Bounding self-dual L-functions: the Conrey–Iwaniec method revisited

Yongxiao Lin (EPFL)
a joint work with
Ramon Nunes (UFC/MPIM) and Zhi Qi (Zhejiang U.)

KSU Number Theory Seminar
February 16, 2022
Content

- The subconvexity problem
- Pioneering work by Conrey–Iwaniec
- Xiaoqing Li’s results revisited
- $q$-aspect case: a result of Blomer’s
Given $F$ an automorphic form on $\text{GL}_d$, let $\lambda_F(n)$ be its associated Hecke eigenvalues. Let

$$L(F, s) = \sum_{n \geq 1}^{\infty} \frac{\lambda_F(n)}{n^s}, \quad \Re s \gg 1$$

be the $L$-function attached to $F$. Define the completed $L$-function

$$\Lambda(F, s) := q_F^{s/2} \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s + \kappa_j}{2}\right)L(F, s),$$

where $q_F$ is the “arithmetic conductor” of $L(F, s)$, and $\kappa_j \in \mathbb{C}$ are local parameters of $L(F, s)$ at infinity.

This satisfies a functional equation

$$\Lambda(F, s) = \varepsilon(F) \Lambda(\tilde{F}, 1 - s),$$

where $\varepsilon(F)$ (of absolute value 1) is the root number and $\tilde{F}$ is the “dual” form of $F$, with $\lambda_{\tilde{F}}(n) = \overline{\lambda_F(n)}$. 
Analytic Conductor

Associated to $L(F, 1/2)$ one defines the “Analytic Conductor” (Iwaniec–Sarnak, 2000)

$$Q(F) := q_F \prod_{j=1}^{d} (1 + |s + \kappa_j|),$$

measuring the “complexity” of $L(F, 1/2)$ as $F$ varies.

Examples:

- if $F = \chi| \cdot |^it$, where $\chi : \text{Dirichlet character modulo } q$ and $| \cdot |^it : n \rightarrow n^it$, then
  $$Q(F) = q(1 + |t|).$$

- if $F$ is a cusp form on $\Gamma_0(q) \backslash \mathbb{H}$ of weight $k_F$ (if $F$ is holomorphic) or of Laplacian eigenvalue $1/4 + k_F^2$ (if $F$ is Maass), then
  $$Q(F) = q(1 + |k_F|^2).$$
The subconvexity problem

The Phragmén–Lindelöf principle $\Rightarrow$ Convexity bound:

$$L(F, 1/2) \ll Q(F)^{1/4+o(1)}.$$ 

GRH $\Rightarrow$ the Generalised Lindelöf hypothesis:

$$L(F, 1/2) \ll Q(F)^{\varepsilon}.$$ 

The subconvexity problem: Find an $\delta > 0$ such that

$$L(F, 1/2) \ll Q(F)^{1/4-\delta}.$$
Subconvexity on $\text{GL}_1$

- $t$-aspect (i.e., $F = | \cdot |^{|it|}$): Weyl (1922)
  \[ \zeta\left(\frac{1}{2} + it\right) \ll (1 + |t|)^{\frac{1}{6} + o(1)}. \]

  Subsequent works by many authors. Bourgain ($1/6 \rightarrow 13/84$).

- $q$-aspect: Burgess (1963), $\chi \mod q$ Dirichlet characters
  \[ L(\chi, \frac{1}{2} + it) \ll_t q^{3/16 + o(1)}; \]

  and Weyl-bound
  \[ L(\chi, \frac{1}{2}) \ll q^{1/6 + o(1)} \]

by Conrey–Iwaniec/Petrow–Young, Nelson, Balkanova–Frolenkov–Wu, etc.
The \( \text{GL}_2 \) case: twist aspect

Let \( \chi \mod q \) be Dirichlet characters, \( f \) be \( \text{GL}_2 \) automorphic forms, and let \( L(f \times \chi, s) = \sum_{n \geq 1} \lambda_f(n) \chi(n) n^{-s} \).

