Bounding self-dual L-functions: the Conrey–Iwaniec method revisited

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a joint work with
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Content

- The subconvexity problem
- Pioneering work by Conrey–Iwaniec
- Xiaoqing Li's results revisited
- q-aspect case: a result of Blomer's

L-functions

Given F an automorphic form on $\mathrm{GL_d}$, let $\lambda_F(n)$ be its associated Hecke eigenvalues. Let

$$L(F,s) = \sum_{n\geq 1}^{\infty} \frac{\lambda_F(n)}{n^s}, \ \Re s \gg 1$$

be the L-function attached to F. Define the completed L-function

$$\Lambda(F,s) := q_F^{\frac{s}{2}} \pi^{-\frac{ds}{2}} \prod_{j=1}^{d} \Gamma(\frac{s+\kappa_j}{2}) L(F,s),$$

where q_F is the "arithmetic conductor" of $L(F,s)$, and $\kappa_j \in \mathbb{C}$ are local parameters of $L(F,s)$ at infinity.
This satisfies a functional equation

$$\Lambda(F,s) = \varepsilon(F)\, \Lambda(\widetilde{F}, 1-s),$$

where $\varepsilon(F)$ (of absolute value 1) is the root number and $\widetilde{F}$ is the "dual" form of F, with $\lambda_{\widetilde{F}}(n) = \overline{\lambda_F(n)}$.

Analytic Conductor

Associated to $L(F, 1/2)$ one defines the "*Analytic Conductor*" (Iwaniec–Sarnak, 2000)

$$Q(F) := q_F \prod_{j=1}^{d} (1 + |s + \kappa_j|),$$

measuring the "complexity" of $L(F, 1/2)$ as F varies.
Examples:

- if $F = \chi |\cdot|^{it}$, where χ : Dirichlet character modulo q and $|\cdot|^{it} : n \to n^{it}$, then

$$Q(F) = q(1 + |t|).$$

- if F is a cusp form on $\Gamma_0(q)\backslash\mathbb{H}$ of weight k_F (if F is holomorphic) or of Laplacian eigenvalue $1/4 + k_F^2$ (if F is Maass), then

$$Q(F) = q(1 + |k_F|^2).$$

The subconvexity problem

The Phragmén–Lindelöf principle $\Rightarrow$ Convexity bound:

$$L(F, 1/2) \ll Q(F)^{1/4+o(1)}.$$

GRH $\Rightarrow$ the Generalised Lindelöf hypothesis:

$$L(F, 1/2) \ll Q(F)^{\varepsilon}.$$

The subconvexity problem: Find an $\delta > 0$ such that

$$L(F, 1/2) \ll Q(F)^{1/4-\delta}.$$

Subconvexity on GL_1

- t-aspect (i.e., $F = |\cdot|^{it}$): Weyl (1922)

$$\zeta\big(1/2 + it\big) \ll (1 + |t|)^{\frac{1}{6}+o(1)}.$$

Subsequent works by many authors. Bourgain ($1/6 \to 13/84$).

- q-aspect: Burgess (1963), χ mod q Dirichlet characters

$$L(\chi, 1/2 + it) \ll_t q^{3/16+o(1)};$$

and *Weyl-bound*

$$L(\chi, 1/2) \ll q^{1/6+o(1)}$$

by Conrey–Iwaniec/Petrow–Young, Nelson, Balkanova–Frolenkov–Wu, etc.

The GL_2 case: twist aspect

Let χ mod q be Dirichlet characters, f be GL_2 automorphic forms, and let $L(f \times \chi, s) = \sum_{n \geq 1} \lambda_f(n)\chi(n)n^{-s}$.

Theorem (Duke–Friedlander–Iwaniec, Bykovskiĭ, ...)

