an open subset of the free boundary, which is smooth. Similarly, Γ_{2m} for $m \geq 1$ are often called singular points, and are those where the contact set has zero measure (see [GR19]). Together with the sets Γ_{2m+2s} and Γ_{2m+1+s} for $m \geq 1$, they are an $(n-1)$-dimensional rectifiable subset of the free boundary, [GR19, FoSp19]. Finally, Γ_* denotes the set containing the remaining homogeneities (except infinite), and has dimension $n - 2$; and Γ_∞ denotes those boundary points where the solution is approaching the obstacle *faster* than any power (i.e., at infinite order). As before, the set Γ_∞ could have dimension even higher than $n - 1$.

The type of result we want to prove in this setting regarding regularity for most solutions is concerned with global perturbations of the obstacle (rather than boundary perturbations, as before). That is, we will consider obstacles fulfilling (6.8).

We define the set of solutions indexed by $\lambda \in [0,1]$ to the fractional obstacle problem as

$$\begin{cases} (-\Delta)^s v_\lambda &= 0 & \text{in } \mathbb{R}^n \setminus \{v_\lambda = \varphi\} \\ (-\Delta)^s v_\lambda &\geq 0 & \text{in } \mathbb{R}^n \\ v_\lambda &\geq \varphi - \lambda & \text{in } \mathbb{R}^n \\ v_\lambda(x) &\to 0 & \text{as } |x| \to \infty. \end{cases} \tag{6.12}$$

Then, our main result reads as follows.

Theorem 6.4. *Let v_λ be any family of solutions solving* (6.12)*, for some obstacle $\varphi \in C^\infty$ fulfilling* (6.8)*. Then, we have*

$$\dim_{\mathcal{H}} \big(\mathrm{Deg}(v_\lambda)\big) \leq n-2, \quad \textit{for a.e.} \quad \lambda \in [0,1],$$

where $\mathrm{Deg}(v_\lambda)$ *is defined by* (6.10)*.*

In other words, for a.e. $\lambda \in [0,1]$, the free boundary $\Gamma(v_\lambda)$ is a C^∞ $(n-1)$-dimensional manifold, up to a closed subset of Hausdorff dimension $n-2$.

As before, we actually prove more precise results (see Theorem 6.21 and Proposition 6.25). We establish an estimate for the Hausdorff dimension of $\Gamma_{\geq\kappa}(v_\lambda)$. We show that, for $2 \leq \kappa - 2s \leq n$, then $\dim_{\mathcal{H}} \Gamma_{\geq\kappa}(v_\lambda) \leq n-\kappa+2s$, and if $\kappa > n+2s$, then $\Gamma_{\geq\kappa}(v_\lambda)$ is empty for almost every $\lambda \in [0,1]$. Similarly, we can also reduce the regularity of the obstacle to $\varphi \in C^{4,\alpha}$ so that, for a.e. $\lambda \in [0,1]$, $\dim_{\mathcal{H}} (\mathrm{Deg}(v_\lambda)) \leq n-2$ (in particular, the free boundary $\Gamma(v_\lambda)$ is $C^{3,\alpha}$ up to a subset of dimension $n-2$ for a.e. $\lambda \in [0,1]$; see [JN17, AR19]).

Theorem 6.4 is analogous to Theorem 6.1. On the other hand, we also have that:

Theorem 6.5. *Let v_λ be any family of solutions solving* (6.12)*, for some obstacle $\varphi \in C^\infty$ fulfilling* (6.8)*. Then, there exists $\mathcal{E} \subset [0,1]$ such that $\dim_{\mathcal{H}} \mathcal{E} = 0$ and*

$$\Gamma_\infty(v_\lambda) = \varnothing,$$

for all $\lambda \in [0,1] \setminus \mathcal{E}$.

Furthermore, for every $h > 0$, there exists some $\mathcal{E}_h \subset [0,1]$ such that $\dim_{\mathcal{M}} \mathcal{E}_h = 0$ and

$$\Gamma_\infty(v_\lambda) \cap B_{1-h} = \varnothing,$$

for every $\lambda \in [0,1] \setminus \mathcal{E}_h$.

That is, analogously to Theorem 6.2, we can also control the size of λ for which the free boundary points of infinite order exist.

6.1.6 Examples of degenerate free boundary points

Let us finally comment on the non-regular part of the free boundary, that is,

$$\mathrm{Deg}(u) = \Gamma_{\mathrm{even}} \cup \Gamma_{\mathrm{odd}} \cup \Gamma_{\mathrm{half}} \cup \Gamma_* \cup \Gamma_\infty. \tag{6.13}$$

The main open questions regarding each of the subsets of the degenerate part of the free boundary are:

Q1: Are there non-trivial examples (e.g., the limit of regular points) of singular points in $\Gamma_{\rm even}$?

Q2: Do points in $\Gamma_{\rm odd}$ exist?

Q3: Can one construct arbitrary contact sets with free boundary formed entirely of $\Gamma_{\rm half}$ (alternatively, do they exist apart from the homogeneous solutions)?

Q4: Do points in Γ_* exist?

Q5: How big can the set Γ_∞ be?

In this paper, we answer questions Q1, Q3, and Q5. (Questions Q2 and Q4 remain open.)

Let us start with Q1. The set $\Gamma_{\rm even} = \bigcup_{m\geq 1}\Gamma_{2m}$, often called the set of singular points, is an $(n-1)$-dimensional subset of the free boundary. Examples of free boundary points belonging to $\Gamma_{\rm even}$ are easy to construct as level sets of homogeneous harmonic polynomials, such as $x_1^2 - x_{n+1}^2$, in which case we have $\Gamma = \Gamma_{\rm even} = \{x_1 = 0\}$. They are also expected to appear in less trivial situations but, as far as we know, none has been constructed so far that appears as limit of regular points (i.e., on the boundary of the interior of the contact set). Here, we show that:

Proposition 6.6. *There exists a boundary data g such that the free boundary of the solution to the Signorini problem* (6.1) *with $\varphi = 0$ has a sequence of regular points (of order $3/2$) converging to a singular point (of order 2).*

The proof of the previous result is given in Section 6.5. In contrast to what occurs with the classical obstacle problem, the construction of singular points does not seem to immediately arise from continuous perturbations of the boundary value under symmetry assumptions. Instead, one has to be aware that there could appear other points (different from regular, but not in $\Gamma_{\rm even}$). Thus, our strategy is based on being in a special setting that avoids the appearance of higher order free boundary points.

On the other hand, regarding question Q3, it is known that examples of such points can be constructed through homogeneous solutions, in which case they can even appear as limit of regular (or lower frequency) points (see [CSV19, Example 1]). Until now, however, it was not clear whether such points could appear in non-trivial (say, non-homogeneous) situations.

We show that, given *any* smooth domain $\Omega \subset \mathbb{R}^n$, one can find a solution to the Signorini problem whose contact set is exactly given by Ω, and whose free boundary is entirely made of points of order $\frac{7}{2}$ (or $\frac{11}{2}$, etc.). More generally, we show that given Ω, the contact set for the fractional obstacle problem can be made up entirely of points belonging to $\bigcup_{m\geq 1}\Gamma_{2m+1+s}$ (the case $s = \frac{1}{2}$ corresponding to the Signorini problem).

Proposition 6.7. *Let $\Omega \subset \mathbb{R}^n$ be any given C^∞ bounded domain, and let $m \in \mathbb{N}$. Then, there exists an obstacle $\varphi \in C^\infty(\mathbb{R}^n)$ with $\varphi \to 0$ at ∞, and a global solution* *to the obstacle problem*

$$\begin{cases} (-\Delta)^s u & \geq & 0 & \text{in } \mathbb{R}^n \\ (-\Delta)^s u & = & 0 & \text{in } \{u > \varphi\} \\ u & \geq & \varphi & \text{in } \mathbb{R}^n, \\ u(x) & \to & 0 & \text{as } |x| \to \infty, \end{cases}$$

such that the contact set is $\Lambda(u) = \{u = \varphi\} = \Omega$, *and all the points on the free boundary* $\partial\Lambda(u)$ *have frequency* $2m + 1 + s$.

The proof of the previous proposition is constructive: we show a way in which such solutions can be constructed, using some results from [Gru15, AR19].

Finally, we also answer question Q5, that deals with the set Γ_∞. Not much has been discussed about it in the literature, though its lack of structure was somewhat known by the community. For instance, the following result is not difficult to prove:

Proposition 6.8. *For any* $\varepsilon > 0$ *there exists a non-trivial solution* u *and an obstacle* $\varphi \in C^\infty(\mathbb{R}^n)$ *such that*

$$\begin{cases} (-\Delta)^s u & \geq & 0 & \text{in } \mathbb{R}^n \\ (-\Delta)^s u & = & 0 & \text{in } \{u > \varphi\} \\ u & \geq & \varphi & \text{in } \mathbb{R}^n, \end{cases}$$

and the boundary of the contact set, $\Lambda(u) = \{u = \varphi\}$, *fulfils*

$$\dim_{\mathcal{H}} \partial\Lambda(u) \geq n - \varepsilon.$$

This shows that, in general, there is no hope to get nice structure results for the full free boundary for C^∞ obstacles. However, thanks to Theorem 6.5 above we know that such behaviour is extremely rare. As before, we are answering question Q5 in the generality of the fractional obstacle problem; the Signorini problem corresponds to the case $s = \frac{1}{2}$.

6.1.7 Organization of the paper

The paper is organised as follows:

In Section 6.2 we study the behaviour of degenerate points under perturbation. In particular, we show how the free boundary moves around them when perturbing monotonically the solution to the obstacle problem. We treat separately general degenerate points, and those of order 2. In Section 6.3 we study the dimension of the set Γ_2 by means of an appropriate application of Whitney's extension theorem. In Section 6.4 we prove the main results of this work, Theorems 6.1, 6.2, 6.4, and 6.5. In Section 6.5 we construct the examples of degenerate points introduced in Subsection 6.1.6, proving Propositions 6.6, 6.7, and 6.8. Finally, in Section 6.6 we deal with the parabolic Signorini problem and prove Theorem 6.3.

6.2 Behaviour of non-regular points under perturbations

Let $B_1 \subset \mathbb{R}^{n+1}$, $B_1' = \{x' \in \mathbb{R}^n : |x'| < 1\} \subset \mathbb{R}^n$ and let

$$\varphi : B_1' \to \mathbb{R}, \quad \varphi \in C^{\tau,\alpha}(\overline{B_1'}), \quad \tau \in \mathbb{N}_{\geq 2},\ \alpha \in (0,1] \tag{6.14}$$

be our obstacle on the thin space. Let us consider the fractional operator

$$L_a u := \mathrm{div}(|x_{n+1}|^a \nabla u) = \mathrm{div}(|x_{n+1}|^{1-2s}\nabla u), \qquad a := 1-2s,$$

with $a \in (-1,1)$, and $(0,1) \ni s = \frac{1-a}{2}$. We will interchangeably use both a and s depending on the situation. (In general, we will use a for the weight exponent, and s for all the other situations.)

Let us suppose that we have a family of *increasing* even solutions u_λ for $0 \leq \lambda \leq 1$ to the fractional obstacle problem

$$\begin{cases} L_a u_\lambda & = & 0 & \text{in } B_1 \setminus (\{x_{n+1} = 0\} \cap \{u_\lambda = \varphi\}) \\ L_a u_\lambda & \leq & 0 & \text{in } B_1 \\ u_\lambda & \geq & \varphi & \text{on } \{x_{n+1} = 0\}, \end{cases} \tag{6.15}$$

for a given obstacle φ satisfying (6.14). In particular, $\{u_\lambda\}_{0\leq\lambda\leq 1}$ satisfy

$$\begin{array}{rcllll} u_\lambda(x', x_{n+1}) & = & u_\lambda(x', -x_{n+1}) & \text{in} & B_1, \ \text{for} \ \ \lambda \geq 0 & \\ u_{\lambda'} & \geq & u_\lambda & \text{in} & B_1, \ \text{for} \ \ \lambda' \geq \lambda & \\ u_{\lambda+\varepsilon} & \geq & u_\lambda + \varepsilon & \text{in} & B_1 \cap \{|x_{n+1}| \geq \frac{1}{2}\}, & \text{for} \ \ \lambda, \varepsilon \geq 0 \\ \|u_\lambda\|_{C^{2s}(B_1)} & \leq & M, & \text{in} & B_1 \ \text{for} \ \ \lambda \geq 0, & \end{array} \tag{6.16}$$

for some constant M independent of λ, that will depend on the obstacle (see (6.19)-(6.20) below). Notice that solutions are $C^{1,s}$ in $B'_{1/2}$ (or in $\overline{B^+_{1/2}}$), but only C^{2s} in B_1 ($C^{0,1}$ when $s = \frac{1}{2}$).

We denote $\Lambda(u_\lambda) := \{x' : u_\lambda(x', 0) = \varphi(x')\} \times \{0\} \subset \mathbb{R}^n$ the contact set, and its boundary in the relative topology of $\mathbb{R}^n$, $\partial\Lambda(u_\lambda) = \partial\{x' : u_\lambda(x', 0) = \varphi(x')\} \times \{0\}$ is the free boundary. Note that, from the monotonicity assumption,

$$\Lambda(u_\lambda) \subset \Lambda(u_{\lambda'}) \quad \text{for} \quad \lambda \geq \lambda'. \tag{6.17}$$

Lemma 6.9. *Let u_λ denote the family of solutions to* (6.15)-(6.16)*. Then, for any $h > 0$ small, $x_\circ \in B_{1-h}$, and $\varepsilon > 0$,*

$$\frac{u_{\lambda+\varepsilon}(x_\circ) - u_\lambda(x_\circ)}{\varepsilon} \geq c\ \mathrm{dist}^{2s}(x_\circ, \Lambda(u_\lambda)),$$

for some constant $c > 0$ depending only on n, s, and h. In particular,

$$\partial^+_\lambda u_\lambda(x_\circ) := \liminf_{\varepsilon \downarrow 0} \frac{u_{\lambda+\varepsilon}(x_\circ) - u_\lambda(x_\circ)}{\varepsilon} \geq c\ \mathrm{dist}^{2s}(x_\circ, \Lambda(u_\lambda)),$$

for some constant $c > 0$ depending only on n, s, and h.

Proof. Fix some $\lambda > 0$ and $\varepsilon > 0$, and define

$$\delta_{\lambda,\varepsilon} u_\lambda(x) = \frac{u_{\lambda+\varepsilon}(x) - u_\lambda(x)}{\varepsilon}.$$

We will show that the result holds for $\delta_{\lambda,\varepsilon} u_\lambda$ for some constant c independent of $\varepsilon > 0$, and in particular, it also holds after taking the lim inf.

Notice that $\delta_{\lambda,\varepsilon} u_\lambda(x) \geq 0$ from the monotonicity of u_λ in λ. Notice, also, that $\delta_{\lambda,\varepsilon} u_\lambda \geq 1$ in $B_1 \cap \{x_{n+1} \geq \frac{1}{2}\}$, form the third condition in (6.16). On the other hand,

$$L_a \delta_{\lambda,\varepsilon} u_\lambda = 0 \quad \text{in} \quad B_1 \setminus \Lambda(u_\lambda),$$

thanks to (6.17). Now, let

$$r := \frac{h}{4} \text{dist}(x_\circ, \Lambda(u_\lambda))$$

and we define the barrier function $\psi : B_1 \to \mathbb{R}$ as the solution to

$$\begin{cases} L_a \psi &= 0 \quad \text{in} \quad B_1 \setminus \{x_{n+1} = 0\} \\ \psi &= 0 \quad \text{on} \quad \{x_{n+1} = 0\} \\ \psi &= 1 \quad \text{on} \quad \partial B_1 \cap \{|x_{n+1}| \geq \frac{1}{2}\} \\ \psi &= 0 \quad \text{on} \quad \partial B_1 \cap \{|x_{n+1}| < \frac{1}{2}\}. \end{cases}$$

Then, by maximum principle,

$$\delta_{\lambda,\varepsilon} u_\lambda \geq \psi \quad \text{in} \quad B_1.$$

Notice that, by the boundary Harnack inequality for Muckenhoupt weights A_2 (see [FJK83]), ψ is comparable to $|x_{n+1}|^{2s}$ (since both vanish continuously at $x_{n+1} = 0$, and both are a-harmonic), and in particular, there exists some $c' > 0$ small depending only on n, s, and h, such that $\psi \geq c'|x_{n+1}|^{2s}$ in $B_r(x_\circ)$. We have that

$$L_a \delta_{\lambda,\varepsilon} u_\lambda = 0, \quad \delta_{\lambda,\varepsilon} u_\lambda \geq \psi \geq c'|x_{n+1}|^{2s} \quad \text{in} \quad B_r(x_\circ).$$

Now, if $x_\circ = (x'_\circ, x_{\circ,n+1})$ is such that $|x_{\circ,n+1}| \geq \frac{r}{4}$, it is clear that $\delta_{\lambda,\varepsilon} u_\lambda(x_\circ) \geq cr^{2s}$. On the other hand, if $|x_{\circ,n+1}| \leq \frac{r}{4}$, then $L_a \delta_{\lambda,\varepsilon} u_\lambda = 0$ in $B_{r/2}((x'_\circ, 0))$, so that applying Harnack's inequality in $B_{r/4}((x'_\circ, 0))$ to $\delta_{\lambda,\varepsilon} u_\lambda$,

$$\delta_{\lambda,\varepsilon} u_\lambda(x_\circ) \geq \inf_{B_{r/4}((x'_\circ,0))} \delta_{\lambda,\varepsilon} u_\lambda \geq \frac{1}{C} \sup_{B_{r/4}((x'_\circ,0))} \delta_{\lambda,\varepsilon} u_\lambda \geq \frac{c' r^{2s}}{4^{2s} C} = cr^{2s},$$

for some c depending only on n, s, and h. Thus,

$$\delta_{\lambda,\varepsilon} u_\lambda(x_\circ) \geq cr^{2s} = c \text{ dist}^{2s}(x_\circ, \Lambda(u_\lambda)),$$

as we wanted to see. □

Let $0 \in \partial\Lambda(u_\lambda)$ be a free boundary point for u_λ. Let us denote $Q_\tau(x')$ the Taylor expansion of $\varphi(x')$ around 0 up to order τ, and we denote $Q^a_\tau(x)$ its unique even a-harmonic extension (see [GR19, Lemma 5.2]) to $\mathbb{R}^{n+1}$ ($L_a Q^a_\tau(x) = 0$, and $Q^a_\tau(x', 0) = Q_\tau(x')$). Let us define

$$\bar{u}_\lambda(x', x_{n+1}) = u_\lambda(x', x_{n+1}) - Q^a_\tau(x', x_{n+1}) + Q_\tau(x') - \varphi(x').$$

Then $\bar{u}_\lambda(x', x_{n+1})$ solves the zero obstacle problem with a right-hand side

$$\begin{cases} L_a \bar{u}_\lambda = |x_{n+1}|^a f & \text{in } B_1 \setminus (\{x_{n+1} = 0\} \cap \{\bar{u}_\lambda = 0\}) \\ L_a \bar{u}_\lambda \leq |x_{n+1}|^a f & \text{in } B_1 \\ \bar{u}_\lambda \geq 0 & \text{on } \{x_{n+1} = 0\}, \end{cases} \tag{6.18}$$

where

$$f = f(x') = \Delta_{x'}(Q_\tau(x') - \varphi(x')). \tag{6.19}$$

In particular, notice that since $Q_\tau(x')$ is the Taylor approximation of φ up to order τ, we have that

$$|f(x')| \leq M|x'|^{\tau+\alpha-2} \tag{6.20}$$

for some $M > 0$ depending only on φ. We take M larger if necessary, so that it coincides with the one of (6.16).