**Theorem (Duke–Friedlander–Iwaniec, Bykovskii, …)**

Let \( f \) be fixed \( \text{GL}_2 \) automorphic forms. Then

\[
L(f \times \chi, 1/2) \ll_f (q^2)^{1/4-\delta+\varepsilon}, \text{ for } \delta = 1/16.
\]

- The saving \( \delta = 1/16 \) represents the **Burgess**-type subconvex bounds:

\[
L(F, 1/2) \ll Q(F)^{1/4-1/16+\varepsilon},
\]

proving ground for new methods: Blomer–Harcos, Han Wu, Munshi, Aggarwal–Holowinsky–L.–Sun, etc.

- Best: \( \delta = 1/12 \) (**Weyl**-type), Conrey–Iwaniec/Petrow–Young:

\[
L(\chi, 1/2) \ll q^{1/4-1/12+\varepsilon}
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\[
0 \leq L(F \times \chi, 1/2) \ll (q^2)^{1/4-1/12+\varepsilon},
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by Nelson, Balkanova–Frolenkov–Wu over other number fields.

Why one cares?
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Let \( \chi \) mod \( q \) be Dirichlet characters, \( f \) be GL₂ automorphic forms, and let \( L(f \times \chi, s) = \sum_{n \geq 1} \lambda_f(n) \chi(n) n^{-s} \).

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Why one cares?
Let \( q \geq 1 \) be an integer. Then \( q = x^2 + y^2 + z^2, (x, y, z) \in \mathbb{Z}^3 \) (Legendre 1798; Gauss 1801) iff \( q \) is not of the form \( 4^k(8\ell - 1) \). Denote \( \mathcal{R}_q := \{(x, y, z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = q\} \), the set of representations.

Gauss showed that

\[
\#\mathcal{R}_q^* = \begin{cases} 
12h(-q) & \text{if } q \equiv 1, 2 \mod 4 \\
8h(-q) & \text{if } q \equiv 3 \mod 8,
\end{cases}
\]

where \( h(-q) \) is the class number of \( \mathbb{Q}(\sqrt{-q}) \).

Gauss plus Dirichlet’s class number formula
\[
h(-q) = \frac{w}{2\pi} L(\chi_{-q}, 1)q^{1/2}
\]
implies

\[
\#\mathcal{R}_q^* = \frac{12}{\pi} L(\chi_{-q}, 1)q^{1/2} \asymp q^{1/2 + o(1)}
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by appealing to Siegel’s theorem: \( L(\chi_{-q}, 1) \asymp q^{o(1)} \).
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Sum of three squares

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Equidistribution of representations

Deeper question: the distribution of representations.
Recall \( \mathcal{R}_q = \{(x, y, z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = q\} \). Then

\[
\frac{\mathcal{R}_q}{\sqrt{q}} = \left\{ \left( \frac{x}{\sqrt{q}}, \frac{y}{\sqrt{q}}, \frac{z}{\sqrt{q}} \right) : (x, y, z) \in \mathcal{R}_q \right\} \subset S^2 = \left\{ x_1^2 + x_2^2 + x_3^2 = 1 \right\}.
\]

**Question:** How do \( \vec{v} \in \frac{\mathcal{R}_q}{\sqrt{q}} \) distribute on \( S^2 \) as \( q \to \infty \)?

**Conjecture (Equidistribution of lattice points on the 2-sphere)**

Let \( q \) be such that \( q \not\equiv 0, 4, 7 \mod 8 \). Let

\[
\mu_q := \frac{1}{#\mathcal{R}_q} \sum_{\vec{v} \in \frac{\mathcal{R}_q}{\sqrt{q}}} \delta_{\vec{v}}.
\]

Then \( \mu_q \) weak * converges to \( \mu_{S^2} \), as \( q \to \infty \). Equivalently, for “nice” \( \Omega \subset S^2 \), \( \mu_q(\Omega) \to \mu_{S^2}(\Omega) \).
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Examples of Equidistribution of integer points on the 2-sphere:

Integer points on spheres of radii 101, 8011, 104851, respectively

Image credit: Ellenberg, Michel, and Venkatesh.