Let f be fixed GL_2 automorphic forms. Then

$$L(f \times \chi, 1/2) \ll_f (q^2)^{1/4-\delta+\varepsilon}, \text{ for } \delta = 1/16.$$

- The saving $\delta = 1/16$ represents the **Burgess**-type subconvex bounds:

$$L(F, 1/2) \ll Q(F)^{1/4-1/16+\varepsilon};$$

  proving ground for new methods: Blomer–Harcos, Han Wu, Munshi, Aggarwal–Holowinsky–L.–Sun, etc.
- Best: $\delta = 1/12$ (**Weyl**-type), Conrey–Iwaniec/Petrow–Young:

$$L(\chi, 1/2) \ll q^{1/4-1/12+\varepsilon}$$

$$0 \leq L(F \times \chi, 1/2) \ll (q^2)^{1/4-1/12+\varepsilon};$$

  by Nelson, Balkanova–Frolenkov–Wu over other number fields.

Why one cares?

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Why one cares?

Sum of three squares

Let $q \geq 1$ be an integer. Then $q = x^2 + y^2 + z^2, (x, y, z) \in \mathbb{Z}^3$ (Legendre 1798; Gauss 1801) iff q is *not* of the form $4^k(8\ell - 1)$.
Denote $\mathcal{R}_q := \{(x, y, z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = q\}$, the set of representations.
Gauss showed that

$$\#\mathcal{R}_q^* = \begin{cases} 12h(-q) & \text{if } q \equiv 1, 2 \bmod 4 \\ 8h(-q) & \text{if } q \equiv 3 \bmod 8, \end{cases}$$

where $h(-q)$ is the class number of $\mathbb{Q}(\sqrt{-q})$.
Gauss plus Dirichlet's class number formula $h(-q) = \frac{w}{2\pi} L(\chi_{-q}, 1) q^{1/2}$ implies

$$\#\mathcal{R}_q^* = \frac{12}{\pi} L(\chi_{-q}, 1) q^{1/2} \asymp q^{1/2+o(1)}$$

by appealing to Siegel's theorem: $L(\chi_{-q}, 1) \asymp q^{o(1)}$.

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Equidistribution of representations

Deeper question: the *distribution* of representations.
Recall $\mathcal{R}_q = \{(x,y,z) \in \mathbb{Z}^3 : x^2+y^2+z^2 = q\}$. Then

$$\frac{\mathcal{R}_q}{\sqrt{q}} = \left\{\left(\frac{x}{\sqrt{q}}, \frac{y}{\sqrt{q}}, \frac{z}{\sqrt{q}}\right) : (x,y,z) \in \mathcal{R}_q\right\} \subset \mathbb{S}^2 = \{x_1^2+x_2^2+x_3^2 = 1\}.$$

Question: How do $\vec{v} \in \frac{\mathcal{R}_q}{\sqrt{q}}$ distribute on $\mathbb{S}^2$ as $q \to \infty$?

Conjecture (Equidistribution of lattice points on the 2-sphere)

Let q be such that $q \not\equiv 0,4,7 \bmod 8$. Let

$$\mu_q := \frac{1}{\#\mathcal{R}_q} \sum_{\vec{v} \in \frac{\mathcal{R}_q}{\sqrt{q}}} \delta_{\vec{v}}.$$

Then μ_q weak $\star$ converges to $\mu_{\mathbb{S}^2}$, as $q \to \infty$. Equivalently, for "nice" $\Omega \subset \mathbb{S}^2$, $\mu_q(\Omega) \to \mu_{\mathbb{S}^2}(\Omega)$.

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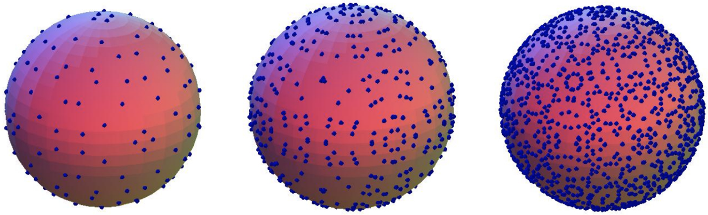
Integer points on $\mathbb{S}^2$

Examples of Equidistribution of integer points on the 2-sphere:



Integer points on spheres of radii $101, 8011, 104851$, respectively

Image credit: Ellenberg, Michel, and Venkatesh.

(screenshot from I. Petrow)

Equidistribution of integer points on $\mathbb{S}^2$

Linnik (1950's-60's): Equidistribution holds if q satisfies an extra congruence condition, by ergodic methods.