We consider the generalized frequency formula, for $\theta \in (0, \alpha)$, and for some C_θ (that is independent of the point around which is taken)

$$\Phi_{\tau,\alpha,\theta}(r, \bar{u}_\lambda) := (r + C_\theta r^{1+\theta}) \frac{d}{dr} \log \max \left\{ H(r), r^{n+a+2(\tau+\alpha-\theta)} \right\}, \tag{6.21}$$

where

$$H(r) := \int_{\partial B_r} \bar{u}_\lambda^2 |x_{n+1}|^a.$$

Then, by [GR19, Proposition 6.1] (see also [CSS08, GP09]) we know that $\Phi_{\tau,\alpha,\theta}(r, \bar{u}_\lambda)$ is nondecreasing for $0 < r < r_\circ$ for some $r_\circ$. In particular, $\Phi_{\tau,\alpha,\theta}(0^+, \bar{u}_\lambda)$ is well defined, and by [GP09, Lemma 2.3.2],

$$n + 3 \leq \Phi_{\tau,\alpha,\theta}(0^+, \bar{u}_\lambda) \leq n + a + 2(\tau + \alpha - \theta).$$

We say that $0 \in \partial\Lambda(u_\lambda)$ is a point of order κ if $\Phi_{\tau,\alpha,\theta}(0^+, \bar{u}_\lambda) = n + 1 - 2s + 2\kappa$. In particular, by the previous inequalities

$$1 + s \leq \kappa \leq \tau + \alpha - \theta$$

Thanks to [GR19, Lemma 6.4] (see, also, [BFR18, Lemma 7.1]) we know that for a point of order greater or equal than κ, for $\kappa < \tau + \alpha - \theta$, then we have

$$\sup_{B_r} |\bar{u}_\lambda| \leq C_M r^\kappa, \tag{6.22}$$

for some constant C_M depending only on M, τ, α, θ.

In general, for any point $x_\circ \in \partial\Lambda(u_\lambda)$, we can define $\bar{u}_\lambda^{x_\circ}$ analogously to before as follows.

Definition 6.1. Let $x_\circ \in \partial\Lambda(u_\lambda)$. We define,

$$\bar{u}_\lambda^{x_\circ}(x) = u_\lambda(x' + x'_\circ, x_{n+1}) - Q_\tau^{a,x_\circ}(x', x_{n+1}) + Q_\tau^{x_\circ}(x') - \varphi(x' + x'_\circ), \tag{6.23}$$

where $Q_\tau^{x_\circ}(x')$ is the Taylor expansion of order τ of $\varphi(x'_\circ + x')$, and $Q_\tau^{a,x_\circ}(x')$ is its unique even harmonic extension to $\mathbb{R}^{n+1}$.

(Notice that, on the thin space, $\bar{u}_\lambda^{x_\circ}(x',0) = \bar{u}_\lambda(x'+x'_\circ, 0)$, but this is not true outside the thin space.) Then, $\bar{u}_\lambda^{x_\circ}(x)$ solves a zero obstacle problem with a right-hand side in $B_{1-|x_\circ|}$ (in fact, in $x_\circ + B_1$). With this, we can define the free boundary points of u_λ of order κ, with $1+s \le \kappa < \tau+\alpha-\theta$, as

$$\Gamma_\kappa^\lambda := \{x_\circ \in \partial\Lambda(u_\lambda) : \Phi_{\tau,\alpha,\theta}(0^+, \bar{u}_\lambda^{x_\circ}) = n+1-2s+2\kappa\},$$

and similarly

$$\Gamma_{\ge\kappa}^\lambda := \{x_\circ \in \partial\Lambda(u_\lambda) : \Phi_{\tau,\alpha,\theta}(0^+, \bar{u}_\lambda^{x_\circ}) \ge n+1-2s+2\kappa\}.$$

Equivalently, one can define $\Gamma_{\ge\kappa}^\lambda$ as those points where (6.22) occurs.

Notice that the previous sets are consistently defined, in the sense that if $x_\circ$ is a free boundary point for u_λ, and $\tau' \in \mathbb{N}$, $\alpha' \in (0,1)$ are such that $\tau'+\alpha' \le \tau+\alpha$, then

$$\Phi_{\tau',\alpha',\theta}(0^+, \bar{u}_\lambda^{x_\circ}) = \min\left\{\Phi_{\tau,\alpha,\theta}(0^+, \bar{u}_\lambda^{x_\circ}), n+1-2s+2(\tau'+\alpha'-\theta)\right\},$$

(cf. [GP09, Lemma 2.3.1]), i.e., the definition of free boundary points of order κ does not depend on which regularity of the obstacle we consider. In particular, for C^∞ obstacles we can define the points of infinite order as

$$\Gamma_\infty^\lambda := \bigcap_{\kappa\ge 2} \Gamma_{\ge\kappa}^\lambda. \tag{6.24}$$

We will need the following lemma, similar to [ACS08, Lemma 4] and analogous to [CSS08, Lemma 7.2].

Lemma 6.10. *Let $w \in C^0(B_1)$, and let $\Lambda \subset B_1 \cap \{x_{n+1} = 0\}$. There exists some $\varepsilon_\circ > 0$, depending only on n and a, such that if $0 < \varepsilon < \varepsilon_\circ$ and*

$$\begin{cases} w \ge 1 & \text{in } B_1 \cap \{|x_{n+1}| \ge \varepsilon\} \\ w \ge -\varepsilon & \text{in } B_1 \\ |L_a w| \le \varepsilon |x_{n+1}|^a & \text{in } B_1 \setminus \Lambda \\ w \ge 0 & \text{on } \Lambda, \end{cases}$$

then $w > 0$ in $B_{1/2}$.

Proof. Suppose that it is not true. In particular, suppose that there exists some $z = (z', z_{n+1}) \in B_{1/2} \setminus \{x_{n+1} = 0\}$ such that $w(z) = 0$. Let us define the cylinder

$$Q := \left\{x = (x', x_{n+1}) \in B_1 : |x'-z'| < \frac{1}{2},\ \ |x_{n+1} - z_{n+1}| < \frac{\sqrt{1+a}}{4}\right\},$$

and let

$$P(x) = P(x', x_{n+1}) := |x'-z'|^2 - \frac{n}{1+a} x_{n+1}^2$$

so that $L_a P = 0$. Let

$$v(x) := w(x) + \frac{1}{n} P(x) - \frac{\varepsilon}{1+a} x_{n+1}^2.$$

Notice that $v(z) = -\frac{n}{n(1+a)} z_{n+1}^2 - \frac{\varepsilon}{1+a} z_{n+1}^2 < 0$. We also have that

$$L_a v = L_a w - 2\varepsilon |x_{n+1}|^a \le -\varepsilon |x_{n+1}|^a < 0 \quad \text{in} \quad B_1 \setminus \Lambda,$$

and

$$v \ge 0 \quad \text{on} \quad \Lambda.$$

That is, v is super- a-harmonic and is negative at $z \in Q$, then it must be negative somewhere on ∂Q. Let us check that this is not the case, to reach a contradiction.

First, notice that, assuming $\varepsilon_\circ < \frac{\sqrt{1+a}}{4}$, on $\partial Q \cap \{|x_{n+1}| \ge \varepsilon\}$ we have

$$v \ge 1 - \frac{n}{16(n+1)} - \frac{\varepsilon}{16} \ge 0.$$

On the other hand, on $\left\{|x' - z'| = \frac{1}{2}\right\} \cap \{|x_{n+1}| \le \varepsilon\}$ we have

$$v \ge -\varepsilon + \frac{1}{n+1}\left(\frac{1}{4} - \frac{n}{1+a}\varepsilon^2\right) - \frac{\varepsilon^3}{1+a} > 0,$$

if ε is small enough depending only on n and a. Thus, $v \ge 0$ on ∂Q and on Λ, and is super- a-harmonic in $Q \setminus \Lambda$, so we must have $v \ge 0$ in Q, contradicting $v(z) < 0$. □

Let us now show the following proposition.

Proposition 6.11. *Let u_λ satisfy* (6.15)-(6.16), *and let φ satisfy* (6.14). *Let $h > 0$ small, and let $x_\circ \in B_{1-h} \cap \Gamma^\lambda_{\ge\kappa}$ with $\kappa \le \tau + \alpha - a$ and $\kappa < \tau + \alpha$. Then,*

$$u_{\lambda + C_* r^{\kappa - 2s}} > \varphi \quad \textit{in} \quad B_r'(x_\circ'), \quad \textit{for all} \quad r < \frac{h}{4},$$

for some C_* depending only on n, s, M, κ, τ, α, and h.

In particular, if $x_\circ \in B_{1-h} \cap \Gamma^\lambda$, then

$$u_{\lambda + C_* r^{1-s}} > \varphi \quad \textit{in} \quad B_r'(x_\circ'), \quad \textit{for all} \quad r < \frac{h}{4}, \tag{6.25}$$

for some C_* depending only on n, s, M, κ, τ, α, and h.

Proof. Let us assume that $r < \frac{h}{4}$, and let us establish some properties of $\bar u^{x_\circ}_{\lambda + C_* r^{\kappa-2s}}$ in $B_r(0)$ (see Definition 6.1), for C_* yet to be chosen.

From Lemma 6.9 we know that, for any $z \in B_{h/2}$,

$$\begin{aligned}\frac{\bar u^{x_\circ}_{\lambda+\varepsilon}(z) - \bar u^{x_\circ}_{\lambda}(z)}{\varepsilon} &= \frac{u_{\lambda+\varepsilon}(x_\circ + z) - u_\lambda(x_\circ + z)}{\varepsilon} \\ &\ge c\, \mathrm{dist}^{2s}(x_\circ + z, \Lambda(u_\lambda)) \\ &= c\, \mathrm{dist}^{2s}(z, \Lambda(\bar u^{x_\circ}_\lambda)).\end{aligned}$$

From the previous inequality applied at $x \in B_r(0) \cap \{|x_{n+1}| \ge r\sigma\}$, for some $\sigma > 0$ to be chosen, for $r < \frac{h}{4}$, and with $\varepsilon = C_* r^{\kappa - 2s}$ for some C_* to be chosen,

$$\bar u^{x_\circ}_{\lambda + C_* r^{\kappa-2s}}(x) \ge c\, C_* r^{\kappa - 2s}(r\sigma)^{2s} + \bar u^{x_\circ}_\lambda(x) \quad \text{for} \quad x \in B_r(0) \cap \{|x_{n+1}| \ge r\sigma\}.$$

On the other hand, notice that 0 is a free boundary point of $\bar{u}_\lambda^{x_\circ}$ of order greater or equal than κ. In particular, from the growth estimate (6.22), we know that

$$\bar{u}_\lambda^{x_\circ} \geq -Cr^\kappa \quad \text{in} \quad B_r(0), \quad \text{for} \quad r < \frac{h}{4},$$

for some C depending only on n, M, s, τ, α, θ, and h. By choosing, for example, $\theta = \min\{\frac{\alpha}{2}, \frac{\tau+\alpha-\kappa}{2}\}$ in the definition of the generalized frequency function, (6.21), we can get rid of the dependence on θ. That is,

$$\bar{u}^{x_\circ}_{\lambda+C_*r^{\kappa-2s}}(x) \geq c\,C_* r^\kappa \sigma^{2s} - Cr^\kappa \quad \text{for} \quad x \in B_r(0) \cap \{|x_{n+1}| \geq r\sigma\}.$$

Moreover, since $\bar{u}^{x_\circ}_{\lambda+C_*r^{\kappa-2s}} \geq \bar{u}^{x_\circ}_\lambda$,

$$\bar{u}^{x_\circ}_{\lambda+C_*r^{\kappa-2s}} \geq -Cr^\kappa \quad \text{in} \quad B_r(0), \quad \text{for} \quad r < \frac{h}{4}.$$

Notice, also, that

$$|L_a \bar{u}^{x_\circ}_{\lambda+C_*r^{\kappa-2s}}| \leq M|x_{n+1}|^a r^{\tau+\alpha-2} \quad \text{in} \quad B_r(0) \setminus \Lambda(\bar{u}^{x_\circ}_{\lambda+C_*r^{\kappa-2s}}).$$

Let us rescale in domain. We denote

$$w(x) := \bar{u}^{x_\circ}_{\lambda+C_*r^{\kappa-2s}}(rx).$$

Then w is a solution to a thin obstacle problem with right-hand side and with zero obstacle in the ball B_1, such that

$$\begin{cases} w \;\geq\; (c\,C_*\sigma^{2s} - C)r^\kappa & \text{in} \;\; B_1(0) \cap \{|x_{n+1}| \geq \sigma\} \\ w \;\geq\; -Cr^\kappa & \text{in} \;\; B_1(0) \\ |L_a w| \;\leq\; M|x_{n+1}|^a r^{\tau+\alpha-a} & \text{in} \;\; B_1 \setminus (\{x_{n+1} = 0\} \cap \{w = 0\}). \end{cases}$$

In particular, if we take $\tilde{w} := \frac{w}{(c\,C_*\sigma^{2s}-C)r^\kappa}$, then

$$\begin{cases} \tilde{w} \;\geq\; 1 & \text{in} \;\; B_1(0) \cap \{|x_{n+1}| \geq \sigma\} \\ \tilde{w} \;\geq\; -\frac{C}{c\,C_*\sigma^{2s}-C} & \text{in} \;\; B_1(0) \\ |L_a \tilde{w}| \;\leq\; \frac{M}{c\,C_*\sigma^{2s}-C}|x_{n+1}|^a r^{\tau+\alpha-a-\kappa} & \text{in} \;\; B_1 \setminus (\{x_{n+1} = 0\} \cap \{\tilde{w} = 0\}). \end{cases}$$

(Notice that $\tau + \alpha - a - \kappa \geq 0$ by assumption.) We now want to apply Lemma 6.10. We need to choose $\sigma < \varepsilon_\circ(n, a)$, and C_* such that

$$\frac{C}{c\,C_*\sigma^{2s} - C} < \varepsilon_\circ, \qquad \frac{M}{c\,C_*\sigma^{2s} - C} < \varepsilon_\circ.$$

By choosing $C_* \gg \varepsilon_\circ^{-1-2s}$ we get that such C_* exists independently of r, depending only on n, M, s, κ, τ, α, and h.

From Lemma 6.10, we deduce that $\tilde{w} > 0$ in $B_{1/2}$, so that $\bar{u}^{x_\circ}_{\lambda+C_*r^{\kappa-2s}} > 0$ in $B_{r/2}(0)$. Since $r < h/4$, we get the desired result, noticing that $\bar{u}^{x_\circ}_{\lambda+C_*r^{\kappa-2s}} = (u_{\lambda+C_*r^{\kappa-2s}} - \varphi)(\cdot + x_\circ)$ on B'_r.

Finally, notice that thanks to the optimal regularity of solutions, if $x_\circ \in \Gamma^\lambda$, then $x_\circ \in \Gamma^\lambda_{\geq 1+s}$, so that applying the previous result we are done. □

The following corollary will be useful below.