(screenshot from I. Petrow)
Equidistribution of integer points on $S^2$

**Linnik** (1950’s-60’s): Equidistribution holds if $q$ satisfies an extra congruence condition, by ergodic methods.

**Theorem (Duke, 1988; Golubeva–Fomenko, 1990)**

Let $q \to \infty$ be such that $q \not\equiv 0, 4, 7 \mod 8$. Then

$$\frac{1}{\# R_q} \sum_{\vec{v} \in \frac{R_q}{\sqrt{q}}} \varphi(\vec{v}) \to \int_{S^2} \varphi(y) d\mu_{S^2},$$

for every $\varphi \in C(S^2)$.

One such proof can be reduced to a subconvex bound of the $\chi$-twisted $GL_2$ $L$-functions: $L(f \times \chi_{-q}, 1/2) \ll_f q^{1/2-\eta'}$. 
**Equidistribution of integer points on \( S^2 \)**

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A fundamental work of Conrey–Iwaniec

Let $\chi \mod q$ be characters. C–I established

$$\frac{1}{q} \sum_{f_j \in B(q, \text{triv})} h(t_j)L(f_j \times \chi, 1/2)^3 + \frac{1}{q} \int_{-\infty}^{\infty} h(t)|L(\chi, 1/2 + it)|^6 dt \ll_{t_j} q^\varepsilon.$$

Rmk: $\text{Cond}(L(f_j \times \chi, 1/2)^3) = q^6$, $\text{size}(B(q, \text{triv})) = q$,

$$\Rightarrow \frac{\log(\text{conductor})}{\log(\text{family})} = 6.$$

Restricting to $\chi = \chi_q$ real and appealing to Lapid:

$$L(f_j \times \chi_q, 1/2) \geq 0,$$

C–I derived

**Theorem (Conrey–Iwaniec, 2000)**

Let $\chi_q \mod q$ be real characters with square-free conductor $q$.

$$L(f_j \times \chi_q, 1/2) \ll_{t_j} q^{1/3+\varepsilon}, \quad L(\chi_q, 1/2 + it) \ll_t q^{1/6+\varepsilon}.$$
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*Let \( \chi_q \mod q \) be real characters with square-free conductor \( q \).*

\[
L(f_j \times \chi_q, 1/2) \ll_t q^{1/3 + \varepsilon}, \quad L(\chi_q, 1/2 + it) \ll_t q^{1/6 + \varepsilon}.
\]
Remarks on the Weyl bound

The bound $L(\chi_q, 1/2) \ll q^{1/6+\varepsilon}$: first improvement of Burgess’s 1963 bound $L(\chi, 1/2) \ll q^{3/16+\varepsilon}$.

The Conrey–Iwaniec bounds were extended to

- hybrid $(\chi_q, t_j)$-aspect by Young;
- all characters $\chi \mod q$, all $q$, over $\mathbb{Q}$ by Petrow–Young;
- Hecke characters $\chi$ over number fields $K$ by P. Nelson and by Balkanova–Frolov–Wu (for $K$ totally real) independently.
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A Motohashi-type formula

A spectral identity is visible from the proof:

\[
\frac{1}{q} \sum_{f_j \in B(q)} h(t_j) L(f_j \times \chi, 1/2)^3 + \frac{1}{q} \int_{-\infty}^{\infty} h(t)|L(\chi, 1/2 + it)|^6 dt
\]

\[\sim MT + \frac{1}{q} \sum_{\psi \bmod q}^* g(\psi, \chi)|L(\psi, 1/2)|^4 \tilde{H}(h),\]

where

\[
g(\psi, \chi) = \frac{1}{q} \sum_{u, v(q)} \chi(u(v + 1))\overline{\chi}(v(u + 1))\psi(uv - 1).
\]