Theorem (Duke, 1988; Golubeva–Fomenko, 1990)

Let $q \to \infty$ be such that $q \not\equiv 0, 4, 7 \bmod 8$. Then

$$\frac{1}{\#\mathcal{R}_q} \sum_{\vec{v} \in \frac{\mathcal{R}_q}{\sqrt{q}}} \varphi(\vec{v}) \to \int_{\mathbb{S}^2} \varphi(y) \mathrm{d}\mu_{\mathbb{S}^2},$$

for every $\varphi \in C(\mathbb{S}^2)$.

One such proof can be reduced to a subconvex bound of the χ-twisted GL_2 L-functions: $L(f \times \chi_{-q}, 1/2) \ll_f q^{1/2-\eta'}$.

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A fundamental work of Conrey–Iwaniec

Let χ mod q be characters. C–I established

$$\frac{1}{q}\sum_{f_j\in\mathcal{B}(q,\mathrm{triv})} h(t_j)L(f_j\times\chi,1/2)^3+\frac{1}{q}\int_{-\infty}^{\infty}h(t)|L(\chi,1/2+it)|^6\mathrm{d}t \ll_{t_j} q^{\varepsilon}.$$

Rmk: $\mathrm{Cond}(L(f_j\times\chi,1/2)^3)=q^6$, $\mathrm{size}(\mathcal{B}(q,\mathrm{triv}))=q$,
$\Rightarrow \frac{\log(\mathrm{conductor})}{\log(\mathrm{family})}=6$.

Restricting to $\chi=\chi_q$ *real* and appealing to Lapid:

$$L(f_j\times\chi_q,1/2)\geq 0,$$

C–I derived

Theorem (Conrey–Iwaniec, 2000)

Let χ_q mod q be real characters with square-free conductor q.

$$L(f_j\times\chi_q,1/2)\ll_{t_j} q^{1/3+\varepsilon},\quad L(\chi_q,1/2+it)\ll_t q^{1/6+\varepsilon}.$$

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Remarks on the Weyl bound

The bound $L(\chi_q, 1/2) \ll q^{1/6+\varepsilon}$: **first** improvement of Burgess's 1963 bound $L(\chi, 1/2) \ll q^{3/16+\varepsilon}$.

The Conrey–Iwaniec bounds were extended to

- hybrid (χ_q, t_j)-aspect by Young;
- **all** characters χ mod q, all q, over $\mathbb{Q}$ by Petrow–Young;
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A Motohashi-type formula

A spectral identity is visible from the proof:

$$\frac{1}{q}\sum_{f_j\in\mathcal{B}(q)} h(t_j)L(f_j\times\chi,1/2)^3+\frac{1}{q}\int_{-\infty}^{\infty} h(t)|L(\chi,1/2+it)|^6\mathrm{d}t$$

$$\rightsquigarrow MT+\frac{1}{q}\sideset{}{^\star}\sum_{\psi \bmod q} g(\psi,\chi)|L(\psi,1/2)|^4\widetilde{H}(h),$$

where

$$g(\psi,\chi)=\frac{1}{q}\sum_{u,v(q)}\chi(u(v+1))\overline{\chi}(v(u+1))\psi(uv-1).$$

The next move of C–I is to ignore possible sign change from the arguments of $g(\psi,\chi)$:

$$RHS \ll \|g(\psi,\chi)\|_\infty\frac{1}{q}\sideset{}{^\star}\sum_{\psi \bmod q}|L(\psi,1/2)|^4$$

and appeal to Deligne's RH $\|g(\psi,\chi)\|_\infty\ll 1$ (interpreted as a "trace function" modulo q, q primes).

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- To improve the Weyl-bound, one wants to detect cancellation from sign changes of ψ in $g(\psi,\chi)$:

$$\sum_{\psi \bmod q}^{\star} g(\psi,\chi)|L(\psi,1/2)|^4 \ll q^{1-\delta};$$

  more generally, to prove

$$\sum_{\psi \bmod q}^{\star} g(\psi,\chi)|L(f\times\psi,1/2)|^2 \ll_f q^{1-\delta},$$

  is related to the *mixing conjecture* of Michel–Venkatesh.
- If $g(\psi,\chi)\equiv 1$, M. Young (2011):

$$\sum_{\psi \bmod q}^{\star} |L(\psi,1/2)|^4 = MT + O(q^{1-\delta}).$$

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GL_3: Xiaoqing Li's work

Recall Conrey–Iwaniec ($E_{\min} = 1 \boxplus 1 \boxplus 1$):

$$\sum_{f_j \in \mathcal{B}(q,\mathrm{triv})} h(t_j) L(f_j \times \chi, 1/2)^3 + \int_{-\infty}^{\infty} h(t) |L(\chi, 1/2+it)|^6 \mathrm{d}t$$