Corollary 6.12. *Let* $u^{(1)}$ *and* $u^{(2)}$ *denote two solutions to*

$$\left\{\begin{array}{rcll} L_a u^{(i)} & = & 0 & \text{in } B_1 \setminus \big(\{x_{n+1}=0\}\cap\{u^{(i)}=\varphi\}\big) \\ L_a u^{(i)} & \le & 0 & \text{in } B_1 \\ u^{(i)} & \ge & \varphi & \text{on } \{x_{n+1}=0\}, \end{array}\right. \qquad \text{for} \quad i\in\{1,2\}. \tag{6.26}$$

Then, for any $\varepsilon_\circ > 0$ *and* $h > 0$*, there exists a* $\delta > 0$ *such that if*

$$u^{(2)} \ge u^{(1)}, \quad \text{and} \quad u^{(2)} \ge u^{(1)} + \varepsilon_\circ \quad \text{in} \quad \{|x_{n+1}| > 1/2\},$$

then

$$\inf\left\{|x_1 - x_2| : x_1 \in \partial\Lambda(u^{(1)})\cap B_{1-h}, x_2 \in \partial\Lambda(u^{(2)})\cap B_{1-h}\right\} \ge \delta.$$

Proof. The proof follows by Proposition 6.11. Let us denote $u^{(1)}_\lambda$ the solution to the thin obstacle problem (6.15) with boundary data equal to $u^{(1)}$ on $\partial B_1 \cap \{|x_{n+1}| \le 1/2\}$, and $u^{(1)}_\lambda + \lambda\varepsilon_\circ$ on $\partial B_1 \cap \{|x_{n+1}| > 1/2\}$. In particular, $u^{(1)} = u^{(1)}_0 \le u^{(1)}_1 \le u^{(2)}$. Moreover, thanks to the Harnack inequality we know that $u^{(1)}_{\lambda+\varepsilon} \ge u^{(1)}_\lambda + c\varepsilon\varepsilon_\circ$ for $\varepsilon > 0$ in $B_1 \cap \{|x_{n+1}| \ge \frac12\}$, for some constant c. Thus, if we define

$$w_\lambda := (c\varepsilon_\circ)^{-1} u^{(1)}_\lambda,$$

then w_λ fulfil (6.16). The result now follows applying Proposition 6.11 to w_λ and using that $u^{(1)} = c\varepsilon_\circ w_0 \le c\varepsilon_\circ w_\lambda \le u^{(2)}$ for $\lambda \in [0,1]$. □

As a direct consequence of Proposition 6.11 (in particular, of (6.25)), we get that if $0 \in \partial\Lambda(u_\lambda)$, then $0 \notin \partial\Lambda(u_{\bar\lambda})$ for $\bar\lambda \ne \lambda$ (since $u_{\lambda + C_*\delta^{1-s}} > \varphi$ in B_δ for $\delta > 0$ small enough).

In particular:

Definition 6.2. We define

$$\Gamma_\kappa := \bigcup_{\lambda\in[0,1]} \Gamma^\lambda_\kappa, \qquad \Gamma_{\ge\kappa} := \bigcup_{\lambda\in[0,1]} \Gamma^\lambda_{\ge\kappa}, \qquad \text{and} \qquad \Gamma := \bigcup_{\lambda\in[0,1]} \Gamma^\lambda.$$

We also define

$$\lambda(x_\circ) := \big\{\lambda \in [0,1] : x_\circ \in \partial\Lambda(u_\lambda)\big\}, \tag{6.27}$$

which is uniquely defined on Γ.

The fact that $\lambda(x_\circ)$ is uniquely defined for $x_\circ \in \Gamma$ follows since $\Gamma_\kappa \cap \Gamma_{\bar\kappa} = \varnothing$ if $\kappa \ne \bar\kappa$. In particular, if $x_\circ \in \Gamma_\kappa$ then $x_\circ \in \Gamma^{\lambda(x_\circ)} = \partial\Lambda(u_{\lambda(x_\circ)})$.

A direct consequence of Proposition 6.11 is that $\Gamma \ni x_\circ \mapsto \lambda(x_\circ)$ is continuous:

Corollary 6.13. *Let* u_λ *satisfy* (6.15)-(6.16)*, and let* φ *satisfy* (6.14)*. The function*

$$\Gamma \ni x_\circ \mapsto \lambda(x_\circ)$$

for $\lambda(x_\circ)$ *defined by* (6.27) *is continuous. Moreover, for each* $h > 0$*,*

$$\Gamma \cap B_{1-h} \ni x_\circ \mapsto \bar u^{x_\circ}_{\lambda(x_\circ)}$$

is continuous in the C^0*-norm.*

Proof. Let us start with the first statement. If $x_1, x_2 \in \Gamma$ are such that $|x_1 - x_2| \le \frac{\delta}{2}$ for $\delta > 0$ small enough, and $\lambda(x_1) \ge \lambda(x_2)$, then

$$u_{\lambda(x_2)+C_*\delta^{1-s}} > \varphi \quad \text{in} \quad B_\delta(x_\circ)$$

by Proposition 6.11. In particular, $\lambda(y) < \lambda(x_2) + C_*\delta^{1-s}$ for any $y \in B_\delta(x_2)$, so that $\lambda(x_1) < \lambda(x_2) + C_*\delta^{1-s}$. That is,

$$|\lambda(x_1) - \lambda(x_2)| \le C_*\delta^{1-s}$$

and $\lambda(x)$ is continuous (in fact, it is $(1-s)$-Hölder continuous).

Let us now show that

$$\Gamma \cap B_{1-h} \ni x_\circ \mapsto \bar{u}^{x_\circ}_{\lambda(x_\circ)}$$

is also continuous (in the C^0-norm). From the definition of $\bar{u}^{x_\circ}_{\lambda(x_\circ)}$, Definition 6.1, and since φ is continuous, it is enough to show that $\Gamma \cap B_{1-h} \ni x_\circ \mapsto u_{\lambda(x_\circ)}(x_\circ + \cdot)$ is continuous. Moreover, since each u_λ is continuous (and in fact, they are uniformly C^{2s}), we will show that $\Gamma \ni x_\circ \mapsto u_{\lambda(x_\circ)}$ is continuous, in the sense that, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that if $x, z \in \Gamma \cap B_{1-h}$ (for some $h > 0$), $|x - z| \le \delta$, then

$$\sup_{B_1} |u_{\lambda(x)} - u_{\lambda(z)}| \le \varepsilon.$$

Let us argue by contradiction. Suppose that it is not true, and that there exist sequences $x_i, z_i \in B_{1-h} \cap \Gamma$ such that $|x_i - z_i| \le \frac{1}{i}$ and

$$\sup_{B_1} |u_{\lambda(x_i)} - u_{\lambda(z_i)}| \ge \varepsilon_\circ > 0,$$

for some $\varepsilon_\circ > 0$. In particular, let us assume that $\lambda(x_i) > \lambda(z_i)$, so that $u_{\lambda(x_i)} \ge u_{\lambda(z_i)}$. After taking a subsequence (by compactness, using also that $\|u_\lambda\|_{C^{2s}(B_1)} \le M$), we can assume that there exists some ball $B_\rho(y) \subset B_1$ such that

$$u_{\lambda(x_i)} \ge u_{\lambda(z_i)} + \frac{\varepsilon_\circ}{2} \quad \text{in} \quad B_\rho(y) \subset B_1$$

for all $i \in \mathbb{N}$. (The radius ρ depends only on n, $\varepsilon_\circ$, and M.) By interior Harnack's inequality, we have that

$$u_{\lambda(x_i)} \ge u_{\lambda(z_i)} + c\frac{\varepsilon_\circ}{2} \quad \text{in} \quad B_{h/2}(z_i) \cap \{|x_{n+1}| \ge h/4\},$$

for some constant c depending on ρ and h. After translating and scaling, we are in a situation to apply Corollary 6.12. In particular, for some $\delta > 0$ (depending on $\varepsilon_\circ$ and h), $|x_i - z_i| \ge \delta > 0$. This is a contradiction with $|x_i - z_i| \le \frac{1}{i}$ for $i \in \mathbb{N}$ large enough. Therefore, $x_\circ \mapsto \bar{u}^{x_\circ}_{\lambda(x_\circ)}$ is continuous. □

The following lemma improves Lemma 6.9 in case $x_\circ \in \Gamma_2$. We denote here $a_- := \max\{0, -a\}$.

Lemma 6.14. *Let u_λ satisfy (6.15)-(6.16), and let φ satisfy (6.14). Let $n \ge 2$, and $h > 0$ small. Let $x_\circ \in B_{1-h} \cap \Gamma_2^\lambda$. Then, for each $\eta > 0$ small, and for $\mu > \lambda$,* *(i) if* $s \geq \frac{1}{2}$,

$$\partial_\lambda^+ \bar{u}_\mu^{x_\circ}(0) = \partial_\lambda^+ u_\mu(x_\circ) \geq c\ \mathrm{dist}^{\eta+a_-}(x_\circ, \Lambda(u_\mu)) = c\ \mathrm{dist}^{\eta-a}(0, \Lambda(\bar{u}_\mu^{x_\circ})),$$

(ii) if $s \leq \frac{1}{2}$,

$$\partial_\lambda^+ \bar{u}_\mu^{x_\circ}(0) = \partial_\lambda^+ u_\mu(x_\circ) \geq c\ \mathrm{dist}^{\eta+a_-}(x_\circ, \Lambda(u_\mu)) = c\ \mathrm{dist}^{\eta}(0, \Lambda(\bar{u}_\mu^{x_\circ})),$$

for some constant $c > 0$ *independent of* λ *and* μ *(but possibly depending on everything else).*

Proof. Fix some $\mu > 0$ and $\varepsilon > 0$ small, and define

$$\delta_{\lambda,\varepsilon} \bar{u}_\mu^{x_\circ}(x) = \frac{\bar{u}_{\mu+\varepsilon}^{x_\circ}(x) - \bar{u}_\mu^{x_\circ}(x)}{\varepsilon} = \frac{u_{\mu+\varepsilon}(x + x_\circ) - u_\mu(x + x_\circ)}{\varepsilon}.$$

As in the proof of Lemma 6.9, we know that $\delta_{\lambda,\varepsilon}\bar{u}_\mu^{x_\circ}(x) \geq 0$, $\delta_{\lambda,\varepsilon}\bar{u}_\mu^{x_\circ} \geq 1$ on $(-x_\circ + \partial B_1) \cap \{|x_{n+1}| \geq \frac{1}{2}\}$, and

$$L_a \delta_{\lambda,\varepsilon} \bar{u}_\mu^{x_\circ} = 0 \quad \text{in} \quad (-x_\circ + B_1) \setminus \Lambda(\bar{u}_\mu^{x_\circ}) \supset (-x_\circ + B_1) \setminus \Lambda(\bar{u}_\lambda^{x_\circ}). \tag{6.28}$$

Let us start by showing that, for every $A > 0$, there exists some $\rho_A > 0$ (independent of μ) such that, after a rotation,

$$\Lambda(\bar{u}_\mu^{x_\circ}) \cap B_{\rho_A} \subset \{|x'|^2 \geq A x_1^2\}. \tag{6.29}$$

In particular, we will show that, for every $A > 0$, there exists some $\rho_A > 0$ such that, after a rotation,

$$\Lambda(\bar{u}_\lambda^{x_\circ}) \cap B_{\rho_A} \subset \{|x'|^2 \geq A x_1^2\}. \tag{6.30}$$

(Notice that now we have taken $\mu \downarrow \lambda$, and since the contact set is decreasing in λ, (6.30) implies (6.29).)

Indeed, by [GR19, Theorem 8.2], we know that

$$\bar{u}_\lambda^{x_\circ}(x) = p_2(x) + o(|x|^2)$$

for some 2-homogeneous, a-harmonic polynomial, such that $p_2 \geq 0$ on $\{x_{n+1} = 0\}$ (recall that we are assuming that $x_\circ \in \Gamma_2^\lambda$) and $p_2 \not\equiv 0$. After a rotation, thus, we may assume that $p_2(x', 0) \geq c x_1^2$. That is,

$$\bar{u}_\lambda^{x_\circ}(x', 0) \geq c x_1^2 + o(|x'|^2) > \frac{c}{A}|x'|^2 + o(|x'|^2) > 0 \quad \text{in} \quad B_{\rho_A} \cap \{|x'|^2 < A x_1^2\}$$

if ρ_A is small enough (depending on A, but also on the point $x_\circ$, and the function $\bar{u}_\lambda^{x_\circ}$). That is, (6.30), and in particular, (6.29), holds. Considering again the x_{n+1} direction, we know that for every $A > 0$ there exists some ρ_A such that, after a rotation,

$$\Lambda(\bar{u}_\mu^{x_\circ}) \cap B_{\rho_A} \subset \{x_1^2 + x_{n+1}^2 \leq A^{-1}|x'|^2\} =: \mathcal{C}_A. \tag{6.31}$$

Notice that $\rho_A \downarrow 0$ as $A \to \infty$. Let us suppose that we are always in the rotated setting so that the previous inclusion holds. Let us denote ψ_A the unique homogeneous solution to

$$\begin{cases} L_a \psi_A = 0 & \text{in} \quad \mathbb{R}^n \setminus \mathcal{C}_{A/2} \\ \psi_A = 0 & \text{in} \quad \mathcal{C}_{A/2} \\ \psi_A \geq 0 & \text{in} \quad \mathbb{R}^n, \end{cases}$$

such that $\sup_{\partial B_1} \psi_A = 1$.

Let $\eta_\circ > 0$ denote the homogeneity of ψ_A (i.e., $\psi_A(tx) = t^{\eta_\circ}\psi_A(x)$). It corresponds to the first eigenvalue on the sphere $\mathbb{S}^n$ of L_a with zero boundary condition on $\mathcal{C}_{A/2}$. Alternatively, it corresponds to the infimum of the corresponding Rayleigh quotient among functions with the same boundary values. Notice that, as $A \to \infty$, $\mathcal{C}_{A/2} \to \{x_1 = x_{n+1} = 0\}$ locally uniformly in the Hausdorff distance, and $\{x_1 = x_{n+1} = 0\}$ has zero a-harmonic capacity when $s \leq \frac{1}{2}$ (see [Kil94, Corollary 2.12]). Thus, when $s \leq \frac{1}{2}$ the infimum of the Rayleigh quotient converges to the first eigenvalue of L_a on the sphere without boundary conditions (namely, 0), and thus, $\eta_\circ \downarrow 0$ as $A \to \infty$ if $a \geq 0$. Alternatively, if $s > \frac{1}{2}$ the first eigenvalue corresponds to the homogeneity $-a$ (attained by the function $(x_1^2 + x_{n+1}^2)^{-a/2}$), so that $\eta_\circ \downarrow -a$ as $A \to \infty$ if $a < 0$. In all, $\eta_\circ \downarrow a_-$, with $a_- = \max\{0, -a\}$.

Let us choose some A large enough such that $\eta_\circ < \eta + a_-$. Now, let

$$r := \operatorname{dist}(x_\circ, \Lambda(u_\mu)) = \operatorname{dist}(0, \Lambda(\bar{u}_\mu^{x_\circ})),$$

and let $\psi_{A,r}$ for $r < \rho_A/2$ denote the solution to

$$\begin{cases} L_a \psi_{A,r} = 0 & \text{in} \quad B_r \cup \left(B_{\rho_A/2} \setminus \mathcal{C}_{A/2}\right) \\ \psi_{A,r} = 0 & \text{in} \quad (B_{\rho_A/2} \cap \mathcal{C}_{A/2}) \setminus B_r \\ \psi_{A,r} = \psi_A & \text{on} \quad \partial B_{\rho_A/2}. \end{cases}$$

Let $\bar{c}$ small enough (depending on ρ_A, A, h, n, s, M) such that $\bar{c}\psi_A \leq \delta_{\lambda,\varepsilon}\bar{u}_\mu^{x_\circ}$ on $\partial B_{\rho_A/2}$. For instance, take

$$\bar{c} = \inf_{x \in \partial B_{\rho_A/2} \cap \mathcal{C}_{A/2}^c} \delta_{\lambda,\varepsilon}\bar{u}_\mu^{x_\circ}(x) > 0,$$

which is positive since $\delta_{\lambda,\varepsilon}u_\mu \geq 0$, $\delta_{\lambda,\varepsilon}u_\mu \geq 1$ on $\partial B_1 \cap \{|x_{n+1}| = 0\}$, and $L_a \delta_{\lambda,\varepsilon}u_\mu = 0$ in $(B_1 \setminus \{x_{n+1} = 0\}) \cup (B_{\rho_A}(x_\circ) \setminus \mathcal{C}_A)$ (recall $\delta_{\lambda,\varepsilon}u_\mu = \delta_{\lambda,\varepsilon}\bar{u}_\mu^{x_\circ}(\cdot - x_\circ)$), and thus, by strong maximum principle (or Harnack's inequality, see [FKS82, Theorem 2.3.8]) we must have $\bar{c} > 0$ depending only on ρ_A, A, h, n, s, M.