The next move of C–I is to ignore possible sign change from the arguments of \(g(\psi, \chi)\):

\[
RHS \ll \|g(\psi, \chi)\|_\infty \frac{1}{q} \sum_{\psi \bmod q}^* |L(\psi, 1/2)|^4
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and appeal to Deligne’s RH \(\|g(\psi, \chi)\|_\infty \ll 1\) (interpreted as a “trace function” modulo \(q, q\) primes).
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Remark II on the Weyl bound

To improve the Weyl-bound, one wants to detect cancellation from sign changes of $\psi$ in $g(\psi, \chi)$:

$$\sum_{\psi \mod q}^* g(\psi, \chi)|L(\psi, 1/2)|^4 \ll q^{1-\delta};$$

more generally, to prove

$$\sum_{\psi \mod q}^* g(\psi, \chi)|L(f \times \psi, 1/2)|^2 \ll_f q^{1-\delta},$$

is related to the mixing conjecture of Michel–Venkatesh.

If $g(\psi, \chi) \equiv 1$, M. Young (2011):

$$\sum_{\psi \mod q}^* |L(\psi, 1/2)|^4 = MT + O(q^{1-\delta}).$$

Kowalski–Michel–Sawin (2017):

$$\sum_{\psi \mod q}^* L(f \times \psi, 1/2)L(g \times \psi, 1/2) = MT_{f,g} + O(q^{1-\delta}).$$
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  \[
  \sum_{\psi \mod q}^* L(f \times \psi, 1/2)\overline{L(g \times \psi, 1/2)} = MT_{f,g} + O(q^{1-\delta}).
  \]
Recall Conrey–Iwaniec ($E_{\text{min}} = 1 \boxplus 1 \boxplus 1$):

$$\sum_{f_j \in B(q, \text{triv})} h(t_j)L(f_j \times \chi, 1/2)^3 + \int_{-\infty}^{\infty} h(t)|L(\chi, 1/2 + it)|^6 \, dt$$

$$= \sum_{f_j \in B(q, \text{triv})} h(t_j)L(E_{\text{min}} \times f_j \times \chi, 1/2) + \int_{\mathbb{R}} h(t)|L(E_{\text{min}} \times \chi, 1/2 + it)|^2 \ll q^{1+\epsilon}.$$ 

Xiaoqing Li replaced $E_{\text{min}}$ by $F \in \text{GL}_3$ cuspidal and obtained:

$$\sum_{|t_j - T| \leq M} h(t_j)L(F \times f_j, 1/2) + \int_{T-M}^{T+M} h(t)|L(F, 1/2 + it)|^2 \, dt \ll T^{1+\epsilon} M$$

under $M > T^{3/8}$; Lindelöf-on-average, according to Weyl’s law:

$$\# \{f_j : t_j \in [T - M, T + M]\} \asymp TM.$$ 

**Theorem (Xiaoqing Li, 2011)**

*Let $F$ be self-dual.*

$$L(F \times f_j, 1/2), |L(F, 1/2 + it)|^2 \ll_F t^{11/8+\epsilon} = t^{3/2 - 1/8 + \epsilon}.$$
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$$L(F \times f_j, 1/2), |L(F, 1/2 + it)|^2 \ll_F t^{11/8+\varepsilon} = t^{3/2-1/8+\varepsilon}.$$
Recall Conrey–Iwaniec ($E_{\min} = 1 \boxplus 1 \boxplus 1$):

$$\sum_{f_j \in B(q, \text{triv})} h(t_j) L(f_j \times \chi, 1/2)^3 + \int_{-\infty}^{\infty} h(t) |L(\chi, 1/2 + it)|^6 \, dt$$

$$= \sum_{f_j \in B(q, \text{triv})} h(t_j) L(E_{\min} \times f_j \times \chi, 1/2) + \int_{\mathbb{R}} h(t) |L(E_{\min} \times \chi, 1/2 + it)|^2 \ll q^{1+\varepsilon}.$$ 