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Xiaoqing Li replaced $E_{\min}$ by $F \in \mathrm{GL}_3$ cuspidal and obtained:

$$\sum_{|t_j - T| \leq M} h(t_j) L(F \times f_j, 1/2) + \int_{T-M}^{T+M} h(t) |L(F, 1/2+it)|^2 \mathrm{d}t \ll T^{1+\varepsilon} M$$

under $M > T^{3/8}$; Lindelöf-on-average, according to Weyl's law:

$$\#\{f_j : t_j \in [T-M, T+M]\} \asymp TM.$$

Theorem (Xiaoqing Li, 2011)

Let F be self-dual.

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Let F be self-dual.

$$L(F \times f_j, 1/2),\ |L(F, 1/2 + it)|^2 \ll_F t^{11/8+\varepsilon} = t^{3/2-1/8+\varepsilon}.$$

Main result

We give an improvement over Li's saving.

Theorem (L.–Ramon Nunes–Zhi Qi, 2021+)

It holds true that

$$\sum_{|t_j - T| \leq M} h(t_j) L(F \times f_j, 1/2) + \int_{T-M}^{T+M} h(t) |L(F, 1/2 + it)|^2 \mathrm{d}t$$
$$\ll T^{1+\varepsilon} M (1 + T^{1/4}/M^{5/4}).$$

Corollary

Let F be self-dual.

$$L(F \times f_j, 1/2),\ |L(F, 1/2 + it)|^2 \ll_F t^{6/5+\varepsilon}.$$

- Previous improvements: Mckee–Sun–Ye, Nunes, etc. The exponent $6/5$ is the natural limit of this method.
- *F not necc.* self-dual: $|L(F, 1/2 + it)|^2 \ll t^{27/20+\varepsilon}$, Aggarwal (building on Munshi).

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Remarks: Motohashi-type formula

An underlying spectral identity

$$\frac{1}{TM}\sum_{|t_j-T|\le M} h(t_j)L(F\times f_j,1/2)+\frac{1}{TM}\int_{T-M}^{T+M} h(t)|L(F,1/2+it)|^2\mathrm{d}t$$

$$\leftrightsquigarrow L(F,1)\widetilde{H}(h)+\int_{-T/M}^{T/M}\widetilde{h}(t)L(F,1/2+it)\zeta(1/2-it)\mathrm{d}t,$$

obtained independently (apart from localizing support of $\widetilde{h}(t)$) by

- **Chung-Hang Kwan** (2021), by period integral approach (via Poincaré series);
- **Humphries–Khan** (forthcoming), via analytic continuation of Dirichlet series.
- Motohashi's original formula (corresp. to $F=1\boxplus 1\boxplus 1$)

$$\int_{\mathbb{R}}|\zeta(1/2+it)|^4 g(t)\mathrm{d}t$$

$$\leftrightsquigarrow \sum_{t_j\in\mathcal{B}(1)}\widetilde{g}(t_j)L(f_j,1/2)^3+\text{holo}+\int_{\mathbb{R}}\widetilde{g}(t)|\zeta(1/2+it)|^6\mathrm{d}t+\text{linear}.$$

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Sketch of our proof

We use classical approach (approximate functional equation, Kuznetsov, Voronoï, stationary phase). Some tools:

- *Approximate functional equation*

$$L(F,1/2) \approx \sum_{n \ll \sqrt{Q(F)}} \frac{\lambda_F(n)}{n^{1/2}} + \varepsilon(F) \sum_{n \ll \sqrt{Q(F)}} \frac{\overline{\lambda_F(n)}}{n^{1/2}}.$$