Now notice that $\bar{c}\psi_{A,r} \leq \delta_{\lambda,\varepsilon}\bar{u}_\mu^{x_\circ}$ on $\partial B_{\rho_A/2}$, $\bar{c}\psi_{A,r} \leq \delta_{\lambda,\varepsilon}\bar{u}_\mu^{x_\circ}$ on $B_{\rho_A/2} \cap \mathcal{C}_{A/2} \setminus B_r$, and both $\bar{c}\psi_{A,r}$ and $\delta_{\lambda,\varepsilon}\bar{u}_\mu^{x_\circ}$ are a-harmonic in $B_r \cup (B_{\rho_A/2} \setminus \mathcal{C}_{A/2})$ (thanks to (6.28)-(6.31)). By comparison principle

$$\bar{c}\psi_A \leq \bar{c}\psi_{A,r} \leq \delta_{\lambda,\varepsilon}\bar{u}_\mu^{x_\circ} \quad \text{in} \quad B_{\rho_A/2}.$$

By Harnack's inequality, there exists a constant C depending only on n and s such that

$$\psi_{A,r}(0) \geq \inf_{B_{r/2}(0)} \psi_{A,r} \geq \frac{1}{C} \sup_{B_{r/2}(0)} \psi_{A,r} \geq \frac{1}{C} \sup_{B_{r/2}(0)} \psi_A \geq cr^{\eta_\circ},$$

where in the last inequality we are using the $\eta_\circ$-homogeneity of ψ_A, and c depends only on n and a. Thus,

$$\delta_{\lambda,\varepsilon}\bar{u}_\mu^{x_\circ}(0) \geq \bar{c}\psi_{A,r}(0) \geq c\bar{c}r^{\eta_\circ} = c\ \mathrm{dist}^{\eta_\circ}(x_\circ, \Lambda(u_\mu)) = c\ \mathrm{dist}^{\eta_\circ}(0, \Lambda(\bar{u}_\mu^{x_\circ})),$$

for some $c > 0$ that might depends on everything, but it is independent of μ and λ, where we assumed $r < \rho_A/2$. We can reach all $r > 0$ by taking a smaller $c > 0$ (independent of λ and μ), thanks to Lemma 6.9. Recalling $\eta_\circ < \eta + a_-$, and letting $\varepsilon \downarrow 0$, this gives the desired result. □

Using the previous lemma, combined with an ODE argument, we find the following.

Proposition 6.15. *Let $x_\circ \in \Gamma_2^\lambda$ be any point of order 2. Then,*

- *If $s \leq \frac{1}{2}$, for every $\varepsilon_\circ > 0$, there exists some $\delta_\circ > 0$ such that*

$$\Gamma_2^{\lambda+\delta^{2-\varepsilon_\circ}} \cap B_\delta(x_\circ) = \varnothing,$$

for all $\delta \in (0, \delta_\circ)$.

- *If $s > \frac{1}{2}$, for every $\varepsilon_\circ > 0$, there exists some $\delta_\circ > 0$ such that*

$$\Gamma_2^{\lambda+\delta^{2\frac{2-s}{1+s}-\varepsilon_\circ}} \cap B_\delta(x_\circ) = \varnothing,$$

for all $\delta \in (0, \delta_\circ)$.

Proof. We use Lemma 6.14. We know that, for each $\eta > 0$ small,

$$\partial_\lambda^+ \bar{u}_\mu^{x_\circ}(0) \geq c\ \mathrm{dist}^{\eta+a_-}(0, \Lambda(\bar{u}_\mu^{x_\circ})) \quad \text{for} \quad \mu > \lambda.$$

On the other hand, from the optimal regularity for the thin obstacle problem, we know that

$$\bar{u}_\mu^{x_\circ}(0) \leq C \mathrm{dist}^{1+s}(0, \Lambda(\bar{u}_\mu^{x_\circ})),$$

which gives

$$\partial_\lambda^+ \bar{u}_\mu^{x_\circ}(0) \geq c(\bar{u}_\mu^{x_\circ}(0))^{\frac{\eta+a_-}{1+s}}.$$

Solving the ODE between λ and μ, this yields

$$\bar{u}_\mu^{x_\circ}(0)^{1-\frac{\eta+a_-}{1+s}} \geq c(\mu-\lambda) \quad \Longleftrightarrow \quad \bar{u}_\mu^{x_\circ}(0) \geq c(\mu-\lambda)^{\frac{2+2s}{3-2\eta-|a|}}.$$

Let us now suppose that there exists some $z_\circ \in B_\delta(x_\circ) \cap \Gamma_2^\mu$. Notice that $\bar{u}_\mu^{z_\circ}$ has quadratic growth around zero (since $z_\circ$ is a singular point of order 2), that is $\bar{u}_\mu^{z_\circ} \leq C\rho^2$ in $B'_\rho \times \{0\}$ for $\rho > 0$. Thus, using that $\bar{u}_\mu^{x_\circ} = \bar{u}_\mu^{z_\circ}(\cdot + x_\circ - z_\circ)$ in B'_1

$$C\delta^2 \geq \bar{u}_\mu^{z_\circ}(x_\circ - z_\circ) = \bar{u}_\mu^{x_\circ}(0) \geq c(\mu-\lambda)^{\frac{2+2s}{3-2\eta-|a|}},$$

that is, $\mu - \lambda \leq C\delta^{\frac{3-2\eta-|a|}{1+s}}$. In particular, whenever $\mu - \lambda > C\delta^{\frac{3-2\eta-|a|}{1+s}}$ then $B_\delta(x_\circ) \cap \Gamma_2^\mu = \varnothing$.

Taking δ and η small enough we get the desired result. □

6.3 Dimension of Γ_2

In this section we prove that $\Gamma_2 = \bigcup_{\lambda\in[0,1]} \Gamma_2^\lambda$ has dimension at most $n-1$.

Proposition 6.16. *Let $m \in \mathbb{N}$, and suppose $2m < \tau + \alpha$. Let us denote $p_{2m}^{x_\circ}$ the blow-up of $\bar{u}_{\lambda(x_\circ)}^{x_\circ}$ at $x_\circ \in \Gamma_{2m}$. Then, the mapping $\Gamma_{2m} \ni x_\circ \mapsto p_{2m}^{x_\circ}$ is continuous. Moreover, for any compact set $K \subset \Gamma_{2m}$ there exists a modulus of continuity σ_K such that*

$$|\bar{u}_{\lambda(x_\circ)}^{x_\circ}(x) - p_{2m}^{x_\circ}(x)| \le \sigma_K(|x|)|x|^{2m}$$

for any $x_\circ \in K$.

Proof. This follows exactly as the proof of [GP09, Theorem 2.8.4] (or [GR19, Theorem 8.2]) using that $\Gamma_{2m} \ni x_\circ \mapsto \lambda(x_\circ)$ and $\Gamma_{2m} \ni x_\circ \mapsto \bar{u}_{\lambda(x_\circ)}^{x_\circ}$ are continuous (see Corollary 6.13). □

Singular points (that is, points of order $2m < \tau + \alpha$) have a non-degeneracy property. Namely, as proved in [GR19, Lemma 8.1], if $x_\circ \in \Gamma_{2m}^\lambda$, then there exists some constant $C > 0$ (depending on the point $x_\circ$) such that

$$C^{-1}r^{2m} \le \sup_{\partial B_r} |\bar{u}_\lambda^{x_\circ}| \le Cr^{2m}.$$

In particular, we can further divide the set Γ_{2m} according to the degree of degeneracy of the singular point. That is, let us define

$$\Gamma_{2m,j} := \{x_\circ \in B_{1-j^{-1}} \cap \Gamma_{2m} : j^{-1}r^{2m} \le \sup_{\partial B_r} |\bar{u}_{\lambda(x_\circ)}^{x_\circ}| \le jr^{2m} \text{ for all } r \le (2j)^{-1}\},$$

so that

$$\Gamma_{2m} = \bigcup_{j\in\mathbb{N}} \Gamma_{2m,j},$$

and each $\Gamma_{2m,j} \subset \Gamma_{2m}$ is compact (see [GP09, Lemma 2.8.2], which only uses the upper semi-continuity of the frequency formula with respect to the point).

In the next proposition we are going to use a Monneau-type monotonicity formula. In particular, we will use that, if we define for $m \in \mathbb{N}$, $x_\circ \in \Gamma_{2m}^\lambda$,

$$\mathcal{M}_m(r, \bar{u}_\lambda^{x_\circ}, p_{2m}) := \frac{1}{r^{n+a+4m}} \int_{\partial B_r} (\bar{u}_\lambda^{x_\circ} - p_{2m})^2 |x_{n+1}|^a, \tag{6.32}$$

for any $2m$-homogeneous, a-harmonic, even polynomial p_{2m} with $p_{2m}(x', 0) \ge 0$, such that $p_{2m} \le C$ for some universal bound C, then

$$\frac{d}{dr}\mathcal{M}_m(r, \bar{u}_\lambda^{x_\circ}, p_{2m}) \ge -C_M r^{\alpha-1} \tag{6.33}$$

for some constant C_M independent of λ. (See [GR19, Proposition 7.2] and [GP09, Theorem 2.7.2].)

Proposition 6.17. *Let $m \in \mathbb{N}$, and suppose $2m < \tau + \alpha$. Let us denote $p_{2m}^{x_\circ}$ the blow-up of $\bar{u}_{\lambda(x_\circ)}^{x_\circ}$ at $x_\circ \in \Gamma_{2m}$. Then, for each $j \in \mathbb{N}$ there exists a modulus of continuity σ_j such that*

$$\|p_{2m}^{x_\circ} - p_{2m}^{z_\circ}\|_{L^2(\partial B_1, |x_{n+1}|^a)} \le \sigma_j(|x_\circ - z_\circ|)$$

for all $x_\circ, z_\circ \in \Gamma_{2m,j}$.

Proof. Suppose it is not true. That is, suppose that there exist sequences $x_k, z_k \in \Gamma_{2m,j}$ with $k \in \mathbb{N}$, such that $|x_k - z_k| \to 0$ and

$$\|p_{2m}^{x_k} - p_{2m}^{z_k}\|_{L^2(\partial B_1, |x_{n+1}|^a)} \geq \delta > 0 \tag{6.34}$$

for some $\delta > 0$. Suppose also that $\lambda(x_k) \leq \lambda(z_k)$.

Let $\rho_k := |x_k - z_k| \downarrow 0$ as $k \to \infty$. Let us define

$$v_x^k(x) := \frac{\bar{u}_{\lambda(x_k)}^{x_k}(\rho_k x)}{\rho_k^{2m}} \qquad \text{and} \qquad v_z^k(x) := \frac{\bar{u}_{\lambda(z_k)}^{z_k}(\rho_k x + x_k - z_k)}{\rho_k^{2m}}.$$

We have that

$$\begin{aligned} v_z^k(x) - v_x^k(x) = \rho_k^{-2m}\big\{&u_{\lambda(z_k)}(\rho_k x + x_k) - u_{\lambda(x_k)}(\rho_k x + x_k) + Q_\tau^{x_k}(\rho_k x') \\ &- Q_\tau^{z_k}(\rho_k x' + x_k' - z_k') - \mathrm{Ext}_a(Q_\tau^{x_k}(\rho_k \cdot) - Q_\tau^{z_k}(\rho_k \cdot + x_k' - z_k'))(x', x_{n+1})\big\}, \end{aligned}$$

where, if $p = p(x') : \mathbb{R}^n \to \mathbb{R}$ is a polynomial, $\mathrm{Ext}_a(p)(x', x_{n+1})$ denotes its unique even a-harmonic extension.

Notice that $u_{\lambda(z_k)} \geq u_{\lambda(x_k)}$ (since $\lambda(z_k) \geq \lambda(x_k)$). On the other hand, let us study the convergence of the degree τ polynomials $P_\tau^k(x') = Q_\tau^{x_k}(\rho_k x') - Q_\tau^{z_k}(\rho_k x' + x_k' - z_k')$. First, observe that

$$|P_\tau^k(0)| = |Q_\tau^{x_k}(0) - Q_\tau^{z_k}(x_k' - z_k')| = |\varphi(x_k') - Q_\tau^{z_k}(x_k' - z_k')| = o(\rho_k^\tau),$$

since $Q_\tau^{x_k}$ and $Q_\tau^{z_k}$ are the Taylor expansions of φ of order τ at x_k and z_k respectively, and $|x_k' - z_k'| = \rho_k$. Similarly, for any multi-index $\beta = (\beta_1, \dots, \beta_{n-1})$ with $|\beta| \leq \tau$,

$$|D^\beta P_\tau^k(0)| = \rho_k^{|\beta|}\left|D^\beta \varphi(x_k) - D^\beta Q_\tau^{z_k}(x_k' - z_k')\right| = o(\rho_k^\tau).$$

Thus, the $P_\tau^k = o(\rho_k^\tau)$ (say, in any norm in B_1'), and so the same occurs with the a-harmonic extension. Notice, also, that by assumption, $2m \leq \tau$. In all, we have that

$$v_z^k(x) - v_x^k(x) \geq o(1). \tag{6.35}$$

On the other hand, we have

$$|v_x^k(x) - p_{2m}^{x_k}(x)| \leq \sigma_{K,j}(\rho_k |x|)|x|^{2m} \tag{6.36}$$

thanks to Proposition 6.16 with $K = \Gamma_{2m,j}$, and for some modulus of continuity $\sigma_{K,j}$ depending on j. Similarly, if we denote

$$\xi_k = \frac{z_k - x_k}{\rho_k} \in \mathbb{S}^n,$$

then

$$|v_z^k(x) - p_{2m}^{z_k}(x - \xi_k)| \leq \sigma_{K,j}(\rho_k |x - \xi_k|)|x - \xi_k|^{2m}. \tag{6.37}$$

From the definition of $\Gamma_{2m,j}$ we know that

$$j^{-1} r^{2m} \leq \sup_{\partial B_r} |p_{2m}^{x_k}| \leq j r^{2m}. \tag{6.38}$$

In particular, up to subsequences, $p_{2m}^{x_k} \to p_x$ uniformly for some $2m$-homogeneous polynomial p_x, a-harmonic, such that $p_x(x',0) \geq 0$, and

$$j^{-1}r^{2m} \leq \sup_{\partial B_r} |p_x| \leq jr^{2m}. \tag{6.39}$$

Notice that both bounds (6.38) are crucial: the bound from above allows a convergence, and the bound from below avoid getting as a limit the zero polynomial. We similarly have that $p_{2m}^{z_k} \to p_z$ for some p_z $2m$-homogeneous polynomial, a-harmonic, with $p_z(x',0) \geq 0$ and such that (6.39) holds for p_z.

Combining the convergences of $p_{2m}^{x_k}$ and $p_{2m}^{z_k}$ to p_x and p_z with (6.36)-(6.37) we obtain that

$$v_x^k \to p_x, \quad v_z^k \to p_z(\cdot - \xi_\circ), \quad \text{uniformly},$$

for some $\xi_\circ = (\xi_\circ', 0) \in \mathbb{S}^n$. On the other hand, from (6.35), we know that $p_x \geq p_z(\cdot - \xi_\circ)$.

Thus, $p_x - p_z(\cdot - \xi_\circ) \geq 0$, and is a-harmonic, therefore by Lioville's theorem is constant. Moreover, both terms are non-negative on the thin space, and both attain the value 0 (since they are homogeneous), therefore, $p_x = p_z(\cdot - \xi_\circ)$. Since both p_x and p_z are homogeneous of the same degree, this implies that $p_x = p_z$.

Let us now use the Monneau-type monotonicity formula, (6.32)-(6.33), with polynomials p_x and p_z:

$$\begin{aligned}\int_{\partial B_1} (v_x^k - p_x)^2 |x_{n+1}|^a &= \mathcal{M}_m(\rho_k, \bar{u}^{x_k}_{\lambda(x_k)}, p_x)\\ &\geq \mathcal{M}_m(0^+, \bar{u}^{x_k}_{\lambda(x_k)}, p_x) - C_M \rho_k^\alpha\\ &= \int_{\partial B_1} (p_{2m}^{x_k} - p_x)^2 |x_{n+1}|^a - C_M \rho_k^\alpha,\end{aligned}$$

where we are using that $\rho^{-2m}\bar{u}_{\lambda(x_k)}(\rho x) \to p_{2m}^{x_k}$ as $\rho \downarrow 0$. Letting $k \to \infty$ (so $\rho_k \downarrow 0$), since $v_x^k \to p_x$ we get that

$$\int_{\partial B_1} (p_{2m}^{x_k} - p_x)^2 |x_{n+1}|^a \to 0.$$

On the other hand, proceeding analogously,

$$\int_{\partial B_1} (v_z^k(\cdot + \xi_k) - p_z)^2 |x_{n+1}|^a \geq \int_{\partial B_1} (p_{2m}^{z_k} - p_z)^2 |x_{n+1}|^a - C_M \rho_k^\alpha,$$

and since $v_z^k \to p_z(\cdot - \xi_\circ)$,

$$\int_{\partial B_1} (p_{2m}^{z_k} - p_z)^2 |x_{n+1}|^a \to 0.$$

Thus, since $p_x = p_z$, we obtain that

$$\int_{\partial B_1} (p_{2m}^{z_k} - p_{2m}^{x_k})^2 |x_{n+1}|^a \to 0,$$

a contradiction with (6.34). □

Finally, we prove the following.

Proposition 6.18. *Let $m \in \mathbb{N}$, and suppose $2m < \tau + \alpha$. Then, Γ_{2m} is contained in a countable union of $(n-1)$-dimensional C^1 manifolds.*

Proof. The proof is now standard, and it follows applying the Whitney extension theorem, which can be applied thanks to Proposition 6.17. We refer the reader to the proof of [GP09, Theorem 1.3.8], which we summarise here for completeness.