Xiaoqing Li replaced $E_{\min}$ by $F \in \text{GL}_3$ cuspidal and obtained:

$$\sum_{|t_j - T| \leq M} h(t_j) L(F \times f_j, 1/2) + \int_{T-M}^{T+M} h(t) |L(F, 1/2 + it)|^2 \, dt \ll T^{1+\varepsilon} M$$

under $M > T^{3/8}$; Lindelöf-on-average, according to Weyl’s law:

$$\# \{f_j : t_j \in [T - M, T + M]\} \asymp TM.$$ 

**Theorem (Xiaoqing Li, 2011)**

Let $F$ be self-dual.

$$L(F \times f_j, 1/2), |L(F, 1/2 + it)|^2 \ll_F t^{11/8+\varepsilon} = t^{3/2-1/8+\varepsilon}.$$
Main result

We give an improvement over Li’s saving.

**Theorem (L.–Ramon Nunes–Zhi Qi, 2021+)**

It holds true that

$$
\sum_{|t_j - T| \leq M} h(t_j) L(F \times f_j, 1/2) + \int_{T-M}^{T+M} h(t) |L(F, 1/2 + it)|^2 dt
\ll T^{1+\varepsilon} M(1 + T^{1/4} / M^{5/4}).
$$

**Corollary**

Let $F$ be self-dual.

$$L(F \times f_j, 1/2), |L(F, 1/2 + it)|^2 \ll_F t^{6/5+\varepsilon}.$$

- Previous improvements: Mckee–Sun–Ye, Nunes, etc. The exponent $6/5$ is the natural limit of this method.
- $F$ not necc. self-dual: $|L(F, 1/2 + it)|^2 \ll t^{27/20+\varepsilon}$, Aggarwal (building on Munshi).
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Remarks: Motohashi-type formula

An underlying spectral identity

\[
\frac{1}{TM} \sum_{|t_j - T| \leq M} h(t_j) L(F \times f_j, 1/2) + \frac{1}{TM} \int_{T-M}^{T+M} h(t) |L(F, 1/2 + it)|^2 dt
\]

\[\leadsto L(F, 1) \tilde{H}(h) + \int_{-T/M}^{T/M} \tilde{h}(t) L(F, 1/2 + it) \zeta(1/2 - it) dt,\]

obtained independently (apart from localizing support of \(\tilde{h}(t)\)) by

- **Chung-Hang Kwan** (2021), by period integral approach (via Poincaré series);

- **Humphries–Khan** (forthcoming), via analytic continuation of Dirichlet series.

- Motohashi’s original formula (corresp. to \(F = 1 \boxplus 1 \boxplus 1\))

\[
\int_{\mathbb{R}} |\zeta(1/2 + it)|^4 g(t) dt
\]

\[\leadsto \sum_{t_j \in \mathcal{B}(1)} \tilde{g}(t_j) L(f_j, 1/2)^3 + \text{holo} + \int_{\mathbb{R}} \tilde{g}(t) |\zeta(1/2 + it)|^6 dt + \text{linear.}\]
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Sketch of our proof

We use classical approach (approximate functional equation, Kuznetsov, Voronoï, stationary phase). Some tools:

- **Approximate functional equation**

  \[
  L(F, 1/2) \approx \sum_{n \ll \sqrt{Q(F)}} \frac{\lambda_F(n)}{n^{1/2}} + \varepsilon(F) \sum_{n \ll \sqrt{Q(F)}} \frac{\lambda_F(n)}{n^{1/2}}.
  \]

- **Kuznetsov trace formula**

  \[
  \sum_{t_j} h(t_j) \lambda_j(n_1) \lambda_j(n_2) + (Eis)
  \]

  \[
  = \delta_{n_1, n_2} H + \sum \sum_{c=1}^{\infty} S(n_1, \pm n_2; c) \frac{4\pi}{c} H^\pm \left( \frac{4\pi \sqrt{n_1 n_2}}{c} \right).
  \]