- *Kuznetsov trace formula*

$$\sum_{t_j} h(t_j)\lambda_j(n_1)\lambda_j(n_2) + (Eis)$$

$$= \delta_{n_1,n_2} H + \sum_{\pm} \sum_{c=1}^{\infty} \frac{S(n_1, \pm n_2; c)}{c} H^{\pm}\Big(\frac{4\pi\sqrt{n_1 n_2}}{c}\Big).$$

- $\mathrm{GL_d}(\mathbb{Z})$-*Voronoï summation*:

$$\sum_{n \sim N} \frac{\lambda_F(n)}{\sqrt{n}} Kl_i(an;c) w\Big(\frac{n}{N}\Big) \approx \sum_{\tilde{n} \ll \frac{c^d}{N}} \frac{\overline{\lambda_F(\tilde{n})}}{\sqrt{\tilde{n}}} Kl_{d-i}(\bar{a}\tilde{n};c)\widehat{w}\Big(\frac{\tilde{n}}{c^d/N}\Big).$$

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$$(\text{off}) = \sum_{c \geq 1} \frac{1}{c^2} \sum_{\tilde{n} \geq 1} A_F(1, \tilde{n}) e\Big(\pm \frac{\tilde{n}}{c}\Big) \mathcal{W}^{\pm}\Big(\frac{N\tilde{n}}{c^3}; \frac{\sqrt{N}}{c}\Big).$$

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GL_3: q-aspect case

Let χ mod q be Dirichlet characters. Let $F \in \mathrm{GL}_3$ be a *fixed* cusp form.
By a similar method, Blomer obtained

Theorem (Blomer, 2012)

For F self-dual and χ_q real characters, we have

$$L(F \times f_j \times \chi_q, 1/2),\ L(F \times \chi_q, 1/2)^2 \ll_{F,f_j} q^{\frac{5}{4}+\varepsilon} = q^{3/2-1/4+\varepsilon}.$$

- The exponent $5/4 < 11/8$ (Li's bound), due to the use of large sieve ineq. in place of second Voronoï in the later case.
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Comparison: Blomer vs Conrey–Iwaniec

Conrey–Iwaniec ($F = 1 \boxplus 1 \boxplus 1$):

$$\frac{1}{q} \sum_{f_j \in \mathcal{B}(q,\mathrm{triv})} h(t_j) L(f_j \times \chi_q, 1/2)^3 + \frac{1}{q} \int_{-\infty}^{\infty} h(t) |L(\chi_q, 1/2 + it)|^6 \mathrm{d}t \ll q^{\varepsilon};$$

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resulting from Cauchy–Schwarz

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$$g(\psi, \chi) = \frac{1}{q} \sum_{u,v(q)} \chi(u(v+1)) \overline{\chi}(v(u+1)) \psi(uv - 1) \ll 1.$$

Remarks on Blomer

To improve Blomer for q primes, one can try to improve

$$\sum_{\psi \bmod q}^{\star} |L(F\times\psi, 1/2)|^2 \ll_F q^{3/2};$$

or (if ambitious) to improve

$$\sum_{\psi \bmod q}^{\star} g(\psi,\chi) L(F\times\psi, 1/2) L(\overline{\psi}, 1/2) \ll q^{5/4},$$

seems **difficult**! e.g. We do not know how to study

$$\sum_{\psi \bmod q}^{\star} L(F\times\psi, 1/2) L(\overline{\psi}, 1/2),$$

not even

$$\sum_{q\sim Q} \sum_{\psi \bmod q}^{\star} L(F\times\psi, 1/2) L(\overline{\psi}, 1/2)\,!$$

We tried the case q composite, then...

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GL_3 revisited: q composite

Theorem (L.–Ramon Nunes, in progress)

Let F be self-dual. Let $q = q_1 q_2$. $\exists\, \delta = \delta(\frac{\log q_1}{\log q_2}) > 0$, s.t.

$$L(F \times f_j \times \chi, 1/2),\ |L(F \times \chi, 1/2)|^2 \ll q^{5/4-\delta}.$$

Rmk: The strongest saving is when $q_1 \asymp q^{1/5}, q_2 \asymp q^{4/5}$, then

$$L(\cdots, 1/2) \ll q^{6/5+\varepsilon},$$

consistent with the t-aspect case.