Indeed, if $x_\circ \in \Gamma_{2m}$, and $\beta = (\beta_1, \dots, \beta_{n+1})$ is a multi-index, we denote

$$p_{2m}^{x_\circ}(x) = \sum_{|\beta|=2m} \frac{a_\beta(x_\circ)}{\beta!} x^\beta$$

so that $a(x_\circ)$ (the coefficients) are continuous on $\Gamma_{2m,j}$ by Proposition 6.17. Arguing as in [GP09, Lemma 1.5.6] (by means of Proposition 6.16) the function f_β defined for the multi-index β, with $|\beta| \le 2m$,

$$f_\beta(x) = \begin{cases} 0 & \text{if } |\beta| < 2m, \\ a_\beta(x) & \text{if } |\beta| = 2m, \end{cases}$$

for $x \in \Gamma_{2m}$, fulfils the compatibility conditions to apply Whitney's extension theorem on $\Gamma_{2m,j}$. That is, there exists some $F \in C^{2m}(\mathbb{R}^{n+1})$ such that

$$\frac{d^{|\beta|}}{dx^\beta} F = f_\beta \quad \text{on} \quad \Gamma_{2m,j},$$

for any $|\beta| \le 2m$.

Now, for any $x_\circ \in \Gamma_{2m,j}$, since $p_{2m}^{x_\circ} \neq 0$, there exists some $\nu \in \mathbb{R}^n$ such that

$$\nu \cdot \nabla_{x'} p_{2m}^{x_\circ}(x', 0) \neq 0 \quad \text{on} \quad \mathbb{R}^n.$$

In particular, for some multi-index $\beta_\circ$ with $|\beta_\circ| = 2m - 1$,

$$\nu \cdot \nabla_{x'} \partial^{\beta_\circ} F(x_\circ) = \nu \cdot \nabla_{x'} \partial^{\beta_\circ} p_{2m}^{x_\circ}(0) \neq 0, \tag{6.40}$$

where $\partial^{\beta_\circ} := \frac{d^{|\beta_\circ|}}{dx^{\beta_\circ}}$. On the other hand,

$$\Gamma_{2m,j} \subset \bigcap_{|\beta|=2m-1} \{\partial^\beta F = 0\} \subset \{\partial^{\beta_\circ} F = 0\},$$

so that, thanks to (6.40), by the implicit function theorem $\Gamma_{2m,j}$ is locally contained in a $(n-1)$-dimensional C^1 manifold. Thus, Γ_{2m} is contained in a countable union of $(n-1)$-dimensional C^1 manifolds. □

6.4 Proof of main results

Finally, in this section we prove the main results. To do so, the starting point is the following GMT lemma from [FRS19].

Lemma 6.19 ([FRS19]). *Consider the family $\{E_\lambda\}_{\lambda\in[0,1]}$ with $E_\lambda \subset \mathbb{R}^n$. and let us denote $\mathbb{R}^n \supset E := \bigcup_{\lambda\in[0,1]} E_\lambda$.*

Suppose that for some $\beta \in (0, n]$ and $\gamma \geq 1$, we have

- $\dim_{\mathcal{H}} E \leq \beta$,
- *for any $\varepsilon > 0$, and for any $x_\circ \in E_{\lambda_\circ}$ for some $\lambda_\circ \in [0,1]$, there exists some $\rho = \rho(\varepsilon, x_\circ, \lambda_\circ) > 0$ such that*

$$B_r(x_\circ) \cap E_\lambda = \varnothing \quad \textit{for all} \quad r < \rho, \ \textit{and}\ \lambda > \lambda_\circ + r^{\gamma-\varepsilon}.$$

Then,

1. *If $\beta < \gamma$, then $\dim_{\mathcal{H}}(\{\lambda : E_\lambda \neq \varnothing\}) \leq \beta/\gamma < 1$.*
2. *If $\beta \geq \gamma$, then for $\mathcal{H}^1$-a.e. $\lambda \in \mathbb{R}$, we have $\dim_{\mathcal{H}}(E_\lambda) \leq \beta - \gamma$.*

We will also use the following lemma, analogous to the first part of Lemma 6.19 but dealing with the upper Minkowski dimension instead (which we denote $\overline{\dim}_{\mathcal{M}}$). We refer to [Mat95, Chapter 5] for more details on the upper/lower Minkowski content and dimension.

Lemma 6.20. *Consider the family $\{E_\lambda\}_{\lambda\in[0,1]}$ with $E_\lambda \subset \mathbb{R}^n$. and let us denote $\mathbb{R}^n \supset E := \bigcup_{\lambda\in[0,1]} E_\lambda$.*

Suppose that for some $\beta \in [1, n]$ and $\gamma > \beta$, we have

- $\overline{\dim}_{\mathcal{M}} E \leq \beta$,
- *for any $\varepsilon > 0$, and for any $x_\circ \in E_{\lambda_\circ}$ for some $\lambda_\circ \in [0,1]$, there exists some $\rho = \rho(\varepsilon) > 0$ such that*

$$B_r(x_\circ) \cap E_\lambda = \varnothing \quad \textit{for all} \quad r < \rho, \ \textit{and}\ \lambda > \lambda_\circ + r^{\gamma-\varepsilon}.$$

Then, $\overline{\dim}_{\mathcal{M}}(\{\lambda : E_\lambda \neq \varnothing\}) \leq \beta/\gamma < 1$.

Proof. Given $A \subset \mathbb{R}^n$, let us denote

$$N(A, r) := \min\left\{k : A \subset \cup_{i=1}^k B_r(x_i) \quad \text{for some } x_i \in \mathbb{R}^n\right\}, \tag{6.41}$$

the smallest number of r-balls needed to cover A. The upper Minkowski dimension of A can then be defined as

$$\overline{\dim}_{\mathcal{M}} A := \inf\left\{s : \limsup_{r\downarrow 0} N(A, r) r^s = 0\right\}$$

(see [Mat95]). Notice that the definition of upper Minkowski dimension does not change if we assume that the balls $B_r(x_i)$ from (6.41) are centered at points in A (by taking, for instance, balls with twice the radius).

Since $\overline{\dim}_{\mathcal{M}} E \le \beta$, we have that for any $\delta > 0$, $N(E,r) = o(r^{\beta+\delta})$. Let us consider $N(E,r)$ balls of radius r centered at E, $B_r(x_i)$, with $x_i \in E$. Thanks to our second hypothesis we have that

$$\bigcup_{\lambda\in[0,1]} \{\lambda\} \times E_\lambda \subset \bigcup_{i=1}^{N(E,r)} (\lambda(x_i) - r^{\gamma-\varepsilon}, \lambda(x_i) + r^{\gamma-\varepsilon}) \times B_r(x_i),$$

where $x_i \in E_{\lambda(x_i)}$. Thus,

$$\{\lambda \in [0,1] : E_\lambda \neq \varnothing\} \subset \bigcup_{i=1}^{N(E,r)} (\lambda(x_i) - r^{\gamma-\varepsilon}, \lambda(x_i) + r^{\gamma-\varepsilon}),$$

where the intervals are balls of radius $r^{\gamma-\varepsilon}$. In particular, using that $N(E,r) = o(r^{\beta+\delta})$, we deduce that

$$\overline{\dim}_{\mathcal{M}} \{\lambda \in [0,1] : E_\lambda \neq \varnothing\} \le \frac{\beta+\delta}{\gamma-\varepsilon}.$$

Since this works for any $\delta, \varepsilon > 0$, we deduce the desired result. □

Remark 6.2. Notice that Lemma 6.19 is somehow a generalization of the coarea formula. Namely, if we consider the case $\gamma = 1$, $\beta = n$, and $\varepsilon = 0$, and we denote E_λ the level sets of a Lipschitz function $f = f(\lambda)$ ($E_\lambda = f^{-1}(\lambda)$), the the coarea formula says that

$$\int_0^1 \mathcal{H}^{n-1}\left(f^{-1}(\lambda)\right) d\lambda = \int_{B_1} |\nabla f| < \infty,$$

since f is Lipschitz by assumption. In particular, $\mathcal{H}^{n-1}\left(f^{-1}(\lambda)\right) < \infty$ for $\mathcal{H}^1$-a.e. $\lambda \in [0,1]$. This is used by Monneau in [Mon03] for the classical obstacle problem.

This observation is also the reason why we do not expect to have a Minkowski analogous to Lemma 6.19 (2), as we did in Lemma 6.20 for part (1).

By applying the previous lemmas together with Proposition 6.11 we obtain the following result.

Theorem 6.21. *Let u_λ solve* (6.15)-(6.16). *Let $\varphi \in C^{\tau,\alpha}$, and let $\kappa < \tau + \alpha$ and $\kappa \le \tau + \alpha - a$.*

If $2 + 2s \le \kappa \le n + 2s$, then,

$$\dim_{\mathcal{H}}(\Gamma^\lambda_{\ge\kappa}) \le n - \kappa + 2s \quad \textit{for a.e.} \quad \lambda \in [0,1],$$

On the other hand, if $\kappa > n + 2s$, then

$$\Gamma^\lambda_{\ge\kappa} = \varnothing \quad \textit{for all} \quad \lambda \in [0,1] \setminus \mathcal{E}_\kappa,$$

where $\mathcal{E}_\kappa \subset [0,1]$ is such that $\dim_{\mathcal{H}}(\mathcal{E}_\kappa) \le \frac{n}{\kappa-2s}$.

Furthermore, for any $h > 0$, if $\kappa > n + 2s$, then

$$\Gamma^\lambda_{\ge\kappa} \cap B_{1-h} = \varnothing \quad \textit{for all} \quad \lambda \in [0,1] \setminus \mathcal{E}_{\kappa,h},$$

where $\mathcal{E}_{\kappa,h} \subset [0,1]$ is such that $\overline{\dim}_{\mathcal{M}}(\mathcal{E}_{\kappa,h}) \le \frac{n}{\kappa-2s}$.

Proof. The proof of this result follows applying Lemmas 6.19 and 6.20 to the right sets. Indeed, we consider the sets

$$E_\lambda := \Gamma^\lambda_{\geq\kappa}, \qquad E := \bigcup_{\lambda\in[0,1]} E_\lambda.$$

Notice that $E = \Gamma_{\geq\kappa}$, and we can take $\beta = n$ in Lemma 6.19. On the other hand, we know that for any $\lambda_\circ \in [0,1]$, $x_\circ \in E_{\lambda_\circ}$, there exists $\rho = \rho(x_\circ, \lambda_\circ) > 0$ such that

$$B_r(x_\circ) \cap E_\lambda = \varnothing \quad \text{for all} \quad r < \rho, \text{ and } \lambda > \lambda_\circ + C_* r^{\kappa-2s}.$$

thanks to Proposition 6.11. That is, for any $\varepsilon > 0$ there exists some $\rho = \rho(\varepsilon, x_\circ, \lambda_\circ) > 0$ such that

$$B_r(x_\circ) \cap E_\lambda = \varnothing \quad \text{for all} \quad r < \rho, \text{ and } \lambda > \lambda_\circ + r^{\kappa-2s-\varepsilon}.$$

and the hypotheses of Lemma 6.19 are fulfilled, with $\beta = n$ and $\gamma = \kappa - 2s$. The result now follows by Lemma 6.19.

The last part of the theorem follows by applying Lemma 6.20 instead of Lemma 6.19. We notice in this case that the dependence of ρ on the point has been removed, but now it depends on $h > 0$. This forces the result to hold only in smaller balls B_{1-h}. □

In particular, we can also deal with the set of free boundary points of infinite order.

Corollary 6.22. *Let u_λ solve* (6.15)-(6.16)*. Let $\varphi \in C^\infty$, and let $\Gamma^\lambda_\infty := \bigcap_{\kappa\geq 2} \Gamma^\lambda_{\geq\kappa}$. Then,*

$$\Gamma^\lambda_\infty = \varnothing \quad \textit{for all} \quad \lambda \in [0,1] \setminus \mathcal{E},$$

where $\mathcal{E} \subset [0,1]$ is such that $\dim_{\mathcal{H}}(\mathcal{E}) = 0$.

Furthermore, for any $h > 0$,

$$\Gamma^\lambda_\infty \cap B_{1-h} = \varnothing \quad \textit{for all} \quad \lambda \in [0,1] \setminus \mathcal{E}_h,$$

where $\mathcal{E}_h \subset [0,1]$ is such that $\dim_{\mathcal{M}}(\mathcal{E}) = 0$.

Proof. Apply Theorem 6.21 to $\Gamma^\lambda_{\geq\kappa}$ and let $\kappa \to \infty$. □

And we get that the free boundary points of order greater or equal than $2 + 2s$ are at most $(n-2)$-dimensional, for almost every $\lambda \in [0,1]$.

Corollary 6.23. *Let u_λ solve* (6.15)-(6.16)*. Let $\varphi \in C^{4,\alpha}$. Then,*

$$\dim_{\mathcal{H}}(\Gamma^\lambda_{\geq 2+2s}) \leq n-2,$$

for almost every $\lambda \in [0,1]$.

Proof. This is simply Theorem 6.21 with $\kappa = 2 + 2s$. □

On the other hand, combining the results from Sections 6.2 and 6.3 with Lemma 6.19 we get the following regarding the free boundary points of order 2.

Theorem 6.24. *Let u_λ solve* (6.15)-(6.16), *and let $n \geq 2$. Then*

$$\dim_{\mathcal{H}}(\Gamma_2^\lambda) \leq n-2 \quad \textit{for a.e.} \quad \lambda \in [0,1].$$

Proof. The proof of this result follows applying Lemma 6.19 to the right sets. We consider

$$E_\lambda := \Gamma_2^\lambda, \qquad E := \bigcup_{\lambda\in[0,1]} E_\lambda = \Gamma_2.$$

Notice that E has dimension $\mathcal{H}(E) = n-1$ by Proposition 6.18, so that we can take $\beta = n-1$ in Lemma 6.19. On the other hand, we know that for any $\lambda_\circ \in [0,1]$, $x_\circ \in E_{\lambda_\circ}$, and any $\varepsilon > 0$, there exists $\rho = \rho(\varepsilon, x_\circ, \lambda_\circ) > 0$ such that

$$B_r(x_\circ) \cap E_\lambda = \varnothing \quad \text{for all} \quad r < \rho, \text{ and } \lambda > \lambda_\circ + r.$$

thanks to Proposition 6.15 (notice that $2\frac{2-s}{1+s} > 1$ for all $s \in (1/2, 1)$). That is, the hypotheses of Lemma 6.19 are fulfilled, with $\beta = n-1$ and $\gamma = 1$. The result now follows by Lemma 6.19. □

In fact, the previous theorem is a particular case of the more general statement involving singular points given by the following proposition. We give it for completeness, although we do not need it in our analysis.

Proposition 6.25. *Let u_λ solve* (6.15)-(6.16). *Let $n \geq 2$ and let $\varphi \in C^{\tau,\alpha}$ for some $\tau \in \mathbb{N}_{\geq 4}$ and $\alpha \in (0,1)$. Then, if $s \leq \frac{1}{2}$,*

$$\dim_{\mathcal{H}}(\Gamma_2^\lambda) \leq n-3 \quad \textit{for a.e.} \quad \lambda \in [0,1].$$

Alternatively, if $s > \frac{1}{2}$,

$$\dim_{\mathcal{H}}(\Gamma_2^\lambda) \leq n-1-2\frac{2-s}{1+s} \quad \textit{for a.e.} \quad \lambda \in [0,1].$$

Finally, if $m \in \mathbb{N}$ is such that $2m \leq \tau$,

$$\dim_{\mathcal{H}}(\Gamma_{2m}^\lambda) \leq n-1-2m+2s \quad \textit{for a.e.} \quad \lambda \in [0,1].$$

Proof. This proof simply follows by analysing the previous results more carefully. The first part follows exactly as Theorem 6.24, using Proposition 6.15 and looking at each case separately.

Finally, regarding general singular points of order $2m$, the proof follows exactly as Theorem 6.21 using that Γ_{2m} has dimension $n-1$ instead of n thanks to Proposition 6.18. □

Finally, in order to control the size of points of homogeneity in the interval $(2, 2+2s)$, we refer to the following result by Focardi–Spadaro, that establishes that points in Γ_* are lower dimensional with respect to the free boundary. The result in [FoSp19] involves higher order points as well, but we state it in the explicit form it will be used below.

Proposition 6.26 ([FoSp19]). *Let u be a solution to the fractional obstacle problem with obstacle $\varphi \in C^{4,\alpha}$ for some $\alpha \in (0,1)$,*

$$\begin{cases} L_a u &= 0 & \text{in } B_1 \setminus (\{x_{n+1}=0\} \cap \{u=\varphi\}) \\ L_a u &\le 0 & \text{in } B_1 \\ u &\ge \varphi & \text{on } \{x_{n+1}=0\}. \end{cases} \tag{6.42}$$

Let $\theta \in (0,\alpha)$ and let us denote

$$\tilde{\Gamma}_* := \bigcup_{\kappa \in (2,2+2s)} \left\{ x_\circ \in \partial\Lambda(u) : \Phi_{\tau,\alpha,\theta}(0^+, \bar{u}^{x_\circ}) = n+1-2s+2\kappa \right\}. \tag{6.43}$$

Then

$$\dim_{\mathcal{H}} \tilde{\Gamma}_* \le n-2.$$

Moreover, if $n=2$, $\tilde{\Gamma}_*$ is discrete.

Combining the previous results we obtain the following.

Corollary 6.27. *Let u_λ solve* (6.15)-(6.16). *Let $\varphi \in C^{4,\alpha}$. Then,*

$$\dim_{\mathcal{H}}(\mathrm{Deg}(u_\lambda)) \le n-2,$$

for almost every $\lambda \in [0,1]$.