- **GL_d(\mathbb{Z})-Voronoï summation:**

  \[
  \sum_{n \sim N} \frac{\lambda_F(n)}{\sqrt{n}} Kl_i(an; c) w \left( \frac{n}{N} \right) \approx \sum_{\tilde{n} \ll \frac{c^d}{N}} \frac{\lambda_F(\tilde{n})}{\sqrt{\tilde{n}}} Kl_{d-i}(\tilde{a} \tilde{n}; c) \hat{w} \left( \frac{\tilde{n}}{c^d / N} \right).
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\]
Proof sketch

Start from

$$\sum_{|t_j - T| \leq M} h(t_j)L(F \times f_j, 1/2) + \int_{T-M}^{T+M} h(t)|L(F, 1/2 + it)|^2dt.$$ 

**Proof steps**: approx functional eq + Kuznetsov + Voronoï + inverse Mellin + functional eq + large sieve ineq.

- **AFE gives**

  $$\sum_{|t_j - T| \leq M} h(t_j) \sum_{m^2 n \leq T^{3+\epsilon}} \frac{A_F(n, m) \lambda_j(n)}{(m^2 n)^{1/2}} + (Eis);$$

- **Kuznetsov gives**

  $$\sum_{n \leq T^{3+\epsilon}} \frac{A_F(n, 1)}{n^{1/2}} \left( MT \delta_{n,1} + \sum_{\pm} \sum_{c \geq 1} \frac{1}{c} S(n, \pm 1; c) B^\pm \left( \frac{4\pi \sqrt{n}}{c} \right) \right);$$

- **Voronoï** transforms the off-diagonal into
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Proof sketch (cont.)

\[
(\text{off}) = \sum_{c \geq 1} \frac{1}{c^2} \sum_{\tilde{n} \geq 1} A_F(1, \tilde{n}) e\left( \pm \frac{\tilde{n}}{c} \right) \mathcal{W}^\pm \left( \frac{N \tilde{n}}{c^3}; \frac{\sqrt{N}}{c} \right).
\]

- M. Young (stationary phase+inverse Mellin):

\[
\mathcal{W}^\pm \left( \frac{N \tilde{n}}{c^3}; \frac{\sqrt{N}}{c} \right) \approx \text{(factor)} \times e\left( \mp \frac{\tilde{n}}{c} \right) \int_{-T_M}^{T_M} \tilde{h}(t) \left( \frac{\tilde{n}}{c} \right)^{it} dt,
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cancelled out!

\[
\implies (\text{off}) = \text{(factor)} \times \int_{-T_M}^{T_M} \tilde{h}(t) \sum_{c \ll T^{1/2}} \frac{1}{c^{1/2+it}} \sum_{\tilde{n} \ll T^{3/2}} A_F(1, \tilde{n}) \tilde{n}^{1/2-it} dt.
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- Functional equation in the \(\tilde{n}\)-variable if \(\tilde{n}\)-sum is long;

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\[
(\text{int}) \ll \left( \int_{-T_M}^{T_M} \left| \sum_{c \ll T^{1/2}} \frac{1}{c^{1/2+it}} \right|^2 dt \right)^{1/2} \left( \int_{-T_M}^{T_M} \left| \sum_{\tilde{n} \ll (T_M)^{3/2}} A_F(1, \tilde{n}) \tilde{n}^{1/2-it} \right|^2 dt \right)^{1/2}.
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GL₃: q-aspect case

Let \( \chi \mod q \) be Dirichlet characters. Let \( F \in \text{GL}_3 \) be a fixed cusp form.