Proof. This follows by combining the previous results. Notice that

$$\mathrm{Deg}(u_\lambda) = \Gamma^\lambda \setminus \Gamma^\lambda_{1+s} = \Gamma^\lambda_2 \cup \tilde{\Gamma}_*(u_\lambda) \cup \Gamma^\lambda_{\ge 2+2s}.$$

The result now follows thanks to Proposition 6.26, Corollary 6.23, and Theorem 6.24. □

Remark 6.3. Following the proofs carefully, one can see that the previous result holds true for obstacles $\varphi \in C^{3,1}$ if $s \le \frac{1}{2}$. The condition $\varphi \in C^{4,\alpha}$ is only used whenever $s > \frac{1}{2}$, since otherwise, in this case the previous methods do not imply the *smallness* of $\tilde{\Gamma}_*$.

We can now prove the main results.

Proof of Theorem 6.1. Notice that, by the Harnack inequality, there exists a constant c such that $u_{\lambda+\varepsilon} \ge g_\lambda + c\varepsilon$ in $\partial B_1 \cap \{|x_{n+1}| \ge \frac{1}{2}\}$. Thus, let us consider $w_\lambda = c^{-1}u_\lambda$, so that w_λ fulfils (6.16) and we can apply Corollary 6.27 to w_λ. Since $\Gamma_\kappa(w_\lambda) = \Gamma_\kappa(u_\lambda)$ for all $\kappa \in [3/2,\infty]$, $\lambda \in [0,1]$,

$$\dim_{\mathcal{H}}(\Gamma(u_\lambda) \setminus \Gamma_{3/2}(u_\lambda)) \le n-2.$$

We finish by recalling that $\Gamma_{3/2}(u_\lambda) = \mathrm{Reg}(u_\lambda)$ is open, and a C^∞ $(n-1)$-dimensional manifold (see [ACS08, KPS15, DS16]). □

Proof of Theorem 6.2. With the same transformation as in the previous proof, the result now follows from Corollary 6.22. □

Proof of Theorem 6.4. Let us suppose that, after a rescaling if necessary, $\{\varphi > 0\} \subset B_1' \subset \mathbb{R}^n$.

We define $w_\lambda = v_\lambda + \lambda$, which fulfils a fractional obstacle problem, with obstacle φ, but with limiting value λ. Take the standard a-harmonic (i.e., with the operator L_a) extension of w_λ, which we denote $\tilde{w}_\lambda$, from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$. Thanks to [CS07], $\tilde{w}_\lambda$ fulfils a problem of the form (6.15) in $B_1 \subset \mathbb{R}^{n+1}$.

Moreover, by the Harnack inequality, $\tilde{w}_{\lambda+\varepsilon} \geq \tilde{w}_\lambda + c\varepsilon$ in $B_1 \cap \{|x_{n+1}| \geq \frac{1}{2}\}$ for some constant c. Now, the functions $c^{-1}\tilde{w}_\lambda$ fulfil (6.16), so that we can apply Corollary 6.27 to $c^{-1}\tilde{w}_\lambda$ to obtain

$$\dim_{\mathcal{H}}(\mathrm{Deg}(v_\lambda)) = \dim_{\mathcal{H}}(\Gamma(v_\lambda) \setminus \Gamma_{1+s}(v_\lambda)) \leq n - 2.$$

The result now follows since $\Gamma_{1+s}(v_\lambda) = \mathrm{Reg}(v_\lambda)$ is open, and a C^∞ $(n-1)$-dimensional manifold (see [ACS08, JN17, KRS19]). □

Proof of Theorem 6.5. With the same transformation as in the previous proof, the result follows from Corollary 6.22. □

6.5 Examples of degenerate free boundary points

Let us consider the thin obstacle problem in a domain $\Omega \subset \mathbb{R}^{n+1}$, with zero obstacle defined on $x_{n+1} = 0$. That is,

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega \setminus (\{x_{n+1} = 0\} \cap \{u = 0\}) \\ -\Delta u &\geq 0 \quad \text{in } \Omega \\ u &\geq 0 \quad \text{on } \{x_{n+1} = 0\} \\ u &= g \quad \text{on } \partial\Omega, \end{cases} \tag{6.44}$$

for some continuous boundary values $g \in C^0(\partial\Omega)$ such that $g > 0$ on $\partial\Omega \cap \{x_{n+1} = 0\}$.

Proof of Proposition 6.6. We will show that there exists some domain Ω and some boundary data g such that the solution to (6.44) has a sequence of regular points (of order 3/2) converging to a non-regular (singular) point (of order 2). Then, the solution from Proposition 6.6 will be the solution here constructed restricted to any ball inside Ω containing such singular point, with its own boundary data (and appropriately rescaled, if necessary).

In order to build such a solution we will use [BFR18, Lemma 3.2], which says that solutions to

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega \setminus (\{x_{n+1} = 0\} \cap \{u = \varphi\}) \\ -\Delta u &\geq 0 \quad \text{in } \Omega \\ u &\geq \varphi \quad \text{on } \{x_{n+1} = 0\} \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6.45}$$

with $\Delta_{x'}\varphi \leq -c_0 < 0$ and Ω convex and even in x_{n+1} have a free boundary containing only regular points (frequency 3/2) and singular points of frequency 2. In particular, they establish a non-degeneracy result stating that for any $x_\circ = (x_\circ', 0) \in \Gamma(u)$ then

$$\sup_{B_r'(x_\circ')} (u - \varphi) \geq c_1 r^2 \quad \text{for all } r \in (0, r_1), \tag{6.46}$$

for some r_1, c_1 that do not depend on the point $x_\circ$. More precisely, they show it around points $x \in \{u > \varphi\}$ and then take the limit $x \to x_\circ \in \Gamma(u)$.

On the other hand, from their proof one can also show that in fact, the convexity on Ω can be weakened to convexity in Ω in the $\boldsymbol{e}_{n+1}$ direction.

Let us fix $n = 2$. Up to subtracting the right obstacle, we consider the problem

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega \setminus (\{x_3 = 0\} \cap \{u = 0\}) \\ -\Delta u &\geq 0 \quad \text{in } \Omega \\ u &\geq \varphi_t \quad \text{on } \{x_3 = 0\} \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6.47}$$

for some analytic obstacle φ_t, and some domain Ω smooth, convex and even in x_3, to be chosen.

Let $\varphi_t(x) = t - (1-x_1^2)^2 - 4x_2^2$. Notice that, in the thin space, $\Delta_{x'}\varphi_t = -12x_1^2 - 4 \leq -4$, so that, by the result in [BFR18], under the appropriate domain Ω, the points on the free boundary $\Gamma(u_t)$ are either regular (with frequency 3/2) or singular (with frequency 2), and we have non-degeneracy (6.46). Let $\Omega' := \{x' \in \mathbb{R}^2 : (1 - x_1^2)^2 + 4x_2^2 \leq 2\}$, and take any bounded, convex in x_3, and even in x_3 extension of Ω', Ω. Then, if $t = 2$, the solution u_2 to (6.47) is exactly equal to the solution to

$$\begin{cases} \Delta u_2 &= 0 \quad \text{in } \Omega \setminus \{x_3 = 0\} \\ u_2 &= 0 \quad \text{on } \partial\Omega \\ u_2 &= \varphi_2 \quad \text{on } \{x_3 = 0\}, \end{cases}$$

so that, in particular, the contact set is full.

Notice that, when $t < 0$, the contact set is empty, $\Lambda(u_t) = \varnothing$, and when $t = 0$ the contact set is two points, $p_\pm = (\pm 1, 0, 0)$ (which, in particular, are singular points). Notice, also, that the contact set is always closed and is monotone in t, in the sense that $\Lambda(u_{t_1}) \subseteq \Lambda(u_{t_2})$ if $t_1 \leq t_2$. Let us say that a set is $p_\pm$-connected if the points p_+ and p_- belong to the same connected component. Then, there exists some $t^* \in (0, 2]$ such that $\Lambda(u_t)$ is not $p_\pm$-connected for $t < t^*$, and is $p_\pm$-connected for $t > t^*$. Notice, also, that since $\Lambda(u_t) \subset \{x' : \varphi_t \geq 0\}$ then $t^* > 1$.

We claim that $\Lambda(u_{t^*})$ is $p_\pm$-connected and has a set of regular points converging to a singular point.

Let us first show that $\Lambda(u_{t^*})$ is $p_\pm$-connected. Suppose it is not. That is, $\Lambda(u_{t^*})$ is a closed set with $p_\pm$ on different connected components. On the other hand, $\Lambda(u_t)$ is compact and $p_\pm$-connected for $t > t^*$, and nested ($\Lambda(u_t) \subset \Lambda(u_{t'})$ for $t < t'$). Take

$$\tilde{\Lambda}_{t^*} := \bigcap_{t \in (t^*, 2]} \Lambda(u_t),$$

then $\tilde{\Lambda}_{t^*}$ is $p_\pm$-connected (being the intersection of compact $p_\pm$-connected nested sets), and $\Lambda(u_{t^*}) \subsetneq \tilde{\Lambda}_{t^*}$, since $\Lambda(u_{t^*})$ is not $p_\pm$-connected. In particular, there exists some $x_\circ \in \Lambda(u_t)$ for all $t > t^*$ such that $x_\circ \notin \Lambda(u_{t^*})$. But, by continuity, this is not possible: $0 < (u_{t^*} - \varphi_{t^*})(x_\circ) = \lim_{t \downarrow t^*}(u_t - \varphi_t)(x_\circ) = 0$. Therefore, $\Lambda(u_{t^*})$ is $p_\pm$-connected.

Take $\Lambda^p(u_{t^*})$ to be the connected component containing both p_+ and p_-. Then, $\partial\Lambda^p(u_{t^*})$ must contain at least one singular point. Indeed, suppose it is not true. In this case, all points in $\partial\Lambda^p(u_{t^*})$ are regular, and in particular, $\Lambda^p(u_{t^*})$ is a compact connected set with smooth boundary, with all points of the boundary having positive density (in $\{x_3 = 0\}$), and therefore $(\Lambda^p(u_{t^*}))^\circ$ is also connected. Let us denote $\Lambda^p_\pm(u_t)$ the corresponding connected components of $\Lambda(u_t)$ containing $p_\pm$ for $t < t^*$ (notice that, by definition of t^*, $\Lambda^p_+(u_t) \neq \Lambda^p_-(u_t)$. Then,

$$\Lambda^{p,\circ}_{t<t^*} := \left(\bigcup_{t<t^*} (\Lambda^p_+(u_t))^\circ\right) \cup \left(\bigcup_{t<t^*} (\Lambda^p_-(u_t))^\circ\right) \subsetneq (\Lambda^p(u_{t^*}))^\circ ,$$

given that the left-hand side is not connected, and the right-hand side is. Take $y_\circ \in (\Lambda^p(u_{t^*}))^\circ \setminus \Lambda^{p,\circ}_{t<t^*}$, so that around $y_\circ$ the non-degeneracy (6.46) holds for any $t < t^*$. Then, there exists some $r_\circ > 0$, $r_1 > r_\circ$ (where r_1 is defined in (6.46)) such that $B'_{r_\circ}(y_\circ) \subset \Lambda^p(u_{t^*})$, so that $u_{t^*} - \varphi_{t^*}|_{B'_{r_\circ}(y_\circ)} \equiv 0$ and

$$0 < c_1 r_\circ^2 \le \lim_{t\uparrow t^*} \sup_{B'_r(x'_\circ)} (u_t - \varphi_t) = \sup_{B'_r(x'_\circ)} (u_{t^*} - \varphi_{t^*}) = 0,$$

a contradiction. That is, not all points on $\partial\Lambda^p(u_{t^*})$ are regular. By [BFR18], then there exist some degenerate (singular) point of frequency 2, $x_D \in \partial\Lambda^p(u_{t^*})$. Now consider Γ_D, the connected component in $\partial\Lambda^p(u_{t^*})$ containing x_D. Since the density of the contact set around singular points is zero, if Γ_D consist exclusively of singular points, then Γ_D itself is the whole connected component $\Lambda^p(u_t)$, and $p_\pm \in \Gamma_D$ are singular points. Nonetheless, for small $t > 0$, $\Lambda(u_t)$ contains a neighbourhood of $p_\pm$, which contradicts the singularity of $p_\pm$. Therefore, Γ_D is not formed exclusively of singular points, and then there exists a sequence of regular points converging to a singular point. □

Now, before proving Proposition 6.7, let us show the following lemma.

Lemma 6.28. *Let $m \in \mathbb{N}_{>0}$, and let $\eta \in C^\infty_c(B_2)$ such that $\eta \equiv 1$ in B_1. Let $u_+ = \max\{u, 0\}$ and $u_- = -\min\{u, 0\}$. Then,*

$$(-\Delta)^s \left[(x_1)_+^{2m+1+s}\eta\right] - C_{m,s}(x_1)_-^{2m+1-s} \in C^\infty(B_{1/2}),$$

for some positive constant $C_{m,s} > 0$ depending only on n, m, and s.

Proof. We consider the extension problem from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$. Namely, let us denote u_1 the extension of $(x_1)_+^{2m+1+s}\eta$, that is, u_1 solves

$$\begin{cases} L_a u_1 = 0 & \text{in } \mathbb{R}^{n+1} \cap \{x_{n+1} > 0\} \\ u_1(x', 0) = (x_1)_+^{2m+1+s}\eta & \text{for } x' \in \mathbb{R}^n \\ u_1(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $a = 1 - 2s$. Then, we know that

$$\left\{(-\Delta)^s \left[(x_1)_+^{2m+1+s}\eta\right]\right\}(x') = \lim_{y\downarrow 0} y^a \partial_{x_{n+1}} u_1(x', y)$$

for $x' \in \mathbb{R}^n$. On the other hand, let u_2 be the unique a-harmonic extension of $(x_1)_+^{2m+1+s}$ from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$. That is, u_2 is homogeneous (of degree $2m+1+s$), and fulfils

$$\begin{cases} L_a u_2 = 0 & \text{in } \mathbb{R}^{n+1} \cap \{x_{n+1} > 0\} \\ u_2(x', 0) = (x_1)_+^{2m+1+s} & \text{for } x' \in \mathbb{R}^n. \end{cases}$$

The fact that such solution exists, and that $\lim_{y\downarrow 0} y^a\partial_{x_{n+1}}u_2(x',y) = 0$ if $x_1 > 0$, follows, for example, from [FoSp18, Proposition A.1]. On the other hand, notice that, since u_2 is $(2m+1+s)$-homogeneous, we have that, $\lim_{y\downarrow 0} y^a\partial_{x_{n+1}}u_2(x',y) = C_{m,s}|x_1|^{2m+1-s}$ for $x_1 < 0$, so that, in all,

$$\lim_{y\downarrow 0} y^a\partial_{x_{n+1}}u_2(x',y) = C_{m,s}(x_1)_-^{2m+1-s}.$$

Again, by [FoSp18, Proposition A.1] u_2 is a solution to the thin obstacle problem with operator L_a, so $C_{m,s} > 0$ (otherwise, it would not be a supersolution for L_a).

Let now $v = u_1 - u_2$. Notice that v fulfils

$$\begin{cases} L_a v &= 0 & \text{in } \mathbb{R}^{n+1}\cap\{x_{n+1}>0\} \\ v(x',0) &= (x_1)_+^{2m+1+s}(\eta-1) & \text{for } x'\in\mathbb{R}^n. \end{cases}$$

In particular, $v(x',0) = 0$ in B_1'. Let us denote $D_{x'}^\alpha v$ a derivative in the $x'\in\mathbb{R}^n$ direction of v, with multi-index $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_n,0)$. Then $D_{x'}^\alpha v$ is such that

$$\begin{cases} L_a D_{x'}^\alpha v &= 0 & \text{in } B_1\cap\{x_{n+1}>0\} \\ D_{x'}^\alpha v(x',0) &= 0 & \text{for } x'\in B_1'. \end{cases}$$

Then, by estimates for the operator L_a, we know that, if we define

$$w_\alpha(x') := \lim_{y\downarrow 0} y^a\partial_{x_{n+1}}D^\alpha v(x',y), \qquad w_0(x') := \lim_{y\downarrow 0} y^a\partial_{x_{n+1}}v(x',y),$$

then w_α satisfies $w_\alpha \in C^\beta(B_{1/2})$ for some $\beta > 0$ (see [CSS08, Proposition 4.3] or [JN17, Proposition 2.3]). In particular, since $w_\alpha = D^\alpha w_0$, we have that $w_0 \in C^{|\alpha|+\beta}(B_{1/2})$. Since this works for all multi-index α, $w_0 \in C^\infty(B_{1/2})$.

Thus, combining the previous steps,

$$\begin{aligned}(-\Delta)^s\left[(x_1)_+^{2m+1+s}\eta\right] - C_{m,s}(x_1)_-^{2m+1-s} &= \lim_{y\downarrow 0} y^a\partial_{x_{n+1}}(u_1(x',y)-u_2(x',y)) \\ &= \lim_{y\downarrow 0} y^a\partial_{x_{n+1}}v(x',y) \\ &= w_0 \in C^\infty(B_{1/2}),\end{aligned}$$

as we wanted to see. □

We are now in disposition to give the proof of Proposition 6.7.

Proof of Proposition 6.7. We divide the proof into two steps. In the first step, we show the results holds up to an intermediate claim, that will be proved in the second step.