By a similar method, Blomer obtained

**Theorem (Blomer, 2012)**

For \( F \) self-dual and \( \chi_q \) real characters, we have

\[
L(F \times f_j \times \chi_q, 1/2), L(F \times \chi_q, 1/2)^2 \ll_{F, f_j} q^{\frac{5}{4}+\varepsilon} = q^{3/2-1/4+\varepsilon}.
\]

- The exponent \( 5/4 < 11/8 \) (Li’s bound), due to the use of large sieve ineq. in place of second Voronoï in the later case.
- extended to \( F \) and \( \chi \) not necessarily self-dual by Munshi and Sharma respectively, by a \( \delta \)-method approach.
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Comparison: Blomer vs Conrey–Iwaniec

Conrey–Iwaniec ($F = 1 \boxplus 1 \boxplus 1$):

$$\frac{1}{q} \sum_{f_j \in B(q, \text{triv})} h(t_j) L(f_j \times \chi_q, 1/2)^3 + \frac{1}{q} \int_{-\infty}^{\infty} h(t) |L(\chi_q, 1/2 + it)|^6 dt \ll q^\varepsilon;$$

Blomer:

$$\frac{1}{q} \sum_{f_j \in B(q, \text{triv})} h(t_j) L(F \times f_j \times \chi_q, 1/2) + \frac{1}{q} \int_{-\infty}^{\infty} h(t) |L(F \times \chi_q, 1/2 + it)|^2 dt$$

$$\sim MT + \frac{1}{q} \sum_{\psi \mod q}^* g(\psi, \chi) L(F \times \psi, 1/2) L(\overline{\psi}, 1/2) \widetilde{H}(h) \ll q^\varepsilon + q^{1/4+\varepsilon},$$

resulting from Cauchy–Schwarz

$$RHS \ll \frac{1}{q} \|g(\psi, \chi)\|_\infty \left( \sum_{\psi \mod q}^* |L(F \times \psi, 1/2)|^2 \sum_{\psi \mod q}^* |L(\overline{\psi}, 1/2)|^2 \right)^{1/2}$$

$$\ll \frac{1}{q} (q^{3/2} \cdot q)^{1/2}.$$

Recall

$$g(\psi, \chi) = \frac{1}{q} \sum_{u, \nu(q)} \chi(u \nu + 1) \overline{\chi}(\nu(u + 1)) \psi(\nu \nu - 1) \ll 1.$$
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\frac{1}{q} \sum_{f_j \in B(q, \text{triv})} h(t_j)L(f_j \times \chi_q, 1/2)^3 + \frac{1}{q} \int_{-\infty}^{\infty} h(t)|L(\chi_q, 1/2 + it)|^6 \, dt \ll q^\varepsilon;
\]

Blomer:

\[
\frac{1}{q} \sum_{f_j \in B(q, \text{triv})} h(t_j)L(F \times f_j \times \chi_q, 1/2) + \frac{1}{q} \int_{-\infty}^{\infty} h(t)|L(F \times \chi_q, 1/2 + it)|^2 \, dt
\]

\[
\sim MT + \frac{1}{q} \sum_{\psi \mod q}^* g(\psi, \chi)L(F \times \psi, 1/2)L(\overline{\psi}, 1/2)\tilde{H}(h) \ll q^\varepsilon + q^{1/4+\varepsilon},
\]

resulting from Cauchy–Schwarz

\[
RHS \ll \frac{1}{q} \|g(\psi, \chi)\|_{\infty} \left( \sum_{\psi \mod q}^* |L(F \times \psi, 1/2)|^2 \sum_{\psi \mod q}^* |L(\overline{\psi}, 1/2)|^2 \right)^{1/2}
\]

\[
\ll \frac{1}{q} \left( q^{3/2} \cdot q \right)^{1/2}.
\]

Recall

\[
g(\psi, \chi) = \frac{1}{q} \sum_{u, \nu(q)} \chi(u(\nu + 1))\overline{\chi}(\nu(u + 1))\psi(\nu u - 1) \ll 1.
\]
Comparison: Blomer vs Conrey–Iwaniec