Step 1. Thanks to [Gru15, Theorem 4] or [AR19, Section 2], we have that $(-\Delta)^s(d^s\eta) \in C^\infty(\overline{\Omega^c})$ for any $\eta \in C^\infty$ with sufficient decay at infinity. Here, d denotes any C^∞ function (with at most polynomial growth at infinity) such that in a neighbourhood of Ω coincides with the distance to Ω, and $d|_\Omega \equiv 0$.

In particular, once d is fixed, we know that for any $k\in\mathbb{N}$,

$$(-\Delta)^s(d^{k+s}) = f \in C^\infty(\overline{\Omega^c}),$$

and, if we make sure that $d > 0$ in Ω^c, with exponential decay at infinity, we get

$$|f(x)| \le \frac{C}{1+|x|^{n+2s}}.$$

Define, for some g with the previous decay, $|g(x)| \le C(1+|x|^{n+2s})^{-1}$, φ_g such that

$$(-\Delta)^s \varphi_g = g,$$

that is, one can take

$$\varphi_g(x) = I_{2s} g(x) := c \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-2s}} dy.$$

Notice that

$$\begin{aligned} |\varphi_g(x)| &\le C \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^{n+2s})|x-y|^{n-2s}} \\ &\le C \int_{|y-x|\ge \frac{|x|}{2}} \frac{dy}{(1+|y|^{n+2s})|x-y|^{n-2s}} + C \int_{|y-x|\le \frac{|x|}{2}} \frac{dy}{(1+|y|^{n+2s})|x-y|^{n-2s}} \\ &\le \frac{C}{|x|^{n-2s}} \int_{|y-x|\ge \frac{|x|}{2}} \frac{dy}{1+|y|^{n+2s}} + \frac{C}{1+|x|^{n+2s}} \int_{|y-x|\le \frac{|x|}{2}} \frac{dy}{|x-y|^{n-2s}}, \end{aligned}$$

where we are using that if $|y-x| \le \frac{|x|}{2}$ then $|y| \ge \frac{|x|}{2}$ by triangular inequality. Notice also that

$$\int_{|y-x|\le \frac{|x|}{2}} \frac{dy}{|x-y|^{n-2s}} = \int_{B_{|x|/2}} \frac{dz}{|z|^{n-2s}} = \int_0^{|x|/2} r^{2s-1} dr = C|x|^{2s}.$$

In all, also using that $\varphi(x)$ is bounded around the origin, we obtain that

$$|\varphi_g(x)| \le \frac{C}{1+|x|^{n-2s}}.$$

Now let us define $v = d^{k+s}$. We claim that, if $k = 2m+1$ for some $m \in \mathbb{N}_{>0}$, then v fulfils

$$\begin{cases} (-\Delta)^s v \ \ge \ \bar{f} & \text{in } \mathbb{R}^n \\ (-\Delta)^s v \ = \ \bar{f} & \text{in } \{v > 0\} \\ \qquad v \ \ge \ 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{6.48}$$

where $\bar{f}$ is some appropriate C^∞ extension of f inside Ω. Then, if we define

$$u := v + \varphi_{-\bar{f}},$$

u fulfils,

$$\begin{cases} (-\Delta)^s u \ \ge \ 0 & \text{in } \mathbb{R}^n \\ (-\Delta)^s u \ = \ 0 & \text{in } \{u > \varphi_{-\bar{f}}\} \\ \qquad u \ \ge \ \varphi_{-\bar{f}} & \text{in } \mathbb{R}^n, \end{cases}$$

and notice that, since $v > 0$ in Ω^c and $v = 0$ in Ω, by definition, we have that the contact set is exactly equal to Ω. Moreover, by the growth of v at the boundary, the free boundary points are of frequency $k+s$. Also, by the decay at infinity of v and $\varphi_{-\bar f}$, $u \to 0$ at infinity.

Step 2. We still have to show that, for an appropriate choice of $\bar f$, (6.48) holds for $k = 2m+1$. Notice that, in fact, in Ω^c we know that f is C^∞. Moreover, we only have to show the claim for a neighbourhood of $\partial\Omega$ inside Ω, given that exactly at the boundary we expect a *unique* extension of f (that is, all derivatives are prescribed at the boundary).

That is, if we let $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$, we have to show that there exists some $\delta > 0$ small enough such that $(-\Delta)^s v \geq \bar f$ in Ω_δ, where we recall that $\bar f$ is a C^∞ extension of $f \in C^\infty(\overline{\Omega^c})$ inside Ω.

Let $z_\circ \in \partial\Omega$. After a translation and a rotation, we assume that $z_\circ = 0$ and $\nu(0,\partial\Omega) = \boldsymbol{e}_1$, where $\nu(0,\partial\Omega)$ denotes the outward normal to $\partial\Omega$ at 0. After rescaling if necessary, let us assume that we are working in B_1, that each point in B_1 has a unique projection onto $\partial\Omega$, and that $d|_{B_1\cap\Omega^c} = \mathrm{dist}(\cdot,\Omega)$. Moreover, again after a rescaling if necessary (since Ω is a C^∞ domain), let us assume that

$$\{y_1 \leq -|(y_2,\dots,y_n)|^2\} \cap B_1 \subset \Omega \cap B_1 \subset \{y_1 \leq |(y_2,\dots,y_n)|^2\} \cap B_1, \tag{6.49}$$

so that, in particular, $\{-t\boldsymbol{e}_1 : t \in (0,1)\} \subset \Omega$.

Let $\eta \in C^\infty_c(B_2)$ such that $\eta \equiv 1$ in B_1, and let $u_+ = \max\{u, 0\}$ denote the positive part, and $u_- = -\min\{u,0\}$ the negative part. Let $\alpha = 2m+1+s$, and define

$$u_1(x) := (x_1)^\alpha_+\eta, \qquad w(x) := v(x) - u_1(x) = d^\alpha(x) - (x_1)^\alpha_+\eta.$$

Notice that, by Lemma 6.28,

$$(-\Delta)^s u_1(x) - C_{m,s}(x_1)_-^{2m+1-s} \in C^\infty(B_{1/2}), \tag{6.50}$$

for some positive constant $C_{m,s} > 0$.

We begin by claiming that

$$w_1(x_1) := [(-\Delta)^s w](x_1, 0, \dots, 0) \in C^{2m+1-s+\varepsilon}((-1/2,1/2)), \tag{6.51}$$

for some $\varepsilon > 0$.

Indeed, let any $z_1 \in (-1/2,1/2)$. Let us denote for $\gamma \in (0,1]$, $\delta^{(\gamma)}_{\boldsymbol{e}_1,h}$ the incremental quotient in the $\boldsymbol{e}_1$ direction of length $0 < h < 1/4$ and order γ; that is,

$$\delta^{(\gamma)}_{\boldsymbol{e}_1,h} F(y_\circ) := \frac{|F(y_\circ + h\boldsymbol{e}_1) - F(y_\circ)|}{|h|^\gamma}.$$

Since $d \equiv (x_1)_+$ on $\{x_2 = \dots = x_n = 0\} \cap B_1$, we have that $w(x_1, 0, \dots, 0) = 0$ on $(-1,1)$. Now notice that, for any $\ell \in \mathbb{N}$, $\gamma \in (0,1]$,

$$\delta^{(\gamma)}_{\boldsymbol{e}_1,h}\frac{d^\ell}{dx_1^\ell} w_1(z_1) = \left\{\delta^{(\gamma)}_{\boldsymbol{e}_1,h}\partial^\ell_{\boldsymbol{e}_1}[(-\Delta)^s w]\right\}(z_1,0,\dots,0) = \int_{\mathbb{R}^n} \frac{\delta^{(\gamma)}_{\boldsymbol{e}_1,h}\partial^\ell_{\boldsymbol{e}_1} w(\bar z_1 + y)}{|y|^{n+2s}}\,dy, \tag{6.52}$$

where $\bar z_1 = \{z_1, 0, \dots, 0\} \in \mathbb{R}^n$, and we are using that $\delta^{(\gamma)}_{\boldsymbol{e}_1,h}\partial^\ell_{\boldsymbol{e}_1} w(\bar z_1) = 0$. In order to show (6.51), we will bound

$$\lim_{h\downarrow 0}\left|\delta^{(\gamma)}_{\boldsymbol{e}_1,h}\frac{d^\ell}{dx_1^\ell} w_1(z_1)\right| \leq C \quad \text{in} \quad B_{1/2}, \tag{6.53}$$

for some C, for $\ell = 2m$ and for $\gamma = 1 - s + \varepsilon$ for some $\varepsilon > 0$.

We need to separate into different cases according to $\bar{z}_1 + y$. Notice that the the integral in (6.52) is immediately bounded in $\mathbb{R}^n \setminus B_{1/2}$ because $w \in C^\alpha$ and the integrand is thus bounded by $C|y|^{-n-2s}$. We can, therefore, assume that $y \in B_{1/2}$ so that $\bar{z}_1 + y \in B_1$.

Let us start by noticing that, from (6.50), together with the fact that $(-\Delta)^s v$ is smooth in Ω^c, we already know that $w_1 \in C^\infty([0, 1/2))$, so that we only care about the case $z_1 < 0$.

Let $z_1 < 0$, so that $\bar{z}_1 \in \Omega$. If $\bar{z}_1 + y \in \Omega \cap \{x_1 < 0\} \cap B_1$, then $w(\bar{z}_1 + y) = 0$. If $\bar{z}_1 + y \in \Omega \cap \{x_1 > 0\} \cap B_1$, then $|w(\bar{z}_1 + y)| = |z_1 + y_1|^\alpha$ and $|\partial^\ell_{\boldsymbol{e}_1} w|(\bar{z}_1 + y) = C|z_1 + y_1|^{\alpha-\ell} \le C|y|^{2(\alpha-\ell)}$; where we are using that $z_1 + y_1 \le |(y_2, \dots, y_n)|^2 \le |y|^2$, see (6.49). Similarly, $\lim_{h\downarrow 0} |\delta^{(\gamma)}_{\boldsymbol{e}_1,h} \partial^\ell_{\boldsymbol{e}_1} w|(\bar{z}_1 + y) \le C|z_1 + y_1|^{\alpha-\ell-\gamma} \le C|y|^{2(\alpha-\ell-\gamma)}$.

Conversely, if $\bar{z}_1 + y \in \Omega^c \cap \{x_1 < 0\} \cap B_1$, $|w(\bar{z}_1 + y)| = d^\alpha(\bar{z}_1 + y)$ and $|\partial^\ell_{\boldsymbol{e}_1} w|(\bar{z}_1 + y) \le C d^{\alpha-\ell}(\bar{z}_1 + y) \le C|y|^{2(\alpha-\ell)}$, where we are using (6.49) again. Taking the incremental quotients, $\lim_{h\downarrow 0} |\delta^{(\gamma)}_{\boldsymbol{e}_1,h} \partial^\ell_{\boldsymbol{e}_1} w|(\bar{z}_1 + y) \le C d^{\alpha-\ell-\gamma}(\bar{z}_1 + y) \le C|y|^{2(\alpha-\ell-\gamma)}$

Finally, if $\bar{z}_1 + y \in \Omega^c \cap \{x_1 > 0\} \cap B_1$, both terms in the expression of w are relevant. Using that $|a^\beta - b^\beta| \le C|a - b||a^{\beta-1} + b^{\beta-1}|$ we obtain that

$$|w(\bar{z}_1 + y)| \le C|d - u_1| \left(d^{\alpha-1} + u_1^{\alpha-1}\right)(\bar{z}_1 + y).$$

Notice that on $\{x_2 = \cdots = x_n = 0\} \cap B_1$, $d = u_1$ and $\partial_i d = \partial_i u = 0$ for $2 \le i \le n$, so that in fact $|d - u_1|(\bar{z}_1 + y) \le C|y|^2$. On the other hand, we also have that $d^{\alpha-1}(\bar{z}_1 + y) \le C|y|^{\alpha-1}$, so that

$$|w(\bar{z}_1 + y)| \le C|y|^{\alpha+1}. \tag{6.54}$$

Notice, also, that $w \in C^\alpha$ (i.e., $\nabla^{\ell+1} w \in C^s$). By classical interpolation inequalities for Hölder spaces (or fractional Sobolev spaces with $p = \infty$) we know that, if $0 < \gamma < 1$,

$$\|\nabla^\ell w\|_{C^\gamma(B_r(\bar{z}_1))} \le C \|\nabla^{\ell+1} w\|^{\frac{\ell+\gamma}{\alpha}}_{C^s(B_r(\bar{z}_1))} \|w\|^{\frac{1+s-\gamma}{\alpha}}_{L^\infty(B_r(\bar{z}_1))}$$

(see, for instance, [BL76, Theorem 6.4.5]). Thus, in our case we have that

$$\lim_{h\downarrow 0} \left| \delta^{(\gamma)}_{\boldsymbol{e}_1,h} \frac{d^\ell}{dx_1^\ell} w \right| (\bar{z}_1 + y) \le C|y|^{(\alpha+1)\frac{1+s-\gamma}{\alpha}}. \tag{6.55}$$

Thus, putting all together we obtain that

$$\lim_{h\downarrow 0} \left| \delta^{(\gamma)}_{\boldsymbol{e}_1,h} \partial^\ell_{\boldsymbol{e}_1} w \right| (\bar{z}_1 + y) \le C \max\left\{ |y|^{2(\alpha-\ell-\gamma)}, |y|^{(\alpha+1)\frac{1+s-\gamma}{\alpha}} \right\}.$$

If we want (6.53) to hold, we need (by checking (6.52))

$$2(\alpha - \ell - \gamma) > 2s \qquad \text{and} \qquad (\alpha + 1)\frac{1 + s - \gamma}{\alpha} > 2s, \tag{6.56}$$

for some $1 - s < \gamma < 1$, and $\ell = 2m$ (recall we need to show $\gamma = 1 - s + \varepsilon$ for some $\varepsilon > 0$). The first inequality holds as long as $\gamma < 1$. The second inequality will hold if

$$\gamma < 1 + s - \frac{2s\alpha}{\alpha + 1} = 1 - \frac{\alpha - 1}{\alpha + 1} s.$$

Thus, we can choose $\gamma = 1 - s + \varepsilon$ with $0 < \varepsilon < \frac{2}{\alpha+1}s$ and (6.51) holds with this ε.

Now, combining (6.51)-(6.50) we obtain that

$$f_v := [(-\Delta)^s v](x_1, 0, \ldots, 0) - C_{m,s}(x_1)_-^{2m+1-s} \in C^{2m+1-s+\varepsilon}((-1/2, 1/2)).$$

In particular, if we recall that $\bar{f} \in C^\infty(B_1)$ is a C^∞ extension of $(-\Delta)^s v$ inside Ω, and noticing that $f_v - \bar{f}(x_1, 0, \ldots, 0) \equiv 0$ for $x_1 > 0$, we have that $\bar{f}(\cdot, 0, \ldots, 0) - f_v \in C^{2m+1-s+\varepsilon}((-1/2, 1/2))$ and

$$f_v - \bar{f}(x_1, 0, \ldots, 0) = o(|x_1|^{2m+1-s+\varepsilon}),$$

or

$$[(-\Delta)^s v](x_1, 0, \ldots, 0) = C_{m,s}(x_1)_-^{2m+1-s} + \bar{f}(x_1, 0, \ldots, 0) + o(|x_1|^{2m+1-s+\varepsilon}).$$

Thus, since $C_{m,s} > 0$, $[(-\Delta)^s v](x_1, 0, \ldots, 0) \geq \bar{f}(x_1, 0, \ldots, 0)$ if $|x_1|$ is small enough (depending only on n, m, s, and Ω), as we wanted to see.

We have that, for a fixed $\bar{f}$ extension of f inside Ω, $(-\Delta)^s v \geq \bar{f}$ in Ω_δ for some small $\delta > 0$ depending only on n, m, s, and Ω. Up to redefining $\bar{f}$ in $\Omega \setminus \Omega_{\delta/2}$, we can easily build an $\bar{f} \in C^\infty$ such that $(-\Delta)^s v \geq \bar{f}$ in Ω, as we wanted to see. □

To finish, we study the points of order infinity. To do that, we start with the following proposition.

Proposition 6.29. *Let $\mathcal{C} \subset B_1 \subset \mathbb{R}^n$ be any closed set. Then, there exists a non-trivial solution u and an obstacle $\varphi \in C^\infty(\mathbb{R}^n)$ such that*

$$\begin{cases} (-\Delta)^s u \geq 0 & \text{in } \mathbb{R}^n \\ (-\Delta)^s u = 0 & \text{in } \{u > \varphi\} \\ u \geq \varphi & \text{in } \mathbb{R}^n, \end{cases}$$

and $\Lambda(u) \cap B_1 = \{u = \varphi\} \cap B_1 = \mathcal{C}$.

Proof. Take any obstacle $\psi \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \psi \subset\subset B_1(2\boldsymbol{e}_1)$, with $\psi > 0$ somewhere, and take the non-trivial solution to

$$\begin{cases} (-\Delta)^s u \geq 0 & \text{in } \mathbb{R}^n \\ (-\Delta)^s u = 0 & \text{in } \{u > \psi\} \\ u \geq \psi & \text{in } \mathbb{R}^n. \end{cases}$$

Notice that $u > \psi$ in B_1 (in particular, $u \in C^\infty(B_1)$). Let $f_{\mathcal{C}}$ be any C^∞ function such that $0 \leq f_{\mathcal{C}} \leq 1$ and $\mathcal{C} = \{f_{\mathcal{C}} = 0\}$.

Now let $\eta \in C_c^\infty(B_{3/2})$ such that $\eta \geq 0$ and $\eta \equiv 1$ in B_1. Consider, as new obstacle, $\varphi = \psi + \eta(u - \psi)(1 - f_{\mathcal{C}}) \in C^\infty(B_1)$. Notice that $u - \varphi \geq 0$. Notice, also, that for $x \in B_1$, $(u - \varphi)(x) = 0$ if and only if $x \in \mathcal{C}$. Thus, u with obstacle φ gives the desired result. □

And now we can provide the proof of Proposition 6.8:

Proof of Proposition 6.8. The proof is now immediate thanks to Proposition 6.29, since we can choose as contact set any closed set with boundary of dimension greater or equal than $n - \varepsilon$ for any $\varepsilon > 0$, and points of finite order are at most $(n-1)$-dimensional. □

6.6 The parabolic Signorini problem

We consider now the parabolic version of the thin obstacle problem. Given $(x_\circ, t_\circ) \in \mathbb{R}^{n+1} \times \mathbb{R}$, we will use the notation

$$\begin{aligned} Q_r(x_\circ, t_\circ) &:= B_r(x_\circ) \times (t_\circ - r^2, t_\circ] \subset \mathbb{R}^{n+1} \times \mathbb{R}, \\ Q_r'(x_\circ', t_\circ) &:= B_r'(x_\circ') \times (t_\circ - r^2, t_\circ] \subset \mathbb{R}^{n} \times \mathbb{R}, \\ Q_r^+((x_\circ', 0), t_\circ) &:= B_r^+((x_\circ', 0)) \times (t_\circ - r^2, t_\circ] \subset \mathbb{R}^{n+1} \times \mathbb{R}. \end{aligned}$$

We will denote, $Q_r = Q_r(0,0)$, $Q_r' = Q_r'(0,0)$ and $Q_r^+ = Q_r^+(0,0)$. We consider the problem posed in $Q_1^+ := B_1^+ \times (-1, 0]$ for some fixed obstacle

$$\varphi : B_1' \to \mathbb{R}, \quad \varphi \in C^{\tau,\alpha}(\overline{B_1'}), \quad \tau \in \mathbb{N}_{\geq 2}, \alpha \in (0,1],$$

that is,

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u &=& 0, & \text{in } \ Q_1^{;} \\ \min\{u - \varphi, \partial_{x_{n+1}} u\} &=& 0, & \text{on } \ Q_1'. \end{array} \right. \tag{6.57}$$

The free boundary for (6.57) is given by

$$\Gamma(u) := \partial_{Q_1'} \{(x', t) \in Q_1' : u(x', 0, t) > \varphi(x')\},$$

where $\partial_{Q_1'}$ denotes the boundary in the relative topology of Q_1'. For this problem, it is more convenient to study the *extended* free boundary, defined by

$$\overline{\Gamma}(u) := \partial_{Q_1'} \{(x', t) \in Q_1' : u(x', 0, t) = \varphi(x'),\ \partial_{x_{n+1}} u(x', 0, t) = 0\},$$

so that $\overline{\Gamma}(u) \supset \Gamma(u)$. This distinction, however, will not come into play in this work.

In order to study (6.57), one also needs to add some boundary condition on $(\partial B_1 \times (-1, 0]) \cap \{x_{n+1} > 0\}$. Instead of doing that, we will assume the additional hypothesis $u_t > 0$ on $(\partial B_1 \times (-1,0]) \cap \{x_{n+1} > 0\}$. That is, there is actually some time evolution, and it makes the solution grow. Recall that such hypothesis is (somewhat) necessary, and natural in some applications (see subsection 6.1.4).

Notice, also, that if $u_t > 0$ on the spatial boundary, by strong maximum principle applied to the caloric function u_t in $Q_1 \cap \{x_{n+1} > \frac{1}{2}\}$, we know that $u_t > c > 0$ for $x_{n+1} > \frac{1}{2}$. Thus, after dividing u by a constant, we may assume $c = 1$, and thus, our problem reads as

$$\left\{ \begin{array}{rcll} u_t - \Delta u &=& 0 & \text{in } \ Q_1^+ \times (-1, 0], \\ \min\{u - \varphi, \partial_{x_{n+1}} u\} &=& 0 & \text{on } \ Q_1', \\ u_t &>& 0 & \text{on } \ (\partial B_1 \times (-1, 0]) \cap \{x_{n+1} > 0\}, \\ u_t &\geq& 1 & \text{in } \ Q_1 \cap \{x_{n+1} > \frac{1}{2}\}. \end{array} \right. \tag{6.58}$$

In order to deal with the order of free boundary points, one requires the introduction of heavy notation, analogous to what has been presented in the elliptic case, but for the parabolic version. We will avoid that by focusing on the main property we require about the order of the extended free boundary points:

Definition 6.3. Let $(x_\circ, t_\circ) \in \overline{\Gamma}(u) \cap Q_{1-h}$ be an extended free boundary point. We define

$$\overline{u}^{x_\circ,t_\circ}(x,t) := u((x+x'_\circ, x_{n+1}), t+t_\circ) - \varphi(x'+x'_\circ) + Q_\tau^{x_\circ}(x') - Q_\tau^{x_\circ,0}(x', x_{n+1}),$$

where $Q_\tau^{x_\circ}$ is the Taylor polynomial of order τ of φ at $x_\circ$, and $Q_\tau^{x_\circ,0}$ is its harmonic extension to $\mathbb{R}^{n+1}$.

We say that $(x_\circ, t_\circ) \in \overline{\Gamma}(u) \cap Q_{1-h}$ is an extended free boundary point of order $\geq \kappa$, $(x_\circ, t_\circ) \in \Gamma_{\geq\kappa}$, where $2 \leq \kappa \leq \tau$, if

$$|\overline{u}^{x_\circ,t_\circ}| \leq Cr^\kappa \quad \text{in} \quad Q_r^+,$$

for all $r < \frac{h}{2}$, and for some constant C depending only on the solution u.

Notice that, in particular, the points of order greater or equal than κ as defined in [DGPT17] fulfil the previous definition. Notice, also, that we have denoted by $\Gamma_{\geq\kappa}$ the set of points of order $\geq \kappa$.

Thus, we can proceed to prove the following proposition, analogous to Proposition 6.11:

Proposition 6.30. *Let $h > 0$ small, and let $(x_\circ, t_\circ) \in Q_{1-h}^+ \cap \Gamma_{\geq\kappa}$ with $t_\circ < -h^2$, where $2 \leq \kappa \leq 3$. Then,*

$$u(\cdot, t_\circ + C_* t^{\kappa-1}) > \varphi \quad \textit{in} \quad B'_t(x'_\circ), \quad \textit{for all} \quad 0 < t < T_h,$$

for some constant C_* depending only on n, h, u, and T_h depending only on n, h, τ, κ, u.

Proof. Let us assume, for simplicity in the notation, that $x_\circ = 0$, and $t_\circ = -\frac{1}{2}$, and we denote $\overline{u} := \overline{u}^{0,-1/2}$. Notice that, by the parabolic Hopf Lemma, since $\overline{u}_t \geq 0$ in Q_1 and $\overline{u}_t \geq 1$ in $Q_1 \cap \{x_{n+1} \geq \frac{1}{2}\}$ we have that for some constant c and for any $\sigma > 0$,

$$\overline{u}_t \geq c\sigma \quad \text{in} \quad (B_{1/2}^+ \cap \{x_{n+1} \geq \sigma\}) \times [-1/2, 0].$$

Notice, also, that since $(0, -1/2) \in \mathbb{R}^{n+1} \times \mathbb{R}$ is an extended free boundary point of order $\geq \kappa$, we have that, for $r > 0$ small enough,

$$\overline{u}(\cdot, -1/2 + s) \geq \overline{u}(\cdot, -1/2) \geq -Cr^\kappa \quad \text{in} \quad B_r^+, \tag{6.59}$$

for $s \geq 0$ by the monotonicity of the solution in time.

On the other hand, since $\overline{u}_t \geq cr\sigma$ in $\{x_{n+1} \geq r\sigma\}$, we have that

$$\overline{u}(\cdot, -1/2 + s) \geq c(r\sigma)s + \overline{u}(\cdot, -1/2) \quad \text{in} \quad \{x_{n+1} \geq r\sigma\} \quad \text{for} \quad s \geq 0.$$

As in (6.59), this gives

$$\overline{u}(\cdot, -1/2 + s) \geq c(r\sigma)s - Cr^\kappa \quad \text{in} \quad \{x_{n+1} \geq r\sigma\} \cap B_r^+ \quad \text{for} \quad s \geq 0.$$

Let $w(y, \zeta) = \overline{u}(ry, -1/2 + r^2\zeta)$. Then we have that

$$w(y, \zeta) \geq -Cr^\kappa, \quad \text{for} \quad y \in B_1^+ \quad \text{for} \quad \zeta \geq 0,$$

and

$$w(y,\zeta) \geq c(r\sigma)r^2\zeta - Cr^\kappa, \quad \text{for} \quad y \in \{y_{n+1} \geq \sigma\} \cap B_1^+ \quad \text{for} \quad \zeta \geq 0.$$

Notice, also, that since

$$|(\partial_t - \Delta)\overline{u}| = o(r^{\tau-2}) \quad \text{in} \quad B_r^+,$$

then

$$|(\partial_\zeta - \Delta_y)w| = o(r^\tau) \quad \text{in} \quad B_1^+.$$

Considering now $\bar{w}(y,\zeta) := \frac{\sigma}{Cr^\kappa} w(y,\zeta)$, we have that

$$\bar{w}(y,\zeta) \geq -\sigma, \quad \text{for} \quad y \in B_1^+ \text{ and } \zeta \geq 0,$$

$$\bar{w}(y,\zeta) \geq cr^{3-\kappa}\sigma^2\zeta - \sigma, \quad \text{for} \quad y \in \{y_{n+1} \geq \sigma\} \cap B_1^+ \text{ and } \zeta \geq 0,$$

and

$$|(\partial_\zeta - \Delta_y)\bar{w}| \leq \sigma \quad \text{in} \quad B_1^+,$$

for $r > 0$ small enough. Let us take $\zeta = C_* r^{\kappa-3}$, for some C_* depending on n and σ such that $cr^{3-\kappa}\sigma^2\zeta - \sigma \geq 1$. Then, by [DGPT17, Lemma 11.5] (which is the parabolic version of Lemma 6.10 for $a = 0$), there exists some $\sigma_\circ > 0$ depending on n such that if $\sigma \leq \sigma_\circ$, then $\bar{w}(\cdot, C_* r^{\kappa-3}) > 0$ in $\overline{B_{1/2}^+}$. In particular, recalling the definition of $\bar{w}$, this yields the desired result. □

As in the elliptic case, the non-regular part of the free boundary is $\Gamma_{\geq 2}$ (see [DGPT17, Proposition 10.8]). Thanks to Proposition 6.30 we will obtain a bound on the dimension of $\Gamma_{\geq\kappa} \cap \{t = t_\circ\}$ for almost every time $t_\circ \in (-1, 0]$ if $\kappa > 2$. For the limiting case, $\kappa = 2$, one has to proceed differently, analogous to what has been done in the elliptic case.

Let us start by defining the set Γ_2. We say that a point $(x_\circ, t_\circ) \in \overline{\Gamma}(u) \cap Q_{1-h}^+$ belongs to Γ_2, $(x_\circ, t_\circ) \in \Gamma_2 \cap Q_{1-h}^+$, if parabolic blow-ups around that point converge uniformly to a parabolic 2-homogeneous polynomial.

Namely, consider a fixed test function $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset B_h$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ in $B_{h/2}$, and $\psi(x', x_{n+1}) = \psi(x', -x_{n+1})$. Then $u^{x_\circ,t_\circ}(x,t)\psi(x)$ can be considered to be defined in $\mathbb{R}_+^n \times (-h^2, 0]$, and we denote

$$H_u^{x_\circ,t_\circ}(r) := \frac{1}{r^2} \int_{-r^2}^0 \int_{\mathbb{R}_+^n} \bar{u}^{x_\circ,t_\circ}(x,t)\psi(x)G(x,t)\,dx\,dt,$$

where $G(x,t)$ is the backward heat kernel in $\mathbb{R}^{n+1} \times \mathbb{R}$,

$$G(x,t) = \begin{cases} (-4\pi t)^{-\frac{n+1}{2}} e^{\frac{|x|^2}{4t}} & \text{if } t < 0, \\ 0 & \text{if } t \geq 0. \end{cases}$$

We then define the rescalings

$$u_r^{x_\circ,t_\circ}(x,t) := \frac{\bar{u}^{x_\circ,t_\circ}(rx, r^2 t)}{H_u^{x_\circ,t_\circ}(r)^{1/2}}.$$

Then, we say that $(x_\circ, t_\circ) \in \Gamma_2$ if for every $r_j \downarrow 0$, there exists some subsequence $r_{j_k} \downarrow 0$ such that

$$u^{x_\circ,t_\circ}_{r_{j_k}} \to p_2^{x_\circ,t_\circ} \quad \text{uniformly in compact sets,}$$

for some parabolic 2-homogeneous caloric polynomial $p_2^{x_\circ,t_\circ} = p_2^{x_\circ,t_\circ}(x,t)$ (i.e., $p_2(\lambda x, \lambda^2 t) = \lambda^2 p_2(x,t)$ for $\lambda > 0$), which is a global solution to the parabolic Signorini problem. The existence of such polynomial, the uniqueness of the limit, and its properties, are shown in [DGPT17, Proposition 12.2, Lemma 12.3, Theorem 12.6]. Moreover, by the classification of free boundary points performed in [DGPT17] we know that

$$\Gamma(u) = \mathrm{Reg}(u) \cup \Gamma_{\geq 2}.$$

In addition, by [Shi18, Proposition 4.5] there are no free boundary points with frequency belonging to the interval $(2, 2+\alpha_\circ)$ for some $\alpha_\circ > 0$ depending only on n. Thus,

$$\Gamma(u) = \mathrm{Reg}(u) \cup \Gamma_2 \cup \Gamma_{\geq 2+\alpha_\circ}. \tag{6.60}$$

Proposition 6.31. *The set* Γ_2 *defined as above is such that*

$$\dim_{\mathcal{H}}(\Gamma_2 \cap \{t = t_\circ\}) \leq n-2, \quad \textit{for a.e.} \quad t_\circ \in (-1, 0].$$

Proof. We separate the proof into two steps.

Step 1. By [DGPT17, Theorem 12.6], we know that

$$\bar{u}^{x_\circ,t_\circ}(x,t) = p_2^{x_\circ,t_\circ}(x,t) + o(\|(x,t)\|^2),$$

where $\|(x,t)\| = (|x|^2 + |t|)^{1/2}$ is the parabolic norm. Here $p_2^{x_\circ,t_\circ}$ is a polynomial, parabolic 2-homogeneous global solution to the parabolic Signorini problem. In particular, it is at most linear in time. On the other, since $u_t \geq 0$ everywhere, the same occurs with the parabolic blow-up up, i.e., $p_2^{x_\circ,t_\circ}$ is non-decreasing in time. All this implies that $p_2^{x_\circ,t_\circ}$ is actually constant in time, so that we have that $p_2^{x_\circ,t_\circ} = p_2^{x_\circ,t_\circ}(x)$ is an harmonic, second-order polynomial in x, non-negative on the thin space $\{x_{n+1} = 0\}$, and we have

$$\bar{u}^{x_\circ,t_\circ}(x,t) = p_2^{x_\circ,t_\circ}(x) + o(\|(x,t)\|^2).$$

On the other hand, also from [DGPT17, Theorem 12.6], $\Gamma_2 \ni (x_\circ, t_\circ) \mapsto p_2^{x_\circ,t_\circ}$ is continuous. These last two conditions correspond to Proposition 6.16 and Proposition 6.17 from the elliptic case. In particular, one can apply Whitney's extension theorem as in Proposition 6.18 to obtain that the set

$$\pi_x\Gamma_2 := \{x \in \mathbb{R}^{n+1} : (x,t) \in \Gamma_2 \text{ for some } t \in (-1, 0]\},$$

is contained in the countable union of $(n-1)$-dimensional C^1 manifolds. That is,

$$\dim_{\mathcal{H}}(\pi_x\Gamma_2) \leq n-1,$$

$\pi_x\Gamma_2$ is $(n-1)$-dimensional.

Step 2. Thanks to Step 1, and by Proposition 6.30 with $\kappa = 2$, proceeding analogously to Theorem 6.21 by means of Lemma 6.19, we reach the desired result. □

Proposition 6.32. *Let $a > 0$. Then,*

$$\dim_{\mathcal{H}}(\Gamma_{\geq 2+a} \cap \{t = t_\circ\}) \leq n - 1 - a, \quad \textit{for a.e.} \quad t_\circ \in (-1, 0],$$

Proof. The result follows by Proposition 6.30 with $\kappa = 2+a$, proceeding analogously to Theorem 6.21 by means of Lemma 6.19. □

We can now give the proof of the main result regarding the parabolic Signorini problem.

Proof of Theorem 6.3. Is a direct consequence of (6.60), Proposition 6.31, and Proposition 6.32 with $a = \alpha_\circ$ depending only on n, given by [Shi18, Proposition 4.5]. The regularity of the free boundary follows from [DGPT17, Theorem 11.6]. □

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