Conrey–Iwaniec ($F = 1 \boxplus 1 \boxplus 1$):

\[
\frac{1}{q} \sum_{f_j \in B(q, \text{triv})} h(t_j) L(f_j \times \chi_q, 1/2)^3 + \frac{1}{q} \int_{-\infty}^{\infty} h(t) |L(\chi_q, 1/2 + it)|^6 dt \ll q^\varepsilon;
\]

Blomer:

\[
\frac{1}{q} \sum_{f_j \in B(q, \text{triv})} h(t_j) L(F \times f_j \times \chi_q, 1/2) + \frac{1}{q} \int_{-\infty}^{\infty} h(t) |L(F \times \chi_q, 1/2 + it)|^2 dt
\]

\[
\sim MT + \frac{1}{q} \sum_{\psi \mod q}^* g(\psi, \chi) L(F \times \psi, 1/2) L(\overline{\psi}, 1/2) \tilde{H}(h) \ll q^\varepsilon + q^{1/4+\varepsilon},
\]

resulting from Cauchy–Schwarz

\[
RHS \ll \frac{1}{q} \sum_{\psi \mod q}^* |L(F \times \psi, 1/2)|^2 \sum_{\psi \mod q}^* |L(\overline{\psi}, 1/2)|^2 \right)^{1/2}
\]

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\ll \frac{1}{q} \left(q^{3/2} \cdot q\right)^{1/2}.
\]

Recall

\[
g(\psi, \chi) = \frac{1}{q} \sum_{u, \nu(q)} \chi(u(\nu + 1)) \overline{\chi}(\nu(u + 1)) \psi(u\nu - 1) \ll 1.
\]
Remarks on Blomer

To improve Blomer for \( q \) primes, one can try to improve

\[
\sum_{\psi \text{ mod } q}^* |L(F \times \psi, 1/2)|^2 \ll F q^{3/2};
\]

or (if ambitious) to improve

\[
\sum_{\psi \text{ mod } q}^* g(\psi, \chi) L(F \times \psi, 1/2) L(\overline{\psi}, 1/2) \ll q^{5/4},
\]

seems difficult! e.g. We do not know how to study

\[
\sum_{\psi \text{ mod } q}^* L(F \times \psi, 1/2) L(\overline{\psi}, 1/2),
\]

not even

\[
\sum q \sim Q \sum_{\psi \text{ mod } q}^* L(F \times \psi, 1/2) L(\overline{\psi}, 1/2) !
\]

We tried the case \( q \) composite, then...
Remarks on Blomer

To improve Blomer for $q$ primes, one can try to improve

$$\sum_{\psi \mod q}^* |L(F \times \psi, 1/2)|^2 \ll_F q^{3/2};$$

or (if ambitious) to improve

$$\sum_{\psi \mod q}^* g(\psi, \chi)L(F \times \psi, 1/2)L(\overline{\psi}, 1/2) \ll q^{5/4},$$

seems difficult! e.g. We do not know how to study

$$\sum_{\psi \mod q}^* L(F \times \psi, 1/2)L(\overline{\psi}, 1/2),$$

not even

$$\sum_{q \sim Q} \sum_{\psi \mod q}^* L(F \times \psi, 1/2)L(\overline{\psi}, 1/2) !$$

We tried the case $q$ composite, then...
Let $F$ be self-dual. Let $q = q_1 q_2$. \(\exists \delta = \delta \left( \frac{\log q_1}{\log q_2} \right) > 0, \text{ s.t.} \)

\[
L(F \times f_j \times \chi, 1/2), \ |L(F \times \chi, 1/2)|^2 \ll q^{5/4-\delta}.
\]

**Rmk:** The strongest saving is when $q_1 \asymp q^{1/5}, q_2 \asymp q^{4/5}$, then

\[
L(\cdots, 1/2) \ll q^{6/5+\varepsilon},
\]

consistent with the $t$-aspect case.